A Note on Stochastic Optimization Problems with Nonlinear Dependence on a Probability Measure

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Abstract. Nonlinear dependence on a probability measure begins to appear (last time) in a stochastic optimization rather often. Namely, the corresponding type of problems corresponds to many situations in applications. The nonlinear dependence can appear as in the objective functions so in a constraints set. We plan to consider the case of static (one-objective) problems in which nonlinear dependence appears in the objective function with a few types of constraints sets. In details we consider constraints sets "deterministic", depending nonlinearly on the probability measure, constraints set determined by second order stochastic dominance and the sets given by mean-risk problems. The last case means that the constraints set corresponds to solutions those guarantee an acceptable value in both criteria. To introduce corresponding assertions we employ the stability results based on the Wasserstein metric and $\mathcal{L}_1$ norm. Moreover, we try to deal also with the case when all results have to be obtained (estimated) on the data base.

Keywords: stochastic optimization problem, nonlinear dependence, static problems, second order stochastic dominance, mean–risk model, stability, empirical estimates

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1 Introduction

Optimization problems with nonlinear dependence on a probability measure begin to appear (in a stochastic literature) (see, e.g., [2], [3], [6], [7]). This type of problems corresponds to many situations from practice.

To introduce the above mentioned type of the problems let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi \; (:= \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F_\xi(z), z \in R^s)$ the distribution function of ξ; P_F, Z_F the probability measure and a support corresponding to F; $X_F \subset X \subset R^n$ a nonempty set generally depending on F; $X \subset R^n$ a nonempty "deterministic" set. If $\bar{g}_0(:= \bar{g}_0(x, z, y))$ is a real–valued function defined on $R^n \times R^s \times R^{m_1}$; $h(:= h(x, z)) = (h_1(x, z), \ldots, h_{m_1}(x, z))$ is an m_1–dimensional vector function defined on $R^n \times R^s$, then stochastic (static) optimization problem with a nonlinear dependence on the probability measure can be introduced in the form:

Find
$$\bar{\varphi}(F, X_F) = \inf\{\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi)) | x \in X_F\}. \tag{1}$$

Evidently a nonlinear dependence can appear as in the objective function so in the constraints set. We consider a few types of X_F :

$$\begin{array}{lll} a. & X_F := & X, \\ b. & X_F := & \{x \in X : \mathsf{E}_F \bar{g}_i(x, \xi, \mathsf{E}_F h(x, \xi)) \le 0, \; i = 1, \ldots, m\}, \\ & & \text{where} \quad \bar{g}_i(x, z, y), \; i = 1, \ldots, m \quad \text{are defined on} \quad R^n \times R^s \times R^{m_1}. \end{array} \tag{2}$$

Of course it is assumed that all finite mathematical expectation in the relations (1), (2) exist.

Problem (1) covers a classical problem with linear dependence in the form:

Find
$$\varphi(F, X_F) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X_F\}, \tag{3}$$

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with $\bar{g}_0(x, z, y) := g_0(x, z)$, $X_F = X$; $g_0(:= g_0(x, z))$ a real-valued function defined on $R^n \times R^s$.

To introduce the next type of X_F we have first to recall a notion of second order stochastic dominance. If $V := V(\xi)$, $Y := Y(\xi)$ are random values for which there exist finite $\mathsf{E}_F V(\xi)$, $\mathsf{E}_F Y(\xi)$ and if

$$F^2_{Y(\xi)}(u) = \int_{-\infty}^{u} F_{Y(\xi)}(y)dy, \quad F^2_{V(\xi)}(u) = \int_{-\infty}^{u} F_{V(\xi)}(y)dy,$$

then $V(\xi)$ dominates in second order $Y(\xi)$ $(V(\xi) \succeq_2 Y(\xi))$ if

$$F^2_{V(\xi)}(u) \le F^2_{Y(\xi)}(u) \quad \text{for every} \quad u \in R^1.$$

To define second order stochastic dominance constraints set X_F, let $g(x, \xi)$ (defined on $R^n \times R^s$) be for every $x \in X$ a random variable with distribution function $F_{g(x, \xi)}$. Let, moreover, for every $x \in X$ there exists finite $\mathsf{E}_F g(x, \xi)$, $\mathsf{E}_F Y(\xi)$ and

$$F^2_{g(x, \xi)}(u) = \int_{-\infty}^{u} F_{g(x, \xi)}(y)dy, \quad F^2_{Y(\xi)}(u) = \int_{-\infty}^{u} F_{Y(\xi)}(y)dy, \quad u \in R^1,$$

then rather general second order stochastic dominance constraints set X_F can be defined by

c.
$$X_F = \{x \in X : F^2_{g(x, \xi)}(u) \le F^2_{Y(\xi)}(u) \quad \text{for every} \quad u \in R^1\}, \tag{4}$$

equivalently by

$$X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in R^1\}. \tag{5}$$

(The equivalence of the constraints sets (4) and (5) can be found in [10]. For definitions of the stochastic dominance of others orders see, e.g., [9].)

To introduce the last considered type of the set X_F we start with classical mean-risk problem:

Find

$$\max \mathsf{E}_F g_0(x, \xi), \quad \min \rho_F(g_0(x, \xi)) \quad \text{s.t.} \quad x \in X; \quad \rho(\cdot) := \rho_F(\cdot) \quad \text{denotes a risk measure.} \tag{6}$$

Evidently to optimize simultaneously both objectives is mostly impossible. However it can happen that there exist two real-valued acceptable constants ν_2, ν_1 and $x_0 \in X$ such that

$$\mathsf{E}_F g_0(x_0, \xi) \ge \nu_2, \quad \rho_F(g_0(x_0, \xi)) \le \nu_1. \tag{7}$$

If furthermore the function g_0 and risk measure ρ_F follow the following definition:

Definition. [4] The mean–risk model (6) is called consistent with the second order stochastic dominance if for every $x \in X$ and $x' \in X$,

$$g_0(x, \xi) \succeq_2 g_0(x', \xi) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x, \xi) \ge \mathsf{E}_F g_0(x', \xi) \quad \text{and} \quad \rho_F(g_0(x, \xi)) \le \rho_F(g_0(x', \xi)); \tag{8}$$

then we can define the set $X_F(x_0)$ by

$$X_F(x_0) = \{x \in X, x \neq x_0 : \mathsf{E}_F(u - g_0(x, \xi))^+ \le \mathsf{E}_F(u - g_0(x_0, \xi))^+ \quad \text{for every} \quad u \in R^1\}. \tag{9}$$

In the case when $X_F(x_0)$ is a nonempty set, then according to the relation (8) we can see that

$$x \in X_F(x_o) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x, \xi) \ge \mathsf{E}_F g_0(x_0, \xi) \quad \text{and simultaneously} \quad \rho_F(g_0(x, \xi)) \le \rho_F(g_0(x_0, \xi)).$$

Evidently, we can set

d. $X_F := X_F(x_0)$.

2 Some Definitions and Auxiliary Assertions

In this section we recall some auxiliary assertions suitable for stability and empirical estimates of the problems (1), (3) with constraints sets a., b., c., d. To this end, first, let $\mathcal{P}(R^s)$ denote the set of all (Borel) probability measures on R^s and let the system $\mathcal{M}_1^1(R^s)$ be defined by the relation:

$$\mathcal{M}_1^1(R^s) \;=\; \{\nu \in \mathcal{P}(R^s) : \int\limits_{R^s} \|z\|_1 d\nu(z) < \infty\}, \quad \|\cdot\|_1 \quad \text{denotes} \quad \mathcal{L}_1 \quad \text{norm in} \quad R^s.$$

We introduce the system of the assumptions:

A.1

1. $g_0(x,z)$ is for $x \in X$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x,
2. $g_0(x,z)$ is either a uniformly continuous function on $X \times R^s$ or there exists $\varepsilon > 0$ such that $g_o(x,\, z)$ is a convex bounded function on $X(\varepsilon)$; ($X(\varepsilon)$ denotes the ε–neighborhood of the set$X$),

B.1 For $P_F,\, P_G \in \mathcal{M}_1^1(R^s)$, there exist $\varepsilon > 0$ such that

1. $\bar{g}_0(x,\, z,\, y)$ is for $x \in X(\varepsilon),\, z \in R^s$ a Lipschitz function of $y \in Y(\varepsilon)$ with a Lips. constant L_y; $Y(\varepsilon) = \{y \in R^{m_1} : y = h(x,\, z) \quad \text{for some} \quad x \in X(\varepsilon),\, z \in R^s\}$, $\mathsf{E}_F h(x,\, \xi),\;\; \mathsf{E}_G h(x,\, \xi) \in Y(\varepsilon)$,
2. for every $x \in X(\varepsilon),\, y \in Y(\varepsilon)$ there exist finite mathematical expectations, $\mathsf{E}_F \bar{g}_0(x,\, \xi,\, \mathsf{E}_F h(x,\, \xi))$, $\mathsf{E}_F \bar{g}_0(x,\, \xi,\, \mathsf{E}_G h(x,\, \xi))$, $\quad \mathsf{E}_G \bar{g}_0(x,\, \xi,\, \mathsf{E}_F h(x,\, \xi))$, $\quad \mathsf{E}_G \bar{g}_0(x,\, \xi,\, \mathsf{E}_G h(x,\, \xi))$,
3. $h_i(x,\, z),\; i = 1,\, \ldots,\, m_1$ are for every $x \in X(\varepsilon)$ Lipschitz functions of z with the Lipschitz constants L_h^i (corresponding to $\mathcal{L}_1$),
4. $\bar{g}_0(x,\, z,\, y)$ is for every $x \in X(\varepsilon),\, y \in Y(\varepsilon)$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L_z (corresponding to $\mathcal{L}_1$ norm),

B.2 $\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi))$, $\mathsf{E}_G \bar{g}_0(x, \xi, \mathsf{E}_G h(x, \xi))$ are continuous functions on X.

If $F_i(z_i), G_i(z_i),\; i = 1,\, \ldots,\, s$ are one dimensional marginal distributions corresponding to $F,\; G$, then we can recall

Proposition 1. [7]. Let $P_F, P_G \in \mathcal{M}_1^1(R^s)$ and let X be a compact set. If

1. Assumption A.1 1 is fulfilled, then for $x \in X$ it holds

$$|\mathsf{E}_F g_0(x,\xi) - \mathsf{E}_F g_0(x,\xi)| \;\leq L \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} F_i(z_i) - G_i(z_i)|dz_i. \tag{10}$$

   If moreover A.1 2 is fulfilled, then also

$$|\varphi(F,\, X) - \varphi(G,\, X)| \leq L \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|dz_i, \tag{11}$$

2. Assumptions B.1 are fulfilled, then there exist $\hat{C} > 0$ such that for $x \in X$

$$|\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi)) \;-\; \mathsf{E}_G \bar{g}_0(x, \xi, \mathsf{E}_G h(x, \xi))| \leq \hat{C} \sum_{i=1}^{s} \int\limits_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)|dz_i. \tag{12}$$

   If moreover B.2 is fulfilled, then also

$$|\bar{\varphi}(F,\, X) - \bar{\varphi}(G,\, X)| \leq \hat{C} \sum_{i=1}^{s} \int\limits_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)|dz_i. \tag{13}$$

To complete auxiliary assertions we recall a very useful inequalities. To this end let $P_F,\ P_G \in \mathcal{M}_1^1(R^s)$ and let problems (1), (3) be well defined, then on employing the triangular inequality we can obtain

$$\begin{aligned} |\varphi(F,\, X_F) - \varphi(G,\, X_G)| &\leq |\varphi(F,\, X_F) - \varphi(G,\, X_F)| + |\varphi(G,\, X_F) - \varphi(G,\, X_G)|, \\ |\bar{\varphi}(F,\, X_F) - \bar{\varphi}(G,\, X_G)| &\leq |\bar{\varphi}(F,\, X_F) - \bar{\varphi}(G,\, X_F)| + |\bar{\varphi}(G,\, X_F) - \bar{\varphi}(G,\, X_G)|. \end{aligned} \tag{14}$$

Evidently, employing the assertion of Proposition 1 and the relations (14) we can bounded the gaps between optimal values of the problems (1), (3) with different distributions F and G. However to this end it is reasonable, first, to define for $\varepsilon \in R^1$ the sets X_F^{ε} by

- $$X_F^{\varepsilon} = \{x \in X:\ \mathsf{E}_F \bar{g}_1(x,\, \xi,\, \mathsf{E}_F h(x,\, \xi)) \leq \varepsilon\} \quad \text{in the case of constraints set } b. \text{ with } m = 1, \tag{15}$$

- $$\begin{aligned} X_F^{\varepsilon} &= \{x \in X:\ \mathsf{E}_F(u - g(x,\, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \leq \varepsilon \quad \text{for every} \quad u \in R^1\} \\ &\quad \text{in the case of constraint set} \quad c. \end{aligned} \tag{16}$$

Employing the last relations, assumptions of Proposition 1 and the approach of the work [8] we can obtain:

1. in the case of constraint b. with $m = 1$ and the function $\bar{g}_1$ fulfilling the assumption B.1 that

$$X_G^{\delta - \varepsilon} \subset X_F^{\delta} \subset X_G^{\delta + \varepsilon} \quad \text{with} \quad \delta \in R^1, \quad \varepsilon = \hat{C} \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i, \tag{17}$$

2. in the case of constraint set c; $g(x,\, z),\ Y(z)$ to be for every $x \in X$ Lipschitz functions of $z \in R^s$ with the Lipschitz constant L_g not depending on $x \in X$, that

$$X_G^{\delta - \varepsilon} \subset X_F^{\delta} \subset X_G^{\delta + \varepsilon} \quad \text{with} \quad \delta \in R^1, \quad \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i. \tag{18}$$

Evidently, the analysis and the results of this section can be employed in the case when the distribution function G is replaced by empirical one.

3 Empirical Estimates

First, in this section, we introduce a new system of the assumptions.

A.2
- $\{\xi^i\}_{i=1}^{\infty}$ is an independent random sequence corresponding to F,
- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}, N = 1, 2, \ldots,$

A.3 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w. r. t. the Lebesgue measure on R^1.

Empirical problem, corresponding to the "underlying" problem (1), can be introduced in the form:

Find

$$\bar{\varphi}(F^N,\, X_{F^N}) = \inf\{\mathsf{E}_{F^N} \bar{g}_0(x, \xi, \mathsf{E}_{F^N} h(x, \xi)) | x \in X_{F^N}\}. \tag{19}$$

Employing the idea of the paper [8], we can obtain.

Proposition 2. Let X be a nonempty compact set, $P_F \in \mathcal{M}_1^1(R^s)$, X_F be defined by the relation (5); $X_F,\ X_{F^N},\ N = 1, \ldots$ be nonempty compact sets. Let, moreover, $g(x,\, z),\ Y(z)$ be for every $x \in X$ Lipschitz functions of $z \in Z_F$ with the Lipschitz constant L_g not depending on $x \in X$. If

1. Assumptions B.1, B.2 , A.2 are fulfilled,
2. $\mathsf{E}_F\bar{g}_0(x,\,\xi,\,\mathsf{E}_F h(x,\,\xi))$ is a Lipschitz function on X,
3. there exists $\varepsilon_0 > 0$ such that X_F^{ε} (defined by the relation (15)) are nonempty compact sets for every $\varepsilon \in \langle -\varepsilon_0,\, \varepsilon_0\rangle$ and, moreover, there exists a constant $\hat{C} > 0$ such that

$$\Delta_n[X_F^{\varepsilon},\; X_F^{\varepsilon'}] \;\le \hat{C}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon, \varepsilon' \in \langle -\varepsilon_0,\, \varepsilon_0\rangle,$$

then

$$P\{\omega :\; |\bar{\varphi}(F,\, X_F) - \bar{\varphi}(F^N,\, X_{F^N})| \longrightarrow_{N\longrightarrow\infty} 0\} = 1. \tag{20}$$

($\Delta[\cdot,\,\cdot] := \Delta_n[\cdot,\,\cdot]$ denotes the Hausdorff distance in the subsets of n-dimensional Euclidean space; for definition see, e.g., [11].)

Proposition 3. Let X be a nonempty compact set, $P_F \in \mathcal{M}_1^1(R^s)$, X_F be defined by the relation (2); $X_F,\, X_{F^N},\, N = 1, \ldots$ be nonempty compact sets.
1. functions $\bar{g}_0,\, \bar{g}_1$ fulfil Assumptions B.1, B.2; Assumption A.2 is fulfilled,
2. $\mathsf{E}_F\bar{g}_0(x,\,\xi,\,\mathsf{E}_F h(x,\,\xi))$ is a Lipschitz function on X,
3. there exists $\varepsilon_0 > 0$ such that X_F^{ε} (defined by the relation (15)) are nonempty compact sets for every $\varepsilon \in \langle -\varepsilon_0,\, \varepsilon_0\rangle$ and, moreover, there exists a constant $\hat{C} > 0$ such that

$$\Delta_n[X_F^{\varepsilon},\; X_F^{\varepsilon'}] \;\le \hat{C}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon, \varepsilon' \in \langle -\varepsilon_0,\, \varepsilon_0\rangle,$$

then

$$P\{\omega :\; |\bar{\varphi}(F,\, X_F) - \bar{\varphi}(F^N,\, X_{F^N})| \longrightarrow_{N\longrightarrow\infty} 0\} = 1. \tag{21}$$

Remark. In the both cases (Proposition 2, Proposition 3) $\bar{\varphi}(F^N,\, X_{F^N})$ is a consistent estimate of $\bar{\varphi}(F,\, X_F)$. Evidently, it is possible also to prove results about the rate of convergence for this estimates. However to present the corresponding assertion is beyond the scope of this contribution.

It remains to deal with an analysis of the constraints set X_F corresponding to the case d. If we can assume that constants $\nu_2,\, \nu_1$ fulfill condition (7) with some $x_0 \in X$, then we can replace distribution function F by F^N and try to find x_0^N and $X_{F^N}(x_0^N)^N$ such that for $N = 1, 2, \ldots$

$$\mathsf{E}_F g_0(x_0^N,\, \xi) \ge \nu_2 - \frac{1}{N}, \quad \rho_F(g_0(x_0^N,\, \xi)) \le \nu_1 + \frac{1}{N}.$$

$$X_{F^N}(x_{0^N}) = \{x \in X, x \neq x_0^N : \mathsf{E}_F(u - g_0(x,\xi))^+ \le \mathsf{E}_F(u - g_0(x_0^N,\xi))^+ \quad \text{for every } u \in R^1\} \tag{22}$$

Let X be a compact set and let there exist a constant $C^1 > 0$ such that

$$|\rho_F(g_0(x,\,\xi)) \;-\; \rho_{F^N}(g_0(x,\,\xi))| \;\le\; C^1 \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i \quad \text{for every} \quad x \in X. \tag{23}$$

If we can assume that $\rho_F(g_0(x,\,\xi))$ is a uniformly continuous on X, assumptions A.1 are fullfilled, then we can obtain

$$\begin{aligned} P\{\omega : |\rho_F(g_0(x_0,\,\xi)) \;-\; \rho_{F^N}(g_0(x_0^N,\,\xi))| \longrightarrow_{N\longrightarrow\infty} 0\,\} &= 1, \\ P\{\omega : |\mathsf{E}_F g_0(x_0,\,\xi) \;-\; \mathsf{E}_{F^N} g_0(x_0^N,\,\xi)| \longrightarrow_{N\longrightarrow\infty} 0\,\} &= 1. \end{aligned} \tag{24}$$

Consequently, we have proven the convergence $\mathsf{E}_{F^N} g_0(x_0^N,\,\xi)$ to $\mathsf{E}_F g_0(x_0,\,\xi)$ and $\rho_{F^N}(g_0(x_0^N,\,\xi))$ to $\rho_F(g_0(x_0,\,\xi))$ (a.s.). This result is important for us. However, our aim is to find out the assumptions under which

$$P\{\omega : |\bar{\varphi}(F, X_F(x_0)) - \bar{\varphi}(F^N, X_{F^N}(x_0^N)))| \longrightarrow_{N\longrightarrow\infty} 0\} \;= 1. \tag{25}$$

Evidently if we can assume that

$$P\{\omega : \Delta[X_F^0(x_0),\, X_{F^N}^0(x_0^N)] \longrightarrow_{N\longrightarrow\infty} 0\} \;= 1 \tag{26}$$

and if we add to all above mentioned assumptions:

- $P_F \in \mathcal{M}_1^1(R^1), X, X_F(x_0), X_{F^N}(x_0^N),\ N = 1, 2, \ldots$ are nonempty compact sets,
- $\mathsf{E}_F g_0(x_0,\ \xi,\ \mathsf{E}_F h(x,\xi))$ is a Lipschitz function on X,
- Assumptions B.1, B.2, A.2 are fulfilled,

then the assertion (25) is valid.

4 Conclusion

The contribution is focused on a special type of the stochastic optimization problems in which dependence on the probability measure is not linear. This type of problems corresponds to real-life situations rather often. A risk given by variance (in the mean-risk problem) is well known example of this class. The aim of this contribution is to show that many properties of these problems (under acceptable assumptions) are similar to them in the "classical" case. However the detailed analysis is beyond the scope of this contribution.

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