# ON QUANTILE OPTIMIZATION PROBLEM BASED ON INFORMATION FROM CENSORED DATA

Petr Volf

Stochastic optimization problem is, as a rule, formulated in terms of expected cost function. However, the criterion based on averaging does not take in account possible variability of involved random variables. That is why the criterion considered in the present contribution uses selected quantiles. Moreover, it is assumed that the stochastic characteristics of optimized system are estimated from the data, in a non-parametric setting, and that the data may be randomly right-censored. Therefore, certain theoretical results concerning estimators of distribution function and quantiles under censoring are recalled and then utilized to prove consistency of solution based on estimates. Behavior of solutions for finite data sizes is studied with the aid of randomly generated example of a newsvendor problem.

*Keywords:* optimization, censored data, product-limit estimator, empirical quantile, newsvendor problem

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## 1. INTRODUCTION

Let us consider an optimization problem with objective function $\varphi(y, v)$, where $v$ is an input variable from certain feasibility set $\boldsymbol{V}$ and values $y$ are the realizations of a random variable $Y$ with distribution function $F$. Standardly, corresponding stochastic optimization problem can be formulated as $\inf_v E_F \varphi(Y, v)$, where $E_F$ stands for the expectation w.r. to $F$. If $F$ is known, we actually deal with a "deterministic" optimization case. However, criterion based on averaging does not take in account possible variability of r.v. $Y$ and is actually reasonable for optimization of actions repeated regularly over a long time period. Even then the variability of solution can be large. That is why the present paper is devoted to the optimization of quantiles of random criterion $Z(v) = \varphi(Y, v)$. Alternatively, we can be interested in a multi-objective optimization task, reducing both the expectation and the variability of solution (measured by its variance, or by certain inter-quantile range). Further, our information on probability distribution could be non-complete, we have to employ nonparametric estimate of $F$. Then, as a rule, the estimate

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is used instead $F$. Hence, we have to analyze both the bias and variability of obtained solution (compared to the ideal solution when $F$ is known).

The investigation of utilization of empirical (estimated) characteristics in stochastic optimization problems started already in 70-ties. A number of papers has dealt with these tasks, let us mention here just Kaňková [4] offering also a historical overview containing a number of other references. In the present paper we consider even more complicated case when the distribution function $F$ should be estimated from the data censored randomly from the right side. Such a situation is quite frequent in the analysis of demographic, survival or insurance data. The lack of information leads to higher variability of estimates and, consequently, to lower accuracy of optimal solutions. The approach to statistical data analysis in the cases when the data are censored or even truncated is also provided by a number of authors. The most of results were derived in the framework of statistical survival analysis and collected in several monographs (c.f. Kalbfleisch and Prentice [3], or Andersen et al. [1] ).

In the next section, theoretical properties of estimates under random right censoring will be recalled briefly. We shall consider the product-limit estimator as a generalization of the empirical distribution function, and a corresponding estimator of quantiles. Their properties in cases with and without censoring will be compared. In Section 3 these properties are utilized to asses the almost sure consistency of optimal solutions, i. e. the convergence of the solution based on a data sample (eventually with random right-censoring) to the solution obtained under full information on probability distribution. It is proved that the asymptotic rate is comparable with the cases without censoring. Finally, in Section 4, it is shown how, in a finite data case, the lack of information leads to higher variability and bias of estimates. Behavior of solutions for finite data sizes is studied and illustrated on a simple case of the newsvendor optimization problem.

## 2. ESTIMATORS OF DISTRIBUTION AND QUANTILE FUNCTIONS

Let us consider a continuous-type random variable $Y$ characterizing for instance a random time to certain event. Let another continuous random variable $Z$ be a censoring variable, both be positive and mutually independent. Further, let $f(y)$, $g(z)$, $F(y)$, $G(z)$, $\overline{F}(y) = 1 - F(y)$, $\overline{G}(z) = 1 - G(z)$ denote the density, distribution and survival functions of both variables. It is assumed that we observe just $X = \min(Y, Z)$ and $\delta = 1[Y \le Z]$, i. e. $\delta$ indicates whether $Y$ is observed or censored from the right side. The data are then given as a random sample $(X_i, \delta_i, i = 1, \ldots, N)$. Notice that the case without censoring is obtained when $G(t) \equiv 0$ on the region where $F(t) < 1$. One of standard assumptions to prevent the censoring variable cutting off a part of domain of $Y$ is that $\sup\{t : F(t) < 1\} \le \sup\{t : G(t) < 1\}$. Here the notation $\sup\{t : \ldots\}$ means the supremum of the set of $t$ having given property. Let us remark here that in some cases we can deal, for instance, with the logarithm of time. Then the domain of data can be the whole real line.

A generalization of the empirical distribution function is the well known Kaplan–Meier "Product Limit Estimate" (PLE) of survival function. Let us first sort (re-index) the data in increasing order, $X_1 \le X_2 \le \cdots \le X_N$, then the PLE of $\overline{F}(t)$ has the form

$$\overline{F}_N(t) = \prod_{i=1}^{N} \left( \frac{N-i}{N-i+1} \right)^{\delta_i \cdot 1[X_i \le t]}. \tag{1}$$

Again, notice that when all $\delta_i = 1$, we obtain the empirical survival function. The following proposition is due to Breslow and Crowley [2]:

**Proposition 1.** Let $T > 0$ be such that still $\overline{F}(T) \cdot \overline{G}(T) > 0$. Then the random process

$$V_N(t) = \sqrt{N} \left( \frac{\overline{F}_N(t)}{\overline{F}(t)} - 1 \right) = \sqrt{N} \, \frac{F(t) - F_N(t)}{\overline{F}(t)} \tag{2}$$

converges, on $[0, T]$, when $N \to \infty$, to the Gauss martingale with zero mean and the variance function

$$C(t) = \int_0^t \frac{\mathrm{d}F(s)}{\overline{F}(s)^2 \, \overline{G}(s)}. \tag{3}$$

Here, $F_N(t) = 1 - \overline{F}_N(t)$. In other words, $V_N(t)$ converges in distribution on $[0, T]$ to the process $W(C(t))$, where $W(\cdot)$ denotes the Wiener process. The asymptotic variance function can be estimated by its empirical version:

$$C_N(t) = \sum_{i=1}^{N} \frac{N\delta_i}{(N-i+1)^2} \cdot 1[X_i \le t].$$

Both the PLE and its estimated variance function $C_N(t)$ are consistent in probability, uniformly w.r. to $t \in [0, T]$ (see again Breslow and Crowley [2]).

Let us now recall also some properties of empirical quantiles. 'True' $p$-quantile, for any $p \in (0, 1)$, is defined as $Q(p) = \min\{x : F(x) \ge p\}$, and is obtained as a unique solution of equation $F(x) = p$ provided $F$ is strictly increasing. Empirical quantile is then defined as $Q_N(p) = \min\{x : F_N(x) \ge p\}$. Let now $p \in (0, F(T))$, where $T$ is from Proposition 1. Notice that $Q_N(p)$ is well defined only if $F_N(x) \ge p$ for some $x$. However, from the consistency of the PLE it follows that with probability tending to one, when N grows to infinity, there exists $x < T$ such that $F_N(x) \ge p$. The following statement can be found for instance in Andersen et al. [1], Ch.IV.3.

**Proposition 2.** Let $f(x) > 0$ in the neighborhood of $Q(p)$. Then the empirical quantile $Q_N(p)$ is P-consistent and asymptotically normal, namely, for each $c < 1/2$

$$N^c \cdot (Q_N(p) - Q(p)) \to^P 0, \quad \sqrt{N}(Q_N(p) - Q(p)) \to^d N(0, S(p)),$$

and the asymptotic variance equals

$$S(p) = \frac{(1-p)^2 \cdot C(Q(p))}{f(Q(p))^2}.$$

It follows that the variance of $(Q_N(p) - Q(p))$ can be estimated by

$$\frac{S_N(p)}{N} = \frac{(1-p)^2 \cdot C_N(Q_N(p))}{N \cdot f_N(Q_N(p))^2}, \tag{4}$$

which is complicated by inevitable estimation of density function, as a rule with the aid of kernel method.

If we denote $D_N(t) = V_N(t)/(1 + C(t))$, then for the case without censoring we obtain that $C(t) = F(t)/\overline{F}(t)$ and $D_N(t) = \sqrt{N}(F(t) - F_N(t))$ leading to the standard Kolmogorov–Smirnov statistics. Notice also that then we obtain a well known result $asvar[\sqrt{N}(Q_N(p) - Q(p))] = \frac{p(1-p)}{f(Q(p))^2}$.

Further, from (3) it is also seen that the variance in the case with censoring (when $\overline{G}(t) \le 1$) is larger than without it (i. e. when $\overline{G}(t) = 1$ on the whole $[0, T]$).

However, in the sequel we shall need results on strong (a.s.) consistency of the P.L.E. and of corresponding sample quantiles. In the case without censoring, the well known results (Glivenko–Cantelli theorem and its consequences) yield that almost surely, for $t \in (-\infty, \infty)$,

$$\sup_t |F_N(t) - F(t)| = \mathcal{O}\left(N^{\frac{-1}{2}}\right).$$

A similar result for quantiles is due to the Bahadur representation of sample quantile. Namely, let $p \in (0, 1)$, F be twice differentiable at $Q(p)$ with $F'(Q(p)) = f(Q(p)) > 0$. Then almost surely

$$|Q_N(p) - Q(p)| = \mathcal{O}\left(N^{\frac{-1}{2}}\right).$$

The case of censoring is complicated by the fact that the censoring rate increases in the right end of the domain of values of censored random variable $Y$. Nevertheless, there are also results on uniform convergence of the PLE on the whole line. Let us quote here the following:

**Proposition 3** (Rejto [9]) Let us consider the random right-censoring model with distribution functions $F, G$ being continuous. Let there exist $\alpha, \beta \in (0, 1]$ and a real $\tau$ such that $\alpha \overline{F}(t)^\beta \le \overline{G}(t)$ on $[\tau, \infty)$. Then almost surely

$$\sup_t |F_N(t) - F(t)| = \mathcal{O}\left(\left[\frac{\log N}{N}\right]^{\frac{1}{2+\beta}}\right).$$

Again, $F_N(t) = 1 - \overline{F}_N(t)$, $\overline{F}_N(t)$ denotes the PLE of survival function $\overline{F}(t)$ of random variable $Y$.

In the sequel, we shall utilize a weaker result, namely the strong uniform consistency of the PLE with rate $\mathcal{O}(N^{-1/2})$ on $(-\infty, T]$, where $T$ is as in Proposition 1. Such an assertion has been proved for instance in Peterson [7].

Let us now derive the strong consistency of empirical quantiles without using the Bahadur representation result mentioned above.

**Lemma 1.** Let $p \in (0,1)$. Further, let there exists an interval $[a,b]$ such that $a < Q(p) < b$, and for $x \in [a,b]$ it holds that $f(x) \geq d > 0$. Then

$$|Q_N(p) - Q(p)| = \mathcal{O}(N^{-1/2}) \text{ a.s.}$$

P r o o f. From assumptions of Lemma 1 it follows that $b \leq T$ and therefore

$$\sup_{x\in[a,b]} |F_N(x) - F(x)| = \mathcal{O}(N^{-1/2}) \text{ a.s.}$$

Let us then consider a sequence $C_N \to \infty$ such that $C_N/\sqrt{N} \to 0$ (for instance $C_N = N^\gamma$, $\gamma < 1/2$). Then, a.s. and uniformly for $x \in [a,b]$, to each $\varepsilon > 0$ there exists such $N_0$ that for $N > N_0$

$$F_N(x) \in [F(x) - \varepsilon/C_N, F(x) + \varepsilon/C_N].$$

Hence if we take $x_1 = Q(p) - 2\varepsilon/(C_N d)$, $x_2 = Q(p) + 2\varepsilon/(C_N d)$ (both in $(a,b)$ for sufficiently large $N$), we get that $F_N(x_1) < p$, $F_N(x_2) > p$. Therefore $Q_N(p) \in (x_1, x_2)$, i. e.

$$Q_N(p) \in [Q(p) - \frac{2\cdot\varepsilon}{d\cdot C_N}, Q(p) + \frac{2\cdot\varepsilon}{d\cdot C_N}].$$

□

**Remark 1.** The same result holds also in the case when $F(y)$ is not continuous but still uniformly increasing, i. e. $F(x_2) - F(x_1) \geq d\cdot(x_2 - x_1)$ for some $d > 0$ and all $x_1, x_2$ from $[a,b]$ such that $x_1 < Q(p) < x_2$. The difference is that then for some $p$ it may occur that $F(Q(p)) > p$. However, in fact, the equality $F(Q(p)) = p$ was not used in the Lemma 1 proof.

Further, notice that Lemma 1 concerns also the case with random right censoring, due the assumption made above that $\sup\{x : F(x) < 1\} \leq \sup\{x : G(x) < 1\}$.

## 3. CONSISTENCY OF SOLUTION BASED ON EMPIRICAL QUANTILES

Let us return to the optimization problem formulated as the minimization of a $p$-quantile of distribution of the random variable $Z(v) = \varphi(Y, v)$.

Let the following assumptions hold. The first just recapitulates assumptions made in previous section, the second is actually a standard one used in stochastic optimization setting (see for instance Kaňková [4]).

**Assumption A1.** The distribution of r.v. $Y$ is of continuous type with its density function continuous and positive on interval $(\inf\{y : F(y) > 0\}, \sup\{y : F(y) < 1\})$. Further, in the case of censoring, $\sup\{y : F(y) < 1\} \leq \sup\{y : G(y) < 1\}$.

**Assumption A2.** Let $v \in \boldsymbol{V}$, $\boldsymbol{V}$ be compact in $R$, and let the function $\varphi(y,v)$ be continuous in $R \times \boldsymbol{V}$. Further, let $\varphi(y,v)$ be Lipschitz w.r. to $y \in R$, with a constant $L > 0$ not depending on $v$, for each $v \in \boldsymbol{V}$.

First, we are interested in the distribution of r.v. $Z(v)$. Let us denote its distribution function for given $v$ $F_v(z)$, its quantiles $Q_v(p)$. Notice that $F_v(z) = P(\varphi(Y,v) \le z)$ and that the set $\{\varphi(y,v) \le z\} = \cup_{k=1}^{K} I_k$, where $I_k$ are disjoint intervals in $R$. Naturally, they, as well as their number $K = K(z,v)$, depend on $z$, $v$.

**Lemma 2.** For each $v \in \boldsymbol{V}$ the distribution function $F_v(z)$ is strictly increasing in $z \in \boldsymbol{Z}_v = (\inf_y \varphi(y,v), \sup_y \varphi(y,v))$, $y \in R$.

Proof. Let $z_2 > z_1$, both in $\boldsymbol{Z}_v$. Then there is at least one couple of points $y_1, y_2$ at which $\varphi(y,v)$ crosses levels $z_1, z_2$. The distance $|y_2 - y_1| \ge (z_2 - z_1)/L$, hence $F_v(z_2) - F_v(z_1) \ge (z_2 - z_1)/L \cdot \min\{f(y) : y \in (y_1, y_2)\}$. As this minimum of $f(y)$ is positive, the assertion is proved. □

Let us denote by $F_{N,v}(z)$ the estimate of distribution function $F_v(z)$, based on the PLE or EDF $F_N(y)$. Namely, $F_{N,v}(z) = \sum_{k=1}^{K} (F_N(R_k) - F_N(L_k)),$, where $L_k$, $R_k$ are left and right endpoints of interval $I_k$ from above.

**Proposition 4.** Let function $\varphi(y,v)$ be such that the number of crossing points of each level $z$, for each $v$, is bounded, i. e. $K(z,v) \le K^* < \infty$. Then, for $z \in \boldsymbol{Z}_v, v \in \boldsymbol{V}$,

$$\sup_{(z,v)} |F_{N,v}(z) - F_v(z)| = \mathcal{O}\left(N^{\frac{-1}{2}}\right) \ a.s.$$

Proof. As the number of crossing points, and therefore of intervals in decomposition of the set $\{\varphi(Y,v) \le z\}$, is bounded, uniformly, then, a.s.,

$$\sup_{(z,v)} |F_{N,v}(z) - F_v(z)| \le 2K^* \sup_{y \le T} |F_N(y) - F(y)| = \mathcal{O}\left(N^{\frac{-1}{2}}\right).$$

The condition of the proposition is fulfilled for instance when functions $\varphi(y,v)$ are convex or concave in $y$; then $K^* = 2$. □

A similar result can be proved for consistency of empirical quantiles, however, at least for now, separately for each $v \in \boldsymbol{V}$. Quantiles of r.v. $Z_v$ are defined in a standard way, i. e. for $p \in (0,1)$ $Q_v(p) = \min\{z : F_v(z) \ge p\}$, while corresponding empirical quantile is $Q_{N,v}(p) = \min\{z : F_{N,v}(z) \ge p\}$. The following statement follows then from Lemma 1 and Remark 1 dealing with non-continuous but strictly increasing distribution functions.

**Proposition 5.** Under conditions of Proposition 4, for each $v \in \boldsymbol{V}$ and $p \in (0,1)$,

$$|Q_{N,v}(p) - Q_v(p)| = \mathcal{O}\left(N^{\frac{-1}{2}}\right) \ a.s.$$

However, in the sequel, in order to prove the main result, we shall need a stronger conditions ensuring that the increase of $F_v(z)$ is uniform w.r. to $v \in \boldsymbol{V}$:

**Assumption A3.** Let $p \in (0,1)$ be such that distribution functions $F_v(z)$ are strictly increasing in a neighborhood of $Q_v(p)$, uniformly w.r. to $v \in \boldsymbol{V}$.

In other words, there exist $z_1, z_2$ such that $z_1 < Q_v(p) < z_2$ and $d > 0$ such that $F_v(z_2) - F_v(z_1) \geq d(z_2 - z_1)$, for all $v \in \boldsymbol{V}$.

**Corollary 1.** If Assumption A3 is added, then the result of Proposition 5 holds uniformly w.r. to $v \in \boldsymbol{V}$.

## Main results

The main objective is to show that the minimum of $Q_v(p)$ over $v \in \boldsymbol{V}$ can be approached by optimal results based on empirical estimates. As the set $\boldsymbol{V}$ is compact, there exist $Q^*(p) = \min_v Q_v(p)$ which is achieved in at least one point $v^* \in \boldsymbol{V}$. Let us denote the set of all such solutions $\boldsymbol{V}^* \subset \boldsymbol{V}$. Further, the same holds for the task of minimization of $Q_{N,v}(p)$ over $v$, for each $N$. Namely, the minimum $Q^*_N(p) = \min_v Q_{N,v}(p)$ is achieved in some set of (random) points $v^*_N$ with values in $\boldsymbol{V}$. Let us denote this set $\boldsymbol{V}^*_N$. Again, it is random, depending on realized data.

**Theorem 1.** Let the assumptions A1, A2, A3, and conditions of Proposition 4 hold, $p \in (0,1)$. Then a.s.

$$|Q^*_N(p) - Q^*(p)| = \mathcal{O}\left(N^{\frac{-1}{2}}\right).$$

P r o o f. Consider any sequence $C_N \to \infty$ such that $C_N/\sqrt{N} \to 0$, any $v^* \in \boldsymbol{V}^*$, any sequence $v^*_N \in \boldsymbol{V}^*_N$. From Corollary 1 it follows that a.s.

$$C_N|Q_{N,v^*}(p) - Q^*(p)| \to 0, \;\; C_N|Q_{N,v^*_N}(p) - Q_{v^*_N}(p)| \to 0.$$

Further, $Q_{N,v^*_N}(p) \leq Q_{N,v^*}(p)$ and $Q^*(p) \leq Q_{v^*_N}(p)$. Hence also $C_N|Q_{N,v^*_N}(p) - Q^*(p)| \to 0$. □

We are also interested whether there exists a sequence of "empirical" solutions $v^*_N$ converging towards "true" solutions in $\boldsymbol{V}^*$. To prove it, we shall utilize the following proposition taken from Kibzun and Kan [5], see also Timofeeva [10]. Its assumptions are in fact covered by assumptions A1 and A2:

**Proposition 6.** Let A1, A2 hold, further let $P\{|\varphi(Y,v) - z| \leq \varepsilon\} > 0$ for each $\varepsilon > 0$, $v \in V$, $y \in R$ and $z \in (\inf_y \varphi(y,v), \sup_y \varphi(y,v))$. Then

$$Q_v(p) \text{ is continuous in } v, \text{ for each } p \in (0,1).$$

Again, from the compactness of $\boldsymbol{V}$ the uniform continuity of $Q_v(p)$ follows. The assumption that $P\{|\varphi(Y,v) - z| \leq \varepsilon\} > 0$ follows from A2, namely from the Lipschitz continuity of $\varphi(y,v)$ in $y$.

**Theorem 2.** Let the conditions of Theorem 1 hold. Then there exists a subsequence $N_k$ such that corresponding optimal solutions $v^*_{N_k} \in \boldsymbol{V}^*_{N_k}$ converge a.s. to a $v_0$ having values in $\boldsymbol{V}^*$.

Proof. Existence of a converging subsequence inside each sequence in $\boldsymbol{V}$ follows from its compactness. Hence there exists also converging subsequence $v^*_{N_k}$ selected from $v^*_N \in \boldsymbol{V}^*_N$. We have to show that its (random) limit $v_0$ belongs a.s. to $\boldsymbol{V}^*$. From Proposition 6, Corollary 1 and Theorem 1 it follows that a.s.

$$|Q_{v^*_{N_k}}(p) - Q_{v_0}(p)| \to 0, \;\; |Q_{N_k, v^*_{N_k}}(p) - Q_{v^*_{N_k}}(p)| \to 0, \;\; |Q_{N_k, v^*_{N_k}}(p) - Q^*(p)| \to 0.$$

Hence $Q_{v_0}(p) = Q^*(p)$ and $v_0 \in \boldsymbol{V}^*$ a.s. □

Thus, except the convergence of optimal values of quantiles we showed also the existence of a random sequence of solutions converging towards the set of optimal solutions $\{v^*\}$. If $v^*$ is unique, then $v_0 = v^*$ a.s.

## 4. EXAMPLE, OPTIMIZATION IN NEWSVENDOR PROBLEM

Let us consider the following rather simple example of optimization problem in standard newsvendor model, in order to show the behavior of optimal solutions based on estimated quantiles and to study the influence of censoring.

Let $D$ be a (nonnegative) random variable with distribution function $F$ representing the demand (of units of certain commodity), let each unit be sold for price $q$ and purchased for price $c < q$, $S$ be the number of units stocked, i. e. purchased to be sold. In the simplest case $c$ and $q$ are fixed, we search the solution to the optimal stocking quantity $S$ which maximizes the profit

$$Z = q \cdot \min(S, D) - c \cdot S.$$

As $D$ and therefore $Z$ are random, we have a choice which criterion to maximize. As the newsvendor's actions are repeated regularly, the expectation of profit could be preferred, however quantiles could also be of interest. It is well known and can be shown easily that to get $\max_S EZ$, the optimal solution is $S = S_E = ((q-c)/q)$-quantile of distribution of $D$. It follows from the maximization of

$$EZ = (q-c)S - q \int_0^S F(x)\,\mathrm{d}x.$$

Hence, the mean profit is achieved through the quantile of underlying distribution. Let us add that for the maximization of $\alpha$-quantile of $Z$ the optimal stock equals $S = S_\alpha = Q_D(\alpha)$, the $\alpha$-quantile of $D$, and achieved $\max_S Q_Z(\alpha) = Q_D(\alpha)(q-c)$. See for instance Petruzzi and Dada [8] and also Kim and Powell [6].

In the simulation study let us assume that the underlying demand distribution is $D \sim N(\mu = 150, \sigma = 20)$, the prices are $q = 15, c = 10$. Optimal value of $S$ maximizing the mean profit is $S_E = Q_D(1/3) = 141.386$, maximal mean profit then achieves $EZ = 640.92$. However, the demand is observed just through data. We can imagine that the data on demand were collected from the past newsvendor's experience, the censoring occurred when the stock was smaller than demand. Let us consider two cases:

A) Fully observed data $X_i = D_i$, $i = 1, \ldots, N$, ($N$ =50 or 200).

B) Randomly right-censored data, censored by a random variable $Z$. In this case we observe $X_i = \min(D_i, Z_i)$. Let the distribution of $Z$ be uniform on $(120, 200)$, such a selection leads to approximately 40% censoring.

Each generation of data was repeated $K = 100$-times in order to obtain a representation (sample distribution) of results. From each sample the PLE (the EDF in the case A) $F_N$ of the distribution function of demand was computed and $S$ taken optimal w.r. to it, i.e. $S = S_N = Q_N(1/3) = \min\{X_i : F_N(X_i) \geq 1/3\}$. Results, namely the values $S_N$ and corresponding achieved mean profits E$Z$ when $S = S_N$ are displayed in Figure 1.

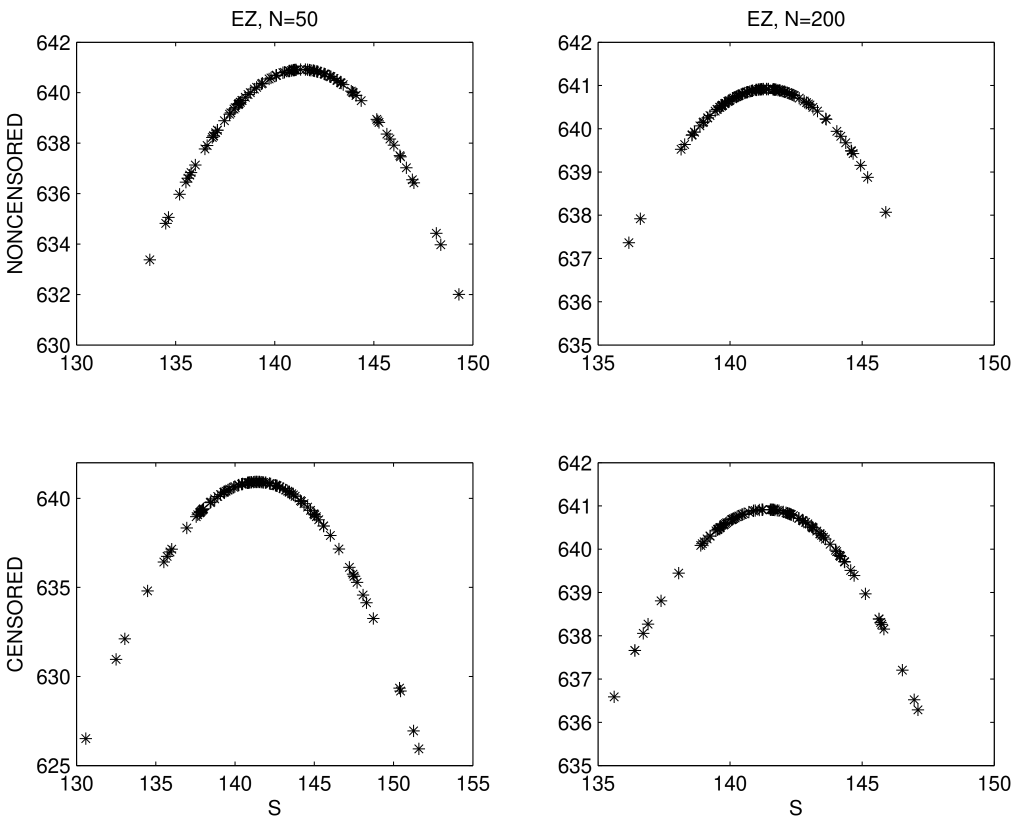


**Fig. 1.** Empirical distribution of mean profits at "sub-optimal" $S_N$ based on estimated distribution of demand, for data extent $N = 50$ and 200, each repeated 100 times.

Two rather expected phenomena are observed, first, that the variability decreases with growing data sample, and second, that the censoring really leads to smaller information and therefore larger uncertainty of achieved expected profit.

## 5. CONCLUSION

We have studied the impact of variability of statistical estimates to uncertainty of solution in a stochastic optimization problem formulated via certain quantile of objective function. We compared two cases: In the first case, stochastic characteristics of the problem were estimated, in a non-parametric way, from fully observed data. In the second case the estimates of the same characteristics were based on randomly right-censored data. Theoretical properties of estimators of distribution function and quantiles from censored data were recalled. Then, the convergence of solutions based on estimated quantiles to optimal solutions was proven, the rate of convergence established. The behavior of estimates in real situations was studied with the aid of a simple optimization problem example and randomly generated data.

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