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# ONE-ADHESIVE POLYMATROIDS

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*Dedicated to the memory of Frantisek Matúš*

Adhesive polymatroids were defined by F. Matúš motivated by entropy functions. Two polymatroids are adhesive if they can be glued together along their joint part in a modular way; and are one-adhesive, if one of them has a single point outside their intersection. It is shown that two polymatroids are one-adhesive if and only if two closely related polymatroids have joint extension. Using this result, adhesive polymatroid pairs on a five-element set are characterized.

*Keywords:* polymatroid, amalgam, adhesive polymatroid, entropy function, polyhedral cone

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## 1. PRELIMINARIES

A *polymatroid* $(f, M)$ is a non-negative, monotone and submodular function $f$ defined on the collection of non-empty subsets of the finite set $M$. Here $M$ is the *ground set*, and $f$ is the *rank function*. The polymatroid is *integer* if all ranks are integer. An integer polymatroid is a *matroid*, if the rank of singletons are either zero or one. Matroids are combinatorial objects which generalize the properties of linear dependence among a finite set of vectors. For an introduction to matroids, see [15]; and about polymatroids consult [9, 11]. The rank function $f$ can be identified with a $(2^{|M|}-1)$-dimensional real vector, where the indices are the non-empty subsets of $M$. In this paper the *distance* of two polymatroids $f$ and $g$ on the same ground set is measured as the usual Euclidean distance of the corresponding vectors, and is denoted as $\|f-g\|$.

Following the usual practice, ground sets and their subsets are denoted by capital letters, their elements by lower case letters. The union sign $\cup$ is frequently omitted as well as the curly brackets around singletons, thus $Aab$ denotes the set $A \cup \{a, b\}$. For a function $f$ defined on the subsets of the finite set $M$ (such as the rank function of a polymatroid) the usual information-theoretical abbreviations are used. Here $I$, $J$, $K$ are disjoint subsets of the ground set:

$$\begin{aligned} f(I,J|K) &= f(IK)+f(JK)-f(IJK)-f(K), \\ f(I,J) &= f(I,J|\emptyset) = f(I)+f(J)-f(IJ)-f(\emptyset), \\ f(I|K) &= f(IK)-f(K). \end{aligned}$$

When $f$ is a rank function, $f(\emptyset)$ is considered to be zero. In cases when the function $f$ is clear from the context, even $f$ is omitted. Additionally, the *Ingleton expression* [8] is

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abbreviated as

$$f[I,J,K,L] = -f(I,J) + f(I,J|K) + f(I,J|L) + f(K,L).$$

Observe that it is symmetrical for swapping $I$ and $J$ as well as swapping $K$ and $L$.

Vectors corresponding to polymatroids on the ground set $M$ form the pointed polyhedral cone $\Gamma_M$ [19]. Its facets are the hyperplanes determined by the basic submodular inequalities $(i,j|K) \ge 0$ with distinct $i,j \in M{-}K$ and $K \subseteq M$ ($K$ can be empty), and the monotonicity requirements $(i|M{-}i) \ge 0$, see [11, Theorem 2]. Much less is known about the extremal rays of this cone. They have been computed for ground sets up to five elements [18], without indicating any structural property.

### 1.1. Entropic, linear, and modular polymatroids

An important class of polymatroids describes the entropy structure of the marginals of finitely many discrete random variables. Assume $\{\xi_i : i \in M\}$ is a collection of (jointly distributed) random variables. For $A \subseteq M$ let $\mathbf{H}(\xi_A)$ be the usual Shannon entropy of the marginal distribution $\xi_A = \{\xi_i : i \in A\}$. The function $f(A) = \mathbf{H}(\xi_A)$ is a polymatroid [7]. Such polymatroids are called *entropic*, and the collection of entropic polymatroids is $\Gamma^*_M \subseteq \Gamma_M$ [19]. The closure of $\Gamma^*_M$ (in the usual Euclidean topology) is the collection of *almost entropic* or *aent* polymatroids. Studying polymatroids is motivated partly by the difficult task of understanding the entropic region as well as solving problems arising in secret sharing [6, 17], network coding [1], and other areas.

Another important subclass is the linear polymatroids. $(f,M)$ is *linearly representable* if there is a vector space $V$ over some finite field, linear subspaces $V_i \subseteq V$ for each $i \in M$, such that $f(A)$ is the dimension of the linear subspace spanned by the vectors in $\bigcup\{V_i : i \in A\}$. Linearly representable polymatroids are integer. A polymatroid is *linear* if it is in the conic hull of linearly representable polymatroids, namely, it can be written as a non-negative linear combination of such polymatroids. Linear polymatroids are almost entropic, see [4, 12, 16].

The polymatroid $(f,M)$ is *modular* if $f(I,J) = 0$ for any two disjoint non-empty subsets $I,J \subset M$, or, equivalently, if for all $A \subseteq M$ we have

$$f(A) = \sum\{f(i) : i \in A\}.$$

Modular polymatroids are entropic and linear [11].

In matroid theory modularity refers to a different notion [15], which will be called *flat-modularity* here. $F \subseteq M$ is a *flat* if its rank is strictly smaller than that of any of its proper extensions. The polymatroid $(f,M)$ is *flat-modular* if every pair $(F_1,F_2)$ of its flats forms a modular pair, namely the submodularity holds with equality:

$$f(F_1) + f(F_2) = f(F_1 \cap F_2) + f(F_1 \cup F_2).$$

Modular polymatroids are flat-modular, but the converse is not true in general.

For a subset $A \subset M$ define the function $\boldsymbol{r}_A$ on (non-empty) subsets of $M$ as follows:

$$\boldsymbol{r}_A(I) = \min\{1, |A \cap I|\} = \begin{cases} 1 & \text{if } A \cap I \text{ is not empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $(\boldsymbol{r}_A, M)$ is a matroid and linearly representable over any vector space. They are linearly independent and span the whole $2^{|M|-1}$-dimensional space. This is immediate from the fact that the linear combination

$$\sum_{B\subseteq A} (-1)^{1+|A-B|}\, \boldsymbol{r}_{N-B}(I) \tag{1}$$

takes one at $A$, and zero anywhere else, see [12, Lemma 3].

It is well known that all polymatroids on two or three elements are linear, moreover a polymatroid $f$ on the four element set $abcd$ is linear if and only if it satisfies all six instances of the Ingleton inequality:

$$\begin{array}{lll} f[a,b,c,d] \ge 0, & f[a,c,b,d] \ge 0, & f[a,d,b,c] \ge 0, \\ f[b,c,a,d] \ge 0, & f[b,d,a,c] \ge 0, & f[c,d,a,b] \ge 0, \end{array}$$

see [14]. Linear polymatroids on a five element set can also be characterized by some finite set of linear inequalities [5]. Polymatroids on ground set of size five or less have the following *simultaneous approximation property*, which will be used in Section 3.

**Proposition 1.** Let $|M| \le 5$, and let $f_1$ and $f_2$ be linear polymatroids on $M$. For each positive $\varepsilon$ and large enough vector space $V$ there is a $\lambda > 0$ and integer polymatroids $\ell_1$ and $\ell_2$ on $M$ linearly representable over $V$, such that $\|f_i - \lambda\ell_i\| < \varepsilon$, additionally $\ell_1(I) = \ell_2(I)$ whenever $f_1(I) = f_2(I)$ $(I \subseteq M)$.

Proof. On ground set $|M| \le 5$ linear polymatroids form a polyhedral cone. Moreover, for every large enough vector space $V$, extremal rays of this cone contain polymatroids linearly representable over $V$, see [5, 14]. Non-negative rational combinations of these polymatroids form a dense subset of linear polymatroids. Let $\ell_1$ and $\ell_2$ be such combinations with $\|f_i - \ell_i\| < \varepsilon$. The linearly representable polymatroids $\boldsymbol{r}_A$ span the whole linear space, thus there are rational coefficients $\alpha_A$ such that

$$\sum_{A\subseteq M} \alpha_A\, \boldsymbol{r}_A(I) = \begin{cases} \ell_1(I) - \ell_2(I) & \text{if } f_1(I) = f_2(I), \\ 0 & \text{otherwise.} \end{cases}$$

As $|\ell_1(I) - \ell_2(I)| < 2\varepsilon$ whenever $f_1(I) = f_2(I)$, (1) implies that all coefficients $\alpha_A$ have absolute value smaller than $2^{|M|+1}\varepsilon$. Using the notation $\alpha^+ = \max\{0, \alpha\}$ and $\alpha^- = \max\{0, -\alpha\}$, the polymatroids

$$\ell_1 + \textstyle\sum_{A\subseteq M} \alpha_A^+\, \boldsymbol{r}_A$$
$$\ell_2 + \textstyle\sum_{A\subseteq M} \alpha_A^-\, \boldsymbol{r}_A$$

are non-negative rational combinations of linearly representable polymatroids; are equal whenever $f_1(I) = f_2(I)$; and are approximating $f_1$ and $f_2$, respectively, better than $2^{2|M|+2}\varepsilon$.

Finally, integer combinations of linearly representable polymatroids over the same vector space $V$ are linearly representable, which implies the claim. $\square$

### 1.2. Amalgam and adhesive extension

Let $M$, $X$, and $Y$ be disjoint sets. Polymatroids $f_X$ and $f_Y$ on the ground sets $M \cup X$ and $M \cup Y$, respectively, with joint restriction on $M$, have an *amalgam*, or *can be glued together*, if there is a polymatroid $f$ on $M \cup X \cup Y$ extending both $f_X$ and $f_Y$ [15]. This extension is *modular* if, in addition, $X$ and $Y$ are independent over $M$, that is, $f(X, Y|M) = 0$. If $f_X$ and $f_Y$ have such a modular extension $f$, then $f_X$ and $f_Y$ are *adhesive*, and $f$ is an *adhesive extension*. Adhesive extensions were defined and studied by F. Matúš in [11]. The main observation is that restrictions of an almost entropic polymatroid are adhesive [11, Lemma 2]. In this paper we investigate adhesive extensions on their own right.

When speaking about amalgam, or adhesive extension, the polymatroids are tacitly assumed to have the same restriction on the intersection of their ground sets.

We have defined the amalgam of $f_X$ and $f_Y$ as a *polymatroid* extending both $f_X$ and $f_Y$. The amalgam of two matroids is traditionally required to be a matroid. It is an interesting problem to decide whether the two different notions of amalgam coincide.

**Problem 1.** Suppose the matroids $f_X$ and $f_Y$ on $M \cup X$ and $M \cup Y$, respectively, have a polymatroid amalgam on $M \cup X \cup Y$. Is it true that then they have a matroid amalgam as well?

If the joint extension is integer valued then it must be a matroid; and if there is a joint extension at all, then there is one with rational values.

Whether two matroids have an amalgam is a combinatorial question; the same question about polymatroids is a *geometrical* one. Polymatroids $f_X$ and $f_Y$ have an amalgam if and only if the point $(f_X, f_Y)$ (merged along coordinates corresponding to subsets of $M$) is in the *coordinatewise projection* of the polymatroid cone $\Gamma_{MXY}$ to the subspace with coordinates $I \subseteq MXY$ where $I \subseteq MX$ or $I \subseteq MY$. The projection is a polyhedral cone whose bounding hyperplanes correspond to (homogeneous) linear inequalities on the projected coordinates. Thus $f_X$ and $f_Y$ have an amalgam if and only if the vector $(f_X, f_Y)$ satisfies all of these inequalities. While theoretically simple, in practice it is unclear how to calculate the facets of the projection efficiently.

The same reasoning applies to adhesive extension. Such an extension satisfies the additional constraint $f(X, Y|M) = 0$, thus the modular extensions form a subcone of dimension one less: the intersection of $\Gamma_{MXY}$ and the hyperplane $f(XM) + f(YM) - f(XYM) - f(Y) = 0$. $f_X$ and $f_Y$ have an adhesive extension if an only if the point $(f_X, f_Y)$ is in the projection of this restricted cone.

The polymatroid $h$ is *sticky* if any two extensions of $h$ have an amalgam. Flat-modular polymatroids are sticky, the proof in [15, Theorem 12.4.10] works in the polymatroid case as well, but see also [11, Theorem 1]. The "sticky matroid conjecture" asserts that all sticky matroids are flat-modular [2]. The same conjecture is stated here for polymatroids.

**Sticky polymatroid conjecture.** Sticky polymatroids are flat-modular.

Factors of sticky polymatroids are sticky, and the collection of sticky polymatroids on a given ground set forms a closed cone, thus to settle the above conjecture it is enough to consider polymatroidal extensions of a matroid. Consequently, if the answer to Problem 1 is *yes* and the sticky matroid conjecture is true, then so is the sticky polymatroid conjecture.

To state some of our results we need one more definition. The polymatroid $(h, M)$ is *$k$-$\ell$-sticky*, if any two of its extensions $(f_X, MX)$ and $(f_Y, MY)$ with $|X| \le k$ and $|Y| \le \ell$ have an amalgam. A polymatroid is *$k$-sticky*, if it is $k$-$k$-sticky. Sticky polymatroids on small ground sets are discussed in Sections 3 and 4.

### 1.3. New polymatroids from old ones

Each polymatroid can be decomposed as a sum of a modular and a tight polymatroid as described in Lemmas 2 and 3; it is a generalization of [4, Lemma 2]. Lemma 4 discusses how one can extend a polymatroid adding a new element to the base set. The method will be used in later sections to create several extensions. Recall that $\boldsymbol{r}_A$ is the polymatroid defined by $\boldsymbol{r}_A(I) = \min\{1, |A \cap I|\}$.

**Lemma 2.** Let $(f, M)$ be a polymatroid and $A \subset M$. Suppose the real number $\lambda$ satisfies the following conditions:

$$\begin{array}{ll} \lambda \le f(x,y|B) & \text{for } \textit{different } x,y \in A \text{ and all } B \subseteq M{-}A\text{; and} \\ \lambda \le f(x|M{-}A) & \text{for every } x \in A. \end{array}$$

Then $(f - \lambda \boldsymbol{r}_A, M)$ is a polymatroid.

Observe that if $A$ has a single member $a$, then the first condition holds vacuously, and the second condition simplifies to $\lambda \le f(a|M{-}a)$.

P r o o f. The claim clearly holds when $\lambda \le 0$, so assume $\lambda > 0$, and let $f^* = f - \lambda \boldsymbol{r}_A$. If $I$ and $A$ are disjoint, then $f(I) - f^*(I) = 0$, in the other cases this difference is $\lambda$. One has to check the monotonicity for the special case $f^*(Cx) - f^*(C) \ge 0$, $Cx \subseteq M$ only. This difference equals to $f(Cx) - f(C)$ except when $A$ and $C$ are disjoint and $x \in A$. But then

$$\begin{aligned} f^*(Cx) - f^*(C) &= f(Cx) - f(C) - \lambda \\ &= f(x|C) - \lambda \ge f(x|M{-}A) - \lambda \ge 0 \end{aligned}$$

by assumption.

To check submodularity, observe that $f^*(x, y|B) = f(x, y|B)$ except when $A$ and $B$ are disjoint and both $x$ and $y$ are in $A$. In the latter case $f^*(x, y|B) = f(x, y|B) - \lambda$, which is non-negative by the fist assumption. □

Let $(f, M)$ be any polymatroid and $a \in M$. By the remark above, $f - \lambda \boldsymbol{r}_a$ is a polymatroid whenever $\lambda \le f(a|M{-}a)$. Choosing $\lambda$ to be this maximal value, the polymatroid $f - \lambda \boldsymbol{r}_a$ is denoted by $f{\downarrow}a$, and called *tightening of $f$ at* (or on) $a$. $f$ is *tight at* $a$ if $f = f{\downarrow}a$, that is, if $f(a|M{-}a) = 0$. Note that $(f{\downarrow}a){\downarrow}a = f{\downarrow}a$, thus $f{\downarrow}a$ is tight at $a$; moreover $(f{\downarrow}a){\downarrow}b = (f{\downarrow}b){\downarrow}a$. Thus one can define the *tight part of $f$* at $A = \{a_1, \ldots, a_k\}$ as $f{\downarrow}a_1{\downarrow}\cdots{\downarrow}a_k$. $f$ is *tight on* $A$, if $f = f{\downarrow}A$, and is *tight* if $f = f{\downarrow}M$. The next lemma summarizes the properties of tightening used in this paper, see [4].

**Lemma 3.** Let $(f, M)$ be a polymatroid and $A \subseteq M$.

- $f$ is tight on $A$ if and only if it is tight on all elements of $A$.
- $f{\downarrow}A$ is tight on $A$.
- $f - f{\downarrow}A = \sum_{a\in A} f(a|M{-}a) \cdot \boldsymbol{r}_a$ is a modular polymatroid.
- $f{\downarrow}M$ is tight, and $f = f{\downarrow}M + (f - f{\downarrow}M)$ is the unique decomposition of $f$ into the sum of a tight and modular part.

In the last part of this section we investigate how to extend the polymatroid $(f, M)$ to the ground set $Mx$ using the *excess function* $e(A) = f(xA) - f(A)$ defined for all subsets $A \subseteq M$ (including the empty set). In agreement with the previous notation, $e(a, b|A)$ abbreviates $e(aA) + e(bA) - e(abA) - e(A)$, in particular, $e(a, b) = e(a, b|\emptyset) = e(a) + e(b) - e(ab) - e(\emptyset)$.

**Lemma 4.** Suppose $x$ is not in the ground set $M$ of the polymatroid $f$. Extend $f$ to the subsets of $Mx$ by $f_x(Ax) = f(A) + e(A)$. Then $f_x$ is a polymatroid on $Mx$ if and only if the following conditions hold:

1. $e$ is non-negative and non-increasing: $e(A) \geq e(B) \geq 0$ for $A \subseteq B \subseteq M$;
2. $e(a|M{-}a) + f(a|M{-}a) \geq 0$ for all $a \in M$;
3. $e(a, b|A) + f(a, b|A) \geq 0$ for all $abA \subseteq M$.

P r o o f. An easy case by case checking. □

As $e$ is non-increasing, $e(A|B) \leq 0$; in particular $e(a|M{-}a) \leq 0$ for all $a \in M$. On the other hand, $e(a, b|A)$ can take both positive and negative values even for the same excess function.

**Example 5.** Let $f$ be a polymatroid on $M$ and $0 \leq u, t$. Define the excess function $e_x$ by

$$e_x(A) = \begin{cases} u + t & \text{if } A = \emptyset, \\ u & \text{otherwise.} \end{cases}$$

If $t \leq f(a, b)$ for all pairs $a, b \in M$, then $f_x$ is a polymatroid .

P r o o f. Conditions 1 and 2 of Lemma 4 trivially hold. As for Condition 3, $e_x(a, b|A)$ is zero except when $A = \emptyset$, and then $e_x(a, b) = -t$. Thus it also holds by the assumption on $t$. □

An easy calculation shows that for this extension $f_x$, for all pairs $a, b \in M$ and non-empty $A \subseteq M{-}a$ we have $f_x(x, a|A) = 0$, and $f_x(a, b|x) = f(a, b) - t$.

**Example 6.** Let $c \in M$ and $0 \le u, t$. Define the excess function $e_x$ by

$$e_x(A) = \begin{cases} u + t & \text{if } A = \emptyset \text{ or } A = \{c\}, \\ u & \text{otherwise.} \end{cases}$$

If $t \le f(a, b)$ and $t \le f(a, b|c)$ for all pairs $a, b \in M{-}c$, then $f_x$ is a polymatroid.

P r o o f. Similar to the previous Example. Conditions 1 and 2 hold, moreover $e_x(a, b|A)$ is either zero or $-t$, and the latter case holds when $A = \emptyset$ and $a, b \in M{-}c$, or when $A = \{c\}$. Thus in all cases Condition 3 holds as well. □

## 2. ADHESIVITY VERSUS AMALGAM

As defined in Section 1.2, polymatroids $f_X$ and $f_Y$ on ground set $MX$ and $MY$, respectively, have an *amalgam* if there is a polymatroid on $MXY$ extending both $f_X$ and $f_Y$. The same polymatroids are *adhesive* if, in addition, they have a modular extension. When $Y$ has a single element $y$, then the polymatroid on $My$ will be denoted by $f_y$. In this special case adhesivity of $f_X$ and $f_y$ is equivalent to the existence of the amalgam of closely related polymatroids. Recall that $f_y$ is *tight on* $y$ if $f_y(y|M) = 0$, and by tightening $f_y$ on $y$ one gets the (tight) polymatroid

$$f_y{\downarrow}y = f_y - f_y(y|M) \cdot \boldsymbol{r}_y.$$

**Theorem 7.** Polymatroids $f_X$ and $f_y$ are adhesive if and only if $f_X$ and $f_y{\downarrow}y$ have an amalgam.

P r o o f. First let $g$ be the modular extension of $f_X$ and $f_y$, that is $g(X, y|M) = 0$. This equality rewrites to

$$g(y|MX) = g(XMy) - g(MX) = g(My) - g(M) = f_y(My) - f_y(M) = f_y(y|M).$$

Let $g^* = g{\downarrow}y$. The above equality means that restricting $g^*$ to $My$ one gets $f_y{\downarrow}y$, and, as $g$ and $g^*$ on $MX$ are the same, restricting $g^*$ to $MX$ one gets $f_X$. Consequently $g^*$ is the required amalgam of $f_X$ and $f_y{\downarrow}y$.

Conversely, let $g^*$ be an amalgam of $f_X$ and $f_y{\downarrow}y$. Then using that $f_y{\downarrow}y$ is tight on $y$, $g^*(My) = f_y{\downarrow}y(My) = f_y{\downarrow}y(M) = g^*(M)$, thus

$$g^*(XMy) - g^*(XM) \le g^*(My) - g^*(M) = 0,$$

which means that $g^*(X, y|M) = 0$. Let $g = g^* + \lambda \boldsymbol{r}_y$ with $\lambda = f_y(y|M)$. Then $g$ extends $f_X$ (as $g{\restriction}MX = g^*{\restriction}MX = f_X$), and $f_y$ (as $g{\restriction}My = g^*{\restriction}My + \lambda \boldsymbol{r}_y = (f_y - \lambda \boldsymbol{r}_y) + \lambda \boldsymbol{r}_y$). Finally, $g(X, y|M) = g^*(X, y|M) = 0$, as required. □

The last step in the proof works in a more general setting.

**Proposition 8.** Suppose $f_X{\downarrow}X$ and $f_Y{\downarrow}Y$ have an amalgam. Then $f_X$ and $f_Y$ have an amalgam as well.

P r o o f. If $g$ is an amalgam of $f_X{\downarrow}X$ and $f_Y{\downarrow}Y$, then $g+(f_X-f_X{\downarrow}X)+(f_Y-f_Y{\downarrow}Y)$ is an amalgam of $f_X$ and $f_Y$. □

In particular, to show that $f$ is sticky, it is enough to consider extensions $f_X$ and $f_Y$ which are tight on $X$ and $Y$, respectively. The condition stated in Proposition 8 is sufficient but not necessary. Polymatroids $f_x$ and $f_y$ in Example 13 have an amalgam but are not adhesive. Thus, by Theorem 7, $f_x{\downarrow}x$ and $f_y{\downarrow}y$ have no amalgam.

## 3. ONE-ELEMENT EXTENSIONS

This section starts with an alternative proof for a result of F. Matúš [11] which claims, using our terminology, that polymatroids on two element sets are 1-sticky. A similar proof to this one will be given for Theorem 17. Theorem 10 gives a sufficient and necessary condition for a pair of one-element extensions of a polymatroid on three elements to have an amalgam. Using Theorem 7, this is turned into sufficient and necessary conditions for such polymatroid pairs to be adhesive, which, in turn, yields new 5-variable non-Shannon entropy inequalities stated in Corollary 12.

The section concludes with several examples. The first one specifies two linearly representable (entropic) polymatroids which have an amalgam, but are not adhesive. Thus there are two linearly representable polymatroids which have a polymatroid extension, but no almost entropic (or linear) extension. Finally, two general examples are presented for 1-sticky and not 1-sticky polymatroids on three elements.

**Theorem 9.** (Matúš [11], Corollary 2.) All Polymatroids $f_x$ and $f_y$ on the ground sets $abx$ and $aby$ with common restriction to $ab$ are adhesive. In particular, such polymatroids have an amalgam, thus every polymatroid on a two element set is 1-sticky.

P r o o f. As discussed in Section 1.2, adhesive polymatroid pairs $(f_x, f_y)$ form a polyhedral cone. Consequently, $(f_x, f_y)$ is adhesive if and only if $(\lambda f_x, \lambda f_y)$ is adhesive for some (or all) positive $\lambda$. The adhesive cone is closed, thus to show that a particular pair $(f_x, f_y)$ is adhesive, it is enough to find, for each positive $\varepsilon$, some adhesive pair $(\ell_x, \ell_y)$ such that $\|f_x-\lambda\ell_x\|<\varepsilon$, and $\|f_y-\lambda\ell_y\|<\varepsilon$. In this particular case $\ell_1$ and $\ell_2$ will be the linearly representable polymatroids guaranteed by Proposition 1. Thus $\ell_1$ and $\ell_2$ are represented over the same vector space $V$, $\lambda\ell_x$ and $\lambda\ell_y$ are $\varepsilon$-close to $f_x$ and $f_y$, respectively, and the linear subspaces in both representations corresponding to subsets of $\{ab\}$ have the same dimensions: $\ell_x(a)=\ell_y(a)$, $\ell_x(b)=\ell_y(b)$ and $\ell_x(ab)=\ell_y(ab)$ as these equalities are true for the polymatroids $f_x$ and $f_y$. To conclude the claim of the theorem it is enough to show that $(\ell_x, \ell_y)$ is an adhesive pair.

The dimensions of subspaces spanned by $V_a$, $V_b$, and $V_a\cup V_b$ are the same in both representations. Choose a base in the first representation which can be partitioned to $B^x_x\cup B^x_a\cup B^x_b\cup B^x_{ab}$ such that $\ell_x(a)=|B^x_a\cup B^x_{ab}|$, $\ell_x(b)=|B^x_b\cup B^x_{ab}|$, and $\ell_x(ab)=|B^x_a\cup B^x_b\cup B^x_{ab}|$, and similarly for $\ell_y$. Identify $B^x_a$ and $B^y_a$, $B^x_b$ and $B^y_b$, $B^x_{ab}$ and $B^y_{ab}$, and take the vector space with base $B^x_x\cup B_a\cup B_b\cup B_{ab}\cup B^y_y$ (that is, glue the representations of $\ell_x$ and $\ell_y$ along their common part). It will be a linear representation of a polymatroid on $abxy$, where $x$ and $y$ are independent given $ab$. Consequently $\ell_x$ and $\ell_y$ have an adhesive extension, which concludes the proof. □

Now we turn to the case of one-point extensions of polymatroids on three-element sets. If not mentioned otherwise, all polymatroids in the rest of this section are extensions of a fixed polymatroid on $M = \{a, b, c\}$.

**Theorem 10.** Polymatroids $f_x$, $f_y$ on the ground sets $abcx$ and $abcy$ have an amalgam if and only if the following eight inequalities and their permutations (permuting $a, b, c$ and $x, y$) hold, where either the top or the bottom expression is chosen from all three pairs in curly brackets:

$$\begin{aligned} & f_x(a,x|c) + f_x(a,b|x) + f_y(a,b|y) + f_y(c,y) \\ & + \begin{Bmatrix} f_x(b,x|ac) \\ f_y(b,y|ac) \end{Bmatrix} + \begin{Bmatrix} f_x(c,x|ab) \\ f_y(c,y|ab) \end{Bmatrix} + 2\begin{Bmatrix} f_x(x|abc) \\ f_y(y|abc) \end{Bmatrix} \geq f_x(a,b). \end{aligned} \tag{2}$$

P r o o f. It is clear that all terms are defined over one of the polymatroids $f_x$ and $f_y$. Also, these inequalities hold for any polymatroid with ground set $abcxy$, which can easily be checked using an automated entropy checker, thus they *must* hold when $f_x$ and $f_y$ have an amalgam. Actually, inequalities in (2) written in basic terms and rearranged, are equivalent to

$$\begin{aligned} & (a,b|xy) + (x,y|a) + (x,y|b) + (c,y|x) + (a,x|cy) \\ & + \begin{Bmatrix} (b,x|acy) \\ (b,y|acx) \end{Bmatrix} + \begin{Bmatrix} (c,x|aby) \\ (c,y|abx) \end{Bmatrix} + 2\begin{Bmatrix} (x|abcy) \\ (y|abcx) \end{Bmatrix} \geq 0, \end{aligned} \tag{3}$$

which evidently holds for any polymatroid on five elements.

The sufficiency can be checked by the method indicated in Section 1.2. Polymatroids $f_x$ and $f_y$ determine 23 out of the 31 coordinates of the polymatroid on $N = abcxy$. The missing 8 variables are indexed by subsets of the form $Axy$ with $A \subseteq \{abc\}$.

The facets of the polymatroid cone $\Gamma_N$ are determined by the basic submodular inequalities $(i, j|K) \geq 0$ and by the monotonicity requirements $(i|N-i) \geq 0$. The strong duality of linear programming says that the facet equations of the projection are non-negative linear combinations of these inequalities in which the combined coefficients of the projected (dropped) variables are zero. Let $\mathcal{M}$ denote the matrix whose columns are indexed by the non-empty subsets of $N$, and whose rows contain the coefficients of the bounding facets of $\Gamma_N$ as discussed above. In each row there are two, three, or four non-zero entries only. When $\mathcal{M}$ is restricted to the eight columns labeled by $xyA$, 27 different non-zero rows remain. Let $\mathcal{M}'$ be this 27 by 8 matrix. Table 1 shows some rows of $\mathcal{M}'$ with the corresponding facet equations (one or two). The matrix $\mathcal{M}'$ can be constructed by hand, or by some interpretative computer program. The next step is to extract the extremal non-negative linear combinations of the rows which give zero sums for all eight columns. This can be done, e. g., by the freely available software package PORTA [3]. The result is 154 extremal non-negative linear combinations. One of them is the combination taking all but the first and last row from Table 1 once, and taking the last row twice. Eight of the corresponding 32 facet combinations give the inequalities in (3), which, after rearranging the terms, give the inequalities in (2). The other $3 \cdot 8$ combinations, when one takes $(c, x|y)$ instead of $(c, y|x)$, or $(a, y|cx)$ instead of $(a, x|cy)$,

| $xy$ | $axy$ | $bxy$ | $cxy$ | $abxy$ | $acxy$ | $bcxy$ | $abcxy$ | | |
|---|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $(x,y)$ | |
| 1 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | $(c,x\|y)$ | $(c,y\|x)$ |
| -1 | 1 | 1 | 0 | -1 | 0 | 0 | 0 | $(a,b\|xy)$ | |
| 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | $(x,y\|a)$ | |
| 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | $(x,y\|b)$ | |
| 0 | 0 | 0 | 1 | 0 | -1 | 0 | 0 | $(a,x\|cy)$ | $(a,y\|cx)$ |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | -1 | $(c,x\|aby)$ | $(c,y\|abx)$ |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | -1 | $(b,x\|acy)$ | $(b,y\|acx)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $(x\|abcy)$ | $(y\|abcx)$ |

**Tab. 1.** A submatrix of $\mathcal{M}_{abcxy}$.

or both, yield supporting hyperplanes to the projected cone, but not facets as they are consequences of the basic (Shannon) inequalities for *abcx* and *abcy*. In other words, these hyperplanes do not cut into the cones $\Gamma_{abcx}$ and $\Gamma_{abcy}$.

All other bounding hyperplanes (inequalities) resulting from the remaining 153 extremal combinations were checked by an interpretative computer program whether they are really facets of the projection. This search resulted in the statement of the Theorem. □

**Corollary 11.** Polymatroids $f_x$ and $f_y$ on the ground sets *abcx* and *abcy* are adhesive if and only if the following four inequalities and their permutations hold:

$$f_x(a,x|c) + f_x(a,b|x) + f_y(a,b|y) + f_y(c,y) + \begin{Bmatrix} f_x(b,x|ac) \\ f_y(b,y|ac) \end{Bmatrix} + \begin{Bmatrix} f_x(c,x|ab) \\ f_y(c,y|ab) \end{Bmatrix} \geq f_x(a,b). \tag{4}$$

P r o o f. By Theorem 7, $f_x$ and $f_y$ are adhesive if and only if $f_x{\downarrow}x$ and $f_y{\downarrow}y$ have an amalgam. All terms in (2) are the same for $f_x$ and $f_x{\downarrow}x$ ($f_y$ and $f_y{\downarrow}y$) except for $(f_x{\downarrow}x)(x|abc) = 0$ and $(f_y{\downarrow}y)(y|abc) = 0$. □

**Corollary 12.** The following are four five-variable non-Shannon information inequalities, that is, they hold in every entropic polymatroid on at least five elements:

$$(a,x|c) + (a,b|x) + (a,b|y) + (c,y) + \begin{Bmatrix} (b,x|ac) \\ (b,y|ac) \end{Bmatrix} + \begin{Bmatrix} (c,x|ab) \\ (c,y|ab) \end{Bmatrix} \geq (a,b).$$

P r o o f. As observed in [11], restrictions of an entropic polymatroid are adhesive, consequently the inequalities (4) in Corollary 11 must hold. □

### 3.1. Examples

**Example 13.** There are linearly representable polymatroids $f_x$ and $f_y$ on $abcx$ and $abcy$ which have an amalgam but are not adhesive.

Proof. Polymatroids $f_x$ and $f_y$ will be extensions of the uniform polymatroid

$$f(A) = \begin{cases} 4 & \text{if } |A| = 1, \\ 6 & \text{otherwise,} \end{cases} \qquad A \subseteq \{abc\}.$$

Clearly $f(i,j) = f(i,j|k) = 2$ for all distinct $i$, $j$, $k$. The excess functions defining $f_x$ and $f_y$ are

$$e_x(A) = \begin{cases} 3 & \text{if } A = \emptyset, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad e_y(A) = \begin{cases} 3 & \text{if } A = \emptyset \text{ or } A = \{c\}, \\ 1 & \text{otherwise.} \end{cases}$$

By Examples 5 and 6 both $f_x$ and $f_y$ are polymatroids. They are *not* adhesive, as all terms on the left hand side of (4) are zero, while $f_x(a,b) = f(a,b) = 2$. To show that they have an amalgam, one can check that all conditions of Theorem 10 hold. The polymatroid $f_{xy}$ specified in Table 2 gives such an extension explicitly. The four groups

| | $A$ | | | $Ax$ | | | $Ay$ | | | $Axy$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 6 | | | 7 | | | 7 | | | 7 | |
| 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 7 | 6 | 6 | 7 |
| | 0 | | | 3 | | | 3 | | | 5 | |

**Tab. 2.** The polymatroid $f_{xy}$ for $A \subseteq \{abc\}$.

contain the values for the subsets indicated at the top line where $A$ runs over all subsets of $abc$. The values are arranged in four lines (from bottom to top) for $A = \emptyset$, one-element subsets $a$, $b$, $c$, two-element subsets $ab$, $ac$, $bc$, and $abc$ at the top.

Finally, the polymatroids $f_x$ and $f_y$ are linearly representable over any field. Choose seven independent vectors $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}$. Subspaces assigned to the ground elements are the ones spanned by the vectors listed below:

| | $abcx$ is linear | | $abcy$ is linear |
|---|---|---|---|
| $a$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2$ | $a$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2$ |
| $b$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{v}_1, \mathbf{v}_2$ | $b$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{v}_1, \mathbf{v}_2$ |
| $c$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2$ | $c$ : | $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ |
| $x$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{r}$ | $y$ : | $\mathbf{s}_1, \mathbf{s}_2, \mathbf{r}$ |

It is easy to check that all generated subspaces have the right dimension. Note that while the dimensions of the subspaces corresponding to subsets of $abc$ are the same, the subspace arrangements are *not* isomorphic. □

It is easy to check that $f_x{\downarrow}x$ and $f_y{\downarrow}y$ are also linearly representable. As $f_x$ and $f_y$ are not adhesive, according to Theorem 7, $f_x{\downarrow}x$ and $f_y{\downarrow}y$ have no amalgam.

Theorem 10 can be used to characterize 1-sticky polymatroids on three-element sets. The following examples show some particular cases.

**Example 14.** Let $f$ be a polymatroid on $\{abc\}$. If $(a,b)$, $(a,b|c)$ are positive, $(a,b)\le(a,c)$, $(a,b)\le(b,c)$, then $f$ is not 1-sticky.

Proof. We specify two extensions $f_x$ and $f_y$ so that one of the inequalities in Theorem 10 fails. Let $t=(a,b)>0$, and $u=\min\{(a,b),\ (a,b|c)\}>0$. Define the excess functions $e_x$, $e_y$ by

$$e_x(A)=\begin{cases} t & \text{if } A=\emptyset,\\ 0 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad e_y(A)=\begin{cases} u & \text{if } A=\emptyset \text{ or } A=\{c\},\\ 0 & \text{otherwise.}\end{cases}$$

According to Examples 5 and 6, $f_x$ and $f_y$ are polymatroids. In this case $f_x(a,x|c)=f_x(b,x|ac)=f_x(c,x|ab)=0$, $f_x(x|abc)=0$, and $f_x(a,b|x)=f(a,b)-t=0$, see the remark following Example 5. Similarly, we have $f_y(a,b|y)=f(a,b)-u=t-u$, $f_y(c,y)=0$, thus the left hand side of the top line in (2) is

$$\begin{aligned} f_x(a,x|c)+(f_x(a,b|x)+f_y(a,b|y)+f_y(c,y)\\ +f_x(b,x|ac)+f_x(c,x|ab)+2f_x(x|abc)=t-u,\end{aligned}$$

while the right hand side is $f(a,b)=t$. Thus no amalgam of $f_x$ and $f_y$ exists. □

**Example 15.** Suppose $(a|bc)=(b|ac)=(c|ab)=0$, and at least one of $(a,b|c)$, $(a,c|b)$, $(b,c|a)$ is zero. Then $f$ is 1-sticky.

Proof. Let $f_x$ and $f_y$ be two extensions of $f$. Our goal is to show that all instances of the inequalities in Theorem 10 hold. From the assumptions it follows that for $|A|\ge 2$ we have $f(A)=f(abc)=t$; moreover at least one of $f(a)$, $f(b)$, $f(c)$ also equals $t$. Suppose $f_x$ and $f_y$ are specified by the excess functions $e_x$ and $e_y$. By Proposition 8 we can assume that $f_x$ is tight on $x$ and $f_y$ is tight on $y$, which gives $e_x(M)=e_y(M)=0$, where $M=\{abc\}$. In our case $f(i|M{-}i)=0$, thus we must also have $e_x(i|M{-}i)=e_y(i|M{-}i)=0$, thus $e_x(A)=e_y(A)=0$ for all two-element subsets of $M$. This means

$$f_x(i,x|M{-}i)=f_x(x|M)=0,\quad f_y(i,y|M{-}i)=f_y(y|M)=0,$$

thus all terms in the second line of (2) are zero. Consequently we only need to show that

$$f_x(a,x|c)+f_x(a,b|x)+f_y(a,b|y)+f_y(c,y)\ge f(a,b),$$

which rewrites to

$$f(a,b)+e_x(a)+e_x(b)+e_x(c)-e(x)+e_y(a)+e_y(b)-e_y(c)\ge 0. \tag{5}$$

The condition that one of $f(a)$, $f(b)$, $f(c)$ equals $t$ was not used yet. If $f(c) = t$, then $e_x(c) = e_y(c) = 0$, and then (5) follows from

$$f(a, b) + e_x(a, b) + e_y(a) + e_y(b) \geq 0,$$

which holds by Lemma 4, Condition 3. When $f(a) = t$ (or, symmetrically, $f(b) = t$), then $e_x(a) = e_y(a) = 0$, $f(a, b) = f(b) = f(b, c) + f(a, b|c)$, and (5) rewrites to

$$f(b, c) + e_x(b, c) + f(a, b|c) + e_y(a, b|c) + e_y(b) \geq 0,$$

which, again, holds by Lemma 4. □

## 4. TWO-ELEMENT EXTENSIONS

Using similar techniques necessary and sufficient conditions for the existence of an amalgam of polymatroids on larger sets can be obtained. Theorem 16 is such an example. It is a consequence of [11, Remark 6] and Theorem 7; we sketch a direct proof. The result is used to get a characterization of 2-sticky polymatroids on two-element sets.

**Theorem 16.** Polymatroids $f_X$ and $f_y$ on $abx_1x_2$ and $aby$, respectively, have an amalgam if and only if the following two inequalities and all of their permutations (permuting $a$ and $b$, and $x_1$ and $x_2$) hold choosing either the top or the bottom line from the list in curly brackets:

$$\begin{Bmatrix} f_X[a, b, x_1, x_2] \\ f_X[a, x_1, b, x_2] \end{Bmatrix} + f_y(y, a|b) + f_y(y, b|a) + f_y(a, b|y) + 3f_y(y|ab) \geq 0.$$

Proof. After expanding and rearranging the above inequalities are equivalent to

$$\begin{aligned} &\begin{Bmatrix} (x_2, y|b) + (x_1, x_2|y) + (a, b|x_2y) + (a, y|x_1x_2) \\ (x_2, b|y) + (x_2, y|x_1) + (a, y|bx_2) + (a, x_1|x_2y) \end{Bmatrix} \\ &+ (x_1, y|a) + (x_2, y|a) + (x_1, y|b) + (a, b|x_1y) \\ &+ (y|abx_1) + (y|abx_2) + (y|ax_1x_2) \geq 0, \end{aligned}$$

thus if $f_X$ and $f_y$ have an amalgam, then the expressions must be non-negative.

The sufficiency can be checked similarly as in Theorem 10 by computing the facets of the projection of the cone $\Gamma_{\{abx_1x_2y\}}$ to the coordinates which are subsets of $abx_1x_2$ and $aby$. There are 12 dropped coordinates: $x_1y$, $x_2y$, $x_1x_2y$, $\ldots$, $abx_1y$, $abx_2y$, $abx_1x_2y$. Restricting the matrix $\mathcal{M}$ describing the facets of the cone $\Gamma_{\{abx_1x_2y\}}$ to these columns, one gets the submatrix $\mathcal{M}'$ with 48 rows and 12 columns. Some of the rows are shown in Table 3. Software Porta [3] found 6938 extremal non-negative linear combinations giving zero sums for the 12 projected variables. One facet of the projection is generated by the linear combination taking all but the first and last rows from Table 3 once, and the last row three times. As in case of Theorem 10 all extremal combinations were expanded to bounding hyperplanes and checked whether it was a facet of the projection. This search confirmed the claim. □

| $x_1y$ | $x_2y$ | $x_1x_2y$ | | $a$ | | | $b$ | | | $ab$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $(a, x_1\|y)$, $(a, y\|x_1)$ |
| -1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | -1 | 0 | 0 | $(a, b\|x_1y)$ |
| 1 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $(x_1, x_2\|y)$ |
| 0 | -1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | -1 | 0 | $(a, b\|x_2y)$ |
| 0 | 0 | 1 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | $(a, y\|x_1x_2)$ |
| 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $(x_1, y\|a)$ |
| 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $(x_2, y\|a)$ |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | $(b, y\|ax_1x_2)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | $(x_1, y\|b)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | $(x_2, y\|b)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | -1 | $(x_2, y\|abx_1)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | $(x_1, y\|abx_2)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $(y\|abx_1x_2)$ |

**Tab. 3.** A submatrix of $\mathcal{M}_{\{abuxy\}}$.

**Theorem 17.** The polymatroid on the two-element set $ab$ is 2-sticky if and only if one of the following cases hold: $(a, b) = 0$ (it is modular); $(a|b) = 0$, or $(b|a) = 0$ (one of them determines the other).

Proof. First we show that these polymatroids are 2-2-sticky. Modular polymatroids are sticky without any restriction, so suppose, e.g., that $f(a|b) = 0$. Let $f_X$ be an extension on $abx_1x_2$. All six Ingleton expressions for $f_X$ are non-negative using the following equalities and their symmetric versions:

$$[a, b, x_1, x_2] + (a|b) = (a, x_1|b) + (a, x_2|b) + (x_1, x_2|a) + (a|x_1x_2);$$
$$[a, x_1, b, x_2] + (a|b) = (a, x_1|b) + (b, x_2|a) + (a, x_2|x_1) + (a|bx_2);$$
$$[b, x_1, a, x_2] + (a|b) = (a, x_1|b) + (a, x_2|b) + (a, x_1|x_1) + (b, x_1|ax_2) + (a|bx_1x_2);$$
$$[x_1, x_2, a, b] + (a|b) = (a, x_1|b) + (a, x_2|x_1) + (a, b|x_2) + (x_1, x_2|ab) + (a|bx_1x_2).$$

It means that $f_X$ is linear, and the same is true for $f_Y$. As in the proof of Theorem 9, using Proposition 1 we may assume that $f_X$ and $f_Y$ are linearly representable over the same field, and the dimensions of the subspaces corresponding to the common subsets $a$, $b$ and $ab$ are the same in both representations. Choose maximal independent set of vectors in both representations which span these subspaces in an equivalent way. Extend this set to be a base in both representations. Glue the two vector spaces together along the equivalent set of base vectors. This gives an amalgam (even an adhesive extension) of $f_X$ and $f_Y$, as required.

In the other direction first we show that given $f$ with $f(a, b) > 0$, $f(a|b) > 0$, $f(b|a) > 0$, it can be extended to a polymatroid $f_X$ on $abx_1x_2$ so that $f_X[a, b, x_1, x_2] < 0$.

For the construction we recall the *natural coordinates* of polymatroids on four elements from [4]. This coordinate system has the additional advantage that points with natural coordinates in the non-negative orthant $\mathbf{R}^{15}_{\ge 0}$ are polymatroids. Let us recall these coordinates below:

$$\begin{aligned}
&-[a,b,x_1,x_2],\\
&(a,b|x_1),\ (a,b|x_2),\ (a,x_1|b),\ (b,x_1|a),\ (a,x_2|b),\ (b,x_2|a),\\
&(x_1,x_2|a),\ (x_1,x_2|b),\ (x_1,x_2),\ (a,b|x_1x_2),\\
&(a|bx_1x_2),\ (b|ax_1x_2),\ (x_1|abx_2), (x_2|abx_1).
\end{aligned}$$

From these coordinates the values $f_X(a)$, $f_X(b)$, and $f_X(ab)$ can be expressed as follows, where only the coefficients of the coordinates in the above order are shown:

$$\begin{aligned}
f_X(a) &= (\mathbf{2},\ 1,1,1,0,1,0,\ \mathbf{1},\mathbf{0},\mathbf{1},1,\ 1,0,0,0)\\
f_X(b) &= (\mathbf{2},\ 1,1,0,1,0,1,\ \mathbf{0},\mathbf{1},\mathbf{1},1,\ 0,1,0,0)\\
f_X(ab) &= (\mathbf{3},\ 1,1,1,1,1,1,\ \mathbf{1},\mathbf{1},\mathbf{1},2,\ 1,1,0,0).
\end{aligned}$$

Choose the coordinates first and from eighth to tenth (typeset in bold) to have the positive values $\varepsilon$, $f(a|b)-\varepsilon$, $f(b|a)-\varepsilon$, and $f(a,b)-\varepsilon$, respectively, for some small enough $\varepsilon$; set all other coordinates to zero. With this choice $f_X$ will be a polymatroid which extends the one given on $ab$ as, e. g., $f_X(a) = 2\varepsilon + f(a,b) - \varepsilon + f(a|b) - \varepsilon = f(a)$, moreover the Ingleton value $f_X[a,b,x_1,x_2]$, as given by the first coordinate, is $-\varepsilon$, which is negative.

Define the other extension $f_y$ by the excess function

$$e_y(A) = \begin{cases} f(a,b) & \text{if } A=\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

By the remark after Example 5, $f_y$ is a polymatroid and $f_y(a,b|y) = f_y(a,y|b) = f_y(b,y|a) = 0$ as well as $f_y(y|ab) = 0$. According to Theorem 16 if $f_X$ and $f_y$ have an amalgam, they must satisfy

$$f_X[a,b,x_1,x_2] + f_y(a,b|y) + f_y(a,y|b) + f_y(b,y|a) + 3f_y(y|ab) \ge 0.$$

This value, however, is $-\varepsilon < 0$, which proves the theorem. □

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