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Relationship between two types of superdecomposition integrals on finite spaces

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Abstract

This paper investigates the relationship between two types of superdecomposition integrals, namely, the convex integral and the pan-integral from above, on finite spaces. To this end, we introduce two new concepts related to monotone measures — *superadditivity with respect to singletons* and *minimal strictly subadditive set* — and discuss some of their properties. In the case that the monotone measure μ is superadditive with respect to singletons, we show that these two types of integrals are equivalent. In other cases, by means of the characteristics of minimal strictly subadditive sets we provide a set of necessary and sufficient conditions for which these two types of integrals coincide with each other.

Keywords: Monotone measure; Superdecomposition integral; Convex integral; Pan-integral from above; Minimal strictly subadditive set

1. Introduction

In non-additive measure and integral theory, the Choquet integral [1], the pan-integral based on $(+, \cdot)$ [22] (see also [21]) and the concave integral ([6,7]) are three kinds of well-known nonlinear integrals. In [3] Even and Lehrer introduced the decomposition integrals forming a common framework of these integrals, i.e., all these integrals can be seen as particular instances for decomposition integrals (see also [9–11]). However, in general case they are significantly different from each other. Recall that the concave integral is the greatest decomposition integral, while the pan-integral and the Choquet integral are incomparable, in general [5].

Recently, we have studied the relationships among above mentioned three specific types of decomposition integrals, namely, the Choquet integral, the concave integral and the $(+, \cdot)$-based pan-integral. By means of *minimal atoms* and *(M)-property* of monotone measures, we obtained necessary and/or sufficient conditions, under which the concave

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integral and the pan-integral, and the Choquet integral and the pan-integral coincide with each other, respectively (see [12–16])

In [11] Mesiar et al. introduced superdecomposition integrals which can be seen as a counterpart of decomposition integrals introduced by Even and Lehrer [3]. Similar to the decomposition integrals, this construction copies the idea of upper integral sums and it is based on a system $\mathcal{H}$ of finite set systems. Although the proposed integrals have several properties that are similar or dual with respect to decomposition integrals, but they also have some significant differences. So, it is a challenging problem to develop the counterparts of [12–14] within the framework of superdecomposition integrals. From [11] we recall the convex integral — a special kind of superdecomposition integral introduced by considering all possible superdecompositions with no constraints on the applied sets. It can be treated as counterpart of the concave integral. The superdecomposition integral corresponding to the pan-integral is pan-integral from above introduced by Wang ([20]). Observe that the Choquet integral, which is a decomposition integral with respect to the system of all finite chains, is also the superdecomposition integral with respect to the same system, i.e., the counterpart of the Choquet integral is itself.

In [8] Lv et al. discussed the relation of the Choquet integral and the pan-integral from above. They proposed the so-called dual (M)-property of a monotone measure and presented a necessary and sufficient condition such that the Choquet integral coincides with the pan-integral from above on finite spaces.

In this paper our main task is to investigate the relation between the convex integral and the pan-integral from above.

The paper is structured as follows. Section 2 recalls some necessary knowledge, including the decomposition integrals and superdecomposition integrals. In Section 3 we propose the concept of *superadditivity with respect to singletons* of a monotone measure, which is a weaker condition than superadditivity. We also consider the case that the monotone measures do not satisfy the superadditivity w.r.t. singletons and introduce the concept of *minimal strictly subadditive set* for monotone measures and study some of their properties. In Section 4 we focus on the equivalence of the convex integral and the pan-integral from above. By means of the characteristics of *superadditivity with respect to singletons* and *minimal strictly subadditive sets* we provide a set of necessary and/or sufficient conditions that these two integrals coincide on finite spaces. Observe that in [13], by means of the characteristics of minimal atoms of monotone measures we presented a set of necessary and sufficient conditions that the concave integral coincides with the pan-integral on finite spaces (Theorem 4.1 in [13]). Although the convex integral as a special superdecomposition integral is a counterpart of the concave integral, and similarly, the pan-integral from above correspond to the pan-integral, as we shall see, the methods presented here are quite different to that of [13]. Section 5 makes some further discussions, in which we consider more general cases and obtain similar results. Finally, Section 6 concludes the paper.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X. A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a monotone measure on measurable space $(X, \mathcal{A})$ if it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;

(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

The class of all monotone measures on $(X, \mathcal{A})$ will be denoted by $\mathcal{M}$. $\mathcal{F}^+$ denotes the set of all $\mathcal{A}$-measurable functions $f : X \to [0, +\infty]$ and $\mathcal{F}_b^+$ denotes the set of all bounded $\mathcal{A}$-measurable functions $f : X \to [0, +\infty)$.

A monotone measure $\mu \in \mathcal{M}$ is said to be *submodular*, if $\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$; *subadditive*, if $\mu(A \cup B) \le \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$; *supermodular*, if $\mu(A \cup B) + \mu(A \cap B) \ge \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$; and *superadditive*, if $\mu(A \cup B) \ge \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ (see [2,17]).

Obviously, if μ is submodular (supermodular, respectively), then it is subadditive (superadditive, respectively), but not vice versa.

2.1. Decomposition integrals

From Even and Lehrer [3], and Mesiar and Stupňanová [10], we recall some results related to decomposition integrals.

For a fixed measurable space $(X, \mathcal{A})$, the set of all systems $\mathcal{H}$ of finite set systems (collections) from $\mathcal{A}\backslash\{\emptyset\}$ will be denoted by $\mathbb{X}$.

Let $\mathcal{H} \in \mathbb{X}$ be fixed. The mapping $I_{\mathcal{H}} : \mathcal{M} \times \mathcal{F}^+ \to [0, +\infty]$ given by

$$I_{\mathcal{H}}(\mu, f) = \sup\left\{ \sum_{i\in J} a_i \mu(A_i) : \{A_i\}_{i\in J} \in \mathcal{H}, \sum_{i\in J} a_i \chi_{A_i} \le f \right\}, \tag{2.1}$$

where all constants $a_i \ge 0$, is called a *decomposition integral*.

Several well-known nonlinear integrals are specific decomposition integrals ([3,10]).

- Let $\mathcal{H}_1 = \{\{A\} : A \in \mathcal{A} \setminus \{\emptyset\}\}$. Then $I_{\mathcal{H}_1}$ is the Shilkret integral ([19]), i.e.,
$$I_{\mathcal{H}_1}(\mu, f) = \sup\left\{t \cdot \mu(\{f \ge t\}) : t \in [0, \infty]\right\}. \tag{2.2}$$
- Let $\mathcal{H}_2 = \{\mathcal{C} : \mathcal{C}$ is a finite chain in $\mathcal{A} \setminus \{\emptyset\}\}$. Then $I_{\mathcal{H}_2}$ is the Choquet integral.
- Let $\mathcal{H}_3 = \{\mathcal{B} : \mathcal{B}$ is a finite subset of $\mathcal{A} \setminus \{\emptyset\}\}$. Then $I_{\mathcal{H}_3}$ is the concave integral.
- Let $\mathcal{H}_4 = \{\mathcal{P} : \mathcal{P}$ is a finite measurable partition of $X\}$. Then $I_{\mathcal{H}_4}$ is the pan-integral introduced in [22] (see also [21]), based on the pair of standard addition and multiplication $(+, \cdot)$.

2.2. Relationships among $I_{\mathcal{H}_2}$, $I_{\mathcal{H}_3}$ and $I_{\mathcal{H}_4}$

We recall the relationships among $I_{\mathcal{H}_2}$, $I_{\mathcal{H}_3}$ and $I_{\mathcal{H}_4}$.

For $I_{\mathcal{H}_2}$ and $I_{\mathcal{H}_3}$, Lehrer and Teper [7] showed the following result: $I_{\mathcal{H}_2}(\mu, f) \equiv I_{\mathcal{H}_3}(\mu, f)$ holds for all $f \in \mathcal{F}^+$ if and only if the underlying monotone measure μ is supermodular.

Recently, we characterized the relationship between $I_{\mathcal{H}_2}$ and $I_{\mathcal{H}_4}$ [12,14] (see also [16]), and the relationship between $I_{\mathcal{H}_3}$ and $I_{\mathcal{H}_4}$ [13] (see also [15]). In the characterization, the concepts of *minimal atom* and *(M)-property* for a monotone measure μ play important roles. For more information about these concepts, we refer to [12–14].

Let $\mu \in \mathcal{M}$. (1) A set $A \in \mathcal{A}$ is called a minimal atom of μ if $\mu(A) > 0$ and for every $B \in \mathcal{A}$ and $B \subset A$ holds either (i) $\mu(B) = 0$, or (ii) $A = B$;

(2) μ is said to have (M)-property, if for any $A, B \in \mathcal{A}$ and $A \subset B$, there exists $C \in \mathcal{A}$ and $C \subset A$ such that

$$\mu(C) = \mu(A) \text{ and } \mu(B) = \mu(C) + \mu(B \setminus C).$$

In [12–14] we have obtained the following results.

Theorem 2.1. [13] *Let X be a finite space and $\mu \in \mathcal{M}$ be fixed. Then, for all $f \in \mathcal{F}^+$*

$$I_{\mathcal{H}_3}(\mu, f) = I_{\mathcal{H}_4}(\mu, f)$$

if and only if the following two conditions hold:

(i) μ possesses the minimal atoms disjointness property, i.e., for every pair of minimal atoms A and B of μ, $A \ne B$ implies $A \cap B = \emptyset$;

(ii) μ is subadditive w.r.t. minimal atoms, i.e., for every set $A \in \mathcal{A}$ with $\mu(A) > 0$, we have

$$\mu(A) \le \sum_{i=1}^{n} \mu(A_i),$$

where $\{A_i\}_{i=1}^n$ is the set of all minimal atoms contained in A.

Theorem 2.2. [12,14] *Let X be a finite space and $\mu \in \mathcal{M}$ be fixed. Then, for all $f \in \mathcal{F}^+$*

$$I_{\mathcal{H}_2}(\mu, f) = I_{\mathcal{H}_4}(\mu, f)$$

if and only if μ has (M)-property.

When X is a general space (not necessarily finite) we proved that the (M)-property of μ is sufficient for the equivalence of $I_{\mathcal{H}_2}$ and $I_{\mathcal{H}_4}$, see [16], and the subadditivity of μ ensures the equivalence of $I_{\mathcal{H}_3}$ and $I_{\mathcal{H}_4}$ [15]. It remains open problem that whether their inverse propositions are true.

2.3. Superdecomposition integrals

In [11] Mesiar et al. introduced the superdecomposition integrals via a dual way, i.e., the superdecomposition of the functions being considered.

Definition 2.3. ([11]) Let $\mathcal{H} \in \mathbb{X}$ be fixed. The mapping $I^{\mathcal{H}} : \mathcal{M} \times \mathcal{F}_b^+ \to [0, +\infty]$ given by

$$I^{\mathcal{H}}(\mu, f) = \inf\left\{\sum_{i\in J} a_i \mu(A_i) : \{A_i\}_{i\in J} \in \mathcal{H}, \sum_{i\in J} a_i \chi_{A_i} \geq f\right\}, \tag{2.3}$$

where all constants $a_i \geq 0$, is called a *superdecomposition integral*.

- Consider $\mathcal{H}_1 = \{\{A\} : A \in \mathcal{A} \setminus \emptyset\}$. Then
$$I^{\mathcal{H}_1}(\mu, f) = \sup\left\{f(x) : x \in X\right\} \cdot \mu\big(\{x : f(x) > 0\}\big) \tag{2.4}$$
(Comparing with (2.2)).
- Observe the equivalence of Eq. (2.1) and Eq. (2.3), for a classical measure m and all $f \in \mathcal{F}_b^+$,
$$I_{\mathcal{H}_3}(m, f) = I^{\mathcal{H}_3}(m, f),$$
yielding the standard Lebesgue integral. In general, for a monotone measure Eq. (2.1) is no longer equivalent to Eq. (2.3). But, if $\mathcal{H}$ is the set of all finite chains contained in $\mathcal{A}$, i.e. $\mathcal{H} = \mathcal{H}_2$, then both equations define the Choquet integral. That is, for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}_b^+$, we have
$$I^{\mathcal{H}_2}(\mu, f) = I_{\mathcal{H}_2}(\mu, f) = \int_0^\infty \mu\big(\{x : f(x) \geq t\}\big)dt.$$
- As a counterpart of concave integral $I_{\mathcal{H}_3}$, $I^{\mathcal{H}_3}$ is also called a convex integral and defined by
$$I^{\mathcal{H}_3}(\mu, f) = \inf\left\{\sum_{i=1}^k a_i \mu(A_i) : \{A_i\}_{i=1}^k \subset \mathcal{A}, a_i \geq 0, \sum_{i=1}^k a_i \chi_{A_i} \geq f\right\}.$$
- The integral $I^{\mathcal{H}_4}$ can be treated as counterpart of the pan-integral $I_{\mathcal{H}_4}$, it is called the common pan-integral from above (see [20]). We simply call it the *pan-integral from above*.
$$I^{\mathcal{H}_4}(\mu, f) = \inf\Bigg\{\sum_{i\in I} a_i \mu(A_i) : \{A_i\}_{i\in I} \text{ is a finite measurable partition of } X, \sum_{i\in I} a_i \chi_{A_i} \geq f\Bigg\}.$$

From the above analysis, we know that $I^{\mathcal{H}_3}(\mu, f)$ is the lower bound of the superdecomposition integrals, i.e., it is the smallest superdecomposition integral.

2.4. Relationships among $I^{\mathcal{H}_2}$, $I^{\mathcal{H}_3}$ and $I^{\mathcal{H}_4}$

In general, for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}_b^+$, we have $I^{\mathcal{H}_3}(\mu, f) \leq I^{\mathcal{H}_2}(\mu, f)$ and $I^{\mathcal{H}_3}(\mu, f) \leq I^{\mathcal{H}_4}(\mu, f)$, but the converse implications may not be true, and $I^{\mathcal{H}_2}$ and $I^{\mathcal{H}_4}$ are incomparable.

Some relationships among the superdecomposition integrals were discussed. In [11] Mesiar et al. proved that $I^{\mathcal{H}_2}(\mu, f) \equiv I^{\mathcal{H}_3}(\mu, f)$ for all $f \in \mathcal{F}_b^+$ (i.e., the Choquet integral coincides with the convex integral) if and only if the underlying monotone measure μ is submodular (comparing with the equivalence of $I_{\mathcal{H}_2}$ and $I_{\mathcal{H}_3}$, see [7]).

Recently, Lv et al. [8] introduced the concept of *dual (M)-property*, as follows: a monotone measure μ is said to have *dual (M)-property* if for any $A, B \in \mathcal{A}$, $A \subset B$, there exists $C \in \mathcal{A}$ with $A \subset C \subset B$ such that

$$\mu(C) = \mu(A) \text{ and } \mu(B) = \mu(C) + \mu(B \setminus C)$$

(comparing with the concept of *(M)-property*). Obviously, the dual (M)-property implies the subadditivity of μ. In [8] it has been proved that if the underlying monotone measure μ has the dual (M)-property then for all $f \in \mathcal{F}^+$ it holds $I^{\mathcal{H}_2}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$. Furthermore, if the considered space X is finite, then the dual (M)-property is also necessary for the equivalence of $I^{\mathcal{H}_2}$ and $I^{\mathcal{H}_4}$ (comparing with the equivalence of $I_{\mathcal{H}_2}$ and $I_{\mathcal{H}_4}$, see [12,14], also see Theorem 2.2 above).

In this paper, we focus on the equivalence between the convex integral $I^{\mathcal{H}_3}$ and the pan-integral from above $I^{\mathcal{H}_4}$ on finite spaces.

3. Superadditivity for singletons and minimal strictly subadditive sets

To discuss the relationship between $I^{\mathcal{H}_3}(\mu, f)$ and $I^{\mathcal{H}_4}(\mu, f)$ on finite spaces, in the rest of sections, if not stating explicitly the form of X, we will consider a fixed space $X = \{1, 2, \ldots, n\}$ and $\mathcal{A} = 2^X$.

We need to pay attention to the following fact: if $i_0 \in X$ with $\mu(\{i_0\}) = 0$, then it will be of zero contribution to the values of $I^{\mathcal{H}_3}(\mu, f)$ and $I^{\mathcal{H}_4}(\mu, f)$.

In fact, for any superdecomposition $\sum_{j=1}^k a_j \chi_{A_j}$ of f, i.e., $\sum_{j=1}^k a_j \chi_{A_j} \geq f$, the expression $\sum_{j=1}^k a_j \chi_{A_j \setminus \{i_0\}} + f(i_0)\chi_{\{i_0\}}$ is also a superdecomposition of f, and we have

$$\begin{aligned}\sum_{j=1}^k a_j \mu(A_j \setminus \{i_0\}) + f(i_0)\mu(\{i_0\}) &= \sum_{j=1}^k a_j \mu(A_j \setminus \{i_0\}) \\ &\leq \sum_{j=1}^k a_j \mu(A_j),\end{aligned}$$

from which we can see that i_0 is of zero contribution to the values of $I^{\mathcal{H}_3}(\mu, f)$ and $I^{\mathcal{H}_4}(\mu, f)$ (i.e., the value $f(i_0)$ has no influence on the resulting integrals $I^{\mathcal{H}_3}(\mu, f)$ and $I^{\mathcal{H}_4}(\mu, f)$).

From the above fact, in the rest of the discussions, we suppose that $\mu(\{i\}) > 0$ for each $i \in X$ (In fact, this assumption can be abandoned, we will give a detailed explanation in later Section 5).

We consider the following characteristic of monotone measures.

Definition 3.1. Let $\mu \in \mathcal{M}$. We say that μ is *superadditive w.r.t. singletons*, if for any $A \in 2^X$ we have

$$\mu(A) \geq \sum_{\{i\} \subset A} \mu(\{i\}).$$

Obviously, if μ is superadditive, then it is superadditive w.r.t. singletons, but not vice versa.

Example 3.2. Let $X = \{1, 2, 3\}$ and μ be defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset; \\ 1 & \text{if } |A| = 1; \\ 3 & \text{if } |A| \geq 2, \end{cases}$$

where $|A|$ stands for the cardinality of A. Then μ is a superadditive w.r.t. singletons, but $\mu(X) < \mu(\{1, 2\}) + \mu(\{3\})$.

In the next section, we shall show that the superadditivity w.r.t. singletons of monotone measure μ ensures the equality $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$. To identify the necessary and sufficient condition for $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$, we need to consider the case that the monotone measure μ defined over X is not superadditive w.r.t. singletons, i.e., there exists at least one subset $A \subset X$ such that $\mu(A) < \sum_{\{i\} \subset A} \mu(\{i\})$. This motivates the following concepts, which are opposed to superadditivity w.r.t. singletons.

Definition 3.3. Let $\mu \in \mathcal{M}$ be given, $A \in 2^X$.

(i) A is said to be *strictly μ-subadditive set w.r.t. singletons* (strictly μ-subadditive set, for short), if

$$\mu(A) < \sum_{\{i\}\subset A} \mu(\{i\}).$$

(ii) A is called *minimal strictly μ-subadditive set w.r.t. singletons*, if A is strictly μ-subadditive set and any nonempty proper subset of A is not strictly μ-subadditive set, i.e., for any $B \subsetneqq A$, $B \neq \emptyset$,

$$\mu(B) \geq \sum_{\{j\}\subset B} \mu(\{j\}).$$

Note that when the monotone measure $\mu \in \mathcal{M}$ is clear from the context, we also briefly refer to "strictly μ-subadditive set w.r.t. singletons" as "strictly μ-subadditive set" or "strictly subadditive set", and "minimal strictly μ-subadditive set w.r.t. singletons" as "minimal strictly μ-subadditive set" or "minimal strictly subadditive set".

Note 3.4. According to the above definition, if $\mu \in \mathcal{M}$ is not superadditive w.r.t. singletons, then there exists at least one strictly μ-subadditive set. Also, it is obvious that a minimal strictly μ-subadditive set contains at least two singletons.

Proposition 3.5. *Let X be finite space and $\mu \in \mathcal{M}$ be given. If $A \in 2^X$ is strictly μ-subadditive set, then it contains at least one minimal strictly μ-subadditive subset.*

Proof. Put

$$\mathcal{S}_A = \{B : B \subset A \text{ is a strictly } \mu\text{-subadditive set}\},$$

then $\mathcal{S}_A$ is nonempty (since $A \in \mathcal{S}_A$). Every minimal element of $\mathcal{S}_A$ is a minimal strictly μ-subadditive set contained in A. □

Example 3.6. Let $X = \{1, 2, 3\}$ and the monotone measure μ be defined as $\mu(A) = 1$ for any nonempty set A. Then X is a strictly μ-subadditive set and

$$\mathcal{S}_X = \Big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, X\Big\}.$$

$\mathcal{S}_X$ contains three minimal elements, thus X has three minimal strictly μ-subadditive subsets w.r.t. singletons, namely $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$.

Definition 3.7. Let $\mu \in \mathcal{M}$ be given and A_1, A_2 be minimal strictly μ-subadditive sets contained in $A \subset X$. If there exists a family $\{C_i\}_{i=1}^k$ of minimal strictly μ-subadditive sets contained in A such that $C_1 = A_1$, $C_k = A_2$ and $C_i \cap C_{i+1} \neq \emptyset$, $1 \leq i \leq k-1$, then we say that A_1 and A_2 are connectable in A, denoted by $A_1 \sim_A A_2$. If $A = X$ then we simply say that A_1 and A_2 are connectable, and denote it by $A_1 \sim A_2$.

Let $\mu \in \mathcal{M}$ be fixed and let $\mathfrak{M}_\mu(X)$ denote the set of all minimal strictly μ-subadditive sets contained in X, i.e.,

$$\mathfrak{M}_\mu(X) = \Big\{M \in 2^X : M \text{ is a minimal strictly } \mu\text{-subadditive set}\Big\}$$

If μ is not superadditive w.r.t. singletons, from Note 3.4 and Proposition 3.5 we know that $\mathfrak{M}_\mu(X) \neq \emptyset$.

The relation "$\sim$" described in Definition 3.7 is an equivalence relation on $\mathfrak{M}_\mu(X)$. Let $A \in \mathfrak{M}_\mu(X)$, and let $[A]$ denote the equivalence class of A with respect to the relation "$\sim$", i.e., $[A] \in \mathfrak{M}_\mu(X)/\sim$,

$$[A] = \Big\{B \in \mathfrak{M}_\mu(X) : B \sim A\Big\}.$$

These equivalence classes of minimal strictly μ-subadditive sets contained in X are illustrated in Fig. 1.

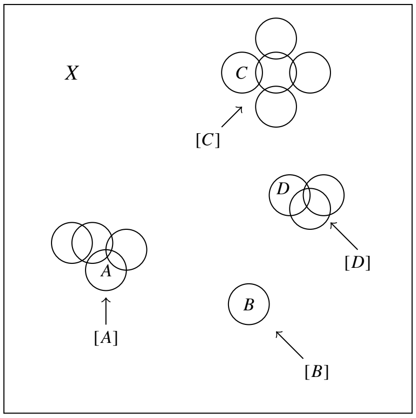

Fig. 1. The equivalence classes of *minimal strictly μ-subadditive sets* contained in X.

For the convenience of discussion, we will denote $\mu\Big(\bigcup_{B\in[A]} B\Big)$ by $\mu([A])$, i.e.,

$$\mu([A]) \triangleq \mu\Big(\bigcup_{B\in[A]} B\Big).$$

(Note: $\mu([A])$ is only a symbol!)

Observe that for any $C \in [A]$ it follows from $[C] = [A]$ that

$$\mu([C]) = \mu([A]).$$

Example 3.8. (i) In Example 3.6, we have

$$\big[\{1,2\}\big] = \big\{\{1,2\}, \{1,3\}, \{2,3\}\big\},$$

and $\mu\big(\big[\{1,2\}\big]\big) = \mu(X) = 1$.

(ii) Let $X = \{1, \dots, 6\}$. The monotone measure μ is defined as $\mu(\{i\}) = 1$ for $i = 1, \dots, 6$, $\mu(\{1,2\}) = \mu(\{2,3\}) = \mu(\{3,4\}) = 1$, $\mu(\{5,6\}) = \frac{3}{2}$ and $\mu(A) = \sum_{\{i\}\subset A} \mu(\{i\})$ otherwise. Then

$$\big[\{1,2\}\big] = \big\{\{1,2\}, \{2,3\}, \{3,4\}\big\}, \mu\big(\big[\{1,2\}\big]\big) = \mu(\{1,2,3,4\}) = 4$$

and

$$\big[\{5,6\}\big] = \big\{\{5,6\}\big\}, \mu\big(\big[\{5,6\}\big]\big) = \mu(\{5,6\}) = \frac{3}{2}.$$

4. Equivalence of the pan-integrals from above and the convex integrals

In this section, we will give a set of necessary and sufficient conditions under which $I^{\mathcal{H}_3}(\mu, f)$ and $I^{\mathcal{H}_4}(\mu, f)$ coincides with each other. We will discuss it in two cases: (1) the considered monotone measures are superadditive w.r.t. singletons; (2) the considered monotone measures are not superadditive w.r.t. singletons.

4.1. The case that μ is superadditive w.r.t. singletons

The following result shows that the superadditivity w.r.t. singletons of monotone measure μ ensures the equality $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$.

Theorem 4.1. *Let X be finite and $\mu \in \mathcal{M}$ be fixed. If μ is superadditive w.r.t. singletons, then for each $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f).$$

Proof. It suffices to prove that for each $f \in \mathcal{F}_b^+$ we have $I^{\mathcal{H}_3}(\mu, f) \geq I^{\mathcal{H}_4}(\mu, f)$. Now let f be given and $\sum_{i=1}^k a_i \chi_{A_i}$ be an arbitrary superdecomposition of f, i.e., $\sum_{i=1}^k a_i \chi_{A_i} \geq f$. Then $\sum_{i=1}^n (\sum_{\{i\} \subset A_j} a_j)\chi_{\{i\}} = \sum_{i=1}^k a_i \chi_{A_i} \geq f$. Moreover,

$$\sum_{j=1}^k a_j \mu(A_j) \geq \sum_{j=1}^k a_j \Big(\sum_{\{i\} \subset A_j} \mu(\{i\}) \Big) = \sum_{i=1}^n \big(\sum_{\{i\} \subset A_j} a_j \big) \mu(\{i\}) \geq I^{\mathcal{H}_4}(\mu, f),$$

from which we conclude that $I^{\mathcal{H}_3}(\mu, f) \geq I^{\mathcal{H}_4}(\mu, f)$. □

Note that under constraints of Theorem 4.1,

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f) = \sum_{i=1}^n f(i)\mu(\{i\}).$$

Corollary 4.2. *Let X be finite and $\mu \in \mathcal{M}$ be fixed. If μ be superadditive, then for each $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f).$$

Remark 4.3. Obviously, any additive measure μ is superadditive, supermodular, submodular and subadditive, simultaneously. Therefore, for an additive measure μ on finite space all discussed integrals coincide with the standard Lebesgue integral, i.e., for any $f \in \mathcal{F}_b^+$,

$$\begin{aligned} & I^{\mathcal{H}_2}(\mu, f) = I_{\mathcal{H}_2}(\mu, f) \\ &= I^{\mathcal{H}_3}(\mu, f) = I_{\mathcal{H}_3}(\mu, f) \\ &= I^{\mathcal{H}_4}(\mu, f) = I_{\mathcal{H}_4}(\mu, f) \\ &= (Leb)\int f d\mu, \end{aligned}$$

where $(Leb)\int f d\mu$ denotes the Lebesgue integral of f with respect to μ.

4.2. The case that μ is not superadditive w.r.t. singletons

In this subsection we suppose that the considered monotone measures are not superadditive w.r.t. singletons.

Proposition 4.4. *Let $\mu \in \mathcal{M}$ be given. If for all $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f),$$

then every minimal strictly μ-subadditive set contains only two singletons.

Proof. Suppose A is a minimal strictly μ-subadditive set. By Note 3.4, it suffices to prove that the case of $|A| \geq 3$ does not occur. For simplicity, we suppose that $\{1\} \subset A$ and let

$$f(x) = \begin{cases} 2, & x = 1, \\ 1, & x \in A \setminus \{1\}, \\ 0, & \text{otherwise.} \end{cases}$$

For an arbitrary partition $\{A_i\}$ of A and $\sum_{i=1}^k a_i \chi_{A_i} \geq f$, there are two cases.

Case (i). If $k = 1$ then $a_1 \geq 2$ and $\sum_{i=1}^k a_i \mu(A_i) \geq 2\mu(A)$.

Case (ii). If $k > 1$ then each A_i is a proper subset of A. Due to the fact that A is a minimal strictly subadditive set, we conclude that

$$\mu(A_i) \geq \sum_{\{j\} \subset A_i} \mu(\{j\})$$

Suppose $1 \in A_1$ then $a_1 \geq 2$ and $a_i \geq 1$ for $i \geq 2$. So

$$\begin{aligned}\sum_{i=1}^{k} a_i \mu(A_i) &\geq 2\mu(A_1) + \sum_{i=2}^{k} \mu(A_i) \\ &\geq 2 \sum_{\{j\}\subset A_1} \mu(\{j\}) + \sum_{i=2}^{k} \sum_{\{j\}\subset A_i} \mu(\{j\}) \\ &\geq \mu(\{1\}) + \sum_{\{j\}\subset A} \mu(\{j\}).\end{aligned}$$

Due to the arbitrariness of the partition $\{A_i\}$, we have that

$$I^{\mathcal{H}_4}(\mu, f) \geq \min\Big(2\mu(A), \mu(\{1\}) + \sum_{\{j\}\subset A} \mu(\{j\})\Big).$$

On the other hand, since $\chi_{\{1\}} + \chi_A = f$, we have that

$$I^{\mathcal{H}_3}(\mu, f) \leq \mu(\{1\}) + \mu(A).$$

Since A is a minimal strictly subadditive set, $\mu(\{1\}) + \mu(A) < \mu(\{1\}) + \sum_{\{j\}\subset A} \mu(\{j\})$. To ensure $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$, the only possibility is $\mu(\{1\}) = \mu(A)$. If $|A| \geq 3$ (we suppose that $\{1, 2, 3\} \subset A$), due to the monotonicity of μ and the fact that A is a minimal strictly subadditive set, then we have that

$$\mu(A) \geq \mu(\{1, 2\}) \geq \mu(\{1\}) + \mu(\{2\}) > \mu(\{1\}),$$

a contradiction. Hence $|A| = 2$. □

From the proof of Proposition 4.4, we can obtain the following.

Corollary 4.5. *Let $\mu \in \mathcal{M}$ be fixed and for each $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$. If $\{i, j\}$ is a minimal strictly μ-subadditive set, then $\mu(\{i\}) = \mu(\{j\}) = \mu(\{i, j\})$.*

The following result is a general version of Corollary 4.5.

Proposition 4.6. *Let $\mu \in \mathcal{M}$ be fixed and for each $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$. If A is minimal strictly μ-subadditive set w.r.t. singletons, then for any singleton $\{t\}$ of $\bigcup_{B\in[A]} B$, we have*

$$\mu([A]) = \mu(\{t\}).$$

The proof is deferred to the “Appendix”.

From Proposition 4.4 and 4.6, we can prove the following result.

Theorem 4.7. *Let X be finite space and $\mu \in \mathcal{M}$ be not superadditive w.r.t. singletons. Then for each $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$$

if and only if the following statement holds: for every minimal strictly μ-subadditive set A, we have $\mu(\{i\}) = \mu([A])$, where $\{i\}$ is an arbitrary singleton of $\bigcup_{B\in[A]} B$.

To prove the above result, we present the following lemma.

Lemma 4.8. *Let $\mu \in \mathcal{M}$ be given and $[A_j]$, $j = 1, \dots, k$ be all of the equivalence classes of minimal strictly μ-subadditive sets. If for each A_j, we have $\mu(\{i\}) = \mu([A_j])$, where $\{i\}$ is an arbitrary singleton of $\bigcup_{B\in[A_j]} B$, then for any $A \subset X$, it holds that*

$$\mu(A) \geq \sum_{\{i\} \subset A \setminus \left(\bigcup_{j=1}^{k}\left(\bigcup_{B \in [A_j]} B\right)\right)} \mu\big(\{i\}\big) + \sum_{j} \mu\big(\{\tilde{j}\}\big)$$

where $\{\tilde{j}\}$ *is an arbitrary singleton of* $A \bigcap \left(\bigcup_{B \in [A_j]} B\right)$.

The proof is postponed to the “Appendix”.

Note 4.9. If the monotone measure μ satisfies the conditions in Lemma 4.8, then for each $A \in 2^X$ it in fact holds that

$$\begin{aligned}
\mu(A) &\geq \mu(\{1, \ldots, s\} \cup \{\tilde{1}, \ldots, \tilde{l}\}) \\
&\geq \sum_{i=1}^{s} \mu(\{i\}) + \sum_{j=1}^{l} \mu(\{\tilde{j}\}) \\
&= \sum_{\{i\} \subset A \setminus \left(\bigcup_{j=1}^{k}\left(\bigcup_{B \in [A_j]} B\right)\right)} \mu\big(\{i\}\big) + \sum_{j} \mu\big(A \cap \big(\cup_{B \in [A_j]} B\big)\big) \\
&= \sum_{\{i\} \subset A \setminus \left(\bigcup_{j=1}^{k}\left(\bigcup_{B \in [A_j]} B\right)\right)} \mu\big(\{i\}\big) + \sum_{A \bigcap \left(\bigcup_{B \in [A_j]} B\right) \neq \emptyset} \mu([B_j]).
\end{aligned}$$

Proof of Theorem 4.7. By Propositions 4.4 and 4.6, the necessity is obvious. Now we prove the sufficiency. We need to show that $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$ for arbitrary but fixed function $f \in \mathcal{F}_b^+$. Suppose all of the equivalence classes of minimal strictly μ-subadditive sets contained in X are $[B_j]$, $j = 1, \ldots, k$ (notice that the sets $\bigcup_{B \in [B_j]} B$, $j = 1, \ldots, k$ are pairwise disjoint). For an arbitrary expression $\sum_{i=1}^{s} a_i \chi_{A_i} \geq f$, if we denote $E_i = A_i \setminus \left(\bigcup_{j=1}^{k}\left(\bigcup_{B \in [B_j]} B\right)\right)$ and $F_{i,j} = A_i \cap \left(\bigcup_{B \in [B_j]} B\right)$, where $i = 1, \ldots, s$ and $j = 1, \ldots, k$, then $E_i \cap F_{i',j} = \emptyset$ for each i, i', j, and $F_{i,j} \cap F_{i,j'} = \emptyset$ for any i whenever $j \neq j'$. Moreover, $A_i = E_i \bigcup \left(\bigcup_{j=1}^{k} F_{i,j}\right)$ for each i, and which implies that

$$\sum_{i=1}^{s} a_i \chi_{E_i} + \sum_{i=1}^{s}\sum_{j=1}^{k} a_i \chi_{F_{i,j}} = \sum_{i=1}^{s} a_i \chi_{A_i} \geq f.$$

For simplicity, we assume that $\bigcup_{i=1}^{s} E_i = \{1, \ldots, t\}$. If we let

$$\delta_{l,i} = \begin{cases} 1, & \text{if } l \in E_i, \\ 0, & \text{otherwise,} \end{cases}$$

where $l = 1, \ldots, t$, $i = 1, \ldots, s$, and

$$\gamma_{i,j} = \begin{cases} 1, & \text{if } A_i \bigcap \left(\bigcup_{B \in [B_j]} B\right) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $i = 1, \ldots, s$, $j = 1, \ldots, k$. Then it is easy to see that

$$\sum_{l=1}^{t}\left(\sum_{i=1}^{s} a_i \delta_{l,i}\right)\chi_{\{l\}} = \sum_{i=1}^{s} a_i \chi_{E_i},$$

and

$$\sum_{j=1}^{k}\left(\sum_{i=1}^{s} a_i \gamma_{i,j}\right)\chi_{\cup_{B \in [B_j]} B} \geq \sum_{i=1}^{s}\sum_{j=1}^{k} a_i \chi_{F_{i,j}}.$$

Hence,

$$\begin{aligned}
\sum_{l=1}^{t}\Big(\sum_{i=1}^{s} a_i\delta_{l,i}\Big)\chi_{\{l\}} + \sum_{j=1}^{k}\Big(\sum_{i=1}^{s} a_i\gamma_{i,j}\Big)\chi_{\cup_{B\in[B_j]}B} \\
&\geq \sum_{i=1}^{s} a_i\chi_{E_i} + \sum_{i=1}^{s}\sum_{j=1}^{k} a_i\chi_{F_{i,j}} \\
&= \sum_{i=1}^{s} a_i\chi_{A_i} \geq f.
\end{aligned}$$

Notice that

$$\Big\{\{1\},\ldots,\{t\}\Big\}\bigcup\Big\{\bigcup_{B\in[B_1]} B,\ldots,\bigcup_{B\in[B_k]} B\Big\}$$

is a partition of X. From the hypothesis in the theorem, we know that the conclusion of Lemma 4.8 holds. So, we have that

$$\begin{aligned}
\mu(A_i) &= \mu\Big(A_i\bigcap\Big(\{1\ldots,t\}\bigcup_{j=1}^{k}\bigcup_{B\in[B_j]} B\Big)\Big) \\
&\geq \mu\Big(\Big(\bigcup_{1\leq l\leq t,\{l\}\subset A_i}\{l\}\Big)\bigcup\Big(\bigcup_{\{\tilde{j}\}\subset A_i\bigcap(\bigcup_{B\in[B_j]}B)}\{\tilde{j}\}\Big)\Big) \\
&\geq \sum_{1\leq l\leq t,\{l\}\subset A_i}\mu(\{l\}) + \sum_{\{\tilde{j}\}\subset A_i\bigcap(\bigcup_{B\in[B_j]}B)}\mu(\{\tilde{j}\}) \\
&= \sum_{1\leq l\leq t,\{l\}\subset A_i}\mu(\{l\}) + \sum_{A_i\bigcap(\bigcup_{B\in[B_j]}B)\neq\emptyset}\mu([B_j]),
\end{aligned}$$

where the first inequality is due to the monotonicity of μ, the second inequality is valid according to Lemma 4.8 and the last equality follows by Note 4.9. Thus,

$$\begin{aligned}
\sum_{i=1}^{s} a_i\mu(A_i) &\geq \sum_{i=1}^{s} a_i\Big(\sum_{1\leq l\leq t,\{l\}\subset A_i}\mu(\{l\}) + \sum_{A_i\bigcap(\bigcup_{B\in[B_j]}B)\neq\emptyset}\mu([B_j])\Big) \\
&= \sum_{l=1}^{t}\Big(\sum_{i=1}^{s} a_i\delta_{l,i}\Big)\mu(\{l\}) + \sum_{j=1}^{k}\Big(\sum_{i=1}^{s} a_i\gamma_{i,j}\Big)\mu([B_j]),
\end{aligned}$$

from which we obtain $I^{\mathcal{H}_3}(\mu, f) \geq I^{\mathcal{H}_4}(\mu, f)$. Since $I^{\mathcal{H}_3}(\mu, f) \leq I^{\mathcal{H}_4}(\mu, f)$ is obvious, it holds that $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$, which concludes the proof. □

4.3. Main result

Combining Theorem 4.1 and Theorem 4.7, we are in a position to present our main result.

Theorem 4.10. *Let X be finite space and $\mu \in \mathcal{M}$ be fixed. Then for each $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$$

if and only if one of the following conditions (i) and (ii) is satisfied:

(i) μ is superadditive w.r.t. singletons.

(ii) For any minimal strictly μ-subadditive set A, we have $\mu(\{i\}) = \mu([A])$, where $\{i\}$ is an arbitrary singleton of $\bigcup_{B\in[A]} B$.

Example 4.11. (1) Continuing in Example 3.6, every *minimal strictly μ-subadditive set* contains two elements, and there is only one equivalence class of minimal strictly μ-subadditive sets, namely $[\{1,2\}]$. Noting that $\bigcup_{B\in[\{1,2\}]} B = X$ and $\mu(X)=\mu(\{i\}), i=1,2,3$. The condition (ii) in Theorem 4.10 is satisfied, so we conclude that $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$ for any $f \in \mathcal{F}_b^+$. In fact, for any $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f) = \max\{f(i) : i \in X\}.$$

(2) Let $X = \{1, \dots, 10\}$ and $\mu\colon 2^X \to [0,\infty]$ be defined as $\mu(\emptyset)=0$, $\mu(A)=1$ if $A \subset \{1,\dots,4\}$ and $A \neq \emptyset$, $\mu(A)=2$ if $A \subset \{5,6\}$ and $A \neq \emptyset$, $\mu(\{i\})=1$ for $i=7,\dots,10$ and $\mu(A)=\sum_{\{i\}\subset A}\mu(\{i\})$ otherwise. Then μ satisfies the condition (ii) in Theorem 4.10, so it holds that $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$ for any $f \in \mathcal{F}_b^+$. In fact, for any $f \in \mathcal{F}_b^+$, we have that

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f) = \max\left\{f(i) : 1 \le i \le 4\right\} + 2\max\left\{f(i) : 5 \le i \le 6\right\} + \sum_{i=7}^{10} f(i).$$

5. Further discussions

Recall that in previous discussion we assumed that the σ-algebra is the power set, i.e., $\mathcal{A} = 2^X$, and $\mu(\{i\}) > 0$ for each $i \in X$. The only aim that we make these assumptions is to simplify our discussion. In fact, these assumptions can all be abandoned.

First, we show that the assumption that $\mu(\{i\}) > 0$ for each $i \in X$ can be ignored.

Let $\mu \in \mathcal{M}$ be given and $X_1 = \{i \in X : \mu(\{i\}) > 0\}$. We assume that $X_1 \neq \emptyset$ (otherwise, $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f) = 0$ for any $f \in \mathcal{F}_b^+$) and $X_1 \neq X$, i.e., there is at least one element $j_0 \in X$ such that $\mu(\{j_0\}) = 0$.

Lemma 5.1. *Let $\mu \in \mathcal{M}$ be given and any $f \in \mathcal{F}_b^+$. Then we have $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_3}(\mu|_{X_1}, f|_{X_1})$ and $I^{\mathcal{H}_4}(\mu, f) = I^{\mathcal{H}_4}(\mu|_{X_1}, f|_{X_1})$.*

The proof is postponed to the "Appendix".

In the case that μ is superadditive w.r.t. singletons, it is clear that $\mu|_{X_1}$ is also superadditive w.r.t. singletons. Thus, Theorem 4.1 remains true. In fact, let $X_1 = \{i \in X : \mu(\{i\}) > 0\}$ and by using the conclusion of Theorem 4.1, we have $I^{\mathcal{H}_3}(\mu|_{X_1}, f|_{X_1}) = I^{\mathcal{H}_4}(\mu|_{X_1}, f|_{X_1})$. Combining this fact with Lemma 5.1, we get that

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$$

holds for each $f \in \mathcal{F}_b^+$. $\square$

In the case that μ is not superadditive w.r.t. singletons, suppose that A is a strictly μ-subadditive set and there is $\{j\} \subset A$ such that $\mu(\{j\}) = 0$, then

$$\mu(A \setminus \{j\}) \le \mu(A) < \sum_{\{i\}\subset A} \mu(\{i\}) = \sum_{\{i\}\subset A, i\neq j} \mu(\{i\}) = \sum_{\{i\}\subset A\setminus\{j\}} \mu(\{i\}),$$

that is, $A \setminus \{j\}$ is also a strictly μ-subadditive set. As a consequence, for any minimal strictly μ-subadditive set $B \subset X$ and any $i \in B$, it must hold that $\mu(\{i\}) > 0$ and hence $B \subset X_1$. From this fact, we know that although we allow $\mu(\{j\}) = 0$ for some $j \in X$, Theorem 4.7 still holds.

Secondly, we show that the assumption that $\mathcal{A} = 2^X$ is also not essential. It is only for the purpose of simplicity.

We recall the concept of *atom* of a σ-algebra of subsets of X. Let X be a finite set and $\mathcal{A}$ be an arbitrary algebra over X. A nonempty set $A \in \mathcal{A}$ is called *an atom of $\mathcal{A}$* [18] (or *an atom of measurable space* $(X, \mathcal{A})$) [23] (see also [4]), if $\emptyset$ and A are the only $\mathcal{A}$-measurable subsets of A, i.e., there is no nonempty proper subset B of A such that $B \in \mathcal{A}$. The atoms of $\mathcal{A}$ possess some of basic properties, as follows: (i) Every two distinct atoms of $\mathcal{A}$ are disjoint; (ii) Let $A_1, \dots, A_k$ be all of atoms of $\mathcal{A}$. Then $A_1, \dots, A_k$ are pairwise disjoint and $X = A_1 \cup A_2 \dots, A_k$, and hence $\{A_1, \ldots, A_k\}$ is a measurable partition of X; (iii) $\mathcal{A} = \sigma(\{A_1, \ldots, A_k\})$, where $\sigma(\{A_1, \ldots, A_k\})$ is the algebra generated by the class of all atoms of $\mathcal{A}$. (iv) Every nonempty set $A \in \mathcal{A}$ is the union of some atoms of $\mathcal{A}$, i.e., $A = A_{i_1} \cup A_{i_2} \ldots, \cup A_{i_s}$, where $\{A_{i_1}, \ldots, A_{i_s}\} \subset \{A_1, A_1, \ldots, A_k\}$; (v) when $\mathcal{A} = 2^X$, A is an atom of $\mathcal{A}$ iff it is a singleton $\{i\}$ of X.

In the following we discuss the case of $\mathcal{A} \neq 2^X$. In order to do so, we need only replace singletons of X by atoms of $\mathcal{A}$. The role that atoms of $\mathcal{A}$ will play is similar to that of singletons of X in Section 4.

Similar to discussion of Theorems 4.1 and 4.7, we propose the following concepts, which are general cases of concepts presented in Definitions 3.1 and 3.3.

Let $(X, \mathcal{A})$ be a finite measurable space, $\mu \in \mathcal{M}$ be fixed, and $A_1, \ldots, A_k$ be all of the atoms of $\mathcal{A}$.

(1) μ is called *superadditive w.r.t. atoms of* $\mathcal{A}$, if for any $A \in \mathcal{A}$,

$$\mu(A) \geq \sum_{A_i \subset A} \mu(A_i), \tag{5.1}$$

(2) A measurable set $A \in \mathcal{A}$ is called a *strictly μ-subadditive set w.r.t. atoms of* $\mathcal{A}$ (being abbreviated as *strictly μ-subadditive set*), if

$$\mu(A) < \sum_{A_i \subset A} \mu(A_i). \tag{5.2}$$

(3) A strictly μ-subadditive set A is said to be *minimal strictly μ-subadditive set w.r.t. atoms of* $\mathcal{A}$ (being abbreviated as *minimal strictly μ-subadditive set*) if for any nonempty proper measurable subset B of A, $\mu(B) \geq \sum_{A_i \subset B} \mu(A_i)$ holds.

(Comparing with Definition 3.1 and 3.3)

In a similar way, we can define the equivalence class of minimal strictly μ-subadditive set w.r.t. atoms of $\mathcal{A}$ (recall Section 3). Similar to discussions in Section 4, we can prove the following results.

Theorem 5.2. *Let $(X, \mathcal{A})$ be a finite measurable space and $\mu \in \mathcal{M}$ be fixed. Then for each $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$$

if and only if one of the following conditions (i) and (ii) is satisfied:

(i) μ is superadditive w.r.t. atoms of $\mathcal{A}$.

(ii) For every minimal strictly μ-subadditive set A w.r.t. atoms of $\mathcal{A}$, we have $\mu(A_i) = \mu([A])$, where A_i is an arbitrary atom of $\mathcal{A}$ and $A_i \subset \bigcup_{B \in [A]} B$.

Example 5.3. Let $A \subsetneqq X$ be a nonempty set and $\mathcal{A} = \{\emptyset, A, A^c, X\}$. Then $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$ for each $f \in \mathcal{F}_b^+$ if and only if the monotone measure μ satisfies one of the two cases: (i) $\mu(X) \geq \mu(A) + \mu(A^c)$; (ii) $\mu(X) = \mu(A) = \mu(A^c)$. For the former,

$$I^{\mathcal{H}_3}(\mu, f) = I^{H_4}(\mu, f) = \lambda_A \cdot \mu(A) + \lambda_{A^c} \cdot \mu(A^c),$$

where λ_A (λ_{A^c}, resp.) stands for the constant value f took on A (A^c, resp.) (to ensure its measurability, f must take constant value on atoms of $\mathcal{A}$, A and A^c). For the latter,

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f) = \max\{\lambda_A, \lambda_{A^c}\} \cdot \mu(X).$$

In fact, if $\mu(X) \geq \mu(A) + \mu(A^c)$, then there are no minimal strictly subadditive sets and the condition (i) in Theorem 5.2 is satisfied. If $\mu(X) = \mu(A) = \mu(A^c)$, then the only minimal strictly subadditive set X contains two atoms of $\mathcal{A}$, which says that the condition (ii) in Theorem 5.2 is also satisfied. If both these cases are not true, i.e., $\mu(X) < \mu(A) + \mu(A^c)$ and $\mu(X) > \mu(A)$, defining $f : X \to [0, \infty)$ by

$$f(x) = \begin{cases} 2, & x \in A, \\ 1, & x \in A^c, \end{cases}$$

then

$$I^{\mathcal{H}_3}(\mu, f) = \mu(A) + \mu(X)$$

and

$$I^{\mathcal{H}_4}(\mu, f) = \min\{2\mu(X),\ 2\mu(A) + \mu(A^c)\},$$

yielding that $I^{\mathcal{H}_3}(\mu, f) < I^{\mathcal{H}_4}(\mu, f)$.

6. Conclusion

In this paper, we have discussed the equality

$$I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$$

in detail, where $\mu \in \mathcal{M}$ is fixed, $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_3}(\mu, f)$ is the convex integral and $I^{\mathcal{H}_4}(\mu, f)$ is the pan-integral from above. As we see, we have proposed the concepts of *superadditivity w.r.t. singletons (resp., w.r.t. atoms of $\mathcal{A}$)* of a monotone measure and *minimal strictly μ-subadditive sets w.r.t. singletons (resp., w.r.t. atoms of $\mathcal{A}$)*, and by means of their characteristics we have presented the necessary and/or sufficient conditions under which these two types of integrals are equivalent. Our main results were shown in Theorems 4.1, 4.7, 4.10 and 5.2.

Remember we assumed that the underlying set X is finite in our discussion. We point out that when X is infinite, X can contain infinite minimal strictly μ-subadditive sets which are pairwise disjoint.

Example 6.1. Let $X_1 = \{1, 2, \ldots\}$, $X_2 = \{1, \frac{1}{2}, \ldots\}$ and $X = X_1 \cup X_2$. Let $\mu\colon 2^X \to [0, \infty]$ be defined by

$$\mu(A) = \begin{cases} 0, & A = \emptyset, \\ 1, & \text{if } |A| = 1 \text{ or } A = \{i, \frac{1}{i}\}, \\ \sum_{x\in A} \mu(\{x\}), & \text{otherwise.} \end{cases}$$

Then $\{i, \frac{1}{i}\}, i = 2, 3, \ldots$ are minimal strictly μ-subadditive sets.

Theorem 4.7 may not hold whenever X is infinite. We open the problem of finding the necessary and sufficient condition (on an infinite space) under which $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$ for any measurable function. Notice that the superadditivity of μ is a sufficient condition for the equality (its proof is similar to which appeared in [15]).

Generalizing the results of these paper to the case of pan-operations different from $(+, \cdot)$ is a future task.

7. Appendix

Proof of Proposition 4.6. We complete the proof by induction.

Observe that by Corollary 4.5 and the definition of $[A]$, it is easy to know that $\mu(\{t_1\}) = \mu(\{t_2\})$ for any $t_1, t_2 \in \bigcup_{B\in[A]} B$.

First of all, we verify that for any two *minimal strictly μ-subadditive sets* $A_1, A_2 \in [A]$ with $A_1 \sim_{A_1\cup A_2} A_2$ (that is, $A_1 \cap A_2 \neq \emptyset$), we have $\mu(A_1 \cup A_2) = \mu(\{r\})$, where r is an arbitrary element in $A_1 \cup A_2$. For simplicity, we suppose that $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$. Let

$$f(x) = \begin{cases} 2, & x = 1, \\ 1, & x = 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have that

$$I^{\mathcal{H}_3}(\mu, f) \le \mu(\{1, 2\}) + \mu(\{1, 3\}) = 2\mu(\{1\})$$

and

$$\begin{aligned} I^{\mathcal{H}_4}(\mu, f) = \min\Big\{&2\mu(\{1\}) + \mu(\{2\}) + \mu(\{3\}), \\ &2\mu(\{1, 2\}) + \mu(\{3\}),\ 2\mu(\{1, 3\}) + \mu(\{2\}), \\ &2\mu(\{1\}) + \mu(\{2, 3\}),\ 2\mu(\{1, 2, 3\})\Big\}. \end{aligned}$$

By Corollary 4.5, we have that

$$\begin{aligned}
2\mu(\{1\})+\mu(\{2\})+\mu(\{3\}) &= 4\mu(\{1\}),\\
2\mu(\{1,2\})+\mu(\{3\}) &= 2\mu(\{1,3\})+\mu(\{2\})\\
&= 3\mu(\{1\}),\\
2\mu(\{1\})+\mu(\{2,3\}) &\geq 2\mu(\{1\})+\mu(\{2\})\\
&= 3\mu(\{1\}).
\end{aligned}$$

Thus, to ensure the equality $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$, the only possibility is $\mu(\{1,2,3\}) = \mu(\{1\})$.

Suppose for any $s(\leq l)$ minimal strictly μ-subadditive sets $A_1, \ldots, A_s$ contained in $[A]$ such that $A_i \sim_{\bigcup_{k=1}^{s} A_k} A_j$ for any $i, j \leq s$, we have that $\mu(\bigcup_{k=1}^{s} A_k) = \mu(\{r\})$, where r is an arbitrary element in $\bigcup_{k=1}^{s} A_k$.

Let $A_1, \ldots, A_l$ be arbitrarily given minimal strictly μ-subadditive sets contained in $[A]$ such that $A_i \sim_{\bigcup_{k=1}^{l} A_k} A_j$ for any $i, j \leq l$. We now add A_{l+1}, an extra minimal strictly μ-subadditive set contained in $[A]$, to $\bigcup_{k=1}^{l} A_k$ such that $A_i \sim_{\bigcup_{k=1}^{l+1} A_k} A_j$ for any $i, j \leq l+1$. If $\bigcup_{k=1}^{l} A_k = \bigcup_{k=1}^{l+1} A_k$ then $\mu(\bigcup_{k=1}^{l+1} A_k) = \mu(\bigcup_{k=1}^{l} A_k) = \mu(\{r\})$ is proved. Otherwise we suppose $A_l \cap A_{l+1} \neq \emptyset$ (we can renumber $A_1, \ldots, A_l$ if needed) and let

$$f(x) = \begin{cases} 2, & x \in A_l, \\ 1, & x \in \big(\bigcup_{k=1}^{l+1} A_k\big) \setminus A_l, \\ 0, & \text{otherwise}. \end{cases}$$

Noticing that $\chi_{\bigcup_{k=1}^{l} A_k} + \chi_{\bigcup_{k=l}^{l+1} A_k} \geq f$, we conclude that

$$I^{\mathcal{H}_3}(\mu, f) \leq \mu\big(\bigcup_{k=1}^{l} A_k\big) + \mu\big(\bigcup_{k=l}^{l+1} A_k\big) = 2\mu(\{r_0\}),$$

where r_0 is an arbitrary fixed element in $\bigcup_{k=1}^{l+1} A_k$.

Suppose $\sum_{j=1}^{n_1} b_j \chi_{B_j} \geq f$, where $\{B_j\}_{j=1}^{n_1}$ is an arbitrary partition of $\bigcup_{k=1}^{l+1} A_k$. If $n_1 = 1$ then $B_1 = \bigcup_{k=1}^{l+1} A_k$ and $b_1 \geq 2$. Thus, $b_1 \mu(B_1) \geq 2\mu(\bigcup_{k=1}^{l+1} A_k)$. If $n_1 > 1$, without loss of generality, we can suppose that $B_1 \cap A_l \neq \emptyset$. Then we have that

$$\sum_{j=1}^{n_1} b_j \mu(B_j) \geq 2\mu(B_1) + \sum_{j=2}^{n_1} \mu(B_j) \geq (n_1+1)\mu(\{r_0\}) \geq 3\mu(\{r_0\}).$$

Due to the arbitrariness of the partition $\{B_j\}_j$, we conclude that

$$I^{\mathcal{H}_4}(\mu, f) \geq \min\left\{3\mu(\{r_0\}),\ 2\mu\big(\bigcup_{k=1}^{l+1} A_k\big)\right\}.$$

To ensure the equality $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_4}(\mu, f)$, the only possibility is $\mu(\{r_0\}) = \mu(\bigcup_{i=1}^{l+1} A_i)$.

Since $[A]$ is the equivalence class of minimal strictly μ-subadditive sets that are connectable with A, for any $C \in [A]$ it necessarily holds that $C \sim_{\bigcup_{B\in[A]} B} A$. As a consequence, for any $A_i, A_j \in [A]$, $A_i \sim_{\bigcup_{B\in[A]} B} A_j$. By induction, we conclude that $\mu([A]) = \mu(\{t\})$, where t is an arbitrary element of $\bigcup_{B\in[A]} B$.

The proof is completed. □

Proof of Lemma 4.8. Suppose $A \bigcap \big(\bigcup_{B\in[A_j]} B\big) \neq \emptyset$ for $j = 1, \ldots, l, l \leq k$ and $A \setminus \Big(\bigcup_{j=1}^{k} \big(\bigcup_{B\in[A_j]} B\big)\Big) = \{1, \ldots, s\}$. We arbitrarily choose one element $\tilde{j}$ from each $\bigcup_{B\in[A_j]} B$, $j = 1, \ldots, l$. We conclude that the set

$$\{1, \ldots, s, \tilde{1}, \ldots, \tilde{l}\}$$

is not a strictly μ-subadditive set and hence

$$\mu(A) \geq \sum_{i=1}^{s} \mu(\{i\}) + \sum_{j=1}^{l} \mu(\{\tilde{j}\}),$$

i.e., the conclusion holds.

In fact, if $\{1, \ldots, s, \tilde{1}, \ldots, \tilde{l}\}$ is a strictly μ-subadditive set, then it contains at least one minimal strictly μ-subadditive set, say, E. If $E \cap \{\tilde{1}, \ldots, \tilde{l}\} \neq \emptyset$, say, $\tilde{1} \in E$, then $E \in [A_1]$, a contradiction (since we only choose one element from each set $\bigcup_{B \in [A_j]} B$, and the minimal strictly μ-subadditive set E contains at least two elements). If $E \cap \{\tilde{1}, \ldots, \tilde{l}\} = \emptyset$, then $\{1, \ldots, s\}$ contains a minimal strictly μ-subadditive set, again a contradiction (since there is another equivalence class of minimal strictly μ-subadditive set which differs from $[A_j]$, $j = 1, \ldots, l, l \leq k$). $\square$

Proof of Lemma 5.1. For any finite set system $\{A_i \subset X_1\}$ and $\sum a_i \chi_{A_i} \geq f|_{X_1}$ there is a finite set system $\{A_i\} \cup \{\{j\} : j \in X \setminus X_1\}$ such that $\sum a_i \chi_{A_i} + \sum_{j \in X \setminus X_1} f(j)\chi_{\{j\}} \geq f$ and $\sum a_i \mu|_{X_1}(A_i) = \sum a_i \mu(A_i) = \sum a_i \mu(A_i) + \sum_{j \in X \setminus X_1} f(j)\mu(\{j\})$. Thus $I^{\mathcal{H}_3}(\mu|_{X_1}, f|_{X_1}) \geq I^{\mathcal{H}_3}(\mu, f)$.

On the other hand, for any finite set system $\{A_i\}$ and $\sum a_i \chi_{A_i} \geq f$, we have a finite set system $\{A_i \cap X_1\}$ with $\sum a_i \chi_{A_i \cap X_1} \geq f|_{X_1}$ such that $\sum a_i \mu(A_i) \geq \sum a_i \mu(A_i \cap X_1) \geq \sum a_i \mu|_{X_1}(A_i \cap X_1)$. Thus $I^{\mathcal{H}_3}(\mu, f) \geq I^{\mathcal{H}_3}(\mu|_{X_1}, f|_{X_1})$ and hence $I^{\mathcal{H}_3}(\mu, f) = I^{\mathcal{H}_3}(\mu|_{X_1}, f|_{X_1})$.

$I^{\mathcal{H}_4}(\mu, f) = I^{\mathcal{H}_4}(\mu|_{X_1}, f|_{X_1})$ can be proved in a similar way. $\square$

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