Theory of SSB Representation of Preferences Revised*

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Abstract

A continuous skew-symmetric bilinear (SSB) representation of preferences has recently been proposed in a topological vector space, assuming a weaker notion of convexity of preferences than in the classical (algebraic) case. Equipping a linear vector space with the so-called inductive linear topology, we derive the algebraic SSB representation on a topological basis, thus weakening the convexity assumption. Such a unifying approach to SSB representation permits also to fully discuss the relationship of topological and algebraic axioms of continuity, and leads to a stronger existence result for a maximal element. By applying this theory to probability measures we show the existence of a maximal preferred measure for an infinite set of pure outcomes, thus generalizing all available existence theorems in this context.

1 Introduction

Many systematic violations of the *expected utility theory* [13] have been observed, see e.g. [12], stimulating the development of alternative decision-making theories [5, 11, 8]. In particular, the axiom of *transitivity* of preferences, nowadays understood as an intuitively appealing cornerstone of rationality, is not always supported by empirical evidence [2]. A concise mathematical model of non-transitive decision-making has been proposed in [3], representing preferences with a skew-symmetric bilinear (SSB) functional. Note that from the mathematical point of view, such representation is closely related to the regret theory [7], see [1].

*This research has been supported by grant GA17-08182S of the Czech Science Foundation.

Denoting   an asymmetric relation of strict preferences on a non-empty convex set P, we say that a functional ϕ on $P \times P$ is an SSB representation of   if ϕ is SSB and $p \quad q \quad \Rightarrow \phi(p,q) > 0$ for all $p, q \in P$. Let $\sim$ and $\gtrsim$ be indifference and preference-or-indifference relations defined in a standard way using . Then, the axioms of (algebraic) SSB representation stated for all $p, q, r \in P$ and all $\lambda \in]0,1[$ are as follows:

$$
\begin{array}{lll}
\text{(C1)} & \text{Continuity:} & p \quad q, q \quad r \Longrightarrow q \sim \alpha p + (1-\alpha) r \text{ for some } \alpha \in]0,1[, \\
\text{(C2)} & \text{Convexity:} & p \quad q, p \gtrsim r \Longrightarrow p \quad \lambda q + (1-\lambda) r, \\
& & p \sim q, p \sim r \Longrightarrow p \sim \lambda q + (1-\lambda) r, \\
& & q \quad p, r \gtrsim p \Longrightarrow \lambda q + (1-\lambda) r \quad p, \\
\text{(C3)} & \text{Symmetry}^1\text{:} & p \quad q, q \quad r, p \quad r \Longrightarrow \Big[q \sim \dfrac{p+r}{2} \Longrightarrow \\
& & \Big(\lambda p + (1-\lambda) r \sim \dfrac{p+q}{2} \quad \Rightarrow \lambda r + (1-\lambda) p \sim \dfrac{r+q}{2} \Big) \Big].
\end{array}
$$

If P is, moreover, a set of probability measures, axioms (C1)–(C3) hold if and only if there exists an SSB representation of , see [3, Theorem 1].

Recently, a variant of an SSB representation of preferences has been proposed in a topological vector space [9]. For a non-empty convex subset P (being equipped with the relative topology) the axioms for all $p, q, r \in P$ and $\lambda \in]0,1[$ are the following:

$$
\begin{array}{lll}
\text{(F1)} & \text{Continuity:} & \text{sets } \{s \in P : p \quad s\} \text{ and } \{s \in P : s \quad p\} \text{ are open}, \\
\text{(F2)} & \text{Convexity:} & p \quad q, p \gtrsim r \Longrightarrow p \quad \lambda q + (1-\lambda) r, \\
& & q \quad p, r \gtrsim p \Longrightarrow \lambda q + (1-\lambda) r \quad p, \\
\text{(F3)} & \text{Balance}^1\text{:} & q \sim \dfrac{p+r}{2}, \lambda p + (1-\lambda) r \sim \dfrac{p+q}{2} \Longrightarrow \\
& & \qquad \lambda r + (1-\lambda) p \sim \dfrac{r+q}{2}.
\end{array}
$$

An asymmetric binary relation   on P satisfies (F1), (F2) and (F3) if and only if there exists an SSB representation of   that is, moreover, separately continuous in each variable, see Theorem 1 below. Further, we have shown that in a compact and convex subset of P there exists a maximal element with respect to , see Theorem 3 below. Consequently, we have generalized the existence result for a maximal element in the case of an infinite set of outcomes, see Theorem 5.

Finally, equipping a linear vector space X with the so-called *inductive linear topology* [6], i.e. the finest topology such that X is a Hausdorff t.v.s. and for any finite-dimensional subspace Y of X canonical injection of Y into X is continuous, the algebraic SSB representation may be considered as an application of the above introduced topological theory [10]. Such an observation leads to a generalization of

[1] We use a slightly adapted variant of axiom (C3) to stress that the conclusion of axiom (C3) is equivalent to axiom (F3). Note that axioms (C3) and (F3) are equivalent given axioms (C1) and (F2), see [10, Theorem 4.2].

the algebraic SSB representation theorem, as well as the theorem for the existence of a maximal element in a linear vector space, see Corollary 2 and Corollary 4, respectively. The proposed technique may be of general interest since it permits one to use topological tools to obtain relatively stronger results that may be finally transposed to a purely algebraic setting employing the inductive linear topology.

The basic notation used is standard. A *topological space* is a set X equipped with a family of subsets $\subset 2^X$ (called *open* sets) satisfying the following conditions: $\emptyset, X \in$; every union of open subsets of X is open; every finite intersection of open subsets of X is open. A topological space is *compact* if each of its open covers has a finite sub-cover; is a *Hausdorff* space if any two distinct points are respectively contained in disjoint open sets; is a real *topological vector space* (t.v.s) if it is moreover a real *linear vector space* (l.v.s.) such that operations of addition and multiplication are continuous. By $\mathscr{P}(X)$ we denote a set of all *regular Borel probability* measures on X equipped with the so-called weak* topology.

2 Main Results

First, we present the topological version of the SSB representation theorem, see [9, Theorem 3.6 and Theorem 5.3].

Theorem 1. *Let P be a non-empty convex subset of a t.v.s. equipped with the relative topology. An asymmetric relation on P satisfies (F1), (F2) and (F3) if and only if there exists a separately continuous SSB functional ϕ on $P \times P$ such that for all $p, q \in P$, $p \quad q \quad \phi(p,q) > 0$.*

Transposing the above theorem in a l.v.s. with the use of inductive linear topology, one obtains the following generalization of [3, Theorem 1], see [10, Theorem 4.2].

Corollary 2. *Let P be a non-empty convex subset of a l.v.s. A binary relation on P satisfies (C1), (F2) and (F3) if and only if there exists an SSB functional ϕ on $P \times P$ such that for all $p, q \in P$, $p \quad q \quad \phi(p,q) > 0$.*

Comparing the statements of Theorem 1 and Corollary 2, we see that axiom (C1) plays two different roles. First, it implies asymmetry of that has to be explicitly assumed in Theorem 1. Besides, it amounts to continuity axiom (F1) in the algebraic setting; indeed, any SSB functional is separately continuous with respect to inductive linear topology, see [6, Proposition 4.1.2 and Proposition 4.5.4].

Next, we show that standard continuity and convexity assumptions imply the existence of a maximal and a minimal element in a t.v.s., see [9, Corollary 3.4 and Theorem 3.6].

Theorem 3. *Let P be a non-empty compact convex subset of a Hausdorff t.v.s., and be an asymmetric relation on P satisfying axioms (F1) and (F2), then there exist a minimal and a maximal element of P with respect to .*

Employing the inductive linear topology again, see [10, Theorem 4.3], one obtains an analogous statement in a l.v.s.

Corollary 4. *Let P be a non-empty compact convex subset of a finite-dimensional l.v.s., and be a relation that satisfies (C1) and (F2) on P, then there exist a minimal and a maximal element of P with respect to .*

Note that previously a similar existence result that has been shown only for a (finitely generated) polyhedral subset of P assuming, moreover, axiom (C3), see [5, Theorem 6.2].

Finally, we generalize [4, Theorem 5] on the basis of Theorem 1 and Theorem 3, for a detailed proof see [9, Theorem 6.2].

Theorem 5. *Let set of outcomes X be a compact Hausdorff space and ϕ be a bounded real function on $X \times X$ that is separately continuous in each variable and satisfies $\phi(x, y) = -\phi(y, x)$ for all $x, y \in X$. Define functional on $\mathscr{P}(X) \times \mathscr{P}(X)$ by*

$$(p, q) \equiv \int_{X \times X} \phi(x, y) dp(x) dq(y).$$

Then is a separately continuous SSB functional on $\mathscr{P}(X)$, and for a closed and convex set $K \subset \mathscr{P}(X)$ there exists $p \in K$ such that $(p, q) \geq 0$ for all $q \in K$.

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