Journal of Mathematical Economics 83 (2019) 84–88

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

SSB representation of preferences: Weakening of convexity assumptions

Miroslav Pištěk

The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 08, Prague 8, Czech Republic

ARTICLE INFO

Article history:
Received 3 January 2019
Received in revised form 3 April 2019
Accepted 9 April 2019
Available online 29 April 2019

Keywords:
SSB representation
Inductive linear topology
Non-transitive preferences

ABSTRACT

A continuous skew-symmetric bilinear (SSB) representation of preferences has recently been proposed in a topological vector space, assuming a weaker notion of convexity of preferences than in the classical (algebraic) case. Equipping a linear vector space with the so-called inductive linear topology, we derive the algebraic SSB representation on such topological basis, thus weakening the convexity assumption. Such a unifying approach to SSB representation leads, moreover, to a stronger existence result for a maximal element and opens a way for a non-probabilistic interpretation of the algebraic theory. Note finally that our method of using powerful topological techniques to derive purely algebraic result may be of general interest.

1. Introduction

Many systematic violations of the *expected utility theory* (Von Neumann and Morgenstern, 1953) have been observed, see e.g. Tversky (1969), stimulating the development of alternative decision-making theories (Fishburn, 1988; Starmer, 2000; Machina et al., 2004). In particular, the axiom of *transitivity* of preferences, nowadays understood as an intuitively appealing cornerstone of rationality, is not always supported by empirical evidence (Bar-Hillel and Margalit, 1988; Butler et al., 2016). A concise mathematical model of non-transitive decision-making has been proposed in Kreweras (1961) and Fishburn (1982), representing preferences with a skew-symmetric bilinear (SSB) functional. Note that from the mathematical point of view, such representation is closely related to the regret theory (Loomes and Sugden, 1982), see Blavatskyy (2006).

Denoting $\succ$ an asymmetric relation of strict preferences on a non-empty convex set P, we say that a functional ϕ on $P \times P$ is an SSB representation of $\succ$ if ϕ is SSB and $p \succ q \Longleftrightarrow \phi(p, q) > 0$ for all $p, q \in P$. Let $\sim$ and $\succsim$ be indifference and preference-or-indifference relations defined in a standard way using $\succ$. Then, the axioms of (algebraic) SSB representation stated for all $p, q, r \in P$ and all $\lambda \in]0, 1[$ are as follows:

(C1) Continuity: $p \succ q, q \succ r \Longrightarrow q \sim \alpha p + (1-\alpha) r$ for some $\alpha \in]0, 1[$,

(C2) Convexity:
$$p \succ q, p \succsim r \Longrightarrow p \succ \lambda q + (1-\lambda) r,$$
$$p \sim q, p \sim r \Longrightarrow p \sim \lambda q + (1-\lambda) r,$$
$$q \succ p, r \succsim p \Longrightarrow \lambda q + (1-\lambda) r \succ p,$$

(C3) Symmetry[1] :
$$p \succ q, q \succ r, p \succ r \Longrightarrow \left[q \sim \frac{p+r}{2} \Longrightarrow \left(\lambda p + (1-\lambda) r \sim \frac{p+q}{2} \Longleftrightarrow \lambda r + (1-\lambda) p \sim \frac{r+q}{2} \right) \right].$$

If P is, moreover, a set of probability measures, axioms (C1)–(C3) hold if and only if there exists an SSB representation of $\succ$, see Fishburn (1982, Theorem 1).

Recently, a variant of SSB representation of preferences has been proposed in a topological vector space. For a non-empty convex subset P (being equipped with the relative topology) the axioms for all $p, q, r \in P$ and $\lambda \in]0, 1[$ are the following:

(F1) Continuity: sets $\{s \in P : p \succ s\}$ and $\{s \in P : s \succ p\}$ are open,

(F2) Convexity:
$$p \succ q, p \succsim r \Longrightarrow p \succ \lambda q + (1-\lambda) r,$$
$$q \succ p, r \succsim p \Longrightarrow \lambda q + (1-\lambda) r \succ p,$$

(F3) Balance:
$$q \sim \frac{p+r}{2}, \lambda p + (1-\lambda) r \sim \frac{p+q}{2} \Longrightarrow \lambda r + (1-\lambda) p \sim \frac{r+q}{2}.$$

An asymmetric binary relation $\succ$ on P satisfies (F1), (F2) and (F3) if and only if there exists an SSB representation of $\succ$ that is, moreover, separately continuous in each variable, see Pištěk (2018, Theorem 3.6 and Theorem 5.3).[2] Further, the existence

[1] We use a slightly adapted variant of axiom (C3) to facilitate the discussion below Theorem 4.2. Note that then the conclusion of axiom (C3) is equivalent to axiom (F3).

[2] In the algebraic SSB representation asymmetry of $\succ$ is implied by axiom (C1). However, using axiom (F1) instead of axiom (C1), asymmetry of $\succ$ has to

E-mail address: pistek@runbox.com.

https://doi.org/10.1016/j.jmateco.2019.04.006
0304-4068/

of a maximal element of $Q \subset P$ with respect to $\succ$ has been shown, assuming compactness and convexity of Q, see Pištěk (2018, Corollary 3.4 and Theorem 3.6). In the algebraic setting of SSB representation, a similar result has been shown only for a (finitely generated) polyhedral set (Fishburn, 1988, Theorem 6.2).

In this article, we show that algebraic SSB representation may be, somehow surprisingly, considered only as an application of the above introduced topological theory. By using the so-called *inductive linear topology* (Jarchow, 1981; Bogachev and Smolyanov, 2017) for an underlying linear vector space, we show that axioms (C1) and (F2) imply axiom (F1) with respect to such topology, see Proposition 4.1. This step is essential to prove that axioms (C1), (F2) and (F3) are equivalent to the existence of algebraic SSB representation of preferences on P in a fully abstract setting, see Theorem 4.2. As a consequence, axioms (C3) and (F3) are equivalent given axioms (C1) and (F2), thus we have generalized Fishburn (1982, Theorem 1) that has been stated for a set of probabilistic measures using a stronger convexity axiom (C2). Further, we propose a generalized existence result for a maximal element w.r.t. $\succ$, see Theorem 4.3. Note finally that the technique used may be of general interest since it permits one to use topological tools to obtain relatively stronger results that may be finally transposed to a purely algebraic setting employing the inductive linear topology.

The article is organized as follows. Section 2 presents the basic notation and preliminary results. In Section 3 we introduce the notion of the inductive linear topology and discuss its basic properties. The main theorem of the algebraic SSB representation is presented in Section 4 together with all the related results. When working on this article, several inaccuracies in Pištěk (2018) have been discovered; all of them are corrected in Appendix.

2. Notation and preliminary results

For a set X we denote 2^X the set of all subsets of X. A *topological space* is a set X equipped with a family of subsets $\tau \subset 2^X$ (called *open* sets) satisfying the following conditions: $\emptyset, X \in \tau$; every union of open subsets of X is open; every finite intersection of open subsets of X is open. A set $Y \subset X$ is a *closed* set if $X \setminus Y \in \tau$. For any $Z \subset X$, we define the *relative topology* of Z as $\tau_Z = \{U \cap Z : U \in \tau\}$. A topological space is *compact* if each of its open covers has a finite sub-cover; is *connected* if it is not a disjoint union of two non-empty open subsets; is a *Hausdorff* space if any two distinct points are respectively contained in disjoint open sets; is a real *topological vector space* (t.v.s) if it is moreover a real *linear vector space* (l.v.s.) such that operations of addition and multiplication are continuous. Note that Pištěk (2018) restricts attention to topological vector spaces that are Hausdorff. By $\mathscr{P}(X)$ we denote a set of all *regular Borel probability* measures on X. Note that once equipped with the so-called weak* topology, $\mathscr{P}(X)$ is a convex subset of a t.v.s., see e.g. Goodearl (2010).

For a real function f satisfying a specified condition, we say that it is *unique up to a similarity transformation* if all functions satisfying the given condition are of the form αf with $\alpha > 0$. Let X be l.v.s. and $Y \subset X$, then we denote by co(Y) and cone(Y) the convex and conic hull of Y, i.e. the smallest convex and conic set in X that contains Y, respectively. For points $p, q \in X$ we use $[p, q] \equiv \{\lambda p + (1-\lambda)q : \lambda \in [0, 1]\}$, and analogously we define $]p, q[$ and $]p, q]$. Point $p \in M \subset X$ is called an *internal point* of M if for each $x \in X$ there exists an $\alpha > 0$ such that $[p, p+\alpha x] \subset M$. For sets $K, L \subset X$ we denote by $K + L$ set $\{k+l : k \in K, l \in L\}$; set $K - L$ is defined analogously.

For a set X and a binary relation S defined on X, $S \subset X \times X$, we write xSy if $(x, y) \in S$, and define *inverse relation* to S as $\{(x, y) \in X \times X : ySx\}$. We say that $x \in X$ is a *maximal element* of X with respect to S if set $\{y \in X : ySx\}$ is empty. Relation S is *asymmetric* if for all $x, y \in X$, xSy implies that ySx is not satisfied. The *preference interior* of X with respect to S is denoted by $X^\star \equiv \{y \in X : xSy, ySz \text{ for some } x, z \in X\}$. Given a (preference) binary relation $\succ$, the *indifference* relation $\sim$ and *preference-or-indifference* relation $\succsim$ are

$$p \sim q \equiv \text{neither } p \succ q \text{ nor } q \succ p,$$
$$p \succsim q \equiv p \succ q \text{ or } p \sim q.$$

Symbols $\prec$ and $\precsim$ denote the inverse relations to $\succ$ and $\succsim$, respectively. Given a topological space X, a binary relation $\succ$ on X is *coherent* (with topology of X) if $\overline{\{y \in X : y \succ x\}} = \{y \in X : y \succsim x\}$ for all $x \in X$ such that $\{y \in X : y \succ x\} \neq \emptyset$.

Let P be a convex subset of a t.v.s., a coherent and asymmetric relation $>$ on P is *upper semi-Fishburn* if $\{q \in P : q > p\}$ is convex for all $p \in P$. A binary relation is *lower semi-Fishburn* if its inverse is upper semi-Fishburn; a *Fishburn* relation is both lower and upper semi-Fishburn. A binary relation $\succ$ defined on P is *balanced* if for all $p, q, r \in P$ and all $\lambda \in]0, 1[$ such that $q \sim \frac{1}{2}p + \frac{1}{2}r$ and $\lambda p+(1-\lambda)r \sim \frac{1}{2}p+\frac{1}{2}q$ it holds that $\lambda r+(1-\lambda)p \sim \frac{1}{2}r+\frac{1}{2}q$. Finally, we say that a functional $\phi : P \times P^\star \to \mathbb{R}$ is a *continuous partial representation* of a binary relation $\succ$ on P if $\phi(p, q)$ is continuous and linear in p for all $q \in P^\star$, and for all $(p, q) \in P \times P^\star$ it holds that $p \succ q \iff \phi(p, q) > 0$ and $p \prec q \iff \phi(p, q) < 0$.

Next, a variant of Fishburn (1982, Lemma 3) with weaker convexity assumptions is provided, allowing us later a respective weakening of axioms in the theory of SSB representation. Note that Lemma 2.1 plays also a key role in Appendix.

Lemma 2.1. *Let P be a non-empty convex subset of a l.v.s., $\succ$ be a relation on P satisfying (C1) and (F2), and $q \in P^\star$. Then there exists a linear functional f on P such that for all $p \in P$ it holds that $p \succ q \iff f(p) > f(q)$ and $q \succ p \iff f(q) > f(p)$.*

Proof. Let $G \equiv \{p \in P : p \succ q\}$, $I \equiv \{p \in P : p \sim q\}$, and $L \equiv \{p \in P : p \prec q\}$. Note that $\{G, I, L\}$ is a disjunctive cover of P since (C1) implies asymmetry of $\succ$, and sets G and L are convex due to (F2) and non-empty. Denoting by Y the linear span of G, we first prove that $P \subset Y$. Take any $p \in I \cup L$ and $r \in G$; axiom (C1) implies that there is $s \in [p, r[$ such that $s \in I$, and (F2) then implies that $]s, r[\subset G$, thus $p \in Y$.

Assuming $0 \in L$ without loss of generality, we denote $V \equiv -(L \cap \text{cone}(G))$ and $U \equiv \text{co}(L \cup V)$. Then $U = \bigcup_{\lambda\in[0,1]}(\lambda L + (1-\lambda)V) \subset L + V$ since L and V are convex and contain 0. Observing $(L+V) \cap G \subset (L \cap (G-V))+V$ and $L \cap (G-V) \subset L \cap (G+\text{cone}(G))$, we see that $U \cap G = \emptyset$ may be shown by proving $L \cap (G+\text{cone}(G)) = \emptyset$. For a contradiction, we take $g, h \in G$ and $\lambda \geq 0$ such that $g + \lambda h \in L$, the convexity of G and L then implies $[g, h] \subset G$ and $[0, g + \lambda h] \subset L$. Since $[g, h] \cap [0, g + \lambda h] \neq \emptyset$, we reached the contradiction with $G \cap L = \emptyset$ and so $U \cap G = \emptyset$.

Further we will show that 0 is an internal point of U. Fixing $y \in Y$ we will find $\alpha > 0$ such that $[0, \alpha y] \subset U$. Since $Y = \text{cone}(G) - \text{cone}(G)$, there are $\lambda_1, \lambda_2 \geq 0$ and $y_1, y_2 \in G$ such that $y = \lambda_1 y_1 - \lambda_2 y_2$. Using (C1) together with (F2), there is $\alpha_1 \in]0, 1[$ such that $[0, \alpha_1 y_1] \subset L$; analogously one may find $\alpha_2 \in]0, 1[$ such that $[0, -\alpha_2 y_2] \subset V$. Note that $[\alpha_1 y_1, -\alpha_2 y_2] \subset U$ due to the convexity of U. Without loss of generality we may assume $y \neq 0$, then $\lambda_1 + \lambda_2 > 0$, and $\alpha^{-1} \equiv \frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}$ is well defined. A short calculation reveals that $\alpha y \in [\alpha_1 y_1, -\alpha_2 y_2]$, and so $[0, \alpha y] \subset U$.

Next, since 0 is an internal point of U, a convex set that is disjunctive with G, there exists a linear functional f on Y such that $G \subset \{p \in P : f(p - q) \geq 0\}$ and $L \subset U \subset \{p \in P :$

be explicitly assumed. This fact has been omitted in Pištěk (2018); see corrected statements in Appendix.

$f(p-q) \leq 0\}$ using the basic separation theorem, see e.g. Dunford and Schwartz (1958, Theorem V.1.12). Observe further that $I \subset \{p \in P : f(p-q) = 0\}$ not to violate (F2). Next, since $\{G, I, L\}$ is a disjunctive cover of P, it necessarily holds that $\{p \in P : f(p-q) > 0\} \subset G$ and $\{p \in P : f(p-q) < 0\} \subset L$. Thus, to satisfy (C1), it holds that $I = \{p \in P : f(p-q) = 0\}$, and then $G = \{p \in P : f(p-q) > 0\}$ and $L = \{p \in P : f(p-q) < 0\}$. □

3. Inductive linear topology

Let us denote by X the underlying l.v.s. In order to derive the theory of algebraic SSB representation based on the continuous SSB representation, we need to provide X with a Hausdorff topology such that any linear functional on X will be continuous. This is non-trivial only for an infinite-dimensional X; a finite-dimensional l.v.s. is a Hausdorff t.v.s. only once equipped with Euclidean topology, and then all linear mappings are continuous.

Next we provide the existence of the *inductive linear topology* τ_ι together with its basic properties; a more detailed survey is to be found in Jarchow (1981) and Bogachev and Smolyanov (2017). In particular, open sets of τ_ι may be explicitly constructed, see e.g. Jarchow (1981, Theorem 4.1.3).

Lemma 3.1. *Let X be a l.v.s., then there exists a finest topology τ_ι on X such that (X, τ_ι) is a t.v.s. and for any finite-dimensional subspace Y of X canonical injection of Y into X is continuous. Moreover, (X, τ_ι) is a Hausdorff space.*

Proof. See Proposition 4.1.1 and Proposition 4.5.4 in Jarchow (1981) (with the subsequent discussion). □

Lemma 3.2. *Every linear mapping from (X, τ_ι) to $\mathbb{R}$ is continuous.*

Proof. See Proposition 4.1.2 and Proposition 4.5.4 in Jarchow (1981). □

A similar concept is the *finite topology* on X; a subset U of X is *finitely open* if, for each finite-dimensional linear subspace Y of X, the set $U \cap Y$ is open in the Euclidean topology of Y. The finite topology is equivalent to the inductive linear topology for a countable-dimensional X; an uncountable-dimensional X equipped with the finite topology may not be a t.v.s. since then vector addition is not necessarily jointly continuous, see Kakutani and Klee (1963).

4. Main results

Next we show that axioms (C1) and (F2) imply axiom (F1) w.r.t. the inductive linear topology.

Proposition 4.1. *Let P be a non-empty convex subset of a l.v.s. equipped with the inductive linear topology τ_ι, and $\succ$ be a relation on P satisfying (C1) and (F2). Then $\succ$ satisfies (F1) with respect to the relative topology of P.*

Proof. Fix any $p \in P$, we will show that sets $G \equiv \{q \in P : q \succ p\}$ and $L \equiv \{q \in P : q \prec p\}$ are open in the relative topology of P. For $L \neq \emptyset$ and $G \neq \emptyset$, i.e. $p \in P^\star$, the statement follows directly from Lemma 2.1. Indeed, any linear functional is continuous with respect to topology τ_ι, see Lemma 3.2. Thus, G and L are open in topology τ_ι restricted to P, and axiom (F1) is satisfied.

Next, consider the case of $G = \emptyset$ and $L \neq \emptyset$, i.e. of p being a maximal element of P. To show that L is open we will prove that any $r \in L$ has an open neighbourhood contained in L. We have $p \succ r$, thus $p \succ \frac{p+r}{2}$ using (F2); and similarly $\frac{p+r}{2} \succ r$. One thus gets $r \in V \equiv \left\{s \in P : \frac{p+r}{2} \succ s\right\}$ as well as $\frac{p+r}{2} \in P^\star$ implying openness of V using the result of the above paragraph. Then $U \equiv \left\{\frac{s+r}{2} : s \in V\right\}$ is an open neighbourhood of r and $U \subset L$ using $p \succ r$ and $p \succsim s$ for all $s \in V$. The case of $L = \emptyset$ and $G \neq \emptyset$ is symmetric, and for $L = G = \emptyset$ the statement is immediate. □

The following generalization of Fishburn (1982, Theorem 1) is our main result.

Theorem 4.2. *Let P be a non-empty convex subset of a l.v.s. and $\succ$ be a binary relation on P. Then the following three statements are equivalent:*

(a) relation $\succ$ satisfies axioms (C1), (F2), and (F3),
(b) relation $\succ$ satisfies axioms (C1), (F2), and (C3),
(c) there exists an SSB functional ϕ on $P \times P$, unique up to a similarity transformation, such that for all $p, q \in P$, $p \succ q \Leftrightarrow \phi(p, q) > 0$.

Proof. First, one may observe that statement (a) directly implies statement (b) since axiom (F3) implies axiom (C3), see Pištěk (2018, Proposition 5.2).

To show that (b) implies (c), we first equip the underlying l.v.s. with the inductive linear topology τ_ι, and denote by τ_P the respective relative topology of P. Given (C1) and (F2), axiom (F1) is satisfied w.r.t. τ_P, see Proposition 4.1. Axioms (F1) and (F2) with asymmetry of $\succ$ due to axiom (C1) together imply that $\succ$ is a Fishburn relation using Pištěk (2018, Theorem 3.6). Now, to show the existence of an SSB representation, it suffices to note that Pištěk (2018, Theorem 5.3) is valid even if one assumes a formally weaker axiom (C3) instead of (F3); indeed, axiom (F3) is used only in the proof of Pištěk (2018, Lemma 5.4) to deduce axiom (C3).

Finally, a short calculation reveals that a relation represented by an SSB functional has to satisfy axioms (C1), (F2) and (F3), thus (c) implies (a). □

We have introduced axiom (F3) for several reasons. The premise of axiom (C3) has been motivated by its interplay with axioms (C1) and (C2), see the first paragraph on Fishburn (1982, p. 44). This is no more the case once axiom (F2) is used instead of axiom (C2). In particular, contrary to Pištěk (2018, Proposition 5.2), it seems to be difficult to show directly that axioms (F3) and (C3) are equivalent given (C1) and (F2). Furthermore, the conclusion of axiom (C3), which is equivalent to axiom (F3), holds generally for any preference relation admitting an SSB representation. Thus axiom (F3) may serve as a more concise variant of axiom (C3). Finally, axiom (F3) appears to be more amenable to empirical verification than axiom (C3).

Previously, the existence of a maximal element has been shown only for a (finitely generated) polyhedral set, see Fishburn (1988, Theorem 6.2). The following result is more general and may be used when set P is non-linearly constrained.

Theorem 4.3. *Let P be a non-empty convex compact subset of a finite-dimensional l.v.s., and $\succ$ be a relation that satisfies (C1) and (F2) on P; then there exist a minimal and a maximal element of P with respect to $\succ$.*

Note that compactness in the above statement is considered w.r.t. the standard, Euclidean topology of the involved finite-dimensional l.v.s.

Proof. Given (C1) and (F2), relation $\succ$ satisfies axiom (F1) w.r.t. the relative (inductive linear) topology of P due to Proposition 4.1. Moreover, relation $\succ$ is asymmetric owing to (C1), thus it is also a Fishburn relation using Pištěk (2018, Theorem 3.6). For a finite-dimensional space the Euclidean topology coincides with the inductive linear topology; thus P is compact in the inductive linear topology as well. Since the inductive linear topology is Hausdorff, see Lemma 3.1, the statement is due to Pištěk (2018, Corollary 3.4). □

Acknowledgements

This research has been supported by grant GA17-08182S of the Czech Science Foundation. The author is grateful to Tomáš Kroupa and Stefan Krömer for helpful discussions about the presented topic. Last but not least, it was a pleasure to follow reviewers' comments and suggestions that definitely increased the quality of this article.

Appendix

In this Appendix we correct several gaps arising in Pištěk (2018), the corrected results being numbered as they were in the original paper. In contrast to the note above (Pištěk, 2018, Definition 3.2), a coherent relation is not necessarily asymmetric. Thus asymmetry has to be assumed in the definition of Fishburn relation as follows.

Definition 3.2 (*Fishburn Relation*)**.** Let a convex subset P of a t.v.s. be equipped with the relative topology, and $\succ$ be an asymmetric and coherent relation on P. We say that $\succ$ is upper semi-Fishburn if $\{q \in P : q \succ p\}$ is convex for all $p \in P$. A binary relation is lower semi-Fishburn if its inverse relation is upper semi-Fishburn. Finally, a binary relation is a Fishburn relation if it is both lower and upper semi-Fishburn.

In the proof of Pištěk (2018, Theorem 3.6) asymmetry of $\succ$ has been used to invoke (Pištěk, 2018, Lemma 2.1); the corrected statement of Pištěk (2018, Theorem 3.6) follows.

Theorem 3.6. *Let P be a non-empty convex subset of a t.v.s., and $\succ$ be a binary relation on P. Then $\succ$ is a Fishburn relation if and only if $\succ$ is asymmetric and satisfies (F1) and (F2).*

Next, asymmetry of $\succ$ is needed also to prove (Pištěk, 2018, Lemma 3.8), the statement of which may be shortened using the above theorem.

Lemma 3.8. *Let a non-empty convex subset P of a t.v.s. be equipped with the relative topology, and $\succ$ be a Fishburn relation on P, then $\succ$ satisfies (C1).*

To prove the above lemma one cannot use the classical separation theorem (Treves, 1967, Proposition 18.1) since P equipped with the relative topology is not a t.v.s.; thus we propose a corrected proof.

Proof. For $p, q, r \in P$ such that $p \succ q$ and $q \succ r$ we define $G \equiv \{t \in P : t \succ q\}$, $I \equiv \{t \in P : t \sim q\}$, and $L \equiv \{t \in P : t \prec q\}$. Note that sets G and L are τ_P-open and convex using (F1) and (F2), respectively. Thus, denoting $K \equiv [p, r]$, there are $p', r' \in]p, r[$ such that $K \cap G = [p, p'[$ and $K \cap L =]r', r]$. Now, since K is connected, set $[p', r']$ is non-empty. Moreover, $[p', r'] \subset I$ since $\{G, I, L\}$ is a disjunctive cover of P using asymmetry of $\succ$. To finish the proof, take any $s \in [p', r']$ and observe that $s = \alpha p + (1 - \alpha) r$ for some $\alpha \in]0, 1[$. □

For the same reason we have to adjust the proof of Pištěk (2018, Theorem 4.2).

Theorem 4.2. *Let P be a non-empty convex subset of a t.v.s., and $\succ$ be a Fishburn relation on P. Then $\succ$ admits a continuous partial representation $\phi : P \times P^\star \to \mathbb{R}$. Moreover, for any fixed $q \in P^\star$, function $p \to \phi(p, q)$ is unique up to a similarity transformation.*

Proof. Using Theorem 3.6 and Lemma 3.8 we see that relation $\succ$ satisfies axioms (F1), (F2) and (C1). Fixing any $q \in P^\star$, we define $\phi(p, q) \equiv f(p - q)$ for all $p \in P$ using functional f due to Lemma 2.1. Functional ϕ represents relation $\succ$, to complete the proof we have to show that $\phi(p, q)$ is continuous in p on P. To this end, note that non-empty sets $\{p \in P : f(p) > f(q)\}$ and $\{p \in P : f(p) < f(q)\}$ are open in the relative topology of P due to (F1), and so $\{p \in P : f(p) = f(q)\}$ is relatively closed. Using (Jamison et al., 1976, Proposition 1)[3] for f and $-f$ we show that f is lower- and upper-semicontinuous on P, respectively. Thus f is continuous on P. □

Finally, the statement of Pištěk (2018, Lemma 6.1) is erroneous as one may not speak about convexity in a topological space. Thus Pištěk (2018, Theorem 6.2) has to be restated as follows.

Theorem 6.2. *Let set of outcomes X be a compact Hausdorff space and ϕ be a bounded real function on $X \times X$ that is separately continuous in each variable and satisfies $\phi(x, y) = -\phi(y, x)$ for all $x, y \in X$. Define functional Φ on $\mathscr{P}(X) \times \mathscr{P}(X)$ by*

$$\Phi(p, q) \equiv \int_{X \times X} \phi(x, y) dp(x) dq(y).$$

Then Φ represents a balanced Fishburn relation $>$ on $\mathscr{P}(X)$, and for a closed and convex set $K \subset \mathscr{P}(X)$ there exists $p \in K$ such that $p \geq q$ for all $q \in K$.

Proof. Note first that separate continuity of ϕ and compactness of X imply that ϕ is Borel measurable on $X \times X$, see Burke and Pol (2005, Proposition 5.2)[4]. Thus, functional Φ is well-defined using boundedness of ϕ. Next, Φ is bilinear on $\mathscr{P}(X) \times \mathscr{P}(X)$ from the definition, and skew-symmetric since $\phi(x, y) = -\phi(y, x)$. Weak*-continuity of Φ in each variable (separately) may be shown as in the proof of Pištěk (2018, Lemma 6.1). Thus, Φ represents a balanced Fishburn relation $>$, see Pištěk (2018, Theorem 5.3). The statement is then due to Pištěk (2018, Corollary 3.4) since $K \subset \mathscr{P}(X)$ is a convex and compact subset of a Hausdorff t.v.s., see Goodearl (2010, Proposition 5.22). □

Note that for such a corrected statement, the discussion following Pištěk (2018, Theorem 6.2) is still valid; in particular, Theorem 6.2 strictly generalizes (Fishburn, 1984, Theorem 5). Furthermore, since a jointly continuous function on a compact set is necessarily bounded and separately continuous, the corrected statement of Theorem 6.2 generalizes the setting of Fishburn (1984, Theorem 5) also in this respect.

References

Arkhangel'skii, A.V., 1992. Topological Function Spaces. Springer Netherlands.
Bar-Hillel, M., Margalit, A., 1988. How vicious are cycles of intransitive choice?. Theory and Decision 24 (2), 119–145.
Blavatskyy, P.R., 2006. Axiomatization of a preference for most probable winner. Theory and Decision 60 (1), 17–33.
Bogachev, V.I., 2007. Measure Theory. Springer Berlin Heidelberg, Berlin, Heidelberg.
Bogachev, V., Smolyanov, O., 2017. Topological vector spaces and their applications. In: Springer Monographs in Mathematics, vol. 157, Springer International Publishing, Cham.
Burke, D.K., Pol, R., 2005. Note on separate continuity and the namioka property. Topology Appl. 152, 258–268, (3 SPEC. ISS.).
Butler, D., Pogrebna, G., Blavatskyy, P.R., 2016. Predictably intransitive preferences. SSRN Electron. J. 13 (3), 217–236.

[3] Note that Hausdorff property is generally assumed for the underlying space in Jamison et al. (1976), however, it is not used to show this particular statement.
[4] We are using the fact that a compact topological space is both a countably compact space as well as Lindelöf Σ-space, see Arkhangel'skii (1992, p. 5 and 6); see also the commentary below (Bogachev, 2007, Theorem 7.14.5).

Dunford, N., Schwartz, J.T., 1958. Linear Operators: General Theory. Pure and Applied Mathematics. Interscience Publishers.
Fishburn, P., 1982. Nontransitive measurable utility. J. Math. Psych. 26 (1), 31–67.
Fishburn, P.C., 1984. Dominance in SSB utility theory. J. Econom. Theory 34 (1), 130–148.
Fishburn, P.C., 1988. Nonlinear Preferences and Utility Theory. The Johns Hopkins University Press.
Goodearl, K.K., 2010. Partially ordered abelian groups with interpolation. In: Mathematical Surveys and Monographs., vol. 20, American Mathematical Society.
Jamison, R., O'Brien, R., Taylor, P., 1976. On embedding a compact convex set into a locally convex topological vector space. Pacific J. Math. 64 (1), 193–205.
Jarchow, H., 1981. Locally Convex Spaces. Mathematische Leitfäden. Vieweg+Teubner Verlag, Wiesbaden.
Kakutani, S., Klee, V., 1963. The finite topology of a linear space. Arch. Math 14 (1), 55–58.
Kreweras, G., 1961. Sur une possibilite de rationaliser les intransitivites. In: La Decision, Colloques Internationaux Du CNRS. Paris, pp. 27–32.
Loomes, G., Sugden, R., 1982. Regret theory: An alternative theory of rational choice under uncertainty. Econ. J. 92 (92), 805.
Machina, M.J., Teugels, J.L., Sundt, B.e., 2004. Nonexpected utility theory. In: Encyclopedia of Actuarial Science, Vol. 2. John Wiley & Sons, Ltd, Chichester, UK, pp. 1173–1179.
Pištěk, M., 2018. Continuous SSB representation of preferences. J. Math. Econom. 77, 59–65.
Starmer, C., 2000. Developements in non-expected utility theory: The hunt for a descriptive teory of choice under risk. J. Econ. Lit. 38 (2), 332–382.
Treves, F., 1967. Topological Vector Spaces, Distributions and Kernels. Dover books on mathematics. Academic Press.
Tversky, A., 1969. Intransitivity of preference. Psychological Rev. 76 (1), 31–48.
Von Neumann, J., Morgenstern, O., 1953. Theory of Games and Economic Behavior. Science: Economics. Princeton University Press, Princeton.