Information Fusion 53 (2020) 165–173

Contents lists available at ScienceDirect

Information Fusion

journal homepage: www.elsevier.com/locate/inffus

Full Length Article

The axiomatization of asymmetric disjunction and conjunction

Miroslav Hudec[a,b,*], Radko Mesiar[c,d]

[a] Faculty of Organizational Sciences, University of Belgrade, Belgrade, Serbia
[b] Faculty of Economic Informatics, University of Economics in Bratislava, Bratislava, Slovakia
[c] Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
[d] Institute of Information Theory and Automation of the Czech Academy of Sciences, Prague, Czech Republic

ARTICLE INFO

Keywords:
Or else operators
And if possible operators
Averaging functions
Continuous and non–continuous fusion functions
Axiomatization

ABSTRACT

In many real–world cases, disjunction is expressed as the fusion of full alternatives and less relevant ones, which leads to an *OR ELSE* connective. Obviously, this connective, so-called intensified disjunction, should provide a solution lower than or equal to the *MAX* operator, and higher than or equal to the projection of the full alternative. Further, to cover the cases when higher satisfaction degrees to the less relevant alternative cause that it becomes the full alternative, non–continuous asymmetric disjunction is required. The dual observation holds for the fusion of constraints (hard conditions) and wishes (soft conditions) expressed by an *AND IF POSSIBLE* connective. In order to cover these requirements, the paper focuses on developing a full axiomatization of asymmetric disjunction and asymmetric conjunction by averaging functions. Next, the necessity and sufficiency for associative behaviour have been proven. Moreover, the non–dual cases are also documented. Finally, the obtained results are illustrated on examples, and their applicability is also discussed.

1. Introduction

In many real–world questions people do not consider *OR* operator as commutative. Instead of, they consider it as the left–right order of predicates (alternatives), i.e. the first predicate is the full alternative, whereas the other ones are less relevant alternatives [1]. An illustrative example is the requirement: "*go to the market and buy broccoli or cauliflower*". The main focus should be on broccoli. If we find broccoli, the score is 1. If we find cauliflower, the score should be less than 1, but better than 0 (more restrictive than disjunction). If we find neither broccoli nor cauliflower, the score is 0. Finally, if we find both, the score is 1 (we are going to cook broccoli and store cauliflower in the larder for the future use, so this task is not considered as the *EXCLUSIVE OR* one). This observation leads to the "or else" interpretation, i.e. P_1 *OR ELSE* P_2, which means that P_2 is not considered as a full alternative to P_1 [2,3] and therefore to the function which values are lower than or equal to the *MAX* function (the lowest disjunctive function), and higher than or equal to the projection of the full alternative. The problem becomes more complex when the intensities of satisfying predicates P_1 and P_2 are considered (in our example, ripeness). Further, the higher satisfaction degree of the optional alternative might cause that it becomes the full alternative, i.e. "*broccoli or else cauliflower, but if cauliflower is very ripe, then it becomes the full alternative*".

The dual observation holds for the *AND* operator, where we shift to the "and if possible" relaxation of conjunction, i.e. P_1 *AND IF POSSIBLE* P_2, where P_2 is not as restrictive as P_1 [2,4]. An example might be: "*searching for a non–expensive and if possible near to the beach hotel*". Clearly, a hotel should meet the fuzzy predicate *is non–expensive* (expressed by a monotone decreasing function) with a degree greater than 0. If a hotel is far from the beach, then it is still acceptable, but with a lower degree than a hotel which is non–expensive and near to the beach. If a hotel is not non–expensive, then it gets score 0, regardless the extreme closeness to the beach. In this case, the function gets a value which might be greater than or equal to the *MIN* function (the highest conjunctive function), and lower than or equal to the projection of P_1. This observation leads to "relaxed conjunction".

In the aforementioned examples, P_1 and P_2 may be any type of predicates, i.e. atomic or compound (e.g., quantified). Thus, we should deeply examine "intensified disjunction" and "relaxed conjunction" to offer the mathematical formalization for covering diverse real–world expectations and promising future topics, e.g., [5–8].

Both conjunctions and disjunctions, belong to the large class of aggregation functions, i.e. functions A: $[0, 1]^n \rightarrow [0, 1]$ which are monotone and satisfy the boundary conditions $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$, $n \in \mathbb{N}$. The standard classification of aggregation functions is due to Dubois and Prade [9]. Namely, conjunctive aggregation functions are

* Corresponding author.
E-mail address: miroslav.hudec@fon.bg.ac.rs (M. Hudec).

https://doi.org/10.1016/j.inffus.2019.06.018
Received 4 December 2018; Received in revised form 7 June 2019; Accepted 9 June 2019
Available online 11 June 2019
1566-2535/

characterized by $A(\mathbf{x}) \leq \min(\mathbf{x})$, disjunctive by $A(\mathbf{x}) \geq \max(\mathbf{x})$, averaging by $\min(\mathbf{x}) \leq A(\mathbf{x}) \leq \max(\mathbf{x})$, and remaining aggregation functions are called hybrid, where $\mathbf{x}$ is a vector, $\mathbf{x} = (x_1, \dots, x_n)$.

Observe that disjunction formalizes reasoning, where satisfaction of any of the predicates suffices, but more than one partially satisfied predicate pushes the total score up [10]. The *MAX* function covers the former: $\max(0, 1) = 1$, or $\max(0.7, 0.8, 0.7) = 0.8$. A lower score than the maximum pushes a function into the averaging class (a solution between the smallest and highest value among the elementary predicates). For the latter, we have $D(0.7, 0.8, 0.7) = 0.982$ by the probabilistic sum to cover the case: "*if someone has high temperature, strong cough and significant weakness, we are almost sure that he has a flu*". In the soft computing propositional logic [11,12], these functions are called hyper-disjunctive (*MAX* is the smallest one among them), whereas functions between the arithmetic mean and *MAX* are regular, or partial disjunctions (the solution is pushed towards the maximal value of atomic predicates, but not beyond it), e.g., $f_D(0, 1) = 0.707$ for the quadratic mean. Due to duality between the conjunction and disjunction the strongest conjunction is *MIN*, or in the soft computing propositional logic it is the strongest hyper–conjunctive aggregator.

To solve the aforementioned problems we are mostly interested in the 2–dimensional averaging aggregation functions. We denote by $\mathcal{A}v_2$ the class of all such functions. For more details we recommend monographs [10,13–15].

The extremal elements of $\mathcal{A}v_2$ are *MAX* (which is also called Zadeh' s disjunction, *OR* operator) and *MIN* (Zadeh' s conjunction, *AND* operator). To characterize the disjunctive (conjunctive) attitude of members of $\mathcal{A}v_2$, one can consider the *ORNESS* measure $ORNESS : \mathcal{A}v_2 \to [0, 1]$ given by

$$ORNESS(A) = 3 \cdot \int_0^1 \int_0^1 A(x, y)\, dy\, dx - 1 \tag{1}$$

Analogously, the *ANDNESS* measure $ANDNESS : \mathcal{A}v_2 \to [0, 1]$ characterizes the conjunctive attitude by

$$ANDNESS(A) = 2 - 3 \cdot \int_0^1 \int_0^1 A(x, y)\, dy\, dx \tag{2}$$

More details regarding these measures can be found in, e.g., [16,17]. Observe that for any n–ary aggregation function A: $[0, 1]^n \to [0, 1]$ its dual A^d: $[0, 1]^n \to [0, 1]$ is given by $A^d(x_1, \dots, x_n) = 1 - A(1 - x_1, \dots, 1 - x_n)$. Then, for any $A \in \mathcal{A}v_2$, it holds $ANDNESS(A) = 1 - ORNESS(A) = ORNESS(A^d)$.

The objective of this paper is a brief overview of the achieved results in this field, and axiomatization of averaging functions in order to cover the large scale of possibilities for asymmetric behaviour of disjunction and conjunction. The main focus is on asymmetric disjunction. Due to duality, the achieved results may be applicable in asymmetric conjunction. The reminder of the paper is organized as follows: Section 2 is dedicated to the *OR ELSE* operators, whereas Section 3 is focused on the *AND IF POSSIBLE* operators. Further, Section 4 demonstrates the results on the illustrative situations and discusses their applicability. Finally, Section 5 concludes the paper.

2. *OR ELSE* operators

In the frame of two–valued logic, the left–right order of predicates $P_1, P_2, \dots, P_n$ has been solved by the Qualitative Choice Logic (QCL) [18]. In that approach, when P_1 is true, the solution gets value 1; when P_2 is true, the solution gets value 2; etc. If not a single predicate is satisfied, the solution is 0. Although this approach is effective, it has to cope with the following problems:

- The comparison among tasks, e.g., when the first task contains four predicates, the second task contains six predicates, and for both the last predicate is met, it is hard to compare these two tasks.
- Generally, it is suitable to work with the unit interval where the worst case assumes value 0, whereas the best case assumes value 1.

In the case of many–valued logics, the literature offers two approaches to manage *OR ELSE* operator: bipolar and asymmetric. A bipolar form of P_1 *OR ELSE* P_2 (the negation to the bipolar *AND IF POSSIBLE* operator) consists of positive pole P_1, which expresses the perfect values (full alternatives) and negative pole P_2 expressing an acceptable value [19]. The ordered pair of grades $[\mu_{P_1}(z), \mu_{P_2}(z)]$ expresses the satisfaction of an element z to the bipolar *OR ELSE* connective. Apparently, the lexicographic order can be applied. The problem is non-considering influence of P_2 for low degrees of P_1. Let us have z_1 with degrees [0.2, 0.95] and z_2 with degrees [0.22, 0.3]. By the lexicographic approach, z_2 is preferred, even though both are very similar in P_1, but z_1 is significantly better in P_2. In this work, we examine this operator as an asymmetric disjunction in the sense of Bosc and Pivert [2], where the solution is an aggregated value of satisfying both: the full alternative and additional alternative.

2.1. *The formalization of* OR ELSE *asymmetric disjunction*

Bosc and Pivert [2] proposed the following six axioms in order to formalize *OR ELSE* operator D, where x and y are the values of predicates P_1 and P_2, respectively:

- A1 D is more drastic than *OR* operator: $D(x, y) \leq \max(x, y)$, i.e. we are crossing the border between averaging and disjunctive functions.
- A2 D is softer than when only P_1 appears, because P_2 opens new choices: $D(x, y) \geq x$.
- A3 D is an increasing function in its first argument.
- A4 D is an increasing function in its second argument.
- A5 D has asymmetric behaviour, i.e. $D(x, y) \neq D(y, x)$ for some $(x, y) \in [0, 1]^2$.
- A6 D is equivalent to x *OR ELSE* (x *OR* y): $D(x, y) = D(x, x \vee y)$.

Note that, for the simplicity, sometimes we use the lattice connectives notation $\vee = MAX$ and $\wedge = MIN$.

The operator which meets these six axioms is expressed by the function

$$D(x, y) = \max(x, A(x, y)) \tag{3}$$

where $A \in \mathcal{A}v_2$.

As a typical example of *OR ELSE* operator (3), Bosc and Pivert [2] have proposed a parametrized class of functions

$$D_{BPk}(x, y) = \max(x, k \cdot x + (1 - k)y) \tag{4}$$

where $k \in]0, 1]$. For the extremal value $k = 0$, we get disjunction expressed by the *MAX* function: $\max(x, y)$.

The *OR ELSE* operators have the next transparent representation.

Theorem 1. *A mapping D: $[0, 1]^2 \to [0, 1]$ is an OR ELSE operator if and only if D is a non–symmetric averaging aggregation function such that $D(x, y) = x$, whenever $x \geq y$.*

Proof. Axioms A1 and A2 ensure for any *OR ELSE* operator D its idempotency, i.e. $D(x, x) = x$ for all $x \in [0, 1]$. This fact together with the monotonicity axioms A3 and A4 ensure $D \in \mathcal{A}v_2$, i.e. D is an averaging aggregation function. Clearly, if $x \geq y$, then $\max(x, y) = x$ and $x \leq D(x, y) \leq x$, where the first inequality is just the axiom A2, while the second inequality follows from A1. Thus, $D(x, y) = x$, whenever $x \geq y$. Finally, A5 concludes that D is a non–symmetric averaging aggregation function, proving the necessity in Theorem 1. The sufficiency is a matter of direct verification of all six axioms A1–A6. □

We denote the class of all *OR ELSE* operators as $\mathcal{D}$, and $\mathcal{D}^* = \mathcal{D} \cup \{\max\}$. The next corollary of Theorem 1 is obvious.

Corollary 1. *$D \in \mathcal{D}^*$ if and only if $D = \max(P_F, A(P_F, P_L))$ where $A \in \mathcal{A}v_2$ and P_F is the first projection, i.e. $P_F(x, y) = x$ and P_L is the last projection.*

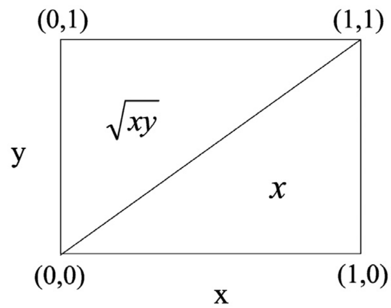

Fig. 1. The graphical interpretation of the asymmetric disjunction D expressed by geometric mean (6).

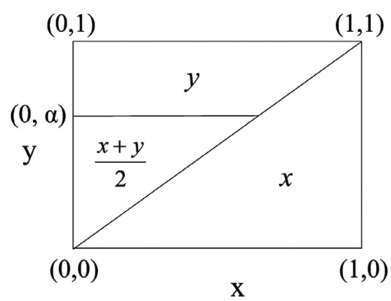

Fig. 2. The graphical interpretation of the operator (7).

By considering the Bosc–Pivert operators (4), we see that $D_{BPk}(x,y) = \max(P_F, W_k)$ where $W_k \in \mathcal{A}v_2$ is the weighted arithmetic mean, $W_k(x,y) = k \cdot x + (1-k)y$.

As another non–trivial example, one can consider

$$D^G_{BPk}(x,y) = \max(P_F, G) \tag{5}$$

where G is the weighted geometric mean. Analogously to (4), we write

$$D^G_{BPk}(x,y) = \max(x, x^k \cdot y^{(1-k)}) \tag{6}$$

where $k \in]0, 1]$. For $k = 0.5$ we get $D^G_{BP0.5}(x,y) = \max(x, \sqrt{xy})$, see also Fig. 1.

The set $\mathcal{D}^*$ can be equipped by the standard partial order of binary functions, and then $(\mathcal{D}^*, \leq)$ is a complete distributive lattice with the top element *MAX*, and the bottom element P_F. Moreover, for any n-ary averaging aggregation form A: $[0,1]^n \to [0,1]$ and $D_1, \dots, D_n \in \mathcal{D}^*$, also the composite $H = A(D_1, \dots, D_n)$, given by $H(x,y) = A(D_1(x,y), \dots, D_n(x,y))$ belongs to $\mathcal{D}^*$. In particular, $\mathcal{D}^*$ is a convex class. Note that $D_{BPk}(x,y) = k \cdot x + (1-k) \cdot \max(x,y)$ is a convex combination of extremal elements of $\mathcal{D}^*$.

Discussing the structure of $\mathcal{D}^*$ and related construction methods, we discuss, in back, *OR ELSE* operators from $\mathcal{D}$, with the only exception when the only symmetric member of $\mathcal{D}^*$, (i.e. *MAX*) appears as the considered output. As an example, consider $D_1, \dots, D_n \in \mathcal{D}^*$, $D_1 \leq \dots \leq D_n$ and $0 = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n = 1$. Let us define the next ordinal sum D: $[0, 1]^2 \to [0, 1]$ given by

$$D(x,y) = D_i(x,y) \text{ whenever } \alpha_{i-1} \leq \max(x,y) \leq \alpha_i, i \in 1, \dots, n \wedge D(0,0) = 0 \tag{7}$$

then, $D \in \mathcal{D}^*$ and if $D \neq MAX$, then D is an *OR ELSE* operator.

Example 1. Let $D_1 = D_{BP0.5}$ (4), $D_2 = MAX, \alpha_1 = \alpha \in]0,1[$. Then the related ordinal sum $D \in \mathcal{D}$ is an *OR ELSE* operator visualized in Fig. 2.

Note that for the high values of x or y, the operator (7) behaves like the standard symmetric *OR* operator (e.g., *MAX*), while when both values x and y are small (not exceeding the threshold α), then *OR ELSE* operator is applied (in this case, the Bosc–Pivert operator (4) or (6)). In Fig. 2 the former is shown. In the next subsection, some particular *OR ELSE* operators are considered.

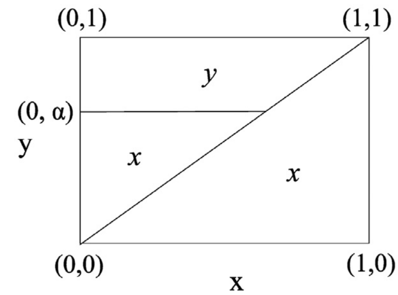

Fig. 3. The graphical interpretation of the non-continuous operator (9).

2.2. Particular OR ELSE operators and their ORNESS parameters

In general, *OR ELSE* operators need not be associative, which requires a deeper discussion on how to aggregate more than two values. However, we have the next important result.

Theorem 2. *An operator $D \in \mathcal{D}$ is associative and continuous if and only if*

$$D = \max(P_F, med_a) \tag{8}$$

where $a = D(0,1) \in [0, 1[$ and $med_a(x, y)$ is the median of values x, a, y.

Proof. Let $D \in \mathcal{D}$ be associative and continuous. Then, denoting $a = D(0,1)$, it holds $D(a,1) = D(D(0,1),1) = D(0, D(1,1)) = D(0,1) = a$. Similarly, $D(0,a) = a$. Due to monotonicity of D, $D(x,y) = a$ whenever $x \leq a \leq y$, and hence $D(x,y) = med_a(x,y)$ on $[0, a] \times [a, 1]$. Next, if $x > a$, $D(z,1) = x$ for some $z \in]a, 1]$ (this is due to the continuity of D), and $D(x,1) = D(D(z,1),1) = D(z, D(1,1)) = D(z,1) = x$. As far as also $D(x,x) = x$, and D is monotone, we have $D(x,y) = x$ whenever $a < x$. Thus, $D(x,y) = med_a(x,y)$ if $a < x \leq y$. Similarly, $D(x,y) = y = med_a(x,y)$ if $x \leq y \leq a$, thus proving that $D = \max(P_F, med_a)$. Note that if $a = 1$, then $D = MAX \notin \mathcal{D}$ and thus $a \in [0, 1[$. On the other hand, consider $D = \max(P_F, med_a)$, $a \in [0,1[$. Clearly, D is continuous due to the continuity of both functions P_F and med_a.

To see the associativity one should check all six possible cases of ordinal structure of elements *x, y, z* $\in [0, 1]$. Some of them are trivial, e.g., if $x \geq y \geq z$, then $D(D(x,y),z) = D(x,z) = x$ and $D(x, D(y,z)) = D(x,y) = x$, i.e. $D(D(x,y),z) = D(x, D(y,z))$. In some other cases, the proof of associativity is much more complicated. Consider $y \geq z \geq x$. Then $D(x, D(y,z)) = D(x,y) = med_a(x,a,y)$. To see the value of $D(D(x, y), z)$, one should consider four different cases related to the position of the constant a:

(i) if $a \leq x \leq z \leq y$, then $D(D(x,y),z) = D(x,z) = x = med(x,a,y) = D(x, D(y,z))$;
(ii) if $x \leq a \leq z \leq y$, then $D(D(x,y),z) = D(a,z) = a = med(x,a,y) = D(x, D(y,z))$;
(iii) if $x \leq z \leq a \leq y$, then $D(D(x,y),z) = D(a,z) = a = med(x,a,y) = D(x, D(y,z))$;
(iv) if $x \leq z \leq y \leq a$, then $D(D(x,y),z) = D(y,z) = y = med(x,a,y) = D(x, D(y,z))$.

Similarly, the validity of the associativity equation in four remaining cases can be shown. □

Note that there are also non–continuous *OR ELSE* operators which are associative. As an example, one can consider the ordinal sum linked to $D_1 = P_F \leq D_2 = MAX$ and $\alpha_1 = \alpha \in]0,1[$ given by

$$D(x,y) = \begin{cases} y & \text{for } y > x \wedge y > \alpha \\ x & \text{otherwise} \end{cases} \tag{9}$$

and visualized in Fig. 3.

Remark 1. (i) Note that Eq. (8) (a continuous associative *OR ELSE* operator) can be seen as the Sugeno integral [20,21] on the space

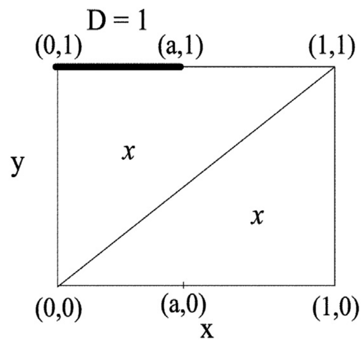

Fig. 4. The graphical interpretation of the smallest *OR ELSE* operator satisfying $D(0,1) = a$ (12).

$X = \{1, 2\}$ with respect to the capacity m_a: $X \to [0, 1]$ given by $m_a(\emptyset) = 0$, $m_a(\{1\}) = 1$, $m_a(\{2\}) = a$, $m_a(\{1, 2\}) = 1$, i.e. $D(x, y) = Su_{m_a}(x, y)$.

(ii) Due to the axiomatic characterization of the Sugeno integral [22], we see that an *OR ELSE* operator D is continuous and associative if and only if it is co–monotone maxitive and min–homogeneous, i.e. if $D(x_1 \vee x_2, y_1 \vee y_2) = D(x_1, y_1) \vee D(x_2, y_2)$ whenever (x_1, y_1), (x_2, y_2) are co–monotone, i.e. $(x_1 - y_1) \cdot (x_2 - y_2) \geq 0$ and $D(c \wedge x, c \wedge y) = c \wedge D(x, y)$ for all $c, x, y \in [0, 1]$.

Klement et al. [23] have introduced discrete universal integrals related to an arbitrary fixed semicopula $\otimes$: $[0, 1]^2 \to [0, 1]$ (i.e. $\otimes$ is a binary aggregation function with neutral element $e = 1$, $x \otimes 1 = 1 \otimes x = x$ for each $x \in [0, 1]$). It is not difficult to see that for any such integral I: $[0, 1]^2 \to [0, 1]$ based on the above defined capacity m_a, $a \in [0, 1[$, we have $I \in \mathcal{D}$ and $I(0, 1) = a$. For example, for a fixed semicopula $\otimes$ the smallest universal integral I related to $\otimes$ and m_a is given by

$$I(x, y) = \max(x, \min(x, y) \otimes a) = \begin{cases} x & \text{for } x \geq y \\ x \vee (y \otimes a) & \text{otherwise} \end{cases} \tag{10}$$

For the greatest semicopula $\otimes = \wedge$ (*MIN* operator), $I_\wedge = Su$, which is the above discussed Sugeno integral. Recall that if $x < y$, then $med_a(x, y) = x \vee (y \wedge a)$.

On the other hand, the smallest semicopula $\circ$*ledS*: $[0, 1]^2 \to [0, 1]$ is given by

$$x Ⓢ y = \begin{cases} x \wedge y & \text{for } x \vee y = 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

i.e. $\circ$*ledS* is the drastic product, $Ⓢ = T_D$ [24], that is

$$T_D(x, y) = \begin{cases} x & \text{for } y = 1 \\ y & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, fixing $a \in [0, 1[$, the related *OR ELSE* operator D is given by

$$D(x, y) = \begin{cases} a & \text{for } y = 1 \wedge x \leq a \\ x & \text{otherwise} \end{cases} \tag{12}$$

Note that D is the smallest *OR ELSE* operator such that $D(0, 1) = a$. This case is shown in Fig. 4.

The next results follow from the axiomatization of the related universal integrals. Let us first recall the definition of co–monotone additivity [14]. A function F: $[0, 1]^2 \to [0, 1]$ is co–monotone additive whenever $F(x_1 + x_2, y_1 + y_2) = F(x_1, y_1) + F(x_2, y_2)$ for all co–monotone pairs (x_1, y_1), $(x_2, y_2) \in [0, 1]^2$ such that $(x_1 + x_2, y_1 + y_2) \in [0, 1]^2$. Also as already observed, F is co–monotone maxitive whenever $F(\max(x_1, x_2), \max(y_1, y_2)) = \max(F(x_1, y_1), F(x_2, y_2))$ for all co–monotone pairs (x_1, y_1), $(x_2, y_2) \in [0, 1]^2$.

Corollary 2. *Let $D \in \mathcal{D}$. Then D is co–monotone additive if and only if $D = D_{BPk}$, see (4), for some $k \in]0, 1]$.*

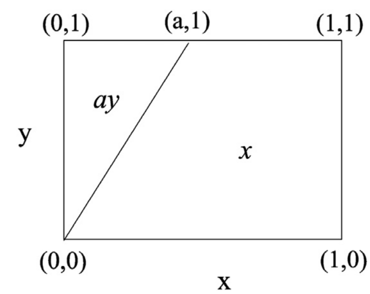

Fig. 5. The graphical interpretation of the *OR ELSE* operator (14).

Proof. The co–monotone additivity of D is equivalent to the fact that it is the Choquet integral with respect to m_a, $a = D(0, 1)$, i.e.

$$D(x, y) = \begin{cases} x & \text{for } x \geq y \\ (1 - a)x + a \cdot y & \text{otherwise} \end{cases} \tag{13}$$

Clearly, if $a = 1$, then $D = MAX \notin \mathcal{D}$, i.e. necessarily $a \in [0, 1[$. Evidently, putting $k = 1 - a$, Eq. (13) defines exactly the Bosc–Pivert operator D_{BPk}. $\square$

Corollary 3. *Let $D \in \mathcal{D}$. Then D is co–monotone maxitive and positively homogeneous if and only if*

$$D(x, y) = x \vee (a \cdot y) \tag{14}$$

for some $a \in [0, 1[$, see Fig. 5.

Proof. Observe that for any $D \in \mathcal{D}$, $D(1, y) = 1$. Then, for any $x \in [0, 1]$, the positive homogeneity of D ensures $D(x, xy) = xD(1, y) = x$, i.e. $D(x, y) = x$ whenever $x \geq y$. Let $D(0, 1) = a \in [0, 1[$. Then, for any $y \in [0, 1]$, $D(0, y) = yD(0, 1) = ay$. Next, for any $x \leq y$ we have $(x, y) = (x, x) \vee (0, y)$, and pairs (x, x) and $(0, y)$ are co-monotone. Then the co-monotone maxitivity of D ensures $D(x, y) = D(x, x) \vee D(0, y) = x \vee ay$. $\square$

Observe that the Eq. (14) is just the Shilkret integral [25] with respect to m_a.

The next introduced *ORNESS* parameter values for particular *OR ELSE* operators are obtained by computation and therefore we summarize our result briefly. Note that for each $D \in \mathcal{D}$, $0.5 \leq ORNESS(D) < 1$.

Fixing the value $a = D(0, 1) \in [0, 1[$, we get the following results:

(1) for $D \in \mathcal{D}$ given by Eq. (8) (Sugeno integral), $ORNESS(D) = 0.5 + 1.5a^2 - a^3$;
(2) for $D \in \mathcal{D}$ given by Eq. (13) (Choquet integral), $ORNESS(D) = (1 + a)/2$, and hence, for the Bosc–Pivert operator (4) we have $ORNESS(D_{BPk}) = 1 - 0.5k$;
(3) for $D \in \mathcal{D}$ given by Eq. (14) (Shilkret integral), $ORNESS(D) = (1 + a^2)/2$.

3. *AND IF POSSIBLE* operators

At the beginning of flexible querying, queries have been generally seen as conjunctions of atomic predicates to restrict the relevant subset of data. These predicates (constraints) were considered as "negative preferences", i.e. failing to meet the conditions disqualifies items. On the other hand, "positive preferences" express wishes or desires that might be satisfied. Obviously, if they are satisfied, it should positively influence the overall matching degree by suitably aggregating "negative preferences" and "positive preferences".

This category of aggregation, similarly to the *OR ELSE* discussed in Section 2.2, can be solved by the bipolar approaches. A way how bipolar queries can handle constraints (negative preferences) and wishes (positive preferences) is explained in, e.g., [26,27]. The aggregation of constraints and wishes of bipolar queries by the Bipolar Satisfaction Degree is examined in, e.g., [4,28].

In this work, we examine *AND IF POSSIBLE* operator (the abbreviated version *AND IF POS* is used throughout this paper) as an asymmetric conjunction suggested by Bosc and Pivert [2], where the solution is an aggregated value of satisfying both: the constraint (must be satisfied) and the wish (is fine if it is satisfied).

Bosc and Pivert [2] proposed the following six axioms in order to formalize *AND IF POS* operator as an asymmetric conjunction C, where x and y are the values of predicates P_1 and P_2 analogously to the axioms A1–A6 of Section 2.1:

- B1 C is less drastic than *AND* operator: $C(x, y) \geq \min(x, y)$, i.e. we are crossing the border between conjunctive and averaging functions.
- B2 C is more drastic than when only P_1 appears, because P_2 should be considered: $C(x, y) \leq x$.
- B3 C is an increasing function in its first argument.
- B4 C is an increasing function in its second argument.
- B5 C has asymmetric behaviour, i.e. $C(x, y) \neq C(y, x)$ for some $(x, y) \in [0, 1]^2$.
- B6 C is equivalent to x *AND IF POS* (x *AND* y): $C(x, y) = C(x, x \wedge y)$.

The operator which meets these axioms is expressed by the function

$$C(x, y) = \min(x, A(x, y)) \tag{15}$$

where $A \in \mathcal{A}v_2$.

As a typical example of *AND IF POS* operator (15), Bosc and Pivert [2] have proposed a parametrized class of functions

$$C_{BPk}(x, y) = \min(x, k \cdot x + (1-k)y) \tag{16}$$

where $k \in]0, 1]$. For the extremal value $k = 0$, we get conjunction expressed by the *MIN* function: $\min(x, y)$. When $k = 0.5$ the second argument becomes the arithmetic mean, i.e. the function $C_{BPk}(x, y) = \min(x, \frac{1}{2}(x + y))$ is obtained.

For A in asymmetric conjunction (15) we can use any weighted geometric mean. Considering Eqs. (5) and (15) we get

$$C^G_{BPk}(x, y) = \min(x, x^k \cdot y^{(1-k)}) \tag{17}$$

where $k \in]0, 1]$. For $k = 0.5$ we get $C^G_{BP0.5}(x, y) = \min(x, \sqrt{xy})$.

The axioms for *OR ELSE* and *AND IF POS* differ in the first, second and sixth axioms only, i.e. in (A1, A2, A6) and (B1, B2, B6). On the other hand, these axioms are related by the duality of aggregation functions, see the next crucial result.

Theorem 3. *Let $C \in \mathcal{A}v_2$. Then C is an AND IF POS operator if and only if its dual C^d is an OR ELSE operator.*

Proof. Consider that C is an *AND IF POS* operator. Then obviously C^d satisfies the axioms A3, A4 and A5. Due to B1, it holds $C^d(x, y) = 1 - C(1-x, 1-y) \geq 1 - \min(1-x, 1-y) = \max(x, y)$, i.e. C^d fulfils A1. Similarly, due to B2, $C^d(x, y) = 1 - C(1-x, 1-y) \leq 1 - (1-x) = x$, showing the validity of A2 for C^d. Finally, considering the axiom B6 to be satisfied for C, we have $C^d(x, y) = 1 - C(1-x, 1-y) = 1 - C(1-x, \min(1-x, 1-y)) = 1 - C(1-x, 1-\max(x, y)) = C^d(x, \max(x, y))$, and hence also A6 is valid for C^d. Summarizing, we see that $C^d \in \mathcal{D}$. The reversed claim, $C \in \mathcal{D}$ implies C^d is an *AND IF POS* operator, can be shown in a similar way. □

Based on Theorem 3, one can directly rewrite all results and examples introduced in Section 2 for *OR ELSE* operators for the case of *AND IF POS* operators. Denoting their class by $\mathcal{C}_{\oint}$, we have:

(1) $C \in \mathcal{C}_{\oint}$ if and only if $C = \min(P_F, A(P_F, P_L))$ for some $A \in \mathcal{A}v_2$, $C \neq MIN$;

(2) $C \in \mathcal{C}_{\oint}$ is associative and continuous if and only if
$$C(x, y) = \begin{cases} x & \text{for } x \leq y \\ med_c(x, y) & \text{otherwise} \end{cases}$$
where $c = C(1, 0) \in]0, 1]$ (i.e. C is the Sugeno integral with respect to the capacity m^c given by $m^c(\{1\}) = c$, $m^c(\{2\}) = 0$; and then $ANDNESS(C) = 1 - 1.5c^2 + c^3$;

(3) $C \in \mathcal{C}_{\oint}$ is co–monotone additive if and only if $C = (D_{BPk})^d$, i.e.
$$C(x, y) = \begin{cases} x & \text{for } x \leq y \\ k \cdot x + (1-k)y & \text{otherwise} \end{cases} \quad k \in]0, 1]$$
(i.e. C is the Choquet integral with respect to the capacity m^k), and then $ANDNESS(C) = 1 - 0.5k$;

(4) $C \in \mathcal{C}_{\oint}$ is co–monotone maxitive and positively homogeneous if and only if
$$C(x, y) = \begin{cases} x & \text{for } x \leq y \\ y \vee c \cdot x & \text{otherwise} \end{cases} \quad c \in]0, 1] \tag{18}$$
(i.e. C is the Shilkret integral with respect to the capacity m^c), and then $ANDNESS(C) = 1 - \frac{c^2}{2}$. Note that this *AND IF POS* operator is not a dual of *OR ELSE* operator given by formula (14), i.e. it differs for all $a \in [0, 1[$, from $D^d(x, y) = x \wedge (1 - a + a \cdot y)$. Obviously, D^d is an *AND IF POS* operator which is neither co–monotone maxitive nor positively homogeneous. Observe that while co–monotone additivity, as well as associativity and continuity are preserved by duality, this is neither the case of co–monotone maxitivity nor of positive homogeneity as is shown in Fig. 6.

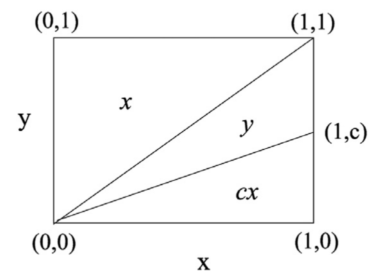

Fig. 6. The graphical interpretation of the *AND IF POSS* operator given by (18).

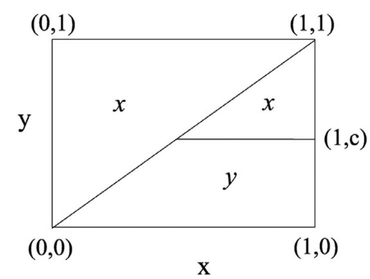

Fig. 7. The graphical interpretation of the non–continuous operator (19).

Similarly, one can introduce ordinal sums for *AND IF POS* operators. As an example, considering duality to formula (9) one gets the class of *AND IF POS* operators given by

$$C(x, y) = \begin{cases} y & \text{for } y < x \wedge y < c \\ x & \text{otherwise} \end{cases} \tag{19}$$

with $c \in [0, 1[$, see Fig. 7. Note that, the above operators are non–continuous, but associative.

4. Illustrative examples and possible applicability

This section provides short illustrative examples as well as real–word situations of the suggested approach, and a brief reflection upon the possible applicability.

4.1. Illustrative examples for OR ELSE

Let us now consider the aforementioned case of the condition "*ripe broccoli (P_1) or ripe cauliflower (P_2)*", where P_1 and P_2 are predicates,

Table 1
OR ELSE connective expressed by the continuous Bosc–Pivert operators.

Item	x	y	*A* is arithmetic mean for $k = 0.5$ (4)	*A* is geometric mean for $k = 0.5$ (6)
I1	0	0	0	0
I2	1	0	1	1
I3	1	1	1	1
I4	0	1	0.5	0
I5	0.1	1	0.55	0.316
I6	0.8	0.2	0.8	0.8
I7	0.2	0.8	0.5	0.4
I8	0.2	0.7	0.45	0.374
I9	0.65	0.35	0.65	0.65
I10	0.75	0.35	0.75	0.75
I11	0.92	0.74	0.92	0.92
I12	0.88	0.76	0.88	0.88

Table 2
OR ELSE operator expressed by the non–continuous function (7).

Item	x	y	non–continuous for $\alpha = 0.75$
I1	0	0	0
I2	1	0	1
I3	1	1	1
I4	0	1	1
I5	0.1	1	1
I6	0.8	0.2	0.8
I7	0.2	0.8	0.8
I8	0.2	0.7	0.45
I9	0.65	0.35	0.65
I10	0.75	0.35	0.75
I11	0.92	0.74	0.92
I12	0.88	0.76	0.88

which take into account the degrees of ripeness of the respective vegetables. Generally, P_1 and P_2 can be any kind of predicates.

In the first example, we consider Bosc–Pivert disjunctions $D_{BP0.5}$ and $D^G_{BP0.5}$, i.e. the cases when *A* is the arithmetic mean (4) or the geometric mean (6). The intensities of *x* and *y*, and the results are in Table 1. Obviously, the geometric mean is a more restrictive function, where in addition, the value 0 is an annihilator, i.e. both values, *x* and *y*, should be greater than 0 (compare *I4* and *I5*). These two functions are continuous. It is clear that all axioms (A1 – A6) hold.

An example might be buying or renting a building for a smaller company owning a 10-cars fleet. An attached garage of capacity for 10 cars is the ideal option ($P_1(10) = 1$). If it is not available, then 10 reserved street parking lots is the less preferable alternative. When a building has a garage for 10 cars ($x = 1$), the solution is 1 regardless the street lots. When a garage has five places ($x = 0.5$) and 10 reserved street lots are available ($y = 1$), then the solution is 0.75. But, when only four reserved street lots are available, then the solution is 0.45 (the driver of the last car should compete for a free lot). To emphasize difficulties of street parking we can apply the geometric mean (6). On the other hand, when street parking is quite comfortable, we can apply $A(x, y) = \sqrt{0.5x^2 + 0.5y^2}$ (the *ORNESS* measure for the quadratic mean is greater than 0.5).

Another possible real–world employment may be in the support for the medical diagnosis queries like *most of* $\{P_1, P_2, P_3, P_5, P_7\}$ *or else most of* $\{P_2, P_4, P_6, P_8\}$ should be met to have a high belief about a particular illness, where P_i is a fuzzy predicate like *high temperature* or *low sedimentation*.

When we consider averaging functions logically [11], then the arithmetic mean is the only neutral function between the conjunction and disjunction ($A(x, y) = [x \wedge y + x \vee y]/2$). Thus, when the optional alternative is more relevant, we can use an averaging function of *ORNESS* measure lower than 0.5 in (3). Otherwise, we can use an averaging function of *ORNESS* measure greater than 0.5.

The next example is focused on the non–continuous case of *OR ELSE* operators. Into the aforementioned example we add "*if cauliflower is very ripe, then it also becomes the full alternative*". The solution is in Table 2 for $\alpha = 0.75$ (7) where $D_1 = D_{BP0.5}$ and $D_2 = MAX$. The non–continuous behaviour is obvious when comparing items *I7* and *I8*. The predicate P_2 is not as relevant as P_1, unless it assigns high intensity and therefore becomes the full alternative. This function also meets the axioms of *OR ELSE* operator.

Recall the aforementioned medical diagnosis query. We can imagine situations, which require non–continuous behaviour like: *most of* $\{P_1, P_2, P_3, P_5, P_7\}$ *or else most of* $\{P_2, P_4, P_6, P_8\}$ should be met to have a high belief about a particular illness, but when *most of* $\{P_2, P_4, P_6, P_8\}$ has (very) high truth value, then it becomes the full alternative.

For the most restrictive case of *OR ELSE* operators we consider the smallest semicopula expressed by the drastic product and managed by a parameter *a* (12). As *a* decreases, the solution decreases when $y = 1$ and $x \le a$. This fulfils our requirement: *if* P_2 *is ideally met but* P_1 *is weakly met or rejected, then a limits the solution*, i.e. when *y* is equal to 1, low values of *x* are compensated by *a* (items *I4* and *I5*), otherwise the solution is *x* (item *I7*), see Table 3. For $a = 0$, $x = 0$ and $y = 1$, the solution is 0.

Table 3
OR ELSE operator related to the drastic product (12).

Item	x	y	for $a = 0.75$	for $a = 0.25$
I1	0	0	0	0
I2	1	0	1	1
I3	1	1	1	1
I4	0	1	0.75	0.25
I5	0.1	1	0.75	0.25
I6	0.8	0.2	0.8	0.8
I7	0.2	0.8	0.2	0.2
I8	0.2	0.7	0.2	0.2
I9	0.65	0.35	0.65	0.65
I10	0.75	0.35	0.75	0.75
I11	0.92	0.74	0.92	0.92
I12	0.88	0.76	0.88	0.88

An optional alternative is weighted when it is fully satisfied. It holds when the full alternative is satisfied with a degree lower than or equal to the value *a*. An example might be in the medical domain. A medicine P_1 is applied, but when it causes a measurable effect lower than *a*, a medicine P_2 is the partial solution. Another example is in evaluating houses. When the distance to the nearest grocery shop is more or less beyond the acceptable walking distance (P_1) the suitability of accessible public transport (P_2) mitigates this drawback.

The next example considers the case when *y* is influenced by the parameter *a* (14). Apparently, when $a = 1$, we get the disjunction, whereas for $a = 0$ we have the first projection. For $a \in]0, 1[$ the influence of *y* is reduced by parameter *a* without considering the common influence of *x* like for instance is the case in the averaging functions in D_{BPk}. The results for different parameters *a* are shown in Table 4. For the low values of *a*, the results are lower or equal than by $D_{BP0.5}$.

The importance of an optional alternative is adjusted by $a \in [0, 1]$. An example might be buying a house where a spacious basement is an ideal solution. But, when no basement is available or it is of a low space, then a spacious attic is an option. Usually, it is a harder task to warehouse items into the attic. Parameter *a* indicates the unpretentiousness for manipulating items into the attic (e.g., it assigns higher value for a younger buyer and less steep stairs, but lower values for an elderly buyer and a steep ladder). The graded predicates assume sizes of the basement and attic. When a basement is insufficiently spacious, then a spacious attic influences the solution. In the extreme case ($a = 1$), both the basement and attic are full alternatives.

Table 4
OR ELSE operator expressed as a co–monotone and homogeneous function (14).

Item	x	y	$a = 0.3$	$a = 0.7$
I1	0	0	0	0
I2	1	0	1	1
I3	1	1	1	1
I4	0	1	0.3	0.7
I5	0.1	1	0.3	0.7
I6	0.8	0.2	0.8	0.8
I7	0.2	0.8	0.24	0.56
I8	0.2	0.7	0.21	0.49
I9	0.65	0.35	0.65	0.65
I10	0.75	0.35	0.75	0.75
I11	0.92	0.74	0.92	0.92
I12	0.88	0.76	0.88	0.88

Table 5
AND IF POS operator expressed by the continuous Bosc–Pivert operators.

Item	x	y	A is the arithmetic mean for $k = 0.5$ (16)	A is the geometric mean for $k = 0.5$ (17)
I1	0	0	0	0
I2	0	1	0	0
I3	1	0	0.5	0
I4	1	0.1	0.55	0.316
I5	1	1	1	1
I6	0.95	0.35	0.65	0.577
I7	0.95	0.41	0.68	0.624
I8	0.41	0.95	0.41	0.41
I9	0.65	0.35	0.5	0.477
I10	0.75	0.35	0.55	0.512
I11	0.9	0.8	0.85	0.848
I12	0.92	0.74	0.83	0.825

Table 6
AND IF POS operator expressed by the non–continuous function dual to (7), where $C_1 = C_{BP0.5}$ and $C_2 = MIN$.

Item	x	y	non–continuous for $\alpha = 0.4$
I1	0	0	0
I2	0	1	0
I3	1	0	0
I4	1	0.1	0.1
I5	1	1	1
I6	0.95	0.35	0.35
I7	0.95	0.41	0.68
I8	0.41	0.95	0.41
I9	0.65	0.35	0.35
I10	0.75	0.35	0.35
I11	0.9	0.8	0.85
I12	0.92	0.74	0.83

Table 7
AND IF POS operator expressed by the drastic product (20).

Item	x	y	for $a = 0.75$
I1	0	0	0
I2	0	1	0
I3	1	0	0.75
I4	1	0.1	0.75
I5	1	1	1
I6	0.95	0.35	0.95
I7	0.95	0.41	0.95
I8	0.41	0.95	0.41
I9	0.65	0.35	0.65
I10	0.75	0.35	0.75
I11	0.9	0.8	0.9
I12	0.92	0.74	0.92

4.2. Illustrative examples for AND IF POS

An example covered by this operator is a problem of searching for a hotel by the condition: "*low price (P_1) and if possible short distance (P_2)*". Generally, P_1 and P_2 can be any kind of predicates assuming values in the unit interval.

In the first example, we consider the Bosc–Pivert conjunction $C_{BP0.5}$, i.e. the cases when A is the arithmetic mean (16) and the geometric mean (17). The results are shown in Table 5. Similarly to asymmetric disjunction, the geometric mean is a more restrictive function where the value 0 is the absorbing element, i.e. both values x and y should be greater than 0, not only the constraint x (compare items *I3* and *I4*). These two functions are continuous. It is clear from Table 5 that axioms B1 – B6 hold for both functions.

Mathematically, we can adopt any averaging function in (15) including the geometric mean. The averaging functions can be interpreted logically [11] and therefore are able to cover various requirements. The geometric mean (similarly to the harmonic mean) covers the most strict behaviour of *AND IF POS* in (0, 0), Fig. 1. When the wish has intensity greater than 0, the asymmetric behaviour is activated. Further, Eq. (17) gives more importance to the wish than (16). For instance, in a romantic trip, the distance of a hotel is not so relevant as the price and amenities, but when the distance is long it disqualifies a hotel. The *ANDNESS* measure of the geometric mean is lower than 0.5, which assigns a bit more relevance to distance, whereas the *ANDNESS* measure of the quadratic mean $A(x, y) = \sqrt{0.5x^2 + 0.5y^2}$ covers the cases when the distance plays a less significant role in the asymmetric conjunction.

The next example is related to the non–continuous case of *AND IF POS* operator as the extension of the aforementioned task by "*low price and if possible short distance, but when distance is not significantly short, it becomes a constraint*". In this example, we have used the dual interpretation of (7), where $C_1 = C_{BP0.5}$ and $C_2 = MIN$ for $\alpha = 0.4$. The solution is in Table 6 for $\alpha = 0.4$, i.e. a distance with the matching degree to the concept *short distance* lower than 0.4 becomes a constraint, that is, if its matching degree is lower than that for a price, then it constraints the solution, see items *I6* and *I7*. This function also meets all axioms of *AND IF POS* operator, e.g., asymmetry is clear from items *I7* and *I8*, non–decreasingness is noticeable when comparing items *I3* and *I4*, or *I9* and *I10*, etc.

The real-world situation may be expressed in the following way. A customer searching for a hotel says that the price must be low and it is desirable that the distance is at least moderately short (e.g., α between 0.4 and 0.6), otherwise the distance should be considered as a constraint.

The next example is focused on the dual case to the smallest *OR ELSE* operator (12) expressed as

$$C(x, y) = \begin{cases} a & \text{for } x = 1 \wedge y \le a \\ x & \text{otherwise} \end{cases} \quad (20)$$

As a decreases, the solution decreases when $x = 1$ and $y \le a$. This fulfils our requirement: *if P_1 is ideally met but P_2 is weakly met (lower than a), then a limits the solution*, i.e. when x is equal to 1, low values of y are compensated by a (items *I3* and *I4*), otherwise the solution is x, which means that in this case this is the highest *AND IF POS* operator (e.g., items *I9* and *I10*). These solutions are in Table 7. Further, for $a = 0$, $x = 1$ and $y = 0$, the solution is 0, whereas for $a = 0.99$, $x = 1$ and $y = 0$, the solution is 0.99.

This case covers the situation when a customer expresses that the price (or a compound predicate covering several key attributes) is a crucial constraint. When it is fully met, then the optional alternative, e.g., distance influences the solution.

The last example considers the case when x is influenced by a parameter c. It is a non–dual case of (14). For $c \in]0, 1[$ the influence of x is reduced by this parameter when $x > y$. The results for $c = 0.5$ and $c = 0.8$ in (18) are shown in Table 8.

Table 8
AND IF POS operator expressed as a co–monotone and homogeneous function (18).

Item	x	y	$c = 0.5$	$c = 0.8$
I1	0	0	0	0
I2	0	1	0	0
I3	1	0	0.5	0.8
I4	1	0.1	0.5	0.8
I5	1	1	1	1
I6	0.95	0.35	0.475	0.76
I7	0.95	0.41	0.475	0.76
I8	0.41	0.95	0.41	0.41
I9	0.65	0.35	0.35	0.52
I10	0.75	0.35	0.375	0.6
I11	0.9	0.8	0.8	0.8
I12	0.92	0.74	0.74	0.74

Let us consider low price and richness of amenities in searching for a hotel. The following situation is covered by this model. When a matching degree to the low price is lower than or equal to the satisfaction degree of offered amenities, the price limits the solution. In the opposite case, we consider richness of amenities or the reduced influence of the price.

4.3. A reflection upon applicability and future perspectives

Both asymmetric disjunction and asymmetric conjunction have a high applicability potential. A possible real–world application area is in database querying and decision support systems, e.g., [29,30]. The diverse situations are shown in Sections 4.1 and 4.2. In this section, we examine further possibilities.

The *AND IF POS* operator has also a high potential to mitigate the *empty answer problem*. For example, let a house builder search for a suitable village for building holiday houses [31]. When he considers a higher number of predicates (e.g., altitude above sea level around 1000 m, small population density, medium area of village, low pollution and so on), they should be aggregated by conjunctive functions. Otherwise, in the aggregation by averaging functions a significant part of atomic predicates need not be satisfied, or is weakly satisfied, but compensated by high values of other predicates; whereas by disjunctive functions one fully satisfied predicate influences the solution (value 1 is the absorbing element). However, by applying conjunctive functions, it is highly presumable that not a single village meets (at least partially) all predicates. It leads to the so–called *empty answer problem* [32]. The literature offers several ways how to solve this problem, e.g., the relaxation of atomic predicates [33]. It is a suitable way, when we have a smaller number of atomic predicates. On the other hand, a customer might divide such number of atomic predicates into the two subsets: hard conditions and soft conditions by the relative fuzzy quantifier *most of* like: *most of* $\{P_i\}$ *and if possible most of* $\{P_j\}$ should be met.

It is worth noting, that situations when the influence of the constraint is reduced by a parameter c for the matching degrees of the constraint higher than for the wish are solved by (18). This case is a non–dual one to the case of asymmetric conjunction (14). The other kinds of asymmetric conjunctions and their respective asymmetric disjunctions are dual.

The last but not the least, this axiomatization might be a valuable support for the emerging field of explainable artificial intelligence. For more details about this topic we recommend [34,35]. Generally, the machine learning methods are very data hungry [36], i.e. they require a large amount of input–output data. The domain experts are usually not familiar with the mathematical formalization of their tasks, but they are able to explain expected aggregation linguistically. From such explanation we can recognize evaluation patterns for the given situation and therefore choose a suitable subclass of aggregation functions. Thus, a smaller amount of training input–output data for learning the most fitted functions and their parameters might suffice. The term explainability is decomposed into [35]: interpretability, comprehensibility and reproducibility (given the data and specific requirements we can model a function that explains the output). Hence, the all aspects of aggregation functions, including asymmetry should be theoretically covered and if possible illustrated on the representative situations.

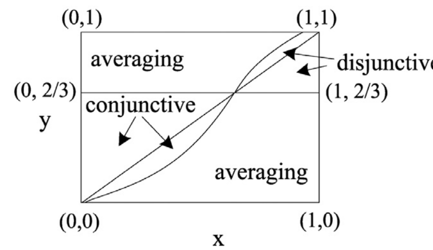

Fig. 8. The graphical interpretation of the hybrid function (21) managing reward and penalty.

This discussion has illustrated the robustness of our approach, in the sense that we can cover diverse requirements. Firstly, we have shown that the asymmetric disjunction and asymmetric conjunction can be expressed by various functions, which cover the variety of real–word situations for fusion of predicates in our data intense society. Secondly, this approach has a lower computation burden in comparison to the approaches which require comparisons among items, because each item in Tables 1–8 is considered independently, i.e. no comparisons among items is required. Thirdly, this approach could improve the explainability of the machine learning solutions when the considered problem requires asymmetric evaluation.

Finally, note that there is also an alternative approach to the asymmetric disjunction and conjunction. Dujmović [11] considers disjunctive and conjunctive partial absorption, in which the influence of the secondary input is expressed in the form of a reward (if it is high), or of a penalty (if it is low). To exemplify this approach, we recall an example from [11], where if the mandatory alternative is fully satisfied and the optional one is fully rejected, we apply penalty of 20%; whereas if the optional alternative is fully satisfied, then the solution is rewarded by 10%. As a numerical model for the above described problem, one can consider the aggregation function

$$A(x, y) = \min(0.8x + 0.3xy, 1) \tag{21}$$

obtained by the linear interpolation from the boundary constraints $A(x, 0) = 0.8x$ and $A(x, 1) = \min(1.1x, 1)$. Put

$$b(t) = \begin{cases} \frac{2}{3} & \text{for } t \in [0, \frac{10}{11}] \\ \frac{t}{5-4t} & t \in [\frac{10}{11}, 1] \end{cases}$$

Then $A(x, y) < x$ if $y < b(x)$ and $A(x, y) > x$ if $y > b(x)$. Note that the aggregation function has a hybrid attitude (it is neither conjunctive nor disjunctive, nor averaging). Further, this function is neither a uninorm (only the right neutral element exists) nor a nullnorm (the absence of an absorbing element), see Fig. 8. Clearly, neither *OR ELSE* nor *AND IF POS* operator can model the above example.

5. Conclusion

The diverse ways of combining predicates is a challenging task in our data intense society, where we face the problems of aggregating satisfaction degrees to evaluate entities. Hence, practice searches for robust mathematical solutions to cover the tasks where aggregation plays the crucial role. In order to contribute, the theoretical part of this work has recognized and formalized diverse requirements for asymmetric disjunction and conjunction, which are illustrated by the real–word situations.

In the asymmetric disjunction, we have extended Bosc–Pivert operators by any averaging function. To cover the cases when a significantly satisfied less relevant alternative becomes the full alternative, we have formalized the non–continuous asymmetric disjunction by composite aggregation of the *MAX* operator and any function of asymmetric behaviour, e.g., the Bosc–Pivert operator. Further, we have proven the requirement for the asymmetric disjunction to be associative and continuous. Next, we explained the case, when an operator is co–monotone maxitive and homogeneous, which is required for the tasks when the influence of an optional alternative is reduced by a parameter assuming values from the unit interval. Finally, the *ORNESS* measure for the examined functions has been introduced.

In the asymmetric conjunction, i.e. the aggregation of constraints and wishes, on the basis of duality to asymmetric disjunction we have extended the Bosc–Pivert operators by any averaging function and developed the formula to fulfil the requirement for non–continuity. Further, we have recognized that duality does not hold for co–monotone maxitivity and homogeneity, when the influence of a constraint is reduced by a parameter assuming the values from the unit interval. Finally, the *ANDNESS* measure for the examined functions has been introduced.

The developed equations are demonstrated on examples to illustrate covering diverse needs. For each formalized asymmetric case we attached an explanation to help the user to choose the most suitable asymmetric connective for given situations. Real–world tasks as medical diagnosis, recommender systems and explainable machine learning might benefit from the results of this work.

Acknowledgments

This paper was partially supported by the project: 002STU–2–1/2018–PV entitled “STU as a leader of the Digital coalition ” by the Ministry of Education, Science, Research and Sport of the Slovak Republic, and partially supported by the project VEGA-1/0373/18 entitled “Big data analytics as a tool for increasing the competitiveness of enterprises and supporting informed decisions” by the Ministry of Education, Science, Research and Sport of the Slovak Republic.

References

[1] R. Singh, On the interpretation of disjunction: asymmetric, incremental, and eager for inconsistency, Linguist. Philos. 31 (2008) 245–260, doi:10.1007/s10988-008-9038-x.
[2] P. Bosc, O. Pivert, On four noncommutative fuzzy connectives and their axiomatization, Fuzzy Sets Syst. 202 (2012) 42–60, doi:10.1016/j.fss.2011.11.005.
[3] L. Karttunen, Presupposition and linguistic context, Theor. Linguist. 1 (1974) 181–193, doi:10.1515/thli.1974.1.1-3.181.
[4] T. Matthé, G.D. Tré, S. Zadrożny, J. Kacprzyk, A. Bronselaer, Bipolar database querying using bipolar satisfaction degrees, Int. J. Intell. Syst. 26 (2011) 890–910, doi:10.1002/int.20505.
[5] M. Hudec, M. Vučetić, Aggregation of fuzzy conformances, in: R.H. R, M. Gagolewski, R. Mesiar (Eds.), New Trends in Aggregation Theory, AGOP 2019, Advances in Intelligent Systems and Computing, vol 981, Springer, Cham, 2019, pp. 302–314.
[6] J.L. Usó-Doménechand, J.A. Nescolarde-Selva, Mathematical logic of notions and concepts, Found. Sci. (First Online). 10.1007/s10699-019-09604-9
[7] P. P., Mulholland, T. Collins, E. Motta, Improving comprehension of knowledge representation languages: a case study with description logics, Int. J. Hum.–Comput. Stud. 122 (2019) 145–167, doi:10.1016/j.ijhcs.2018.08.009.
[8] R. Yager, M. Reformat, N. To, Drawing on the ipad to input fuzzy sets with an application to linguistic data science, Inf. Sci. 479 (2019) 277–291, doi:10.1016/j.ins.2018.11.048.
[9] D. Dubois, H. Prade, On the use of aggregation operations in information fusion processes, Fuzzy Sets Syst. 142 (2004) 143–161, doi:10.1016/j.fss.2003.10.038.
[10] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: a Guide for Practitioners, Springer-Verlag, Berlin Heidelberg, 2007.
[11] J. Dujmović, Soft Computing Evaluation Logic: the LSP Decision Method and Its Applications, Wiley–IEEE Computer Society, Hoboken, 2018.
[12] J. Dujmović, Continuous preference logic for system evaluation, IEEE Trans. Fuzzy Syst. 15 (2007) 1082–1099, doi:10.1109/TFUZZ.2007.902041.
[13] G. Beliakov, H. Bustince, T. Calvo, A Practical Guide to Averaging Functions, Springer, Berlin, 2016.
[14] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions. Encyclopedia of Mathematics and its Applications, No. 127. Cambridge University Press, Cambridge, 2009.
[15] V. Torra, Y. Narukawa, Modeling Decisions. Information Fusion and Aggregation Operators, Springer, Heidelberg, 2007.
[16] J. Dujmović, Weighted conjunctive and disjunctive means and their application in system evaluation, in: Series Mathematics and Physics, 471–497, Univ. Beograd. Publ. Elektrotechn. Fak., Belgrade, 1974, pp. 147–158.
[17] A. Kolesárová, R. Mesiar, Parametric characterization of aggregation functions, Fuzzy Sets Syst. 160 (2009) 816–831, doi:10.1016/j.fss.2008.08.002.
[18] G. Brewka, S. Benferhat, D. Berre, Qualitative choice logic, Artif. Intell. 157 (2004) 203–237, doi:10.1016/j.artint.2004.04.006.
[19] L. Lietard, T. Nouredine, D. Rocacher, Fuzzy bipolar conditions of type “or else”, in: 20th IEEE International Conference on Fuzzy Systems (FUZZ–IEEE’ 11), 2011, pp. 2546–2551. Taiwan
[20] M. Grabisch, Set Functions, Games and Capacities in Decision Making, Springer, Berlin, 2016.
[21] M. Sugeno, Theory of fuzzy integrals and its applications, Tokyo Institute of Technology, Tokyo, 1974 Ph.D. thesis.
[22] J.-L. Marichal, An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework, IEEE Trans. Fuzzy Syst. 9 (2001) 164–172, doi:10.1109/91.917122.
[23] E. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (2010) 176–187, doi:10.1109/TFUZZ.2009.2039367.
[24] E. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer, Dordrecht, 2000.
[25] N. Shilkret, Maxitive measure and integration, Indagationes Mathematicae 33 (1971) 109–116, doi:10.1016/S1385-7258(71)80017-3.
[26] S. Benferhat, D. Dubois, S. Kaci, H. Prade, Bipolar possibility theory in preference modeling: representation, fusion and optimal solutions, Inf. Fusion 7 (2006) 135–150, doi:10.1016/j.inffus.2005.04.001.
[27] D. Dubois, H. Prade, Handling bipolar queries in fuzzy information processing, in: J. Galindo (Ed.), Handbook of Research on Fuzzy Information Processing in Databases, Information Science Reference, Hershey, 2008, pp. 97–114.
[28] P. Bosc, O. Pivert, On a fuzzy bipolar relational algebra, Inf. Sci. 219 (2013) 1–16, doi:10.1016/j.ins.2012.07.018.
[29] R.D. Mol, A. Tapia-Rosero, G.D. Tré, An approach for uncertainty aggregation using generalised conjunction/disjunction aggregators, in: 16th World Congress of the International Fuzzy Systems Association (IFSA), Gijon, 2015, pp. 1499–1506.
[30] A.Tapia-Rosero, A.Bronselaer, R. Mol, G. Tré, Fusion of preferences from different perspectives in a decision–making context, Inf. Fusion 29 (2016) 120–131, doi:10.1016/j.inffus.2015.07.006.
[31] M. Hudec, Constraints and wishes in quantified queries merged by asymmetric conjunction, in: 1st International Conference Fuzzy Management Methods (ICFM-square 2016), Fribourg, 2016, pp. 25–35.
[32] P. Bosc, A. Hadjali, O. Pivert, Empty versus overabundant answers to flexible relational queries, Fuzzy Sets Syst. 159 (2008) 1450–1467, doi:10.1016/j.fss.2008.01.007.
[33] P. Bosc, A. Hadjali, O. Pivert, Weakening of fuzzy relational queries: and absolute proximity relation–based approach, Mathware Soft Comput. 14 (2007) 35–55.
[34] R. Goebel, A. Chander, K. Holzinger, F. Lecue, Z. Akata, S.Stumpf, P. Kieseberg, A. Holzinger, Explainable AI: The new 42?, in: A. Holzinger, P. Kieseberg, A. Tjoa, E. Weippl (Eds.) Machine Learning and Knowledge Extraction. CD-MAKE 2018. Lecture Notes in Computer Science, vol 11015, Springer, Cham, 2018, pp. 295–303.
[35] A. Holzinger, B. Malle, P. Kieseberg, P. Roth, H. Muller, R. Reihs, K. Zatloukal, Towards the augmented pathologist: challenges of explainable AI in digital pathology, 2017. ArXiv: /1712.06657
[36] S. Singh, M. Ribeiro, C. Guestrin, Programs as black–box explanations, in: NIPS 2016 Workshop on Interpretable Machine Learning in Complex Systems, 2016, pp. 1–5. Barcelona