Automatica 103 (2019) 159–165

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Optimal control problems with oscillations, concentrations and discontinuities[☆]

Didier Henrion[a,b,*], Martin Kružík[c,d], Tillmann Weisser[a]

[a] LAAS-CNRS, Université de Toulouse, France
[b] Faculty of Electrical Engineering, Czech Technical University in Prague, Czechia
[c] Czech Academy of Sciences, Institute of Information Theory and Automation, Prague, Czechia
[d] Faculty of Civil Engineering, Czech Technical University in Prague, Czechia

ARTICLE INFO

Article history:
Received 29 May 2018
Received in revised form 19 December 2018
Accepted 9 January 2019
Available online xxxx

Keywords:
Optimal control
Functional analysis
Optimization

ABSTRACT

Optimal control problems with oscillations (chattering controls) and concentrations (impulsive controls) can have integral performance criteria such that concentration of the control signal occurs at a discontinuity of the state signal. Techniques from functional analysis (anisotropic parametrized measures) are applied to give a precise meaning of the integral cost and to allow for the sound application of numerical methods. We show how this can be combined with the Lasserre hierarchy of semidefinite programming relaxations.

1. Introduction

As a consequence of optimality, various limit behaviors can be observed in optimal control: minimizing control law sequences may feature increasingly fast variations, called oscillations (chattering controls Young, 1969), or increasingly large values, called concentrations (impulsive controls Luenberger, 1969). The simultaneous presence of oscillations and concentrations in optimal control needs careful analysis and specific mathematical tools, so that the numerical methods behave correctly. Previous work of two of the authors (Claeys, Henrion, & Kružík, 2017) combined tools from partial differential equation analysis (DiPerna–Majda measures DiPerna & Majda, 1987) and semidefinite programming relaxations (the moment-sums-of-squares or Lasserre hierarchy Lasserre, Henrion, Prieur, & Trélat, 2008) to describe a sound numerical approach to optimal control in the simultaneous presence of oscillations and concentrations. To overcome difficulties in the analysis, a certain number of technical assumptions were made, see Claeys et al. (2017, Assumption 1, Section 2.2), so as to avoid the simultaneous presence of concentrations (in the control signals) and discontinuities (in the system trajectories).

In the present contribution we remove these technical assumptions and accommodate the simultaneous presence of concentrations and discontinuities, while allowing oscillations as well. For this, we exploit a recent extension of the notion of DiPerna–Majda measures called anisotropic parametrized measures (Kałamajska, Krömer, & Kružík, 2018), so that it makes sense mathematically while allowing for an efficient numerical implementation with semidefinite programming relaxations.

To motivate further our work, let us use an elementary example to illustrate the difficulties that may be faced in the presence of discontinuities and concentrations. Consider the optimal control problem

$$\begin{aligned} &\inf_u \int_0^1 (t+y(t))u(t)dt \\ &\text{s.t. } \dot{y}(t)=u(t), \quad y(0)=0, \quad y(1)=1, \\ &\quad 1\geq y(t)\geq 0, \quad u(t)\geq 0, \quad t\in[0,1] \end{aligned} \tag{1}$$

where the infimum is with respect to measurable controls of time. The trajectory y should move the state from zero at initial time to one at final time, yet for the non-negative integrand to be as small as possible, the control u should be zero all the time, except maybe at time zero. We can design a sequence of increasingly large controls u that drive y from zero to one increasingly fast. We observe that this sequence has no limit in the space of measurable functions but it tends (in a suitable weak sense) to the Dirac measure at

☆ This work was supported by a CNRS PICS project. The research of D. Henrion was also supported by the GAČR grant 16-19526S. The research of M. Kružík was also supported by the GAČRgrant 17-04301S. The research of T. Weisser was also supported by the ERC Advanced Grant Taming 666981. The material in this paper was partially presented at the 11th International Conference on Parametric Optimization and Related Topics (ParaOptXI), September 19–22, 2017, Prague, Czechia and the 23rd International Symposium on Mathematical Theory of Networks and Systems, July 16–20, 2018, Hong Kong, China. This paper was recommended for publication in revised form by Associate Editor Franco Blanchini under the direction of Editor Ian R. Petersen.

* Corresponding author.
E-mail addresses: henrion@laas.fr (D. Henrion), kruzik@utia.cas.cz (M. Kružík), tweisser@lanl.gov (T. Weisser).

https://doi.org/10.1016/j.automatica.2019.01.030
0005-1098/

time zero. We speak of control signal concentration or impulsive control. The integrand contains the product yu of a function whose limit becomes discontinuous at a point where the other function has no limit, hence requiring careful analysis. Here however, this product can be written $y\dot{y} = \frac{d}{dt}\frac{y^2}{2}$ and hence the integral term is well defined since $\int_0^1 y\dot{y}dt = \frac{y(1)^2-y(0)^2}{2} = \frac{1}{2}$. Consequently the cost in (1) is equal to $\int_0^1 tu(t)dt + \frac{1}{2}$ and independent of the actual trajectory.

This reasoning is valid because $\dot{y}(t) = u(t)$ in problem (1), but this integration trick cannot be carried out for more general differential equations. For example we cannot solve analytically the following modified optimal control problem

$$\begin{aligned} &\inf_u \int_0^1 (t+y(t))u(t)dt \\ &\text{s.t. } \dot{y}(t) = \sqrt{\varepsilon^2 + u^2(t)}, \quad y(0)=0, \quad y(1)=1, \\ &\quad 1 \geq y(t) \geq 0, \quad u(t) \geq 0, \quad t \in [0,1] \end{aligned} \tag{2}$$

where ε is a given real number. Providing a mathematically sound framework for the analysis of this kind of phenomenon combining concentration and discontinuity, and possibly also oscillation (not illustrated by the simple example above), is precisely the purpose of our paper.

Contribution

The contribution of our paper with respect to previous work can be summarized as follows:

- we propose a unified approach for handling the simultaneous presence of oscillations, concentrations and discontinuities, where previous work considered either oscillations without concentrations (see Fattorini, 1999; Gaitsgory & Quincampoix, 2009; Roubíček, 1997; Vinter, 1993 and references therein), or concentrations without oscillations (see Claeys, Arzelier, Henrion, & Lasserre, 2014 and references therein), or oscillations and concentrations without discontinuities (see Claeys et al., 2017 and references therein);
- we remove the technical assumptions of Claeys et al. (2017) to allow for the simultaneous presence of concentration (of the control) and discontinuity (of the trajectory);
- as in Claeys et al. (2017), our approach allows for a constructive solution via the Lasserre hierarchy (Lasserre et al., 2008); this now provides a unified numerical scheme to deal with oscillations, concentrations and discontinuities;
- we make a connection between anisotropic measures and the occupation measures, which are classical objects in dynamical systems and Markov decision processes, and which have been used in linear reformulations of nonlinear optimal control problems (Gaitsgory & Quincampoix, 2009; Lasserre et al., 2008; Vinter, 1993); the notion of occupation measure was extended in Claeys (2013), Claeys et al. (2014) to cope with concentration (also called impulsive controls); it was pointed out in Zidani (2013) that this extension allows for optimization over all possible graph completions, a tool introduced in Bressan and Rampazzo (1988) – see also Bressan and Piccoli (2007) – to deal with differential equations with discontinuous solutions. Anisotropic measures allow for a further generalization of these approaches.

Some of the results reported in our paper were communicated orally at the 11th International Conference on Parametric Optimization and Related Topics (ParaOptXI), Prague, Czechia, Sept. 19–22, 2017 and at the 23rd International Symposium on Mathematical Theory of Networks and Systems, Hong Kong, China, July 16–20, 2018. A full version of this paper with more detailed numerical examples is available as a LAAS-CNRS Technical Report under number 18218, on the HAL depository under number 01802883, and on the arXiv depository under number 1807.04199.

Outline

The outline of the paper is as follows. In Section 2 we describe the limit phenomena typical of optimal control, namely oscillations, concentrations and discontinuities, as well as the linear formulation of optimal control problems using measures. In Section 3 we introduce the anisotropic parametrized measures, illustrating their use with elementary examples. We show how these measures can cope with concentrations and discontinuities, giving a meaning to otherwise ill-defined integrals. In Section 4 we apply the anisotropic parametrized measures to optimal control, and in Section 5 we describe their relationship with occupation measures, a classical tool in dynamical systems and Markov decision processes. In Section 6 we describe how the Lasserre hierarchy can be applied to our problem, and in Section 7 we provide a simple illustrative example that can be solved numerically, and then analytically. Finally, concluding remarks are gathered in Section 8.

2. Relaxing optimal control

Let $L : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $F : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuous functions. For initial y_0 and final conditions y_1 in $\mathbb{R}^n$ and some integer $1 \leq p \leq \infty$, the formulation of the classical optimal control problem is

$$\begin{aligned} v^* := \inf_u &\int_0^1 L(t, y(t), u(t))dt \\ &\text{s.t. } \dot{y}(t) = F(t, y(t), u(t)), \quad y(0) = y_0, \quad y(1) = y_1, \\ &\quad y \in \mathscr{W}^{1,1}([0,1]; \mathbb{R}^n), \ u \in \mathscr{L}^p([0,1]; \mathbb{R}^m) \end{aligned} \tag{3}$$

where $\mathscr{W}^{1,p}([0,1]; X)$ is the space of functions from $[0,1]$ to X whose weak derivative belongs to $\mathscr{L}^p([0,1]; X)$, the space of functions from $[0,1]$ to X whose pth power is Lebesgue integrable.

A pair (u, y) with a control $u \in \mathscr{L}^p([0,1]; \mathbb{R}^m)$ and the corresponding trajectory $y \in \mathscr{W}^{1,1}([0,1]; \mathbb{R}^n)$ satisfying the differential equation of problem (3) is called admissible. Given a minimizing admissible sequence $(u_k, y_k)_{k\in\mathbb{N}}$, the infimum in (3) might not be attained because $(u_k)_{k\in\mathbb{N}}$ might not converge in $\mathscr{L}^p$ and $(y_k)_{k\in\mathbb{N}}$ might not converge in $\mathscr{W}^{1,1}$ as $\mathscr{L}^1$ is not reflexive. To overcome this issue, it has been proposed to relax the regularity assumptions on u and y. We discuss some of the approaches now in detail.

2.1. Oscillations

The limit of a minimizing sequence for (3) might fall out of the feasible space because of oscillation effects of $(u_k)_{k\in\mathbb{N}}$. Consider for example the optimal control problem

$$\begin{aligned} &\inf_u \int_0^1 (u(t)^2 - 1)^2 + y(t)^2 dt \\ &\text{s.t. } \dot{y}(t) = u(t), \ y(0) = 0, \ y(1) = 0, \\ &\quad y \in \mathscr{W}^{1,4}([0,1]), \quad u \in \mathscr{L}^4([0,1]). \end{aligned} \tag{4}$$

As the integrand in the cost is a sum of squares, the value is at least zero. To see that actually it is equal to zero, consider the sequence of controls $(u_k)_{k\in\mathbb{N}} \subseteq \mathscr{L}^4([0,1])$ equal to 1 if $t \in \left[\frac{2l+1}{2^k}, \frac{l+1}{2^{k-1}}\right]$ for $0 \leq l \leq k-1$ and equal to -1 otherwise, with $u_1 := 0$. For the corresponding sequence of trajectories $(y_k)_{k\in\mathbb{N}}$ defined by $y_k(t) := \int_0^t u_k(s)ds$ it holds that $y_k \in \mathscr{W}^{1,4}([0,1])$ and $y_k(1) = 0$ as desired. Hence, $(u_k)_{k\in\mathbb{N}}$ is a sequence of feasible controls. A short calculation shows that using this sequence the cost in (4) converges to zero. The sequence $(y_k)_{k\in\mathbb{N}}$ converges to $y_\infty := 0$ in $\mathscr{W}^{1,4}$, but the sequence $(u_k)_{k\in\mathbb{N}}$ does not converge to $u_\infty := 0$ in $\mathscr{L}^4$.

In contrast to that, the sequence of measures defined by $d\nu_k(t,u) := \delta_{u(t)}(du|t)dt$ converges weakly to $d\nu(t,u) := \frac{1}{2}(\delta_{-1} + \delta_1)(du)dt$

in the sense that for all $f \in \mathscr{C}([0,1])$ and $g \in \mathscr{C}_p(\mathbb{R})$, it holds $\lim_{k\to\infty}\int_0^1\int_{\mathbb{R}} f(t)g(u)d\nu_k(t,u) = \int_0^1\int_{\mathbb{R}} f(t)g(u)d\nu(t,u)$ where $\mathscr{C}_p(\mathbb{R}) := \{g \in \mathscr{C}(\mathbb{R}) : g(u) = o(|u|^p) \text{ for } |u| \to \infty\}$ is the set of continuous functions of less than pth growth. Integration then yields $y_\infty(1) = \int_0^1\int_{\mathbb{R}} u d\nu(t,u) = \int_0^1\int_{\mathbb{R}} u\frac{1}{2}(\delta_{-1}+\delta_1)(du)dt = 0$. A similar reasoning shows that the cost with respect to ν is zero.

More generally, this observation motivates to relax the regularity assumptions on the control u in (3) and also allow for limits $d\nu(t,u) = d\omega(u|t)dt$ of control sequences $(u_k)_{k\in\mathbb{N}} \subseteq \mathscr{L}^p([0,1];\mathbb{R}^m)$. In general the measure ω depends on time, i.e., we have a family of probability measures $\omega(.|t)_{t\in[0,1]} \subset \mathscr{P}(\mathbb{R}^m)$, where $\mathscr{P}(X)$ denotes the set of probability measures on X, i.e. non-negative Borel regular measures with unit mass. Such parametrized measures obtained as limits of a sequence of functions $(u_k)_{k\in\mathbb{N}} \subseteq \mathscr{L}^p([0,1];\mathbb{R}^m)$ have been called L^p-Young measures. For an explicit characterization of these measures see e.g. Kružík and Roubíček (1999). For a comprehensive reference on Young measures and their use in the control of ordinary and partial differential equations, see Fattorini (1999, Part III).

The relaxed version of (3) that now takes into account oscillating control sequences can be written as

$$\begin{aligned}\inf_{\omega}\ &\int_0^1\int_{\mathbb{R}^m} L(t,y(t),u)\,\omega(du|t)dt\\ &\text{s.t.} \int_0^1\int_{\mathbb{R}^m} F(t,y(t),u)\,\omega(du|t)dt = y_1 - y_0\\ &\quad y \in \mathscr{W}^{1,1}([0,1];\mathbb{R}^n),\ \ \omega(.|t) \in \mathscr{P}(\mathbb{R}^m)\end{aligned} \tag{5}$$

where the constraint is a reformulation of the differential equation $\dot y(t) = \int_{\mathbb{R}^m} F(t,y(t),u)\omega(du|t)$, $t \in [0,1]$ with the boundary conditions $y(0) = y_0$ and $y(1) = y_1$.

2.2. Concentrations

Oscillation of the control sequence due to nonconvexity of the functional is not the only reason that prevents the infimum in (3) of being attained. As a second example consider the following problem of optimal control:

$$\begin{aligned}\inf_u\ &\int_0^1 (t-\tfrac{1}{2})^2 u(t)dt\\ &\text{s.t. } \dot y(t) = u(t) \ge 0,\ \ y(0)=0,\ \ y(1)=1,\\ &\quad y \in \mathscr{W}^{1,1}([0,1]),\quad u \in \mathscr{L}^1([0,1]).\end{aligned} \tag{6}$$

Note that the control enters into the problem linearly. The value is zero as the integrand is positive and using the sequence of controls u_k equal to k if $t \in \left[\frac{k-1}{2k},\frac{k+1}{2k}\right]$ and to 0 otherwise, the cost converges to zero. Neither $(u_k)_{k\in\mathbb{N}}$ nor any subsequence converges in $\mathscr{L}^1$ as this space is not reflexive. In contrast to the previous example this time $(y_k)_{k\in\mathbb{N}}$ does not converge in $\mathscr{W}^{1,1}([0,1])$ neither because $\mathscr{W}^{1,1}$ is not reflexive. We hence use the extension $\mathscr{BV}([0,1])$, the space of functions with bounded variation, as a relaxed space for the trajectory. Following the same approach as before we consider the control as a measure $d\nu_k(t,u) := \delta_{u_k(t)}(du)dt$. As u appears linearly in (6) we can directly integrate with respect to u and define a sequence of probability measures $(\tau_k)_{k\in\mathbb{N}} \subseteq \mathscr{P}([0,1])$ by $\tau_k(dt) := \int_{\mathbb{R}} u d\nu_k(t,u)$. A short calculation shows that this sequence has the weak limit $\tau := \delta_{\frac{1}{2}}$, i.e. for all $f \in \mathscr{C}([0,1])$ it holds $\lim_{k\to\infty}\int_0^1 f(t)\tau_k(dt) = \int_0^1 f(t)\tau(dt)$. Note that by integrating before passing to the limit we transfer the unboundedness of the control into the measurement of time and only keep the direction (i.e. +1 in this example) of the control. Whereas we observed a superposition of two different controls in the previous example, here we see a concentration of the control in time. For optimal control problems with linear growth in the control:

$$\begin{aligned}\inf_u\ &\int_0^1 L(t,y(t))u(t)dt\\ &\text{s.t. } \dot y(t) = F(t,y(t))u(t),\quad y(0)=y_0,\quad y(1)=y_1,\\ &\quad y \in \mathscr{W}^{1,1}([0,1];\mathbb{R}^n),\quad u \in \mathscr{L}^1([0,1];\mathbb{R}^m)\end{aligned}$$

we can therefore build the following relaxation that can take into account concentration effects of the control:

$$\begin{aligned}\inf_\tau\ &\int_0^1 L(t,y(t))\tau(dt)\\ &\text{s.t.} \int_0^1 F(t,y(t))\tau(dt) = y_1 - y_0,\\ &\quad y \in \mathscr{BV}([0,1];\mathbb{R}^n),\quad \tau \in \mathscr{P}([0,1]).\end{aligned} \tag{7}$$

See Claeys et al. (2014) for an application of the Lasserre hierarchy for solving numerically non-linear control problems in the presence of concentration.

2.3. Oscillation and concentration

The relaxations proposed so far allow to consider controls that are either oscillating in value or concentrating in time. However it is possible that both effects appear in the same problem. Consider for example

$$\begin{aligned}\inf_u\ &\int_0^1 \frac{u(t)^2}{1+u(t)^4} + (y(t)-t)^2\,dt\\ &\text{s.t. } \dot y(t) = u(t) \ge 0,\ \ y(0)=0,\ \ y(1)=1,\\ &\quad y \in \mathscr{W}^{1,1}([0,1]),\quad u \in \mathscr{L}^1([0,1]).\end{aligned} \tag{8}$$

The infimum value zero of (8) can be approached arbitrarily close by a sequence of controls $(u_k)_{k\in\mathbb{N}}$ equal to k if $t \in \left[\frac{l}{k}-\frac{1}{2k^2},\frac{l}{k}+\frac{1}{2k^2}\right]$ for $1 \le l < k$, and equal to 0 otherwise, with $u_1 := 1$. The idea to capture the limit behavior of this sequence is to combine a Young measure on the control and replacing the uniform measure on time by a more general measure on time. Note that due to linearity it was possible in Section 2.2 to transfer the limit behavior of the control into the measurement of time. In the present example the control enters non-linearly in the cost, which is why we will need to allow the control to take values at infinity. We consider a metrizable compactification $\beta_{\mathcal{U}}\mathbb{R}$ of the control space corresponding to the ring $\mathcal{U}$ of complete and separable continuous functions (see Section 3.1 for more details). Then the sequence of measures $d\nu_k(t,u) := \delta_{u_k(t)}(du|t)dt$ converges to $d\nu(t,u) := \omega(du)\tau(dt)$ with $\omega(du) := \frac{1}{2}(\delta_0+\delta_\infty)(du)$ and $\tau(dt) := 2dt$ understood in the following weak sense for all $f \in \mathscr{C}([0,1])$ and $g_0 \in \mathcal{U}$:

$$\begin{aligned}\lim_{k\to\infty}&\int_0^1\int_{\mathbb{R}} f(t)g_0(u)(1+|u|^p)d\nu_k(t,u) =\\ &\int_0^1\int_{\beta_{\mathcal{U}}\mathbb{R}} f(t)g_0(u)d\nu(t,u) = \int f\, g_0\, \nu.\end{aligned} \tag{9}$$

In the remainder of the paper, we will sometimes use the above right hand side compact notation whenever the variables and domains of integration are clear from the context.

Measures $\nu \in \mathscr{P}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m)$ obtained as limits of sequences $(u_k)_{k\in\mathbb{N}} \subseteq \mathscr{L}^p([0,1];\mathbb{R}^m)$ in the sense of (9) have been called DiPerna–Majda measures. They will be discussed in more detail in Section 3.1. It turns out that every DiPerna–Majda measure $\nu \in \mathscr{P}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m)$ can be disintegrated into a measure τ on time and an L^p-Young measure ω on $\beta_{\mathcal{U}}\mathbb{R}^m$, i.e. $d\nu(t,u) = d\omega(du|t)d\tau(t)$ for some $\tau \in \mathscr{P}([0,1])$ and $\omega(.|t) \in \mathscr{P}(\beta_{\mathcal{U}}\mathbb{R}^m)$.

A relaxed version of (3) taking into account both oscillation and concentration effects can hence be stated as

$$\begin{aligned}
&\inf_{\nu} \int L_0(t,y(t),u)\,d\nu(t,u)\\
&\quad \text{s.t.} \int F_0(t,y(t),u)d\nu(t,u) = y_1 - y_0,\\
&\qquad \nu \in \mathscr{P}([0,1]\times \beta_{\mathcal{U}}\mathbb{R}^m)
\end{aligned} \tag{10}$$

where

$$L_0(t,y,u) := \frac{L(t,y,u)}{1+|u|^p}, \quad F_0(t,y,u) := \frac{F(t,y,u)}{1+|u|^p}. \tag{11}$$

In Claeys et al. (2017), the Lasserre hierarchy is adapted to compute numerically DiPerna–Majda measures and solve optimal control problem featuring oscillations and concentrations. However, the approach is valid under a certain number of technical assumptions on the data L and F, see Claeys et al. (2017, Assumption 1, Section 2.2). These assumptions are enforced to prevent the simultaneous presence of concentration and discontinuity.

2.4. Oscillations, concentrations and discontinuities

As mentioned in the introduction, the integrals in (3) might not be well defined, as concentration effects of the control are likely to cause discontinuities in the trajectory occurring at the same time. In view of the previous examples we propose to generalize the DiPerna–Majda measures, which themselves are a generalization of Young measures, even further and now also relax the trajectory to a measure valued function depending on time and control. In the sequel we describe accordingly the set of anisotropic parametrized measures. Then we provide a linear formulation of optimal control problem (3) that can cope with oscillations, concentrations and discontinuities in a unified fashion.

3. Anisotropic parametrized measures

In the following we describe the generalized DiPerna–Majda measures. For this it will be instructive to review first the classical DiPerna–Majda measures.

3.1. DiPerna-Majda measures

Let $\mathcal{U}$ be a complete[1] and separable subring of continuous bounded functions from $\mathbb{R}^m$ to $\mathbb{R}$. It is known (Engelking, 1989, Sect. 3.12.22) that there is a one-to-one correspondence between such rings and metrizable compactifications of $\mathbb{R}^m$. By a compactification we mean a compact set, denoted by $\beta_{\mathcal{U}}\mathbb{R}^m$, into which $\mathbb{R}^m$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^m$ and its image in $\beta_{\mathcal{U}}\mathbb{R}^m$. Similarly, we will not distinguish between elements of $\mathcal{U}$ and their unique continuous extensions defined on $\beta_{\mathcal{U}}\mathbb{R}^m$.

DiPerna and Majda (1987), see also Roubíček (1997), have shown that every bounded sequence $(u_k)_{k\in\mathbb{N}}$ in $\mathscr{L}^p([0,1];\mathbb{R}^m)$ with $1 \le p < \infty$ has a subsequence (denoted by the same indices) such that there exists a probability measure $\tau \in \mathscr{P}([0,1])$ and an L^p-Young measure $\omega(.|t) \in \mathscr{P}(\beta_{\mathcal{U}}\mathbb{R}^m)$ satisfying for all $f \in \mathscr{C}([0,1])$ and $g_0 \in \mathcal{U}$:

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^1 f(t)g_0(u_k(t))(1+|u_k(t)|^p)dt\\
&\quad = \int_0^1 \int_{\beta_{\mathcal{U}}\mathbb{R}^m} f(t)g_0(u)\omega(du|t)\tau(dt)\\
&\quad = \int_0^1 \int_{\beta_{\mathcal{U}}\mathbb{R}^m} f(t)g_0(u)d\nu(t,u) = \int f\, g_0\, \nu,
\end{aligned} \tag{12}$$

[1] A ring of functions is complete if it contains all constant functions, it separates points from closed subsets and it is closed with respect to the supremum norm.

compare with (9). The limit measure $d\nu(t,u) := \omega(du|t)\tau(dt)$ of such a sequence, or sometimes the pair (τ,ω), is called a DiPerna–Majda measure.

3.2. Generalization

The drawback of DiPerna–Majda measures is that g in (12) must be a continuous function. This does not fit to our aim to study interactions of discontinuities and concentrations. To go further the simplistic illustration of the introduction, let us consider the following example.

Example 1. Consider a sequence $(y_k)_{k\in\mathbb{N}} \subset \mathscr{W}^{1,1}([0,1])$ such that $\lim_{k\to\infty} y_k = y$ in $\mathscr{L}^q([0,1])$ for every $1 \le q < +\infty$. We are interested in the integral $\lim_{k\to\infty}\int_0^1 g(u_k(t))h(y_k(t))dt$ for continuous functions g and h such that $|g(u)| \le C(1+|u|)$ with some constant $C > 0$, and where $u_k := \dot{y}_k \in \mathscr{L}^1([0,1])$ is the weak derivative of y_k. If g is the identity then the calculation is easy, namely the limit equals $\liminf_{k\to\infty} H(y_k(1)) - H(y_k(0))$ where H is the primitive of h. In the case of a more general function g, the situation is more involved. For example for $k \ge 2$ let $u_k(t)$ be equal to 0 if $0 \le t \le \frac{1}{2}$, to k if $\frac{1}{2} \le t \le \frac{1}{2}+\frac{1}{k}$ and to 0 if $\frac{1}{2}+\frac{1}{k} \le t \le 1$. Its primitive $y_k(t)$ is equal to 0 if $0 \le t \le \frac{1}{2}$, to $k(t-\frac{1}{2})$ if $\frac{1}{2} \le t \le \frac{1}{2}+\frac{1}{k}$ and to 1 if $\frac{1}{2}+\frac{1}{k} \le t \le 1$. A short calculation shows that $\lim_{k\to\infty}\int_0^1 g(u_k(t))h(y_k(t))dt = \frac{1}{2}g_0(0)(h(0)+h(1)) + (H(1)-H(0))\lim_{k\to\infty}\frac{g(k)}{k}$. The sequence $(u_k)_{k\in\mathbb{N}}$ concentrates at $\frac{1}{2}$ which is exactly the point of discontinuity of the pointwise limit of $(y_k)_{k\in\mathbb{N}}$. Also notice that u_k converges weakly to $\delta_{\frac{1}{2}}$ in $\mathscr{P}([0,1])$ when $k \to \infty$. The factor $H(1)-H(0)$ in the previous equation suggests that we should refine the definition of the pointwise limit of $(y_k)_{k\in\mathbb{N}}$ at $\frac{1}{2}$ by enforcing that it is the Lebesgue measure supported on the interval of the jump. We will make this rigorous in the following. The other term in the factor also shows that the limit of $g(k)/k$ should exist when k tends to infinity.

To cope with the simultaneous presence of oscillations, concentrations and discontinuities, a new tool was recently introduced in Kałamajska et al. (2018), namely anisotropic parametrized measures generated by pairs $(y_k,u_k)_{k\in\mathbb{N}}$ where u_k is the control and y_k the corresponding state trajectory. Let us describe now what we need in our optimal control context. First, since the function $t \mapsto y(t)$ is the integral of a Lebesgue integrable function on a bounded time interval, it is bounded. Hence any admissible trajectory of optimal control problem (3) is such that $y \in \mathscr{L}^\infty([0,1];Y)$ for some compact set $Y \subset \mathbb{R}^n$, e.g. a ball of sufficiently large radius.. Then, the following result is a special case of Kałamajska et al. (2018, Theorem 2):

Theorem 2. *Let $1 \le p < +\infty$. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $\mathscr{L}^p([0,1];\mathbb{R}^m)$ and $(y_k)_{k\in\mathbb{N}}$ a bounded sequence in $\mathscr{W}^{1,1}([0,1];\mathbb{R}^n)$. Then there is a (non-relabeled) subsequence $(u_k,y_k)_{k\in\mathbb{N}}$, a measure $\tau \in \mathscr{P}([0,1])$, a measure $\omega(.|t) \in \mathscr{P}(\beta_{\mathcal{U}}\mathbb{R}^m)$ parametrized in $t \in [0,1]$ and a measure $\upsilon(.|t,u) \in \mathscr{P}(Y)$ parametrized in $t \in [0,1]$ and $u \in \beta_{\mathcal{U}}\mathbb{R}^m$ such that for every $f \in \mathscr{C}([0,1])$, $g_0 \in \mathcal{U}$, $h_0 \in \mathscr{C}(Y)$, it holds*

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^1 f(t)g_0(u_k(t))(1+|u_k(t)|^p)h_0(y_k(t))dt\\
&= \int_0^1 \int_{\beta_{\mathcal{U}}\mathbb{R}^m}\int_Y f(t)g_0(u)h_0(y)\upsilon(dy|t,u)\omega(du|t)\tau(dt)\\
&= \int_0^1 \int_{\beta_{\mathcal{U}}\mathbb{R}^m}\int_Y f(t)g_0(u)h_0(y)d\mu(t,y,u) = \int f\, g_0\, h_0\, \mu.
\end{aligned} \tag{13}$$

The measure $d\mu(t,u,y) := \upsilon(dy|t,u)\omega(du|t)\tau(dt)$, or sometimes the triplet (τ,ω,υ), is called an anisotropic parametrized measure. Moreover, the DiPerna–Majda measure (τ,ω) is generated by $(u_k)_{k\in\mathbb{N}}$.

Example 3. Let us revisit Example 1 and the calculations of the integrals there. Let $f \in \mathscr{C}([0,1])$, let $h \in \mathscr{C}(\mathbb{R})$ be bounded with primitive denoted by H, and let $g := (1+|.|)g_0$ where $g_0 \in \mathcal{U}$ corresponding to the two-point (or sphere) compactification $\beta_{\mathcal{U}}\mathbb{R}^m = \mathbb{R} \cup \{\pm\infty\}$, i.e. such that $\lim_{u\to\pm\infty} g_0(u) =: g_0(\pm\infty) \in \mathbb{R}$. Then after some calculations it can be checked that $\lim_{k\to\infty} \int_0^1 f(t)g(u_k(t))h(y_k(t))dt = \int_0^1 \int_{\beta_{\mathcal{U}}\mathbb{R}^m} \int_Y f(t)g_0(u)h(y)\upsilon(dy|t,u)\omega(du|t)\tau(dt)$ where $\tau(dt) = \lambda_{[0,1]} + 2\delta_{\frac{1}{2}}$, $\omega(du|t)$ is equal to $\delta_{+\infty}$ if $t = \frac{1}{2}$, and to δ_0 otherwise, and $\upsilon(dy|t,u)$ is equal to δ_0 if $t \in [0,\frac{1}{2})$, to $\lambda_{[0,1]}$ if $t = \frac{1}{2}$ and to δ_1 if $t \in (\frac{1}{2},1]$, where λ_X denotes the Lebesgue measure on X, and $Y = [0,1]$.

4. Relaxed optimal control with oscillations, concentrations and discontinuities

To the classical optimal control problem (3) we associate the relaxed optimal control problem

$$\begin{aligned} v_R^* := \inf_{\mu} & \int L_0\, \mu \\ \text{s.t.} & \int F_0\, \mu = y_T - y_0, \\ & \mu \in \mathscr{P}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m \times Y) \end{aligned} \tag{14}$$

which is *linear* in the unknown measure μ. In contrast, classical problem (3) is non-linear in the unknown trajectory y and control u.

Since optimal control problem (14) is a relaxation of the optimal control problem (3), it may happen that the infimum in (14) is strictly less than the infimum in (3), i.e. $v_R^* < v^*$. Formulating necessary and sufficient conditions on the problem data F and L such that $v_R^* = v^*$, i.e. there is no relaxation gap is an open problem. However, if we know that the probability measure in problem (14) is generated by limits of functions, then there is no relaxation gap. Let us explain this now.

Assumption 4 (*Regularity of the Data*)**.** Let L and F be such that in (11) it holds $L_0 \in \mathscr{C}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m\times Y)$ and $F_0 \in \mathscr{C}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m\times Y;\mathbb{R}^n)$. Moreover, there is a constant $c_L > 0$ such that $L(t,u,y) \geq c_L|u|^p$ for all t,u,y and there is a constant $c_F > 0$ such that $|F(t,u,y_1) - F(t,u,y_2)| \leq c_F(|u|^p+1)|y_1-y_2|$ for all t,u,y_1,y_2.

The following result follows from classical existence and uniqueness results for differential equations, see e.g. Bressan and Piccoli (2007, Theorem 3.1):

Lemma 5. *Assume that $p \geq 1$, $u \in \mathscr{L}^p([0,1];\mathbb{R}^m)$ and $y_0 \in \mathbb{R}^n$ are given. Let further $F : [0,1]\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^n$ satisfy Assumption 4. Then*

$$dy(t) = F(t,u(t),y(t))dt, \quad y(0) = y_0 \tag{15}$$

has a unique solution $y \in \mathscr{L}^\infty([0,1];Y)$ with values in a compact subset Y of $\mathbb{R}^n$.

Assume that there is a bounded sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{L}^p$ and that $\{y_k\}_{k\in\mathbb{N}} \subset \mathscr{W}^{1,1}$ is a sequence of corresponding solutions obtained in Lemma 5. Then $\{y_k\}$ is uniformly bounded in $\mathscr{W}^{1,1}$. Indeed, due to the regularity assumption on F we see that $\frac{d|y_k(t)|}{dt} \leq \left|\frac{dy_k(t)}{dt}\right| = |F(t,u_k(t),y_k(t))| \leq c_F(1+|u_k(t)|^p+|y_k(t)|)$. Then the Gronwall inequality (Evans, 1998, Appendix B.2.j) implies that $\sup_{k\in\mathbb{N}}\|y_k\|_{W^{1,1}} < \infty$ and since y_k is the integral of an integrable function on a bounded time interval, it holds that $y_k \in \mathscr{L}^\infty([0,1];Y)$ for $Y \subset \mathbb{R}^n$ a ball of radius $\sup_{k\in\mathbb{N}}\|y_k\|_{L^\infty} < \infty$. The limit of the right-hand side of (15) can then be expressed in terms of an anisotropic parametrized measure μ:

$$\lim_{k\to\infty} F(t,u_k(t),y_k(t))dt = \int_{\beta_{\mathcal{U}}\mathbb{R}^m}\int_Y F_0(t,u,y)d\mu(t,u,y). \tag{16}$$

As explained in Kałamajska et al. (2018, Theorem 7), the integral (13) in the definition of the anisotropic parametrized measure can be decomposed as follows

$$\begin{aligned} &\int_0^1\int_{\beta_{\mathcal{U}}\mathbb{R}^m}\int_Y f(t)g_0(u)h_0(y)d\mu(t,y,u) = \\ &\quad \int_0^1\int_{\mathbb{R}^m} f(t)g_0(u)(1+|u|^p)h_0(y(t))\tilde{\omega}(du|t)dt + \\ &\quad \int_0^1\int_{\beta_{\mathcal{U}}\mathbb{R}^m\setminus\mathbb{R}^m}\int_Y f(t)g_0(u)h_0(y)\upsilon(dy|t,u)\omega(du|t)\tau(dt) \end{aligned} \tag{17}$$

where $\tilde{\omega}$ is a classical Young measure on $\mathbb{R}^m$. Using the decomposition (17), we get the following differential equation

$$\begin{aligned} dy(t) = {} & F(t,u,y(t))\tilde{\omega}(du|t)dt + \\ & \int_{\beta_{\mathcal{U}}\mathbb{R}^m\setminus\mathbb{R}^m}\int_Y F_0(t,u,y)\upsilon(dy|t,u)\omega(du|t)\tau(dt). \end{aligned} \tag{18}$$

Lemma 6. *Given an anisotropic parametrized measure μ and an initial condition y_0, the solution y to (18) is unique.*

Proof. Assume that it is not the case, i.e., that there are two solutions $y_1, y_2 \in \mathscr{L}^\infty([0,1];Y)$. Disintegrating $d\mu(t,y,u) = \upsilon(dy|t,u)\omega(du|t)\tau(dt)$, we get the following relationship for the difference $y_d := y_1 - y_2$ because of the regularity Assumption 4, it holds $|\dot{y}_d| \leq \int_{\mathbb{R}^m}|F(t,u,y_1(t)) - F(t,u,y_2(t))|\omega_t(du) \leq \int_{\mathbb{R}^m} c_F(|u|^p+1)\omega_t(du)|y_d(t)|$. The right hand side belongs to $\mathscr{L}^1([0,1])$, therefore the measure $dy_d(t)$ is absolutely continuous with respect to the uniform measure dt. As $y_d(0) = 0$ we have $y_d(t) = 0$ for all $t \in [0,1]$, by the Gronwall inequality (Evans, 1998, Appendix B.2.j). □

In relaxed optimal control problem (14) we use an integral formulation of (18) incorporating the initial and terminal conditions: $\int_0^1\int_{\beta_{\mathcal{U}}\mathbb{R}^m}\int_Y F_0(t,u,y)d\mu(t,u,y) = \int F_0\,\mu = y_1 - y_0$. For each anisotropic parametrized measure μ, we can therefore associate a sequence of trajectories $\{y_k\} \subset \mathscr{W}^{1,1}$ and controls $(u_k) \subset \mathscr{L}^p$ satisfying (15) and such that (16) holds. Conversely, the limit of each such sequence of trajectories and controls can be modeled by an anisotropic parametrized measure. The following result of absence of relaxation gap then follows immediately from the construction of problem (14).

Proposition 7 (*No Relaxation Gap*)**.** *Let Assumption 4 hold and let μ solve problem (14). If there is an admissible sequence $(u_k, y_k)_{k\in\mathbb{N}}$ such that (13) holds then $v_R^* = v^*$.*

5. Relaxed optimal control with occupation measures

In the previous section, we proposed a linear reformulation of non-linear optimal control, thanks to the introduction of anisotropic parametrized measures. In the current section, we describe another linear reformulation proposed in Lasserre et al. (2008) and relying on the notion of occupation measure. The relation between this linear reformulation and the classical Majda–DiPerna measures was investigated in Claeys et al. (2017), with the help of a graph completion argument. In the sequel we show that the generalized Majda–DiPerna measures also fit naturally this framework.

Let $v \in \mathscr{C}^1([0,1]\times Y)$. For any admissible trajectory y and control u solving (15), it holds $\int_0^1 dv(t,y(t)) = v(1,y(1)) - v(0,y(0)) =$

$\int_0^1 \left(\frac{\partial v}{\partial t}(t,y(t)) + \frac{\partial v}{\partial y}(t,y(t)) \cdot \dot{y}(t)\right) dt$. Optimal control problem (3) can then be rewritten as

$$
\begin{aligned}
v^* = \inf_u \int_0^1 & L(t,u(t),y(t))dt \\
\text{s.t.} \int_0^1 & \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial y} \cdot F\right)(t,u(t),y(t))dt \qquad (19)\\
& = v(1,y_1) - v(0,y_0),\ \forall v \in \mathscr{C}^1([0,1]\times\mathbb{R}^n) \\
& y \in \mathscr{W}^{1,1}([0,1];\mathbb{R}^n), \quad u \in \mathscr{L}^p([0,1];\mathbb{R}^m).
\end{aligned}
$$

Definition 8 (*Occupation Measure*). Given a control u and a trajectory y solving (15), we define the occupation measure $\mu_{u,y} \in y \in \mathscr{P}([0,1]\times\mathbb{R}^n\times\mathbb{R}^m)$ by $d\mu_{u,y}(t,u,y) := \delta_{y(t)}(dy)\delta_{u(t)}(du)dt$.

Geometrically $\mu_{u,y}(A\times B\times C)$ is the time spent by the trajectory $(t,u(t),y(t))$ in any Borel subset $A\times B\times C$ of $[0,1]\times\mathbb{R}^m\times Y$. Analytically, integration with respect to $\mu_{u,y}$ is the same as integration along $(u(t),y(t))$ with respect to time. In particular $\int_0^1 L(t,u(t),y(t))dt = \int_0^1\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} L(t,u,y)d\mu_{u,y}(t,u,y) = \int L\,\mu_{u,y}$ and for all test functions $v \in \mathscr{C}^1([0,1]\times Y)$, it holds that $\int_0^1 \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial y}\cdot F\right)(t,u(t),y(t))dt = \int_0^1\int_{\mathbb{R}^m}\int_Y \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial y}\cdot F\right)(t,u,y)d\mu_{u,y}(t,u,y) = \int \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial y}\cdot F\right)\mu_{u,y}$. Using the same arguments as in Claeys et al. (2017, Proposition 4), we can reformulate optimal control problem (19) as a linear problem on measures, leading to the following relaxed formulation:

$$
\begin{aligned}
v_M^* := \inf_\mu \int & L_0\,\mu \\
\text{s.t.} \int & \left(\frac{\partial v}{\partial t}(1+|u|^p)^{-1} + \frac{\partial v}{\partial y}\cdot F_0\right)\mu \qquad (20)\\
& = v(1,y_1) - v(0,y_0) \quad \forall v \in \mathscr{C}^1([0,1]\times Y), \\
& \mu \in \mathscr{P}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m\times Y).
\end{aligned}
$$

Note that μ in the above problem is not necessarily an occupation measure in the sense of Definition 8, but a general probability measure in $\mathscr{P}([0,1]\times\beta_{\mathcal{U}}\mathbb{R}^m\times Y)$. For this reason, the infimum in relaxed problem (20) can be strictly less than the infimum in classical problem (3), i.e. $v_M^* < v^*$.

Proposition 9 (*No Relaxation Gap*). *It holds $v_R^* \le v_M^* \le v^*$ and hence if there is no relaxation gap in relaxed problem (14) then there is no relaxation gap in relaxed problem (20).*

Proof. Just observe that problem (14) corresponds to the particular choice of test functions $v(t,y) := y_k$, $k = 1,\dots,n$ in problem (20). Hence the infimum in (14) is smaller than the infimum in (20), which is in turn smaller than the infimum in (3), i.e. $v_R^* \le v_M^*$. Now if $v_R^* = v^*$ then obviously $v_M^* = v^*$. □

6. The Lasserre hierarchy

Once we get to the measure linear problem (20), we follow the same strategy as in Claeys et al. (2017, Section 4):

(1) compactify the control space by using a change of variables and homogenization;
(2) since all the data are polynomial, construct an equivalent moment linear problem where the unknown are moments of the occupation measure supported on a compact semialgebraic set;
(3) use the moment-sums-of-squares hierarchy as in Lasserre et al. (2008) to obtain a sequence of approximate moments at the price of solving numerically semidefinite programming problems;
(4) from the approximate moments, construct an approximate solution to the optimal control problem.

7. Illustrative example

Let us illustrate this strategy on our introductory example (2). The trajectory y should move the state from zero at initial time to one at final time, yet for the non-negative integrand to be as small as possible, the control u should be zero all the time, except maybe at time zero. If $\varepsilon = 1$ this problem has a trivial optimal solution $u(t) = 0$. For $\varepsilon = 0$ as explained already we can solve the problem by integration by parts because $\dot{y}(t) = u(t)$. The integration trick cannot be carried out in the case of $\varepsilon \in (0,1)$.

We use the relaxation proposed in Section 5 to formulate problem (2) as a measure LP:

$$
\begin{aligned}
\inf_\mu \int & \frac{(t+y)u}{1+u}\mu \\
\text{s.t.} \int & \left(\frac{\partial v}{\partial t}\frac{1}{1+u} + \frac{\partial v}{\partial y}\frac{\sqrt{\varepsilon^2+u^2}}{1+u}\right)\mu \qquad (21)\\
& = v(1,1) - v(0,0),\ \text{for all } v \in \mathscr{C}^1([0,1]^2) \\
& \mu \in \mathscr{P}([0,1]\times\beta\mathbb{R}_+\times[0,1]).
\end{aligned}
$$

Note that we can omit the absolute value in the denominator, as u is constrained to be non-negative. Since in problem (2) the growth of the Lagrangian and the dynamics is at most linear, we expect the control to concentrate. Therefore let $u(t) := \frac{r(t)}{1-r(t)}$ with $r(t) \in [0,1]$. Then the dynamic of y reads $\dot{y}(t) = \sqrt{\left(\frac{r(t)}{1-r(t)}\right)^2 + \varepsilon^2} = \frac{\sqrt{r(t)^2+\varepsilon^2(1-r(t))^2}}{1-r(t)}$. Introduce the auxiliary variable $w(t)$ such that $w(t)^2 = r(t)^2 + \varepsilon^2(1-r(t))^2$. By knowledge of bounds for ε and $r(t)$ we can conclude that $0 \le w(t) \le 1$. The linear problem on moments then reads

$$
\begin{aligned}
\inf_\gamma \int & (t+y)\,r\,\gamma \\
\text{s.t.} \int & \left(\frac{\partial v}{\partial t}(1-r) + \frac{\partial v}{\partial y}w\right)\gamma \qquad (22)\\
& = v(1,1) - v(0,0),\ \text{for all } v \in \mathbb{R}[t,y], \\
& \gamma \in \mathscr{P}([0,1]^3).
\end{aligned}
$$

We could solve the problem numerically for different values of the parameter ε. From these numerical solutions we could guess the analytic optimal solution. The measure $d\mu(t,y,u) = \tau(dt)\omega(du|t)\,\upsilon(dy|t,u)$ with $\tau(dt) = \lambda_{[0,1]} + (1-\varepsilon)\delta_0$, $\omega(du|t)$ equal to δ_∞ for $t = 0$, and equal to δ_0 for $t > 0$, and $\upsilon(dy|t,u)$ equal to $\frac{1}{1-\varepsilon}\lambda_{[0,1-\varepsilon]}$ for $t = 0$ and equal to $\delta_{1-\varepsilon+\varepsilon t}$ for $t > 0$ solves the relaxation (14) and hence yields a lower bound $\frac{(1-\varepsilon)^2}{2}$ on the optimum. Moreover, this optimum is attained by the sequences $u_k(t)$ equal to $\sqrt{(k(1-\varepsilon)+\varepsilon)^2-\varepsilon^2}$ for $t \in [0,\frac{1}{k}]$ and equal to 0 for $t > \frac{1}{k}$, and $y_k(t)$ equal to $(k(1-\varepsilon)+\varepsilon)t$ for $t \in [0,\frac{1}{k}]$ and equal to $\varepsilon t + 1 - \varepsilon$ for $t > \frac{1}{k}$, which proves that we got the optimal solution according to Proposition 7.

The numerical results obtained with the GloptiPoly interface and the SeDuMi semidefinite solver for the 6th relaxation (i.e. moments of degree up to 12) are matching to 4 significant digits with the analytic moments.

8. Conclusion

In this paper we have described a unified methodology to cope with limit phenomena typical of optimal control, namely oscillations, concentrations and discontinuities. Our approach relies numerically on the Lasserre hierarchy of semidefinite programming relaxations, which allows for the application of off-the-shelf computer software and hence opens the possibility for engineering applications. The key mathematical tool are anisotropic parametrized measures, an extension of DiPerna–Majda measures, themselves an extension of Young measures, objects familiar to PDE analysts.

From the numerical solution and the nature of the measure computed, we can deduce whether an integrable optimal control function exists or not:

- if the control measure is concentrated on the graph of a function, then there exists an integrable optimal control law, and hence there is no oscillation of the control;
- if the time measure is absolutely continuous w.r.t. the Lebesgue measure, then there is no concentration of the control, and hence no discontinuity of the trajectory.

Our approach is global, and hence closer in spirit to the Hamilton–Jacobi–Bellman approach rather than the Pontryagin Maximum Principle. We are not aware of first-order optimality conditions for optimal control problems at the level of generality considered in our paper.

Beyond providing a numerical solution to optimal control problems that are potentially troublesome for alternative numerical methods, we believe that our work can pave the way for the application of the Lasserre hierarchy to other problems of calculus of variations and optimal control, especially subject to PDE constraints. Indeed, we would like to emphasize that the theory of anisotropic parametrized measures can also deal with vector-valued multi-dimensional problems.

References

Bressan, A., & Piccoli, B. (2007). Introduction to the mathematical theory of control. *AIMS Series Applied Mathematics*.

Bressan, A., & Rampazzo, F. (1988). On differential systems with vector-valued impulsive controls. *Bollettino della Unione Matematica ItalianaB (7)*, *2*(3), 641–656.

Claeys, M. (2013). *Mesures d'occupation et relaxation semi-définies pour la commande optimale* (Ph.D. thesis), INSA, Univ. Toulouse.

Claeys, M., Arzelier, D., Henrion, D., & Lasserre, J. B. (2014). Measures and LMIs for non-linear optimal impulsive control. *IEEE Transactions on Automatic Control*, *59*, 1374–1379.

Claeys, M., Henrion, D., & Kružík, M. (2017). Semi-definite relaxations for optimal control problems with oscillation and concentration effects. *ESAIM Control Optimization Calcul Variable*, *23*, 95–117.

DiPerna, R. J., & Majda, A. J. (1987). Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Communications in Mathematical Physics*, *108*(4), 667–689.

Engelking, R. (1989). *General topology*. Berlin: Heldermann.

Evans, L. C. (1998). *Partial differential equations*. American Mathematical Society.

Fattorini, H. O. (1999). *Infinite dimensional optimization and control theory*. Cambridge, UK: Cambridge University Press.

Gaitsgory, V., & Quincampoix, M. (2009). Linear programming approach to deterministic infinite horizon optimal control problems with discounting. *SIAM Journal on Control and Optimization*, *48*(4), 2480–2512.

Kałamajska, A., Krömer, S., & Kružík, M. (2018). Weak lower semicontinuity by means of anisotropic parametrized measures. In E. Rocca, U. Stefanelli, L. Truskinovsky, & A. Visintin (Eds.), *Trends in applications of mathematics to mechanics* (pp. 23–51). Symposium on Trends on Applications of Mathematics to Mechanics.

Kružík, M., & Roubíček, T. (1999). Optimization problems with concentration and oscillation effects: relaxation theory and numerical approximation. *Numerical Functional Analysis Optimization*, *20*(5–6), 511–530.

Lasserre, J. B., Henrion, D., Prieur, C., & Trélat, E. (2008). Nonlinear optimal control via occupation measures and LMI-relaxations. *SIAM Journal on Control and Optimization*, *47*(4), 1643–1666.

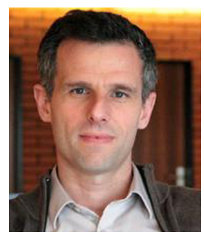
Luenberger, D. G. (1969). *Optimization by vector space methods*. New York: Wiley.

Roubíček, T. (1997). In W. de Gruyter (Ed.), *Relaxation in optimization theory and variational calculus*. Berlin.

Vinter, R. (1993). Convex duality and nonlinear optimal control. *SIAM Journal on Control and Optimization*, *31*(2), 518–538.

Young, L. C. (1969). *Lectures on the calculus of variations and optimal control theory*. Philadelphia: W. B. Saunders Co..

Zidani, H. (2013). *Rapport sur le manuscrit de thèse [3]*. Palaiseau: Ensta ParisTech.



Didier Henrion Didier Henrion received a Ph.D. Degree from the Academy of Sciences of the Czech Republic in 1998 and a Ph.D. degree from INSA Toulouse in 1999. Since 2000, he has been a CNRS researcher at LAAS, the Laboratory of Analysis and Architecture of Systems in Toulouse, as well as a research associate at the Faculty of Electrical Engineering of the Czech Technical University in Prague. He is interested in polynomial optimization for systems control, focusing on the development of constructive tools for addressing mathematical problems arising from systems control theory.

Martin Kružík Martin Kružík received a Ph.D. in mathematical modeling from Charles University in 1997. Since 2004 he has been a head of the department of Decision-Making Theory at the Institute of Information Theory and Automation of the Czech Academy of Sciences and he is an associate professor at Charles University and the Czech Technical University. His research focuses on calculus of variations in applications to continuum mechanics of solids and on various relaxation techniques for variational and control problems which do not possess solutions in Lebesgue spaces due to oscillatory behavior and/or due to concentrations.

Tillmann Weisser Tillmann Weisser is a postdoctoral fellow at Los Alamos National Laboratory. He absolved his master studies at University of Konstanz. In 2018 he received his Ph.D. from Paul Sabatier University in Toulouse, where he was also with the French national research center CNRS. Weisser's research interests include relaxations of (partial) differential equations and approximations of stochastic (in-)equalities using tools from real algebraic geometry.