On Irreducible Min-Balanced Set Systems

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Abstract. Non-trivial minimal balanced systems (= collections) of sets are known to characterize through their induced linear inequalities the class of the so-called balanced (coalitional) games. In a recent paper a concept of an *irreducible* min-balanced (= minimal balanced) system of sets has been introduced and the irreducible systems have been shown to characterize through their induced inequalities the class of totally balanced games. In this paper we recall the relevant concepts and results, relate them to various contexts and offer a catalogue of permutational types of non-trivial min-balanced systems in which the irreducible ones are indicated. The present catalogue involves all types of such systems on sets with at most 5 elements; it has been obtained as a result of an alternative characterization of min-balanced systems.

Keywords: Balanced set system · Irreducible min-balanced system · Totally balanced games · Exact games

1 Introduction

A central notion of this note, namely the concept of a minimal balanced set system, shortened as a *min-balanced* (set) *system*, is basically a combinatorial concept. Nonetheless, the concept itself has been introduced in the context of cooperative game theory, where it plays quite an important role. Specifically, the well-known Shapley-Bondareva theorem [2,12] says that *balanced* systems of subsets of a non-empty finite basic set N covering N induce linear inequalities characterizing coalitional games over the set of players N with a non-empty *core*. Note that the concept of a core (polytope) is a substantial concept in cooperative game theory [12]. The least class of inequalities characterizing the non-emptiness of the core consists of those inequalities, which are induced by non-trivial *minimal balanced systems*, where the minimality is understood with respect to inclusion of set systems covering N.

For analogous reasons the inequalities induced by min-balanced systems are important in the context of the theory of imprecise probabilities [15]. In that context the basic set N can be interpreted as the sample space for probabilities,

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(normalized non-negative) games over N correspond to *lower probabilities* and the cores to *credal sets* of probabilities. Thus, (minimal) balanced set systems on N correspond to the inequalities characterizing lower probabilities *avoiding sure loss*, which are the lower probabilities with non-empty credal sets [15, Sect. 3.3.4]. We refer the reader to [7] for further details about the correspondence between game-theoretical concepts and those in imprecise probabilities.

Here is a formal definition: we say that a system $\mathcal{B} = \{S_1, \ldots, S_\ell\}$, $\ell \geq 1$, of non-empty subsets of N is a *balanced* system *on* a non-empty subset $M \subseteq N$ if there exist strictly positive real coefficients $\lambda_i > 0$, $i = 1, \ldots, \ell$, such that

$$\chi_M = \sum_{i=1}^{\ell} \lambda_i \cdot \chi_{S_i}, \text{ where } \chi_S \in \mathbb{R}^N \text{ denotes the incidence vector of } S \subseteq N. \quad (1)$$

In particular, the sets S_i must be subsets of M and $\lambda_i \leq 1$ for $i = 1, \ldots, \ell$. Thus, the concept of a balanced set system on M generalizes a classic combinatorial concept of a *partition* of M (consider $\lambda_i = 1$ for all $i = 1, \ldots, \ell$). For example, in case $N = \{a, b, c, d\}$ the partition $\mathcal{B} = \{\,\{a\}, \{b, c\}\,\}$ of $M = \{a, b, c\}$ is an example of a balanced system on M. On the other hand, one can also find links to fuzzy set theory with a little bit of imagination: balanced systems can perhaps be regarded as *fuzzy partitions* [1] of a crisp set M with fuzzy subsets having allowed only two grades, namely 0 and $\lambda_i \in (0, 1]$.

Note that balanced systems, called *balanced collections* in game-theoretical literature, do have some applications in combinatorics and topology. Shapley [13] generalized Sperner's celebrated topological lemma concerning triangulations of a simplex and balanced collections of sets play a crucial role in his generalization [4]. On the other hand, we would like to warn the reader that a combinatorial concept of a *balanced hypergraph* from [11, Sect. 83.1] has apparently nothing common with the concept of a balanced set system; these are different notions.

As explained below, in case of a min-balanced system $\mathcal{B}$ the coefficients in (1) are uniquely determined and the class of *min-balanced systems* on a given basic set is finite. Every permutational type of non-trivial min-balanced systems can be viewed as a combinatorial object: it represents a particular way in which a finite set M can be composed from its proper subsets. Thus, questions of natural interest are what are the permutational types of such systems, whether one can classify/categorize them or even whether an enumeration method generating all these types exists. Note in this context that Peleg [8] proposed an algorithm for inductive generating min-balanced systems on a given basic set. Nonetheless, as far as we know, no public available catalogue of their permutational types has been generated as an output of that algorithm.

1.1 Totally Balanced and Exact Games

Because of the above mentioned Shapley-Bondareva theorem, games with non-empty cores are named *balanced games*. There are two important subclasses of the class of balanced games over a player set N. One of them is the class of *totally balanced games*: these are such games m over N that, for every non-empty subset

$M \subseteq N$, the subgame of m for M is a balanced game over M. An even smaller class is the class of *exact games*: these are such games that each bound defining the core polyhedron is tight (a precise definition can be found below). Since these classes of games play an important role in cooperative game theory [9,10], some effort has been exerted to characterize them in terms of linear inequalities.

Special set systems play an important role in this context, too. Csóka et al. [3] characterized exact games over N by means of an infinite set of linear inequalities which could be associated with the so-called *exactly balanced* set systems on N. Lohman et al. [6] then refined that result and showed that the exact games over N can be characterized by means of a finite set of linear inequalities. Specifically, these are inequalities assigned to *min-balanced* set systems on non-empty subsets $M \subseteq N$ and to the so-called minimal *negatively balanced* set systems on N. Nonetheless, the reader was warned in [6] that this set of inequalities is not the the least possible set of inequalities characterizing the exact games.

In a recent paper [5] the least possible set of inequalities (up to a positive multiple) that characterizes the totally balanced games over N was found. These inequalities are induced by special *irreducible min-balanced* set systems on non-empty subsets $M \subseteq N$ (a formal definition is placed below). Another interesting observation from [5] is as follows: if $\mathcal{B}$ is a non-trivial min-balanced system on $M \subseteq N$ then its *complementary system* relative to M, that is,

$$\mathcal{B}^* := \{M \setminus S \,:\, S \in \mathcal{B}\}$$

is also a non-trivial min-balanced system on M. In particular, the non-trivial min-balanced systems on a fixed non-empty set $M \subseteq N$ come in pairs of mutually complementary systems. Moreover, the inequality induced by $\mathcal{B}^*$ is a *conjugate* inequality *with respect to* M to the one induced by $\mathcal{B}$ (also to be defined below).

Finally, a *conjecture* about the least possible set of inequalities characterizing the exact games was formulated in [5]. It says that a game over N is exact if and only if it satisfies the inequalities induced by non-trivial *irreducible min-balanced* systems on non-empty strict subsets $M \subset N$ and their conjugate inequalities with respect to N. The conjecture is known to be true in case $|N| \leq 5$.

Therefore, the question of classifying permutational types of non-trivial irreducible min-balanced systems over a fixed basic set is of great importance for the study of totally balanced and exact games. This is a topic of this note.

1.2 Structure of the Rest of the Paper

We provide a catalogue [14] of permutational types of non-trivial min-balanced systems on small sets in which the irreducible types are indicated. Its initial version describes all such types on sets with at most five elements. Nonetheless, we intend to upgrade it later into an interactive web platform and possibly extend it to involve all types of min-balanced systems on a six-element set. Now, we describe the structure of the rest of the paper.

In Sect. 2 we recall basic concepts and facts. In particular, we describe the way linear inequalities are induced by min-balanced systems and introduce the concept of an irreducible min-balanced system. Section 3 deals with conjugate inequalities and complementary set systems. Section 4 then describes the way our catalogue [14] was obtained. Note that our computations were not based on Peleg's iterative algorithm [8] but on an alternative characterization of the min-balanced systems in terms of linear independence of certain vectors. In Sect. 5 our tools to classify the permutational types are discussed. In last Sect. 6 we mention possible future research directions and open tasks.

2 Basic Concepts and Facts

Let N be a non-empty finite *basic set*. The symbol $\mathcal{P}(N)$ will denote its power set, that is, the collection $\{S : S \subseteq N\}$ of all its subsets. The symbol $\mathbb{R}^N$ will be used to denote the Euclidean space of real vectors $[x_i]_{i\in N}$ whose components are indexed by elements of N. Given $S \subseteq N$, the symbol χ_S will denote the *incidence vector* of S in $\mathbb{R}^N$, that is, its zero-one identifier in $\mathbb{R}^N$ defined by

$$(\chi_S)_i \ := \ \begin{cases} 1 & \text{for } i \in S, \\ 0 & \text{for } i \in N \setminus S, \end{cases} \qquad\qquad \text{whenever } i \in N.$$

2.1 Game-Theoretical Notions

In this context, elements of the basic set N correspond to players and subsets of N to coalitions. A (*transferable-utility coalitional*) *game over* N is modeled by a real function $m\colon \mathcal{P}(N) \to \mathbb{R}$ such that $m(\emptyset) = 0$. If $\emptyset \neq M \subseteq N$ then the restriction of m to $\mathcal{P}(M)$ is called a *subgame* of m *for* M.

The *core* $C(m)$ of a game m over N is a polyhedron in $\mathbb{R}^N$ defined by

$$C(m) := \{\, [x_i]_{i\in N} \in \mathbb{R}^N \ : \ \sum_{i\in N} x_i = m(N) \ \ \& \ \ \sum_{i\in S} x_i \geq m(S) \text{ for any } S \subseteq N \}\,.$$

We say that a game m over N is *balanced* if $C(m) \neq \emptyset$. It is called *totally balanced* if every its subgame is balanced. Finally, a game over N is *exact* if, for each coalition $S \subseteq N$, a vector $[x_i]_{i\in N} \in C(m)$ exists such that $\sum_{i\in S} x_i = m(S)$. This basically means that every inequality defining the core of m is tight.

A well-known fact is that every exact game is totally balanced (see Remark 1.19 in [10, Sect. V.1]); by definition, every totally balanced game is balanced.

2.2 Min-Balanced Set Systems

Any subset $\mathcal{B}$ of $\mathcal{P}(N)$ is called a *set system*; the union of sets in $\mathcal{B}$ will be denoted by $\bigcup\mathcal{B}$. A set system having at most one set is considered to be trivial; thus, set systems $\mathcal{B} \subseteq \mathcal{P}(N)$ with $|\mathcal{B}| \geq 2$ will be named *non-trivial*.

We say that $\mathcal{B}$ *composes to* a non-empty set $M \subseteq N$ if $M = \bigcup\mathcal{B}$ and the vector χ_M belongs to the conic hull of $\{\chi_S \in \mathbb{R}^N : \ S \in \mathcal{B}\}$, that is, there exist *non-negative* coefficients $\lambda_S \geq 0$, $S \in \mathcal{B}$, such that $\chi_M = \sum_{S\in\mathcal{B}} \lambda_S \cdot \chi_S$.

Given $\mathcal{B} \subseteq \mathcal{P}(N)$ and $\emptyset \neq M \subseteq N$ we say $\mathcal{B}$ is *min-balanced on* M if it is a minimal set system in $\mathcal{P}(N)$ which composes to M. That means, $\mathcal{B}$ composes to M and, moreover, there is no $\mathcal{C} \subset \mathcal{B}$ such that $\mathcal{C}$ composes to M. The following basic observation was done in [5, Lemma 2.1].

Lemma 1. *A non-empty set system* $\mathcal{B} \subseteq \mathcal{P}(N)$ *is min-balanced on a non-empty set* $M \subseteq N$ *iff the following two conditions hold:*

(i) *there exist strictly positive* $\lambda_S > 0$, $S \in \mathcal{B}$, *such that* $\chi_M = \sum_{S\in\mathcal{B}} \lambda_S \cdot \chi_S$,
(ii) *the incidence vectors* $\{\chi_S \in \mathbb{R}^N : \; S \in \mathcal{B}\}$ *are linearly independent.*

The condition (i), mentioned already with (1), means that $\mathcal{B}$ is *balanced*, which is usual terminology in game-theoretical literature. The condition (ii), equivalent to minimality, then implies the uniqueness of the so-called *balancing coefficients* λ_S in (i). One can observe using Lemma 1 that a non-empty $\mathcal{B} \subseteq \mathcal{P}(N)$ is min-balanced on M iff it is a minimal set system satisfying (i), which is a standard definition of a *minimal balanced collection* in game-theoretical literature.

Note that it follows from [5, Lemma 2.2] that any *non-trivial* min-balanced system $\mathcal{B}$ on $M \subseteq N$ consists of least two proper subsets of M. Moreover, the intersection of all sets in $\mathcal{B}$ must be empty and one has at most $|M|$ sets in $\mathcal{B}$.

As mentioned earlier, every non-trivial min-balanced system on $M \subseteq N$ induces a unique non-trivial inequality (up to a positive multiple) for games m over N. More specifically, we know by Lemma 1 that *unique* balancing coefficients $\lambda_S > 0$, $S \in \mathcal{B}$, exist such that $\chi_M = \sum_{S\in\mathcal{B}} \lambda_S \cdot \chi_S$. The *induced inequality* for games m over N has then the form

$$m(M) \geq \sum_{S\in\mathcal{B}} \lambda_S \cdot m(S)\,. \tag{2}$$

One can show that the balancing coefficients λ_S must be rational [5, Sect. 3.3], which allows one to multiply (2) by a positive factor so that the (balancing) coefficients become integers with no common prime divisor. Moreover, it is convenient to introduce a conventional coefficient with the empty set which plays no role in (2) because $m(\emptyset) = 0$ for any game m. The convention is such that one gets, after a re-arrangement, a unique *standardized form* of the inequality

$$\alpha(M) \cdot m(M) + \sum_{S\in\mathcal{B}} \alpha(S) \cdot m(S) + \alpha(\emptyset) \cdot m(\emptyset) \geq 0\,, \tag{3}$$

where $\alpha(S)$, $S \in \mathcal{B}$, are negative integers with no common prime divisor and $\alpha(M), \alpha(\emptyset)$ are positive integers determined by the standardization conditions:

$$\sum_{S\subseteq N} \alpha(S) = 0 \quad \text{and} \quad \forall\, i \in N \quad \sum_{S\subseteq N:\, i\in S} \alpha(S) = 0\,. \tag{4}$$

The point of this particular convention will be revealed in Sect. 3.

Example 1. Consider $N = \{a, b, c, d\}$ and a set system $\mathcal{B} = \{a, bc, bd, cd\}$, where abbreviations like ab stand for sets like $\{a, b\}$. One has

$$\chi_N = 1 \cdot \chi_a + \frac{1}{2} \cdot \chi_{bc} + \frac{1}{2} \cdot \chi_{bd} + \frac{1}{2} \cdot \chi_{cd}\,,$$

which allows one to observe using Lemma 1 that $\mathcal{B}$ is min-balanced on N. The respective inequality (2) is multiplied by factor 2, which gives $\alpha(N)=2$, $\alpha(a)=-2$ and $\alpha(S)=-1$ for remaining $S\in\mathcal{B}$. The convention in the first formula of (4) gives $\alpha(\emptyset)=+3$, which leads to the standardized inequality (3)

$$2\cdot m(abcd)-2\cdot m(a)-m(bc)-m(bd)-m(cd)+3\cdot m(\emptyset)\geq 0 \qquad (5)$$

for games m over N.

Shapley-Bondareva theorem can be re-formulated as follows [5, Lemma 3.5]:

Proposition 1. *If $|N|\geq 2$ then the least possible set of standardized inequalities characterizing the balanced games m over N is the set of inequalities (3) induced by non-trivial min-balanced systems $\mathcal{B}$ on N.*

2.3 Irreducible Min-Balanced Systems

Let $\mathcal{B}$ be a min-balanced system on $\emptyset\neq M\subseteq N$. We say that $\mathcal{B}$ is *reducible* if there exist a set $\emptyset\neq A\subset M$ such that $\mathcal{B}_A:=\{S\in\mathcal{B}\,:\,\,S\subset A\}$ composes to A. Note that one can assume without loss of generality that both $|A|\geq 2$ and $A\notin\mathcal{B}$ because otherwise $\mathcal{B}$ cannot be min-balanced. A min-balanced system $\mathcal{B}\subseteq\mathcal{P}(N)$ that is not reducible is called *irreducible*.

The meaning of the reducibility condition is that the induced inequality (for games over N) is a conic combination of inequalities induced by other min-balanced systems, in particular by the irreducible ones.

Lemma 2. *Given a min-balanced system $\mathcal{B}$ on $\emptyset\neq M\subseteq N$, the reducibility condition with a set $\emptyset\neq A\subset M$ is equivalent to the existence of min-balanced systems $\mathcal{C}$ on A and $\mathcal{D}$ on M such that $A\in\mathcal{D}$, $\mathcal{C}\setminus\mathcal{D}\neq\emptyset$ and $\mathcal{B}=\mathcal{C}\cup\mathcal{D}\setminus\{A\}$.*

Proof. The sufficiency of the condition is easy as $\mathcal{C}\subseteq\mathcal{B}_A$. For the necessity realize using Lemma 1 that $\{\chi_S\,:\,\,S\in\mathcal{B}_A\}$ are linearly independent. Hence, uniquely determined coefficients $\mu_S\geq 0$, $S\in\mathcal{B}_A$, exist such that $\chi_A=\sum_{S\in\mathcal{B}_A}\mu_S\cdot\chi_S$. Put $\mathcal{C}:=\{S\in\mathcal{B}_A\,:\,\,\mu_S>0\}$ and $\mu_T:=0$ for $T\in\mathcal{B}\setminus\mathcal{B}_A$. Again by Lemma 1, unique coefficients $\lambda_S>0$, $S\in\mathcal{B}$, exist such that $\chi_M=\sum_{S\in\mathcal{B}}\lambda_S\cdot\chi_S$. Let us put $\varepsilon:=\min_{C\in\mathcal{C}}\frac{\lambda_C}{\mu_C}$ and introduce $\kappa_A:=\varepsilon$, $\kappa_S:=\lambda_S-\varepsilon\cdot\mu_S$ for $S\in\mathcal{B}$. Then one has $\chi_M=\sum_{S\in\mathcal{B}}\lambda_S\cdot\chi_S+\varepsilon\cdot(\chi_A-\sum_{S\in\mathcal{B}}\mu_S\cdot\chi_S)=\sum_{S\in\mathcal{B}\cup\{A\}}\kappa_S\cdot\chi_S$ with $\kappa_S\geq 0$ for $S\in\mathcal{B}\cup\{A\}$. One can put $\mathcal{D}:=\{S\in\mathcal{B}\cup\{A\}\,:\,\,\kappa_S>0\}$ and verify the conditions from Lemma 2.

One can extend the arguments used in the above proof to show that the min-balanced systems $\mathcal{C}$ and $\mathcal{D}$ mentioned above are uniquely determined by the set A. Lemma 2 also allows one to observe that the reducibility condition for a min-balanced system $\mathcal{B}$ is equivalent to the original one from [5, Definition 4.1]. The next example illustrates the fact that the "decomposition" of $\mathcal{B}$ into systems $\mathcal{C}$ and $\mathcal{D}$ leads to conic combination of the induced inequalities.

Example 2. The set system $\mathcal{B} = \{a, bc, bd, cd\}$ from Example 1 is reducible. Put $A := \{b, c, d\}$; then $\mathcal{B}_A = \{bc, bd, cd\}$ composes to A because of

$$\chi_A \equiv \chi_{bcd} = \frac{1}{2} \cdot \chi_{bc} + \frac{1}{2} \cdot \chi_{bd} + \frac{1}{2} \cdot \chi_{cd} \, .$$

One gets $\mathcal{C} = \{bc, bd, cd\}$ and $\mathcal{D} = \{a, bcd\}$ in this particular case. The inequality (5) induced by $\mathcal{B}$ is then a conic combination of inequalities induced by $\mathcal{C}$ and $\mathcal{D}$:

$$\begin{aligned} 1 \ \times \ \{ \ 2 \cdot m(bcd) - m(bc) - m(bd) - m(cd) + m(\emptyset) \ \} &\geq 0 \\ 2 \ \times \ \{ \ m(abcd) - m(a) - m(bcd) + m(\emptyset) \ \} &\geq 0 \, , \end{aligned}$$

where the (conic) coefficient for $\mathcal{C}$ is 1 and the coefficient for $\mathcal{D}$ is 2.

Thus, reducible systems are superfluous for describing totally balanced games. Nonetheless, the irreducible ones are substantial as shown in [5, Theorem 5.1]:

Proposition 2. *Assume $|N| \geq 2$. The least set of standardized inequalities that characterizes totally balanced games m over N is the set of inequalities* (3) *induced by non-trivial irreducible min-balanced systems $\mathcal{B}$ on non-empty subsets $M \subseteq N$.*

3 Conjugate Inequalities and Complementary Systems

Every (standardized) inequality (3) for games m (over N) can be viewed as

$$\sum_{S \subseteq N} \alpha(S) \cdot m(S) \geq 0 \, , \quad \text{where the coefficients outside } \mathcal{B} \cup \{\emptyset, M\} \text{ are zeros,}$$

and assigned its *conjugate inequality* for games m (over N) *with respect to* N:

$$\sum_{T \subseteq N} \alpha^*(T) \cdot m(T) \geq 0 \, , \qquad \text{where} \ \ \alpha^*(T) := \alpha(N \setminus T) \quad \text{for any } T \subseteq N.$$

The importance of this concept for balanced and exact games is apparent from [5, Lemma 3.4], which can be re-phrased as follows.

Proposition 3. *The least set $\mathcal{S}$ of standardized inequalities that characterizes balanced games over N is closed under conjugacy: whenever* (3) *is in $\mathcal{S}$ then its conjugate inequality is in $\mathcal{S}$. The same holds for the least set of standardized inequalities characterizing exact games over N.*

We know from Proposition 1 that the inequalities in the set $\mathcal{S}$ characterizing balanced games over N correspond to non-trivial min-balanced systems on N. Thus, the conjugate inequality to (3) for a system $\mathcal{B}$ on N also corresponds to a non-trivial min-balanced system on N, which is nothing but the complementary system to $\mathcal{B}$ relative to N. The following is a re-formulation of [5, Corollary 3.1].

Proposition 4. *Let $\mathcal{B}$ be a non-trivial min-balanced system on N inducing* (3)*. Then its complementary system $\mathcal{B}^* := \{N \setminus S \,:\, S \in \mathcal{B}\}$ relative to N is also a non-trivial min-balanced system on N, inducing the conjugate inequality to* (3)*.*

The reader can now comprehend the convention from Sect. 2.2, where the coefficient $\alpha(\emptyset)$ with the empty set was introduced. It is just the coefficient $\alpha^*(N)$ in the conjugate inequality, that is, the coefficient with N for the complementary system $\mathcal{B}^*$. This phenomenon is illustrated by the next example.

Example 3. Consider again the system $\mathcal{B} = \{a, bc, bd, cd\}$ on $N = \{a, b, c, d\}$ from Example 1. Its complementary system relative to N is $\mathcal{B}^* = \{ab, ac, ad, bcd\}$. The standardized inequality induced by $\mathcal{B}^*$ is then

$$3 \cdot m(abcd) - m(ab) - m(ac) - m(ad) - 2 \cdot m(bcd) + 2 \cdot m(\emptyset) \geq 0\,,$$

which is, as the reader can check, the conjugate inequality to the inequality (5) induced by $\mathcal{B}$ (see Example 1). Note that $\mathcal{B}^*$ is an irreducible min-balanced system unlike its complementary system $\mathcal{B}^{**} = \mathcal{B}$ (see Example 2).

4 Catalogue: Procedure

Here we describe the method our catalogue [14] has been obtained. As explained in Sect. 2.2, a min-balanced system on a basic set N contains at most $n := |N|$ sets. Thus, a non-trivial such system $\mathcal{B} \subseteq \mathcal{P}(N)$ can be represented by a zero-one $n \times r$-matrix, where $2 \leq r \leq n$, namely by a matrix whose (distinct) columns are incidence vectors of (all) sets $S \in \mathcal{B}$. To get one-to-one correspondence between such matrices and non-trivial set systems one can choose and fix an order of elements in N and also choose and fix an order of elements in $\mathcal{P}(N)$.

Moreover, by Lemma 1(ii), the columns of a matrix which represents a min-balanced system must be linearly independent, which means that the rank of the matrix is r, the number of its columns. This is something one can easily test using linear algebra computational tools. Thus, the first step of our procedure was computing a list of representatives of permutational types of non-trivial set systems $\mathcal{B} \subseteq \mathcal{P}(N)$ such that $\{\chi_S \,:\, S \in \mathcal{B}\}$ are linearly independent in $\mathbb{R}^N$. In our case $n = 5$ we have obtained 1649 such (non-trivial) type representatives.

The second necessary condition for a min-balanced system $\mathcal{B} \subseteq \mathcal{P}(N)$ is that $\mathcal{B}$ *composes to* $M := \bigcup \mathcal{B}$, that is, non-negative coefficients $\lambda_S \geq 0$, $S \in \mathcal{B}$, exist such that $\chi_M = \sum_{B \in \mathcal{B}} \lambda_S \cdot \chi_S$. If $\mathbb{A}$ is the $n \times r$-matrix representing $\mathcal{B}$ then this condition is equivalent to the existence of a non-negative column vector $\lambda \in \mathbb{R}^r$ such that $\mathbb{A} \cdot \lambda = \chi_M$, which is a standard feasibility task in linear programming. Again, this can be tested computationally by means linear programming software packages. In our case $n = 5$ we found that 934 representatives of those 1649 ones mentioned above describe systems $\mathcal{B}$ composing to $\bigcup \mathcal{B}$.

In case of an $n \times r$-matrix $\mathbb{A}$ of the rank r the solution λ of a linear system $\mathbb{A} \cdot \lambda = \chi_M$ is uniquely determined. Hence, the criterion to decide whether the corresponding system $\mathcal{B}$ is min-balanced is immediate: all the components of the unique solution λ must be strictly positive. This gave massive reduction: only 57 representatives of above mentioned 934 ones describe min-balanced systems.

Testing irreducibility of min-balanced systems using their matrix computer representations comes from the definition in Sect. 2.3: testing whether a set subsystem composes to its union is a linear programming feasibility task. In our case $n = 5$ we recognized 23 irreducible types of 57 of min-balanced ones. The particular numbers are given in Table 1, the details can be found in [14].

Table 1. The numbers of non-trivial min-balanced systems in case $|N| \leq 5$.

Variables	Systems	Permutational types	Irreducible systems	Irreducible types
$\|N\| = 2$	1	1	1	1
$\|N\| = 3$	5	3	4	2
$\|N\| = 4$	41	9	18	5
$\|N\| = 5$	1291	44	288	15
$\|N\| \leq 5$	1338	57	311	23

5 Towards Classification of Permutation Types

Here we mention some characteristics which can be used to classify permutational types of non-trivial (irreducible) min-balanced set systems.

5.1 Numerical Characteristics of Permutational Types

Let $\mathcal{B}$ be a non-trivial min-balanced set system on N with $n := |N| \geq 2$. Introduce (set) *cardinality characteristics* of $\mathcal{B}$ by $c_k := |\{S \in \mathcal{B} : |S| = k\}|$ for $k = 1, \ldots, n-1$. Note that $\sum_{k=1}^{n-1} c_k = |\mathcal{B}|$ is the number of sets in $\mathcal{B}$. The *cardinality vector* $[c_1, \ldots, c_{n-1}]$ can then serve as a characteristic of any permutational type of non-trivial min-balanced systems. Cardinality vectors cannot, however, distinguish between some different permutational types.

An alternative idea comes from *multiplicity characteristics* which are defined by $m_i := |\{S \in \mathcal{B} : i \in S\}|$ for elements $i \in N$. One can order the numbers m_i in an increasing way, say, and get a *multiplicity vector* of the length $|N|$, which can serve as a characteristic of the permutational type of $\mathcal{B}$. The sum of its components $\sum_{i \in N} m_i$ can be viewed as a kind of *multiplicity index* for $\mathcal{B}$. Multiplicity vectors cannot, however, distinguish between different partitions.

5.2 Archetypes

Let $\mathcal{B}$ be a set system on a basic set N. It defines an equivalence relation on N: given $i, j \in N$, $i \sim j$ will mean that, for every $S \in \mathcal{B}$, one has $i \in S \;\Leftrightarrow\; j \in S$. For any $i \in N$ put $[i] := \{j \in N : i \sim j\}$ and denote by $\widetilde{N} := \{[i] : i \in N\}$ the *factor set* of $\sim$, that is, the set of equivalence classes of $\sim$. Analogously, any $S \in \mathcal{B}$ can be identified with a subset of $\widetilde{N}$, namely with $\widetilde{S} := \{[i] : i \in S\}$; note that the inverse relation is $S = \bigcup \{[i] : [i] \in \widetilde{S}\}$. The system $\mathcal{B}$ itself can be identified with $\widetilde{\mathcal{B}} := \{\widetilde{S} : S \in \mathcal{B}\}$, which is a set system on $\widetilde{N}$.

Given a set system $\mathcal{B}$ on $N \neq \emptyset$ and a set system $\mathcal{C}$ on $L \neq \emptyset$ we will say that they belong to the *same archetype* if there exists a one-to-one mapping $\psi : \widetilde{N} \to \widetilde{L}$ from $\widetilde{N}$ onto $\widetilde{L}$ which maps $\widetilde{\mathcal{B}}$ to $\widetilde{\mathcal{C}}$, that is, $\widetilde{\mathcal{C}} = \{\, \psi(\widetilde{S}) \,:\, \widetilde{S} \in \widetilde{\mathcal{B}}\}$.

It is easy to see that this is an equivalence relation on set systems coarsening their permutational equivalence. Trivial set systems form one equivalence class of this archetypal equivalence; however, such systems are not of our interest.

Lemma 3. *Let $\mathcal{B}$ be a set system on N and $\mathcal{C}$ a set system on L which belong to the same archetype. Then $\mathcal{B}$ is min-balanced iff $\mathcal{C}$ is min-balanced. Moreover, $\mathcal{B}$ is a non-trivial irreducible min-balanced system iff $\mathcal{C}$ is so.*

Proof. It is enough to verify the claims for a set system $\mathcal{B}$ on N and the system $\widetilde{\mathcal{B}}$ on $\widetilde{N}$ in place of the system $\mathcal{C}$ on L. The claim about min-balanced systems follows easily from Lemma 1: realize that one has $\chi_N = \sum_{S\in\mathcal{B}} \lambda_S \cdot \chi_S$ if and only if $\chi_{\widetilde{N}} = \sum_{S\in\mathcal{B}} \lambda_S \cdot \chi_{\widetilde{S}}$ for arbitrary real coefficients λ_S and similar consideration works with zero vectors in place of χ_N and $\chi_{\widetilde{N}}$.

As concerns the irreducible systems it is more convenient to show that $\mathcal{B}$ is reducible iff $\widetilde{\mathcal{B}}$ is reducible. As mentioned in Sect. 2.3 the set A in the definition of reducibility of $\mathcal{B}$ has the form $A = \bigcup \mathcal{B}_A$ with $\mathcal{B}_A = \{S \in \mathcal{B} \,:\, S \subset A\}$. Such a set A is composed of equivalence classes of $\sim$ and can be identified with a subset of $\widetilde{N}$: one has $\widetilde{A} := \{\,[i] \,:\, i \in A\}$ and $A = \bigcup\{[i] \,:\, [i] \in \widetilde{A}\}$. Hence, one has $\widetilde{A} \subset \widetilde{N}$ and $\chi_A = \sum_{S\in\mathcal{B}_A} \lambda_S \cdot \chi_S$ iff $\chi_{\widetilde{A}} = \sum_{S\in\mathcal{B}_A} \lambda_S \cdot \chi_{\widetilde{S}}$ for arbitrary real coefficients λ_S. This implies the claim about reducible systems. The claim about trivial/non-trivial systems is evident.

Lemma 3 implies that permutational types can be classified by their archetypes. Any archetype can be canonically represented by an *archetypal* set system, which is such a system $\mathcal{B}$ on N that, for any $i, j \in N$, one has $i \sim j$ iff $i = j$.

Example 4. Consider an irreducible min-balanced system $\mathcal{B} = \{ab, acd, bcd\}$ on $N = \{a, b, c, d\}$. One has $c \sim d$ in this case and the system $\mathcal{B}$ belongs to the same archetype as $\mathcal{C} = \{ab, ac, bc\}$ on $M = \{a, b, c\}$. Clearly, $\mathcal{C}$ is an archetypal system.

6 Conclusions

We would like to find out whether our method of generating (all) types of min-balanced systems based on Lemma 1 can be modified and can lead to some iterative algorithm, which would be able to produce catalogues for $|N| \geq 6$.

One of our open tasks is whether the numerical characteristics from Sect. 5.1 are able to distinguish between any distinct types of min-balanced systems. If this is so then an alternative method of generating types could possibly be designed.

This is also related to the question of finding lower and upper estimates for the numbers (of types) of min-balanced systems and irreducible min-balanced system in terms of $|N|$. The asymptotic behavior of these numbers with increasing $|N|$ would be of our interest, too.

References

1. Bodjanova, S., Kalina, M.: Coarsening of fuzzy partitions. In: The 13th IEEE International Symposium on Intelligent Systems and Informatics, September 17–19, 2015, Subotica, Serbia, pp. 127–132 (2015)
2. Bondareva, O.: Some applications of linear programming methods to the theory of cooperative games (in Russian). Problemy Kibern. **10**, 119–139 (1963)
3. Csóka, P., Herings, P.J.-J., Kóczy, L.Á.: Balancedness conditions for exact games. Math. Methods Oper. Res. **74**(1), 41–52 (2011)
4. Ichiishi, T.: On the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem. J. Math. Anal. Appl. **81**, 297–299 (1981)
5. Kroupa, T., Studený, M.: Facets of the cone of totally balanced games. Math. Methods Oper. Res. (2019, to appear). https://link.springer.com/article/10.1007/s00186-019-00672-y
6. Lohmann, E., Borm, P., Herings, P.J.-J.: Minimal exact balancedness. Math. Soc. Sci. **64**, 127–135 (2012)
7. Miranda, E., Montes, I.: Games solutions, probability transformations and the core. In: JMLR Workshops and Conference Proceedings 62: ISIPTA 2017, pp. 217–228 (2017)
8. Peleg, B.: An inductive method for constructing minimal balanced collections of finite sets. Nav. Res. Logist. Q. **12**, 155–162 (1965)
9. Peleg, B., Sudhölter, P.: Introduction to the Theory of Cooperative Games. Theory and Decision Library, series C: Game Theory, Mathematical Programming and Operations Research. Springer, Heidelberg (2007)
10. Rosenmüller, J.: Game Theory: Stochastics, Information, Strategies and Cooperation. Kluwer, Boston (2000)
11. Schrijver, A.: Combinatorial Optimization: Polyhedra and Efficiency. Springer, Berlin (2003)
12. Shapley, L.S.: On balanced sets and cores. Nav. Res. Logist. Q. **14**, 453–460 (1967)
13. Shapley, L.S.: On balanced games without side payments. In: Hu, T.C., Robinson, S.M. (eds.) Mathematical Programming, pp. 261–290. Academic Press, New York (1973)
14. Studený, M., Kratochvíl, V., Vomlel, J.: Catalogue of min-balanced systems, June 2019. http://gogo.utia.cas.cz/min-balanced-catalogue/
15. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London (1991)