Belief Functions and Degrees of Non-conflictness

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Abstract. A hidden conflict of belief functions in the case where the sum of all multiples of conflicting belief masses being equal to zero was observed. To handle that, degrees of non-conflictness and full non-conflictness are defined. The family of these degrees of non-conflictness is analyzed, including its relation to full non-conflictness. Further, mutual non-conflictness between two belief functions accepting internal conflicts of individual belief functions are distinguished from global non-conflictness excluding both mutual conflict between belief functions and also all internal conflicts of individual belief functions. Finally, both theoretical and computational issues are presented.

Keywords: Belief functions · Dempster-Shafer theory · Uncertainty · Conflicting belief masses · Internal conflict · Conflict between belief functions · Hidden conflict · Degree of non-conflictness · Full non-conflictness

1 Introduction

When combining belief functions (BFs) by the conjunctive rules of combination, some conflicts often appear (they are assigned either to $\emptyset$ by non-normalised conjunctive rule $\textcircled{\cap}$ or distributed among other belief masses by normalization in Dempster's rule of combination $\oplus$). Combination of conflicting BFs and interpretation of their conflicts are often questionable in real applications.

Sum of all multiples of conflicting belief masses (denoted by $m_{\textcircled{\cap}}(\emptyset)$) was interpreted as a conflict between BFs in the classic Shafer's approach [19]. Nevertheless, non-conflicting BFs with high $m_{\textcircled{\cap}}(\emptyset)$ have been observed already in 90's examples. Classification of a conflict is very important in the combination of BFs from different belief sources. Thus a series of papers related to conflicts of BFs was published, e.g. [1,6,7,10,11,13–15,18,21].

Supported by grant GAČR no. 19-04579S.

G. Kern-Isberner and Z. Ognjanović (Eds.): ECSQARU 2019, LNAI 11726, pp. 125–136, 2019.
https://doi.org/10.1007/978-3-030-29765-7_11

A new interpretation of conflicts of belief functions was introduced in [4]: an important distinction of an internal conflict of individual BF (due to its inconsistency) from a conflict between two BFs (due to conflict/contradiction of evidence represented by the BFs). Note that zero-sum of all multiples of conflicting belief masses $m_{\textcircled{\cap}}(\emptyset)$ is usually considered as non-conflictness of the belief functions in all the above mentioned approaches.

On the other hand, when analyzing the conflict between BFs based on their non-conflicting parts[1] [7] a positive value of conflict was observed even in a situation when the sum of all multiples of conflicting belief masses equals to zero. The observed conflicts—hidden conflicts [9]—are against the generally accepted classification of BFs, i.e. to be either mutually conflicting or mutually non-conflicting. Above that, different "degrees" of non-conflictness were observed. This also arose a question of what is a sufficient condition for full non-conflictness of BFs.

Section 5 presents the entire family of "non-conflictness" of different degrees between $m_{\textcircled{\cap}}(\emptyset) = 0$ and a full non-conflictness. Results for both general BFs and special classes of BFs are included. Relations to other approaches to non-conflictness are analysed in Sect. 6. Further computational complexity and other computational aspects are presented in Sect. 7.

2 Preliminaries

We assume classic definitions of basic notions from theory of *belief functions* [19] on finite exhaustive frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is a *power-set* of Ω.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed.

A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, such that $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. Because there is a unique correspondence among m and corresponding Bel and Pl, we often speak about m as of a belief function.

A *focal element* is a subset of the frame of discernment $X \subseteq \Omega$, such that $m(X) > 0$; if $X \subsetneq \Omega$ then it is a *proper focal element*. If all focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function*; in fact, it is a probability distribution on Ω. If there are only focal elements such that $|X| = 1$ or $|X| = n$ we speak about *quasi-Bayesian BF*. In the case of $m(\Omega) = 1$ we speak about *vacuous BF* and about *a non-vacuous BF* otherwise. In the case of $m(X) = 1$ for $X \subset \Omega$ we speak about *categorical BF*. If all focal elements have a non-empty intersection, we speak about *a consistent BF*; and if all of them are nested, about *a consonant BF*.

Dempster's (normalized conjunctive) rule of combination $\oplus$: $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X)\ m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa =$

[1] Conflicting and non-conflicting parts of belief functions originally come from [5].

$\sum_{X \cap Y = \emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [19]. Putting $K = 1$ and $(m_1 \textcircled{\cap} m_2)(\emptyset) = \kappa = m_{\textcircled{\cap}}(\emptyset)$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$, see e.g. [20].

Smets' *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_i \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1-m(\emptyset)}$, see e.g. [20]. *Normalized plausibility of singletons*[2] of Bel is a probability distribution Pl_P such that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [2,3]. Sometimes we speak about *pignistic* and *plausibility transform* of respective BF.

3 Conflicts of Belief Functions

Original Shafer's definition of the conflict measure between two belief functions [19] is the following: $\kappa = \sum_{X \cap Y = \emptyset} m_1(X)m_2(Y) = (m' \textcircled{\cap} m'')(\emptyset) = m_{\textcircled{\cap}}(\emptyset)$, more precisely its transformation $log(1/(1-\kappa))$.

After several counter-examples, W. Liu's approach [14] appeared in 2006 followed by a series of other approaches and their modifications. W. Liu suggested a two-dimensional conflict measure composed from $m_{\textcircled{\cap}}(\emptyset)$ and $DifBetP_{m_j}^{m_i}$—a maximal difference of $BetP(\omega)$ for m_i, m_j over singletons $\omega \in \Omega$ (as kind of a distance); as it was shown, neither $m_{\textcircled{\cap}}(\emptyset)$ nor any distance of BFs alone may be used as a convenient measure of conflict of BFs.

Further, we have to mention two axiomatic approaches to conflict of BFs by Desterke and Burger [11] and by Martin [15]. In 2010, Daniel distinguished internal conflict inside an individual BF from the conflict between them [4] and defined three new approaches to conflict; the most prospective of them - *plausibility conflict* - was further elaborated in [6,10]. Finally, Daniel's *conflict based on non-conflicting parts of BFs* was introduced in [7]. This last-mentioned measure motivated our research of hidden conflict [9], hidden auto-conflict [8] and also current research of degrees of non-conflictness.

Among the other approaches, we can mention e.g. Burger's geometric approach [1].

A conflict of BFs Bel', Bel'' based on their non-conflicting parts Bel'_0, Bel''_0 is defined by the expression $Conf(Bel', Bel'') = (m'_0 \textcircled{\cap} m''_0)(\emptyset)$, where non-conflicting part Bel_0 (of a BF Bel) is unique consonant BF such that $Pl\_P_0 = Pl\_P$ (normalized plausibility of singletons corresponding to Bel_0 is the same as that corresponding to Bel); m_0 is a bba related to Bel_0. For an algorithm to compute Bel_0 see [7].

This measure of conflict analogously to Daniel's approaches from [4] does not include internal conflict of individual BFs in conflict between them. Similarly to plausibility conflict, it respects plausibilities equivalent to the BFs; and it better generalises the original idea to general frame of discernment.

[2] Plausibility of singletons is called *contour function* by Shafer in [19], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

4 Hidden Conflict

Example 1. **Introductory example**: Let us assume two simple consistent belief functions Bel' and Bel'' on $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$.

For the better understanding of the problem, see Fig. 1: The only focal element of m'' has a non-empty intersection with both focal elements of m', thus $\sum_{(X \cap Y)=\emptyset} m'(X)m''(Y) = (m' \textcircled{\scriptsize$\cap$} m'')(\emptyset)$ is an empty sum. Considering the conflict based on non-conflicting parts, respective consonant BFs with the same plausibility transform has to be found. Because Bel'' is consonant then $Bel''_0 = Bel''$, $m''_0 = m''$. In case of m' we can easily calculate that $Pl'(\{\omega_1\}) = 1$, $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$, thus $m'_0(\{\omega_1\}) = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = 0.2$, $m'_0(\{\omega_1, \omega_2, \omega_3\}) = 0.4$, hence $Conf(Bel', Bel'') = (m'_0 \textcircled{\scriptsize$\cap$} m''_0)(\emptyset) = m'_0(\{\omega_1\}) \cdot m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1 = 0.4$. Let us recall that the computational algorithm has been published in [7]—we are not putting it here because of the lack of space.

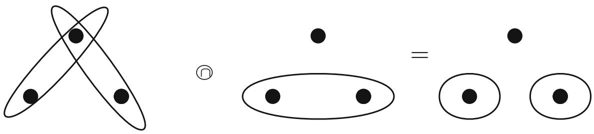

Fig. 1. Introductory Example: focal elements of m', m'', and of $m' \textcircled{\scriptsize$\cap$} m''$.

Then $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$. This seems—and it is usually considered—to be a proof of non-conflictness of m' and m''. Nevertheless, the conflict based on non-conflicting parts $Conf(Bel', Bel'') = (m'_0 \textcircled{\scriptsize$\cap$} m''_0)(\emptyset) = 0.4 > 0$ (which holds true despite of Theorem 4 from [7] which should be revised in future).

Observation of a Hidden Conflict in Example 1

The following questions arise: Does $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ represent non-conflictness of respective BFs as it is usually assumed? Is the definition of conflict based on non-conflicting parts correct? Are m' and m'' conflicting? What does $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ mean?

For the moment, suppose that Bel' and Bel'' are non-conflicting. Thus both of them should be non-conflicting with the result of their combination as well. Does it hold for BFs from Example 1? It does if one combines $m' \textcircled{\scriptsize$\cap$} m''$ with m'' one more time (assuming two instances of m'' coming from two independent belief sources). It follows from the idempotency of categorical m'': $m' \textcircled{\scriptsize$\cap$} m'' \textcircled{\scriptsize$\cap$} m'' = m' \textcircled{\scriptsize$\cap$} m''$ and therefore $(m' \textcircled{\scriptsize$\cap$} m'' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ again. On the other hand, we obtain positive $(m' \textcircled{\scriptsize$\cap$} m'' \textcircled{\scriptsize$\cap$} m')(\emptyset) = (m' \textcircled{\scriptsize$\cap$} m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0.48$ (assuming m' coming from two independent belief sources again). See Table 1 and Fig. 2. When m'' and m' are combined once, then we observe $m_{\textcircled{\scriptsize$\cap$}}(\emptyset) = 0$. When combining m'' with m' twice then $m_{\textcircled{\scriptsize$\cap$}}(\emptyset) = 0.48$. We observe some kind of *a hidden* *conflict*. Moreover, because both individual BFs are consistent, there are no internal conflicts. Thus our hidden conflict is a *hidden conflict between the BFs* and we have an argument for correctness of positive value of $Conf(Bel', Bel'')$.

Table 1. Hidden conflict in the introductory example

X	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$	$\emptyset$
$m'(X)$	0.0	0.0	0.0	0.60	0.40	0.00	0.00	–
$m''(X)$	0.0	0.0	0.0	0.00	0.00	1.00	0.00	–
$(m' \textcircled{\cap} m'')(X)$	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m' \textcircled{\cap} m'' \textcircled{\cap} m'')(X)$	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m' \textcircled{\cap} m'' \textcircled{\cap} m')(X)$	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48
$(m' \textcircled{\cap} m'' \textcircled{\cap} m' \textcircled{\cap} m'')(X)$	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48

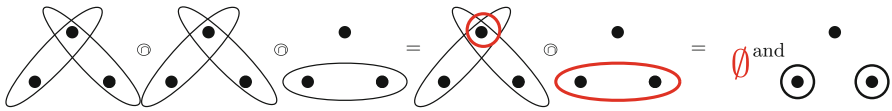

Fig. 2. Arising of a hidden conflict between BFs in the Introductory Example: focal elements of m', m', m''—$m' \textcircled{\cap} m', m''$ and of $(m' \textcircled{\cap} m') \textcircled{\cap} m''$.

What is a decisional interpretation of our BFs? Since *contours* (plausibilities of singletons) are $Pl' = (1.0, 0.6, 0.4)$ and $Pl'' = (0.0, 1.0, 1.0)$, then by normalization we obtain $Pl\_P' = (0.5, 0.3, 0.2)$ and $Pl\_P'' = (0.0, 0.5, 0.5)$. This can be interpreted in a way that ω_1 is significantly preferred by Bel', while it is the opposite in case of Bel''. This is also an argument for a positive value of mutual conflict of the BFs.

Note that in this special case, Smets' pignistic transform and plausibility transform lead to the same result. We obtain $BetP' = (0.5, 0.3, 0.2)$ and $BetP'' = (0.0, 0.5, 0.5)$. Both the probabilistic approximations $BetP$ and Pl_P (in general different) give the highest value to a different singleton for Bel' and Bel''. Thus the argument for mutual conflictness of the BFs is strengthened and we obtain the same pair of incompatible decisions based on the BFs in both frequent decisional approaches: using either normalized contour (which is compatible with the conjunctive combination of BFs) or pignistic probability (designed for betting).

Hence $(m' \textcircled{\cap} m'')(\emptyset)$ does not mean real non-conflictness of the BFs. It means simple or partial compatibility of their focal elements only. Or we can accept it as some weak version of non-conflictness.

5 Degrees of Non-conflictness

A case of a hidden conflict could be seen in the introductory example: Note that the example describes a situation when $(m' \odot m'')(\emptyset) = 0$ while $(m' \odot m' \odot m'' \odot m'')(\emptyset) > 0$. I.e. there is some type of non-conflictness, but weak as both $Conf(m', m'') > 0$ and $(m' \odot m' \odot m'' \odot m'')(\emptyset) > 0$.

Thus the following question arises now: Is $(m' \odot m' \odot m'' \odot m'')(\emptyset) = 0$ sufficient for full non-conflictness of belief functions? The answer is of course "no".

Example 2. **Little Angel example:** Assume for example the following bbas defined over $\Omega_5 = \{\omega_1, \ldots, \omega_5\}$—as described in Table 2 (the example and its title comes from [9], the title is inspired by graphical visualization of respective focal elements structure).

Table 2. Little Angel Example

X	$A = \{\omega_1, \omega_2, \omega_5\}$	$B = \{\omega_1, \omega_2, \omega_3, \omega_4\}$	$C = \{\omega_1, \omega_3, \omega_4, \omega_5\}$	$D = \{\omega_2, \omega_3, \omega_4, \omega_5\}$
$m'(X)$	0.10	0.30	0.60	0.00
$m''(X)$	0.00	0.00	0.00	1.00

Indeed, while we can observe both $(m' \odot m'')(\emptyset) = 0$ and $(m' \odot m'' \odot m' \odot m'')(\emptyset) = 0$ here, note that $(m' \odot m' \odot m' \odot m'' \odot m'' \odot m'')(\emptyset) = 0.108 > 0$, which witnesses some kind of a hidden conflict again. Nevertheless, one can feel that the *degree* of the non-conflictness is higher than in the case described by Example 1.

To make our findings more formal, note that due to associativity and commutativity of conjunctive combination rule $\odot$ we can write $(m' \odot m' \odot m' \odot m'' \odot m'' \odot m'')(\emptyset) = ((m' \odot m'') \odot (m' \odot m'') \odot (m' \odot m''))(\emptyset) = (\bigodot_{i=1}^{3} (m' \odot m''))(\emptyset)$. Thus, in case of Example 2, one can say that while $m_{\odot}(\emptyset) = (\bigodot_1^1 (m' \odot m''))(\emptyset) = (\bigodot_1^2 (m' \odot m''))(\emptyset) = 0$, there is $(\bigodot_1^3 (m' \odot m''))(\emptyset) = 0.108 > 0$. See Table 3.

Table 3. Hidden conflict in the Little Angel Example—Example 2

X	$A \cap D$	$B \cap D$	$C \cap D$	$A \cap B \cap D$	$A \cap C \cap D$	$B \cap C \cap D$	$\emptyset$
$(m' \odot m'')(X)$	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(\bigodot_1^2 (m' \odot m''))(X)$	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(\bigodot_1^3 (m' \odot m''))(X)$	0.001	0.027	0.216	0.036	0.126	0.486	0.108

Definition 1. *(i) Let Bel' and Bel'' be BFs defined by bbms m' and m''. We say that the BFs are non-conflicting in k-th degree if $(\bigodot_1^k (m' \odot m'')(\emptyset) = 0$.*
(ii) BFs Bel' and Bel'' are fully non-conflicting *if they are non-conflicting in any degree.*

Thus we can say that BFs from Table 2 are non-conflicting in the second degree, nevertheless, they are still conflicting in the third degree due to the observed hidden conflict.

Utilizing our results on hidden conflicts we obtain the following theorem.

Theorem 1. *Any two BFs on n-element frame of discernment Ω_n non-conflicting in the n-th degree are fully non-conflicting.*

Idea of the Proof: When combining two conflicting BFs defined over Ω_n repeatedly then, because of set intersection operator properties, we either obtain the least focal element of a cardinality lower than in the previous step, or a stable structure of focal elements as the least focal element is already contained in all others. Hence the empty set will appear as a focal element either in n steps or it will not appear at all.

The theorem offers an upper bound for a number of different degrees of non-conflictness of BFs. If a pair of BFs is non-conflicting in n-th degree then it is non-conflicting in any degree. Note that it is possible to find a pair of BFs non-conflicting in $(n-2)$-th degree but conflicting in $(n-1)$-th degree, as it is shown in the general example below.

Example 3. Assume n-element Ω_n and BFs m^i and m^{ii} are given by:

$m^i(\{\omega_1, \omega_2, ..., \omega_{n-1}\}) = \frac{1}{n-1}$,
$m^i(\{\omega_1, \omega_2, ..., \omega_{n-2}, \omega_n\}) = \frac{1}{n-1}$,
$m^i(\{\omega_1, \omega_2, ..., \omega_{n-3}, \omega_{n-1}, \omega_n\}) = \frac{1}{n-1}$,
$\ldots$,
$m^i(\{\omega_1, \omega_3, \omega_4, ..., \omega_n\}) = \frac{1}{n-1}$, and
$m^{ii}(\{\omega_2, \omega_3, ..., \omega_n\}) = 1$.

There is $(\textcircled{\cap}_1^k(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$ for $k \leq n-2$, $(\textcircled{\cap}_1^2(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0.5$ on Ω_3 and e.g. $(\textcircled{\cap}_1^{15}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 2.98 \cdot 10^{-6}$ on Ω_{16}.

Following the proof of Theorem 1, we can go further in the utilization of results on hidden conflicts and obtain the following theorem, which decreases the number of different degrees of BFs.

Theorem 2. *Any two non-vacuous BFs on any finite frame of discernment non-conflicting in degree c are fully non-conflicting for $c = min(c', c'') + |sgn(c' - c'')|$, where c', c'' are maximal cardinalities of proper focal elements of BFs Bel', Bel'' and $sgn()$ stands for signum.*

Idea of Proof. The smaller is the maximal cardinality of a proper focal element the faster an empty set—as a result of repeated combination of the BFs—may appear.

Corollary 1. *(i) There is only one degree of non-conflictness of any BFs on any two-element frame of discernment Ω_2. In the other words, all degrees of non-conflictness of BFs are equivalent on any two-element frame Ω_2.*

(ii) There is only one degree of non-conflictness of any quasi-Bayesian BFs on any finite frame of discernment Ω_n.
(iii) There are at most two different degrees of non-conflictness of a quasi-Bayesian BF an any other BF on any finite frame of discernment Ω_n.

6 Relation to Other Approaches to Non-conflictness

6.1 Degrees of Non-conflictness and $Conf = 0$.

We have described that there are $n-1$ different degrees of non-conflictness on Ω_n in the previous section. Besides that, we can observe also different types of non-conflictness. Note that $(m' \circledcirc m'')(\emptyset) = 0$ and $Conf(m', m'') > 0$ in both Examples 1 and 2. On the other hand, the opposite situation can be found—as follows:

Example 4. Let us recall W. Liu's Example 2 from [14] on Ω_5 where $m_i(\{\omega_j\}) = 0.2$ for $i = 1, 2$ and $j = 1, 2, \ldots, 5$ and $m_i(X) = 0$ otherwise (i.e. Bayesian bbas corresponding to uniform probability distributions). Note that while $Conf(Bel_1, Bel_2) = 0$, then $(m_1 \circledcirc m_2)(\emptyset) = 0.8$ and $(\circledcirc_1^k (m_1 \circledcirc m_2))(\emptyset) > 0.8$ for any $k > 1$. Specifically, $0.9922, 0.99968, \ldots$.

Example 5. Similarly, we can present more general example on frame Ω_n for an arbitrary $n \geq 3$ – see Table 4.

Table 4. BFs from Example 5

X	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1, \omega_2\}$	Ω	$\emptyset$
$m^i(X)$	0.4	0.2	0.2	0.2	–
$m^{ii}(X)$	0.3	0.1	0.1	0.5	–
$(m^i \circledcirc m^{ii})(X)$	0.48	0.18	0.14	0.10	0.10
$(\circledcirc_1^2 (m^i \circledcirc m^{ii}))(X)$	0.4608	0.1188	0.0476	0.0100	0.3628

Our $n-1$ degrees of non-conflictness are related to conjunctive combination of BFs, it covers general/global non-conflictness. If $(\circledcirc_1^k (m' \circledcirc m''))(\emptyset) = 0$ hold for any $k < n$ then there is neither internal conflict of any of individual BFs nor a mutual conflict between the two BFs. On the other hand, $Conf(m', m'') = 0$ is related only to mutual conflict between the BFs. Indeed, both the BFs in Example 4 are identical. There is no mutual conflict between them, but both of them are highly internally conflicting. Therefore there is also high conflict $(\circledcirc_1^k (m_1 \circledcirc m_2))(\emptyset)$ for any k.

In Example 5 (Table 4), there are two different BFs with the same order of bbms of proper focal elements. Their $\circledcirc$ combination has the same order of bbms as well. Thus, there is no mutual conflict between them, but, there is an internal conflict inside both of them. We can obtain analogous results also in the case when the internal conflict is hidden in only one of the BFs.

6.2 A Comparison of the Approaches

From the above examples, we can simply see that the 1-st degree of non-conflictness is not comparable with $Conf(m', m'') = 0$.

A relation of the other degrees of non-conflictness to $Conf(m', m'') = 0$ is an open issue for further investigation. We can only see that full non-conflictness is stronger than $Conf(m', m'') = 0$. This is nicely illustrated by the following theorem. We can also see the full non-conflictness is equivalent to strong non-conflictness and that the 1-st degree of non-conflictness is equivalent to non-conflictness both from Destercke & Burger approach [11]. A relation of $Conf(m', m'') = 0$ to Destercke & Burger approach is also an open problem for future.

Theorem 3. *(i) Non-conflictness of the 1-st degree is equivalent to Destercke-Burger non-conflictness ($(m_1 \copyright m_2)(\emptyset) = 0$, see [11]).*

(ii) Full non-conflictness is equivalent to Destercke-Burger strong non-conflictness (non-empty intersection of all focal elements of both BFs, see [11]).

(iii) If BFs m' and m'' are fully non-conflicting then $Conf(m', m'') = 0$ as well.

Idea of Proof:

(i) The first statement just follows the definition of the of the 1-st degree of non-conflictness.

(ii) Computing $\bigcirc\!\!\!\!\cap_1^n (m' \copyright m'')$, the intersection of all focal elements of both the BFs appears among the resulting focal elements.

(iii) The intersection of all focal elements of both the BFs is non-empty in the case of full non-conflictness. Thus the intersection of sets of elements with maximal plausibility is non-empty.

7 Computational Complexity and Computational Aspects

When looking for maximal degree of non-conflictness m of two BFs Bel^i and Bel^{ii} on general frame of discernment Ω_n we need to compute $\bigcirc\!\!\!\!\cap_1^m (m^i \copyright m^{ii})$. Following Theorem 1, we know that $m \leq n$. Based on this we obtain complexity $O(n)$ of $\copyright$ operations. Analogously to the case of complexity of looking for hidden conflict [9] we can reduce the complexity to $O(log_2(n))$ of $\copyright$ operations utilizing a simplification of computation based on $\bigcirc\!\!\!\!\cap_{j=1}^{2k} (m^i \copyright m^{ii}) = \bigcirc\!\!\!\!\cap_{j=1}^{k} (m^i \copyright m^{ii})$ $\copyright$ $\bigcirc\!\!\!\!\cap_{j=1}^{k} (m^i \copyright m^{ii})$. Note that the complexity of $\copyright$ operation depends on the number and the structure of focal elements. Utilizing Theorem 3 we can go further in reduction of computational complexity to $O(n)$ of intersection operations $\cap$.

Beside theoretical research of properties degrees of non-conflictness we have also performed a series of example computations on frames of discernment of cardinality from 5 to 16. A number of focal elements rapidly grows up to $|\mathcal{P}(\Omega)| = 2^{|\Omega|} - 1$ when conjunctive combination $\copyright$ is repeated. Note that there are 32.766 and 32.767 focal elements on Ω_{16} in Example 3. Because the conflictness/non-conflictness of BFs depends on the number and the structure of their focal elements not on their bbms, we have frequently used same bbms for all focal elements of BFs in our computations on frames of cardinality greater than 10.

All our experiments were performed in Language R [16] using R Studio [17]. We are currently developing an R package for dealing with belief functions on various frames of discernment. It is based on a relational database approach - nicely implemented in R, in a package called data.table [12].

8 An Important Remark

Repeated applications of the conjunctive combination $\textcircled{\cap}$ of a BF with itself is used here to simulate situations where different independent believers have numerically the same bbm. Thus this has nothing to do with idempotent belief combination (where, of course, no conflict between two BFs is possible).

Our study was motivated by the investigation of conflict $Conf$ of BFs based on their non-conflicting parts [7], thus we were interested in independent BFs when a hidden conflict was observed. But we have to note that conflictness/non-conflictness of BFs has nothing to do with dependence/independence of the BFs. Repeated computation of several (up to n) numerically identical BFs, when looking for hidden conflict is just a technical tool for computation of $m(\emptyset)$ or more precisely say for computation of $\kappa = \sum_{X \cap Y = \emptyset} m_j(X) m_j(Y)$. We are not interested in entire result of repeated application of $\textcircled{\cap}$, we are interested only in $m_{\textcircled{\cap}}(\emptyset)$ or, more precisely, in $\kappa = \sum_{X_1 \cap X_2 \cap \ldots \cap X_k = \emptyset} m_j(X_1)\ m_j(X_2) \ldots m_j(X_k)$. Thus our computation has nothing to do with any idempotent combination of BFs. We can look for non-conflictness of higher degrees using $\textcircled{\cap}_1^k$ (or κ) in the same way for both dependent and independent BFs. It is also not necessary to include any independence assumption in Definition 1.

9 Summary and Conclusion

Based on existence and observation of hidden conflicts (when the sum of all multiples of conflicting belief masses is zero) a family of degrees of non-conflictness has been observed. Number of non-equivalent/different degrees of non-conflictness depends on the size of the corresponding frame of discernment.

Maximal size of degrees of non-conflictness is $n - 1$ for belief functions on a general finite frame of discernment Ω_n. Nevertheless, for special types of BFs or for particular BFs, a size of the family may be reduced in accordance to the sizes of the focal elements of the BFs in question. The highest degree of non-conflictness (different from lower ones) is equivalent to full non-conflictness and also to strong non-conflictness defined by Destescke and Burger [11]. The family of non-conflictness is further compared with non-conflictness given by Daniel's $Conf(Bel^i, Bel^{ii}) = 0$ [7].

The presented approach to non-conflictness includes both the internal non-conflictness of individual BFs and also mutual non-conflictness between them.

Presented theoretical results move us to a better understanding of the nature of belief functions in general. Due to the important role of conflictness/non-conflictness of BFs within their combination, the presented results may consequently serve as a basis for a better combination of conflicting belief functions and better interpretation of the results of belief combination whenever conflicting belief functions appear in real applications.

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