# On Subjective Expected Value under Ambiguity

**Radim Jiroušek & Václav Kratochvíl**
Faculty of Management, University of Economics, and
Institute of Information Theory and Automation, Academy of Sciences
Jindřichův Hradec and Prague, Czech Republic
[radim, velorex]@utia.cas.cz

**Abstract**

The paper describes decision-making models based on a newly introduced notion of personal expected value. Such models exhibit the ambiguity aversion, which is controlled by a subjective parameter with the semantics of "the higher the aversion, the higher the coefficient". For negative values of this parameter the models thus manifest a positive attitude to ambiguity. If this parameter equals zero, then the respective model turns into the usual belief function without the aversion to ambiguity. In this case, the personal expected value equals the recently introduced Shenoy's expectation. Finally, the behavior of these models is briefly compared with experimental data.

## 1. Introduction

It is well known and has also been confirmed by our experiments that most people prefer lotteries in which they know the content of the drawing drum to situations when that content is unknown. In our experiments, the participants were asked to choose one out of six predetermined colors, and they won a prize when the color of a randomly drawn ball was that of their choice. It appeared that the participants were, on average, willing to pay by about 30% more to take part in games when they knew that the urn contained the same number of balls of all six colors, in comparison with the situation when they only knew that the urn contained balls of specified colors but their proportions were unknown. This well-known, seemingly paradoxical phenomenon can hardly be explained by different subjective utility functions. To explain this fact, we accepted the hypothesis of Savage [29] that humans are willing to pay no more than what they expect to get back from the lottery, i.e., they are willing to pay at most their subjective expected value. However, to (unconsciously) estimate this value they do not use their subjective probability measures but just *capacity functions* that do not sum up to one [22]. Roughly speaking, the subjective probability of drawing a red ball is $\frac{1}{6}$ if they know that there is the same number of balls of each color in the drum. However, the respective "subjective probability" in the case of insufficient knowledge is usually $\varepsilon < \frac{1}{6}$. *The lack of knowledge psychologically decreases the subjective chance of drawing the selected color – it decreases the subjective chance of success.* This attitude is subjective and differs from person to person, depending on the *personal intensity of ambiguity aversion*. This is why we will distinguish between two notions in this paper. The *expected value* will be used for the value computed when the uncertainty is formalized in a normative way, such as by a probability measure or a belief function. The notion of *personal expected value* will be used when also the intensity of the personal subjective *ambiguity aversion* is taken into consideration. The former notion is a part of the mathematical apparatus, the latter exceeds the border of mathematics into psychology. Note that the concept of ambiguity aversion goes through all other theories used in decision making, such as game theory [2] or quantum probability [1].

The paper is one of many related to the ambiguity aversion, due to which human behavior violates Savage's expected utility theory [29]. There are many different approaches to ambiguity aversion, like Maxmin-Expected Utility, Choquet-Expected Utility with convex capacities, Smooth Ambiguity-Averse preferences, Variational and Multiplier preferences – recently axiomatized by [5]. For a nice survey, see [14]. Lang considers models with first-order and second-order ambiguity aversion [27], which differ by the inclination to risk-averse behavior.

We present a new model manifesting the same ambiguity aversion as human decision-makers. For this, we construct a reduced capacity function that can, similarly to probability function, be used to compute the personal expected value of a reward in the case that the description of the situation is ambiguous. It is clear from the literature [12, 24, 13, 15] that it need not be a probability function. When considering all elementary events, this function does not necessarily sum up to one because people usually expect smaller rewards under total ignorance than in the case of knowing that all alternatives are of equal probabilities. As we will see later (when discussing the Ellsberg's experiments), this function is generally not additive. Thus, the considered function belongs to the class of functions called *capacities*.

In this paper, we take advantage of the fact that situations with ambiguity are well described by tools of theory of belief functions. This theory distinguishes between two types of uncertainty: the uncertainty connected with the fact that we do not know the result of a random experiment (in our examples a result of a random lottery) and the ignorance arising when we do not know the content of the drawing urn. Namely, we believe that the phenomenon of ambiguity aversion is closely connected with the fact that probability theory has difficulties with representing ignorance or vagueness [30]. We start with describing the situation by belief functions that can be interpreted as *generalized probability* [16], i.e., each belief function corresponds to a set of probability functions [16]. We then adopt the decision-theoretic framework, also used by other authors, based on the transformation of the belief function into a probability function. Our model manifests the required properties for most of the known probability transforms [10]. From a range of transforms discussed in Section 3, we have given preference to the probability measure consistent with the recently proposed Shenoy's expectation [32] because it is the only one having certain theoretical support directly within the theory of belief functions (see [32]), and simultaneously yielding an element from the respective credal set. However, we do not use the achieved probabilistic measure to compute the expected reward (like, e.g., Smets [36]) and use it to support the required decision. In fact, we add one additional step. Before computing the personal expected reward, we reduce the probabilities to account for ambiguity aversion. This is the only point in which our approach differs from Smets' decision-making framework [35], which is based on the Dempster-Shafer theory of belief functions [11, 30].

Before describing the outlined process in more detail, let us stress that our aim is not as ambitious as developing a mathematical theory to describe the ambiguity aversion within the theory of belief functions. It was already done by Jaffray [19], who shows how to compute the generalized expected utility for a belief function. We do not even consider all elements from a credal set with all the preference relations as, for example, in [8]. Our ambition is to provide models simulating human decision-makers. The behavior of these models is dependent on a parameter called a *personal coefficient of ambiguity aversion*. Such a coefficient of ambiguity aversion is also considered by Rajendra Srivastava [38] and our suggested approach repeats some of his basic ideas. For example, we use an almost identical idea to identify the amount of ambiguity connected with individual states of the considered state space.

This paper is an extended and completely rearranged version of the paper presented at ECSQARU 2019. The next Section is devoted to a very brief introduction to belief function theory (in fact, the main purpose is to introduce the notation) and a recently introduced Shenoy's expected value. Newly inserted Section 3 explains the role of this expected value in our models and presents a range of probability transforms that may be used to compute approximations of Shenoy's expected value. Among them, we introduce two transforms that have not appeared in the belief function literature before (Perez's barycenter, and a convex combination of belief and plausibility functions). Section 4 explains the precise meaning of the personal expected value, which depends on the personal coefficient of ambiguity aversion $\alpha$. A possibility of how to find the value of this coefficient for experiment participants is discussed in Section 5. Finally, Section 6 presents an example showing why we should consider non-additive capacity functions. A summary of results achieved in our experimental session, which were realized after the ECSQARU conference, is presented in Section 7.

## 2. Belief Functions

The basic concepts and notations are taken over from [22], where the first ideas of this approach were introduced. We consider only a finite *state space* $\Omega$. In the examples described below, $\Omega$ is a set of six considered colors: $\Omega = \{red, black, white, yellow, green, azure\}$ ($\Omega = \{r, b, w, y, g, a\}$ for short). Similar to probability theory, where a probability measure is a set function defined on an *algebra* of the considered events, belief functions are represented by functions defined on the set of all subsets of $\Omega$ (denoted $2^{\Omega}$) [11, 30].

The fundamental notion is that of a *basic probability assignment* (or, basic assignment), which describes all the information we have about the considered situation. It is a function $m : 2^\Omega \to [0,1]$, such that $\sum_{\mathbf{a}\in 2^\Omega} m(\mathbf{a}) = 1$ and $m(\emptyset) = 0$.

For basic probability assignment $m$, $\mathbf{a} \in 2^\Omega$ is said to be a *focal element* of $m$ if $m(\mathbf{a}) > 0$. This enables us to distinguish between the following special classes of basic assignments representing the extreme situations:

- $m$ is said to be *vacuous* if $m(\Omega) = 1$, i.e., it has only one focal element, $\Omega$. A vacuous basic assignment is denoted by $m_\iota$. It corresponds to total ignorance. In our examples, $m_\iota$ represents situations when we do not have any information regarding the proportions of colors in the drawing drum.

- $m$ is said to be *Bayesian*, if all its focal elements are singletons, i.e., for Bayesian basic assignment $m$, $m(\mathbf{a}) > 0$ implies $|\mathbf{a}| = 1$. Bayesian basic assignments represent exactly the same knowledge as probability functions. Since all focal elements of a Bayesian basic assignment $m$ are singletons, we can define probability measure $P_m$ for $\Omega$ such that
$$P_m(x) = m(\{x\}) \tag{1}$$
for all $x \in \Omega$. In our examples, Bayesian basic assignments thus represent situations when the proportions of colors in the drawing drum are known. We use a uniform Bayesian basic probability assignment $m_u$, for which $m_u(\{r\}) = m_u(\{b\}) = \ldots = m_u(\{a\}) = \frac{1}{6}$.

- $m$ is said to be *quasi-Bayesian* if each of its focal elements is either a singleton or the whole $\Omega$. That is, for quasi-Bayesian basic assignment $m$, $m(\mathbf{a}) > 0$ implies $|\mathbf{a}|$ equals either 1 or $|\Omega|$. As an example of a quasi-Bayesian basic probability assignment, consider the situation when we know that the drawing urn contains $n$ balls and at least one of them is red. In this case, $m(\{r\}) = \frac{1}{n}$ and $m(\Omega) = \frac{n-1}{n}$.

The same knowledge that is expressed by a basic probability assignment $m$ can also be expressed by a *belief function*, or by a *plausibility function*, or by a *commonality function*.

$$Bel_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega:\, \mathbf{b}\subseteq \mathbf{a}} m(\mathbf{b}), \tag{2}$$

$$Pl_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega:\, \mathbf{b}\cap \mathbf{a}\neq\emptyset} m(\mathbf{b}), \tag{3}$$

$$Q_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega:\, \mathbf{b}\supseteq \mathbf{a}} m(\mathbf{b}). \tag{4}$$

Let us point out that, whenever one of these functions is given, it is always possible to reconstruct the corresponding basic probability assignment $m$. For example,

$$m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega:\, \mathbf{b}\supseteq \mathbf{a}} (-1)^{|\mathbf{b}\setminus\mathbf{a}|} Q_m(\mathbf{b}). \tag{5}$$

Recalling the basic situations mentioned above, we can see that the vacuous basic assignment $m_\iota$ describes the case when the composition of the content of the drawing drum is completely unknown. Then, $Bel_{m_\iota}(\mathbf{a}) = 0$ for all $\mathbf{a} \subsetneq \Omega$, and $Bel_{m_\iota}(\Omega) = 1$, $Pl_{m_\iota}(\mathbf{a}) = 1$ for all $\mathbf{a} \neq \emptyset$, and $Q_{m_\iota}(\mathbf{a}) = 1$ for all $\mathbf{a} \in 2^\Omega$. When the drawing drum contains the same number of balls of each color, the situation is described by the uniform Bayesian basic assignment $m_u$. For this basic probability assignment, $Q_{m_u}(\mathbf{a}) = m_u(\mathbf{a})$ for all $\mathbf{a} \neq \emptyset$, and $Bel_{m_u}(\mathbf{a}) = Pl_{m_u}(\mathbf{a}) = \frac{|\mathbf{a}|}{6}$ for all $\mathbf{a} \in 2^\Omega$. For the quasi-Bayesian basic assignment, $m(\{r\}) = \frac{1}{n}$, $m(\Omega) = \frac{n-1}{n}$, one gets $Bel_m(\Omega) = 1$, $Bel_m(\mathbf{a}) = \frac{1}{n}$ whenever $r \in \mathbf{a} \neq \Omega$, and $Bel_m(\mathbf{a}) = 0$ for all the remaining subsets of $\Omega$. For this basic assignment, $Pl_m(\mathbf{a}) = 1$ if $r \in \mathbf{a}$, $Pl_m(\mathbf{a}) = \frac{n-1}{n}$ if $r \notin \mathbf{a} \neq \emptyset$, and $Q_m(\{r\}) = 1$, $Q_m(\mathbf{a}) = \frac{n-1}{n}$ for all the remaining non-empty subsets of $\Omega$.

The interpretation of the belief function theory as a generalization of the probability theory is based on the fact that each basic probability assignment determines a convex set of probability measures $P$ on $\Omega$, the so-called *credal* *set*, which is defined as follows ($\mathscr{P}$ denotes the set of all probability measures on $\Omega$):

$$\mathscr{P}(m) = \left\{ P \in \mathscr{P} : \sum_{x \in \mathbf{a}} P(x) \geq Bel_m(\mathbf{a}) \text{ for } \forall \mathbf{a} \in 2^{\Omega} \right\}.$$

Notice that $P_m$ defined by Equation (1) for a Bayesian basic assignment $m$ is such that $\mathscr{P}(m) = \{P_m\}$. Therefore $\mathscr{P}(m_u) = \{P_u\}$, where $P_u$ is the uniform probability measure on $\Omega = \{r, b, w, y, g, a\}$, i.e., $P_u = (\frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6})$. For the vacuous $m_\iota$, $\mathscr{P}(m_\iota) = \mathscr{P}$. For each basic assignment $m$, it is easy to show that, for all $P \in \mathscr{P}(m)$,

$$Bel_m(\mathbf{a}) \leq P(\mathbf{a}) \leq Pl_m(\mathbf{a}),$$

for all $\mathbf{a} \in 2^{\Omega}$. Thus, if $Bel_m(\mathbf{a}) = Pl_m(\mathbf{a})$ then we are sure that the probability of $\mathbf{a}$ equals $Bel(\mathbf{a})$. Otherwise, the larger the difference $Pl_m(\mathbf{a}) - Bel_m(\mathbf{a})$, the more uncertain we are about the value of the probability of $\mathbf{a}$. This is also why Srivastava, when the knowledge is encoded by basic probability assignment $m$, takes the difference $Pl_m(\mathbf{a}) - Bel_m(\mathbf{a})$ as a measure of ambiguity connected with an event $\mathbf{a}$ [38]. Other ways to measure ambiguity appearing in the literature are based on the total aggregated uncertainty connected with the respective belief function. For example, the authors of [23] suggest to measure the total ambiguity as the entropy value of the pignistic probability transform – this approach was later modified in [31] to avoid drawbacks identified in [25]. For an overview of various measures of uncertainty for belief functions see, e.g., [3].

In this paper, the apparatus of belief functions is used to describe the uncertain information about the situation under which the decision is to be made. As said in the Introduction, our model is based on the idea that a certain probability distribution is reduced as per the ambiguity aversion of the decision-maker. It can easily be done when the recently introduced notion of Shenoy is incorporated into the model instead of the usually employed Choquet integral.

For a real valued function $\hat{g} : 2^{\Omega} \longrightarrow \mathbb{R}$, Shenoy [32] suggests to compute its expected value under uncertainty expressed by basic probability assignment $m$ according to the following formula:

$$E_m(\hat{g}) = \sum_{\mathbf{a} \in 2^{\Omega}} (-1)^{|\mathbf{a}|+1} \hat{g}(\mathbf{a}) Q_m(\mathbf{a}). \tag{6}$$

In the cited paper, he shows that this value meets a long list of required properties. Principally, there are two ways of applying this definition to compute the expected value of the function $g : \Omega \to \mathbb{R}$. The first possibility is to extend the function $g$ to a function $\hat{g}$ defined on $2^{\Omega}$. To preserve the validity of properties of the expected values proven by Shenoy, we have to follow his philosophy and define, for all $\mathbf{a} \in 2^{\Omega}$,

$$\hat{g}(\mathbf{a}) = \frac{\sum\limits_{x \in \mathbf{a}} g(x) Q_m(\{x\})}{\sum\limits_{x \in \mathbf{a}} Q_m(\{x\})} \tag{7}$$

(in the case that $\sum_{x \in \mathbf{a}} Q_m(\{x\}) = 0$, the value $\hat{g}(\mathbf{a})$ does not influence the resulting expected value of $g$, and we can therefore choose any reasonable value; for example, $\hat{g}(\mathbf{a}) = (\min_{x \in \mathbf{a}}\{g(x)\} + \max_{x \in \mathbf{a}}\{g(x)\})/2)$. Now it holds:

$$E_m(g) = \sum_{\mathbf{a} \in 2^{\Omega}} (-1)^{|\mathbf{a}|+1} \hat{g}(\mathbf{a}) Q_m(\mathbf{a}),$$

which, after the elimination of the extension function $\hat{g}$, equals

$$E_m(g) = \sum_{\mathbf{a} \in 2^{\Omega}} (-1)^{|\mathbf{a}|+1} \frac{\sum\limits_{x \in \mathbf{a}} g(x) Q_m(\{x\})}{\sum\limits_{x \in \mathbf{a}} Q_m(\{x\})} Q_m(\mathbf{a}). \tag{8}$$

Alternatively, as shown in [21], for each basic probability assignment $m$, there exists a probability measure $Sh\_P_m$ on $\Omega$ such that

$$E_m(g) = \sum_{x \in \Omega} g(x) Sh\_P_m(x). \tag{9}$$

And this is why we base our approach on the application of Shenoy's expected value. It offers a possibility to compute a "subjective probability" by applying the reduction principle mentioned in the Introduction.

### 3. Probability Transforms

The computation of the probability measure $Sh\_P_m$ for given $m$ is not a computationally easy task. For this, one has to consider $|\Omega|$ simple real functions $w_x : \Omega \to \mathbb{R}$ defined for each $x \in \Omega$

$$w_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Then, for all $x \in \Omega$, $Sh\_P_m(x) = E_m(w_x)$. However, one has to realize that computing $E_m(w_x)$ requires extending each function $w_x$ to $\hat{w}_x : 2^\Omega \to \mathbb{R}$ according to Formula (7). When considering the situation with six colors in the drawing drum, it means that $|\Omega| = 64$, and that the function $\hat{w}_x$ is defined on the set $2^\Omega$ of cardinality $2^{64}$. Because of these computational problems, in [21] we studied a possibility to approximate probability measure $Sh\_P_m(x)$ by other probability measures, which are sometimes considered as representatives of basic belief assignment $m$, and the computation of which is much simpler.

In literature (for a survey paper see [10]) several approaches were designed for representing a basic probability assignment $m$ by a probability measure . In [21], we studied three of them (pignistic, plausibility and maximum entropy transforms), as well as two newly introduced ones. Let us briefly recall all of these five transforms. For this, consider a basic probability assignment $m$ on $\Omega$.

The most famous is a *pignistic transform* (introduced in [33], for a survey see [37]), defined for all $x \in \Omega$ by the formula

$$Bet\_P_m(x) = \sum_{\mathbf{a} \in 2^\Omega : x \in \mathbf{a}} \frac{m(\mathbf{a})}{|\mathbf{a}|}. \tag{11}$$

This transform distributes the mass $m(\mathbf{a})$ uniformly to all elements of $\mathbf{a}$. For decision-making, it was strongly advocated by Philippe Smets [34, 36].

Another transform, which is, as shown in [7] by Cobb and Shenoy, the only transform compatible with Dempster's rule of combination, is the so-called plausibility transform. It is a normalized plausibility function on singletons. It is formally defined, for all $x \in \Omega$, by

$$Pl\_P_m(x) = \frac{Pl_m(\{x\})}{\sum\limits_{y \in \Omega} Pl_m(\{y\})}.$$

Notice also that it is the only one from among the studied transforms for which the resulting probability measure need not be an element of the corresponding credal set.

The third probability transform selects the *maximum entropy* element from the credal set $\mathscr{P}(m)$, i.e.,

$$Me\_P_m(x) = \arg \max_{P \in \mathscr{P}(m)} H(P),$$

where $H(P)$ is the Shannon entropy of probability measure $P$

$$H(P) = - \sum_{x \in \Omega : P(x) > 0} P(x) \log_2 P(x).$$

The fourth transform is the Perez' barycenter [28], which has undeservedly fallen into oblivion:

$$Bac\_P_m(x) = \arg \min_{P \in \mathscr{P}(m)} \max_{Q \in \mathscr{P}(m)} Div(Q;P),$$

where $Div(Q;P)$ denote the well-known Kullback-Leibler divergence [26]

$$Div(Q;P) = \begin{cases} +\infty, & \text{if } \exists\, x \in \Omega : P(x) > 0 = Q(x); \\ \sum\limits_{x \in \Omega : P(x) > 0} P(x) \log\left(\frac{P(x)}{Q(x)}\right), & \text{otherwise.} \end{cases}$$

As the last transform, we propose to consider the element of the credal set that can be expressed as a convex combination of belief and plausibility functions:

$$Cs\_P_m(x) = \delta Bel_m(\{x\}) + (1-\delta)Pl_m(\{x\}),$$

where

$$\delta = \frac{(\sum_{x\in\Omega} Pl_m(\{x\})) - 1}{(\sum_{x\in\Omega} Pl_m(\{x\})) - (\sum_{x\in\Omega} Bel_m(\{x\}))}.$$

It is not difficult to show that, for quasi-Bayesian basic belief assignment's $m$, this last transform coincides with the pignistic transform: $Cs\_P_m(x) = Bet\_P_m(x)$.

All of the above-introduced probability transforms are probabilistic representatives of belief functions (basic probability assignments). As mentioned at the end of the previous section, the expected value of real function $g : \Omega \to \mathbb{R}$ computed using the probability measure $Sh\_P_m$ equals Shenoy's expected value under the knowledge represented by the basic assignment $m$. Since, in practical problems, it is usually impossible to compute probability measure $Sh\_P_m$, a natural question arises: do some of the remaining transforms yield similar results? This question was studied in [21] using randomly generated basic probability assignments on a five-element set $\Omega$ (reflecting the fact that, in practical applications, basic assignments usually have limited numbers of focal elements - for details see [21]), and the results can briefly be summarised in the following observations.

- For simple (in a way symmetric) basic assignments, like those described in the following Sections, all five probabilistic transforms usually yield the same probability measure as $Sh\_P_m$.
- Even for asymmetric situations, when all five probabilistic transforms yield different probability measures, the resulting measures are close to each other (measured by Kullback-Leibler divergence, and/or total variance). Therefore, the expected values computed concerning the different probability transforms do not substantially differ from each other.
- In general, none of the transforms is dominant in approximating the Shenoy's transform, each of them yields the best approximations of the Shenoy's transform for some basic assignments.
- An exception from the preceding observation occurs for quasi-Bayesian basic assignments. For this class of basic assignments, the best approximations of the Shenoy's transform are usually yielded by Perez' barycenter.

## 4. Personal Expected Value

In this Section, we assume that the considered situation is well described by a belief function with a basic probability assignment $m$ on $\Omega$. Considering a function $g : \Omega \to \mathbb{R}$, one can compute its Shenoy's expected value using Formula (8), An alternative way, as explained in Section 3, is to find the probability measure given by the Shenoy's transform $Sh\_P_m$ and compute the respective expected value using Formula (9). If it is computationally unfeasible, one can obtain a reasonable approximation of the considered expected value replacing the Shenoy's transform $Sh\_P_m$ with any of the transforms described in Section 3.

As showed by Ellsberg [12] and his followers, human decision-makers do not merely go along with the expected gain. To model their decision making, we introduce a personal expected value of the considered gain function $g$. To compute the latter, we propose to apply the reduction principle mentioned in the Introduction. For each event $\mathbf{a} \subseteq \Omega$, the considered probability distribution $Sh\_P_m$ is reduced proportionally to the *personal coefficient of ambiguity aversion* $\alpha$. The personal expected value of the gain function is then computed using the resulting capacity function. Let us again point out that we construct a capacity function, which is generally not a probability measure because it neither is additive nor sums up to one on the elements of $\Omega$. Now, we describe this process of model construction in more detail.

Consider basic probability assignment $m$, the corresponding probability measure obtained by the Shenoy's transform $Sh\_P_m$, and the respective belief and plausibility functions $Bel_m$ and $Pl_m$. Let us recall that the higher $Pl_m(\mathbf{a}) - Bel_m(\mathbf{a})$, the higher is the ambiguity about the probability of event $\mathbf{a} \subseteq \Omega$. Our intuition says the higher the ambiguity about the probability of an event, the greater reduction of the respective probability should be done in our model. We therefore define a *reduced capacity function* $r_{m,\alpha}$ for all $\mathbf{a} \subseteq \Omega$ as follows:

$$r_{m,\alpha}(\mathbf{a}) = (1-\alpha)Sh\_P_m(\mathbf{a}) + \alpha Bel_m(\mathbf{a}), \tag{12}$$

where $\alpha$ denotes a *personal coefficient of ambiguity aversion*. Its introduction is inspired by the Hurwicz's *optimism-pessimism* coefficient [17]. Similarly to Hurwicz's coefficient, we first assumed $\alpha \in [0,1]$. Contrary to Hurwicz, who suggests that everybody *can choose* a personal coefficient expressing her optimism, we assume that each person *has* a personal coefficient of ambiguity aversion. The higher the aversion, the higher the coefficient's value. Therefore we propose ways of detecting the value of this coefficient for experiment participants (see the next Section). However, based on our experiments (and in agreement with observations of other authors [18]), it has appeared that an experiment participant may, even if exceptionally, exhibit a positive attitude to ambiguity (for a review of the relevant literature see [39]). For such a participant, $\alpha$ may be negative.

Notice that the amount of modification of $Sh\_P_m$ realized in Formula (12) depends on the ambiguity aversion coefficient $\alpha$, and the amount of ignorance associated with the event $\mathbf{a}$. If we are certain about the probability of an event $\mathbf{a}$, it means that $Sh\_P_m(\mathbf{a}) = Bel_m(\mathbf{a})$, and the corresponding probability is not changed: $r_{m,\alpha}(\mathbf{a}) = Sh\_P_m(\mathbf{a})$. On the other hand, the maximum reduction is achieved for the states connected with maximal ambiguity, i.e., for the events for which $Bel_m(\mathbf{a}) = 0$.

Some trivial properties of the function $r_{m,\alpha}$ are as follows:

1. **Subnormality.** For $\alpha \in [0,1]$, $\sum_{x\in\Omega} r_{m,\alpha}(x) \leq 1$.
2. **Bayesian assignment.** $m$ is Bayesian if and only if $m(\{x\}) = Sh\_P_m(x) = r_{m,\alpha}(x)$ for all $x \in \Omega$.
3. **Monotonocity.** For $\mathbf{a} \subseteq \mathbf{b}$, $r_{m,\alpha}(\mathbf{a}) \leq r_{m,\alpha}(\mathbf{b})$.
4. **Superadditivity** For $\alpha \in [0,1]$, and $\mathbf{a} \cap \mathbf{b} = \emptyset$, $r_{m,\alpha}(\mathbf{a} \cup \mathbf{b}) \geq r_{m,\alpha}(\mathbf{a}) + r_{m,\alpha}(\mathbf{b})$.

As can be expected, the reduced capacity function is used to compute the personal expected value of functions $g : \Omega \to \mathbb{R}$. For this purpose, denote $\Gamma = \{g(x) : x \in \Omega\} \setminus \{0\}$, and $g^{-1}(\gamma) = \{x \in \Omega : g(x) = \gamma\}$. Then the value

$$R_{m,\alpha}(g) = \sum_{\gamma\in\Gamma} \gamma r_{m,\alpha}(g^{-1}(\gamma)) \tag{13}$$

is called a *personal expected value* of function $g$. Notice that most authors use the Choquet integral [6, 8] in this context – which is, in our opinion, not as intuitive as the proposed formula, and its value is always smaller than or equal to the introduced $R_{m,\alpha}$ for $\alpha \in [0,1]$. Let us note that, for $\alpha > 0$, betting the amount $R_{m,\alpha}$ guarantees a sure gain [4, 24].

## 5. Measuring Personal Coefficient of Ambiguity Aversion

As we show in this and the following Sections, the decision based on the personal expected value computed using Formula (13) manifests the same ambiguity aversion as most of the human decision-makers. In this context, one can ask to what extent this approach can be used to predict human behavior. Assume that human decision-makers manifest the same strength of ambiguity aversion in different situations, i.e., they have their personal coefficients of ambiguity aversion. Under this assumption, there should be an experimental way of estimating it. For this, the arrangement of our experiments (described in Section 7 in more detail) provides two ways of achieving this goal. The first possibility of assessing the personal coefficient of ambiguity aversion is based on comparing the bets of a person when facing the following two decision problems.

**F1** The drawing urn contains 30 balls, five of each of the following colors: red, black, yellow, white, green, and azure. How much is the maximum bet you are willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball has the color of your choice?

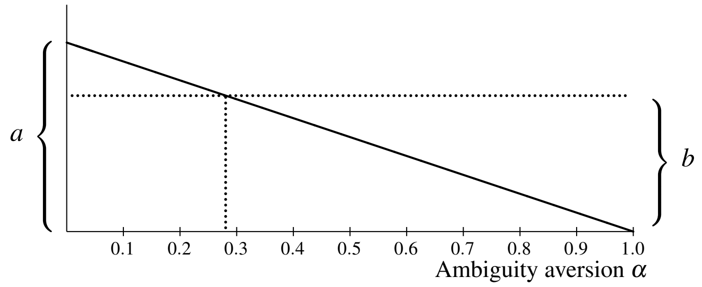


Figure 1: Maximum bets in dependence on the coefficient of ambiguity aversion. $a$ - the bet in situation F1, $b$ - the bet in situation F2.

**F2** The drawing urn contains an unknown number of balls (but at least one); they may be any or all of the following six colors: red, black, yellow, white, green, and azure. You know nothing more, you even do not know how many colors there are in the urn. How much is the maximum bet you are willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

Let us explain how to deduce the coefficient of ambiguity aversion $\alpha$ from the amounts of money the person is willing to pay in these two situations. For either lottery, consider $\Omega = \{r, b, y, w, g, a\}$. Naturally, the knowledge about the content of the drawing urn differs. In the case of lottery F1, the content of the drawing drum is described by the uniform Bayesian basic probability assignment defined

$$m_u(\mathbf{a}) = \begin{cases} \frac{1}{6}, & \text{if } |\mathbf{a}| = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

In the case of lottery F2, the situation is described by the vacuous bpa $m_\iota$.

It may not be surprising that the Shenoy's probability transforms (as well as all other probabilistic transforms introduced in Section 3) coincide for both lotteries: $Sh\_P_{m_u}(x) = Sh\_P_{m_\iota}(x) = \frac{1}{6}$ for all colors $x \in \Omega$. However, the respective reduced capacity functions differ from each other because the respective belief functions do: $Bel_{m_u}(\{x\}) = \frac{1}{6}$ for all $x \in \Omega$, whilst $Bel_{m_\iota}(\{x\}) = 0$ for all $x \in \Omega$. Therefore, using Formula (12), $r_{m_u,\alpha}(\{x\}) = \frac{1}{6}$, and $r_{m_\iota,\alpha}(\{x\}) = \frac{1-\alpha}{6}$ for all $x \in \Omega$.

Consider that a player chose, let us say, the red color. Let $g(x)$ denote the gain received in a case when color $x$ is drawn, i.e., $g(r) = 100$, and for $x \neq r$, $g(x) = 0$. The personal expected rewards are as follows (see Formula (13)):

$$R_{m_u,\alpha}(g) = 100 \cdot r_{m_u,\alpha}(g^{-1}(100)) = 100 \cdot r_{m_u,\alpha}(\{r\}) = \frac{100}{6},$$

$$R_{m_\iota,\alpha}(g) = 100 \cdot r_{m_\iota,\alpha}(g^{-1}(100)) = 100 \cdot r_{m_\iota,\alpha}(\{r\}) = \frac{100 \cdot (1-\alpha)}{6},$$

for F1 and F2, respectively. This can be interpreted as follows. In the case of lottery F1, the fact that the person is willing to pay at most $a \neq \frac{100}{6}$ CZK is explained by her risk attitude and her current need for 100 CZK. Nevertheless, the difference between the amounts the person is willing to pay for F1 and F2 can be explained only by her ambiguity aversion expressed (in our model) by the coefficient $\alpha$. In Formula (12), we introduced a linear dependence of the expected value on the coefficient $\alpha$, which now gives us a possibility to estimate the value of $\alpha$. If a person is willing to pay $a$ CZK for taking part at lotteries F1 and $b$ CZK for taking part at F2, one can assume that her personal coefficient of ambiguity aversion is about (see Figure 1)

$$\alpha = \frac{a-b}{a}. \tag{15}$$

The other possibility of estimating the personal coefficient of ambiguity aversion is based on the observation of experiment participants' behavior when they face a range of the following decision problems.

**Rn** The drawing urn contains $n$ balls, each of which may be red, black, yellow, white, green, or azure. You know that one and only one of them is red, nothing more. You even do not know how many colors there are in the urn. How much is the maximum bet you are willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball has the color of your choice?

This series of lotteries is designed to test the decrease of a person's "subjective probability" in comparison with the combinatorial probability. For this, in our experiments, the participants describe their behavior when facing eight such situations, which differ from each other just in the total number of balls in the drawing drum – the number $n$. We included lotteries with $n = 5, 6, 7, 8, 9, 10, 11, 12$.

To describe these situations, consider again $\Omega = \{r, b, y, w, g, a\}$, and the uncertainty is described by the basic probability assignment $m_\rho$

$$m_\rho(\mathbf{a}) = \begin{cases} \frac{1}{n}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{n-1}{n}, & \text{if } \mathbf{a} = \{b, g, o, y, w\}; \\ 0, & \text{otherwise,} \end{cases}$$

with belief function

$$Bel_{m_\rho}(\mathbf{a}) = \begin{cases} 0, & \text{for } r \notin \mathbf{a} \neq \{b, g, o, y, w\}; \\ \frac{1}{n}, & \text{for } r \in \mathbf{a} \neq \Omega; \\ \frac{n-1}{n}, & \text{for } \mathbf{a} = \{b, g, o, y, w\}; \\ 1, & \text{for } \mathbf{a} = \Omega. \end{cases}$$

Applying to this basic assignment all the probability transforms described in Section 3 , we get the same probability measure:

$$Sh\_P_{m_\rho}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ \frac{n-1}{5n}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

When computing the personal expected gain, we only need the values of the respective reduced capacity function $r_{m_\rho,\alpha}$ for singletons:

$$r_{m_\rho,\alpha}(\{x\}) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ (1-\alpha) \cdot \frac{n-1}{5n}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

Considering (for the sake of simplicity just two) gain functions $g^r(x)$, and $g^w(x)$ corresponding to the instances in which the participant bets on red and white color, respectively, i.e., $g^r(r) = 100$, $g^r(x) = 0$ for $x \neq r$, $g^w(w) = 100$, and $g^w(x) = 0$ for $x \neq w$, the personal expected rewards are as follows. When betting on red it equals

$$R_{m_\rho,\alpha}(g^r) = 100 \cdot r_{m_\rho,\alpha}((g^r)^{-1}(100)) = 100 \cdot r_{m_\rho,\alpha}(\{r\}) = \frac{100}{n},$$

and analogously, for betting on white

$$R_{m_\rho,\alpha}(g^w) \quad = \quad 100 \cdot r_{m_\rho,\alpha}((g^w)^{-1}(100)) = 100 \cdot r_{m_\rho,\alpha}(\{w\}) = \frac{100(1-\alpha)(n-1)}{5n}.$$

Let us explain how we use the results of the corresponding experiments to estimate a personal ambiguity aversion coefficient $\alpha$. The reasoning is based on the hypothesis that humans are willing to pay no more than the amount close to their personal expected reward. If a person chooses red color and the drawing drum contains $n$ balls then, as a rule, her personal expected rewards are such that $R_{m_\rho,\alpha}(g^r) = \frac{100}{n} \geq R_{m_\rho,\alpha}(g^w) = \frac{100(1-\alpha)(n-1)}{5n}$, i.e., $\alpha \geq \frac{5}{n-1}$. In other words, if a person bets on red color for $n = 5, 6, 7$, and on other colors for $n \geq 8$, it is justifiable to assume that it holds $\alpha \in [0.167, 0.286]$ for the value of her personal coefficient of ambiguity aversion.

Table 1: Lottery Rn: Personal expected reward as a function of personal $\alpha$, and the number of balls $n$.

| | $R_{m_\rho,\alpha}(g^r)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ | $n=10$ | $n=11$ | $n=12$ |
| | 20.00 | 16.67 | 14.29 | 12.50 | 11.11 | 10.00 | 9.09 | 8.33 |
| | $R_{m_\rho,\alpha}(g^w)$ | | | | | | | |
| | $n=5$ | $n=6$ | $n=7$ | $n=8$ | $n=9$ | $n=10$ | $n=11$ | $n=12$ |
| $\alpha=-0.250$ | **20.00** | 20.83 | 21.43 | 21.86 | 22.22 | 22.50 | 22.28 | 22.92 |
| $\alpha=0.000$ | 16.00 | **16.67** | 17.14 | 17.50 | 17.78 | 18.00 | 18.18 | 18.33 |
| $\alpha=0.167$ | 13.33 | 13.89 | **14.29** | 14.58 | 14.82 | 15.00 | 15.15 | 15.28 |
| $\alpha=0.286$ | 11.43 | 11.91 | 12.25 | **12.50** | 12.70 | 12.86 | 12.99 | 13.10 |
| $\alpha=0.375$ | 10.00 | 10.42 | 10.71 | 10.94 | **11.11** | 11.25 | 11.34 | 11.46 |
| $\alpha=0.444$ | 8.89 | 9.26 | 9.52 | 9.72 | 9.88 | **10.00** | 10.10 | 10.18 |
| $\alpha=0.500$ | 8.00 | 8.33 | 8.57 | 8.75 | 8.89 | 9.00 | **9.09** | 9.17 |
| $\alpha=0.545$ | 7.27 | 7.58 | 7.79 | 7.95 | 8.08 | 8.18 | 8.26 | **8.33** |

Some of the values of functions $R_{m_\rho,\alpha}(g^r)$ and $R_{m_\rho,\alpha}(g^w)$ are tabulated in Table 1. From this Table we see that, for example, a person with $\alpha=0.3$ should bet on red color for $n\leq 8$, because for these $R_{m_\rho,\alpha}(g^r)>R_{m_\rho,\alpha}(g^x)$ $(x\neq r)$, and bet on any other color for $n\geq 9$, because for these $n$, $R_{m_\rho,\alpha}(g^r)\leq R_{m_\rho,\alpha}(g^x)$ $(x\neq r)$.

Before closing this Section, let us repeat that also in our experiments we found participants "seeking for ambiguity", which is in agreement with observations of other authors [9]. For these respondents, the coefficient of ambiguity aversion is negative, and therefore we included in Table 1 a row for $\alpha=-0.25$.

## 6. Ellsberg Example

In this Section, we want to show that the described model corresponds to what is observed by other authors. The following situations, which are modifications of the original Ellsberg's experiments ([12], pp. 653–654), are included among our experiments.

**E1** The drawing urn contains 15 red, black, and yellow balls. You know that exactly 5 of them are red, you do not know the proportion of the remaining black and yellow balls. How much is the maximum bet you are willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball has the color of your choice?

**E2** The drawing urn contains 15 red, black, and yellow balls. You know that exactly 5 of them are red, you do not know the proportion of the remaining black and yellow balls. How much is the maximum bet you are willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is either yellow or has the color of your choice?

The reader familiar with the cited Ellsberg's paper can see the difference between his experiments and those described here as E1 and E2. The primary goal of our experiments is focused on the question of measuring the strength of the ambiguity aversion. Ellsberg proved just its existence. He asked the respondents which of the two lotteries they prefer. We believe that our respondents will express their preference by paying more to participate in preferable lotteries.

For situations E1 and E2 consider $\Omega=\{r,b,y\}$, and the basic probability assignment

$$m_\varepsilon(\mathbf{a})=\begin{cases}\frac{1}{3}, & \text{if } \mathbf{a}=\{r\};\\ \frac{2}{3}, & \text{if } \mathbf{a}=\{b,y\};\\ 0, & \text{otherwise.}\end{cases}$$

The Shenoy's transform (as well as all other probability transforms from Section 3) yields a uniform probability measure $Sh\_P_{m_\varepsilon}(x) = \frac{1}{3}$ for all $x \in \Omega$. The corresponding belief function is $Bel_{m_\varepsilon}(\{r\}) = \frac{1}{3}$, and $Bel_{m_\varepsilon}(\{b\}) = Bel_{m_\varepsilon}(\{y\}) = 0$, $Bel_{m_\varepsilon}(\{r,b\}) = Bel_{m_\varepsilon}(\{r,y\}) = \frac{1}{3}$, $Bel_{m_\varepsilon}(\{b,y\}) = \frac{2}{3}$, and $Bel_{m_\varepsilon}(\Omega) = 1$. Therefore,

$$r_{m_\varepsilon,\alpha}(\mathbf{a}) = \begin{cases} \frac{1}{3}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{(1-\alpha)}{3}, & \text{for } \mathbf{a} = \{b\}, \{y\}; \\ \frac{(2-\alpha)}{3}, & \text{for } \mathbf{a} = \{r,b\}, \{r,y\}; \\ \frac{2}{3}, & \text{if } \mathbf{a} = \{b,y\}. \end{cases}$$

For E1, we consider two gain functions: $g^r(x)$ and $g^b(x)$ for betting on red and black balls, respectively. These functions are as follows:

$$g^r(r) = 100, \;\; g^r(b) = g^r(y) = 0,$$
$$g^b(b) = 100, \;\; g^b(r) = g^b(y) = 0.$$

Using Formula (13), the personal expected reward for betting on a red ball is

$$R_{m_\varepsilon,\alpha}(g^r) = 100\; r_{m_\varepsilon,\alpha}((g^r)^{-1}(100)) = 100\; r_{m_\varepsilon,\alpha}(\{r\}) = \frac{100}{3},$$

and analogously, for betting on a black ball the personal expected reward is

$$R_{m_\varepsilon,\alpha}(g^b) = 100\; r_{m_\varepsilon,\alpha}((g^b)^{-1}(100)) = 100\; r_{m_\varepsilon,\alpha}(\{b\}) = \frac{100(1-\alpha)}{3}.$$

For positive $\alpha$ (i.e., for a person with ambiguity aversion), we thus get $R_{m_\varepsilon,\alpha}(r) > R_{m_\varepsilon,\alpha}(b)$, which is consistent with Ellsberg's observation that "very frequent pattern of response is that betting on red is preferred to betting on black" [13].

Let us consider lottery E2, which involves betting on a couple of colors. In comparison with the first experiment, the situation changes only in the respective gain functions. Denote them by $g^{ry}(x)$ and $g^{by}(x)$ for betting on red and yellow, and for betting on black and yellow balls, respectively:

$$g^{ry}(r) = g^{ry}(y) = 100, \;\; g^{ry}(b) = 0,$$
$$g^{by}(b) = g^{by}(y) = 100, \;\; g^{by}(r) = 0.$$

The expected subjective rewards are then as follows:

$$R_{m_\varepsilon,\alpha}(g^{ry}) = 100\; r_{m_\varepsilon,\alpha}((g^{ry})^{-1}(100)) = 100\; r_{m_\varepsilon,\alpha}(\{r,y\}) = 100\, \frac{(2-\alpha)}{3},$$

$$R_{m_\varepsilon,\alpha}(g^{by}) = 100\; r_{m_\varepsilon,\alpha}((g^{by})^{-1}(100)) = 100\; r_{m_\varepsilon,\alpha}(\{b,y\}) = 100\, \frac{2}{3}.$$

In this setup, we observe that, for positive $\alpha$, $R_{m_\varepsilon,\alpha}(g^{by}) > R_{m_\varepsilon,\alpha}(g^{ry})$ holds, which is consistent with Ellsberg's observations that "betting on black and yellow is preferred to betting on red and yellow balls" [13].

## 7. Experimental Results

In the preceding Sections, we have described a mathematical model showing the same characteristics as those observed by other authors performing experiments with human decision-makers. Naturally, it does not mean that the behavior of an individual decision-maker corresponds to this model. To support or refute such hypothesis, we organized experimental sessions where the participants (usually faculty staff and students) were asked to answer the above-presented questions F1, F2, R5, R6, R7, R8, R9, R10, R11, R12, E1, E2, and the following two questions G1, G2 (similar to F1, F2, respectively).

**G1** The drawing urn contains 30 balls, five of each of the following colors: red, black, yellow, white, green, and azure. How much is the maximum bet you are willing to pay to take part in the lottery in which you get 100 CZK if the randomly drawn ball is red?

**G2** The drawing urn contains an unknown number (but at least one) of balls. They may be of the following six colors: red, black, yellow, white, green, and azure. You know nothing more, you do not even know how many colors are in the urn. How much is the maximum bet you are willing to pay to take part in the lottery in which you get 100 CZK if the randomly drawn ball is red?

The participants first answered the questions, and all the lotteries were realized only afterwards. At the very beginning of each session, the participants received a 50 CZK show-up payment, and were informed about the goal of the research, and that not all of them would be allowed to take part in all the lotteries. Before each draw, the computer selects participants admitted to taking part in the respective draw; one person at random and several others (about 20 % of the number of participants in the session) who had bet the highest amounts. This rule should make the participants really offer what they were asked for: "the maximum amount they are willing to pay to take part in the lottery". If they offered a deliberately small amount, they would willfully decrease their chances to take part in the game, and consequently they would decrease their chances to win money.

In eleven sessions, we obtained data from 192 participants. We have to admit that not all the participants provided a serious reflection of their behavior. For example, there was a person who bet 0 CZK on red in all 14 situations. In total, 8 persons always bet on red. When doing a detailed analysis, one can also reveal other "strange" patterns of behavior. To avoid the temptation to delete those participants who contradict our model, in what follows we describe the observations based on analyzing all the obtained 192 data records (and deleting only those whose behavior did not allow the respective computations).

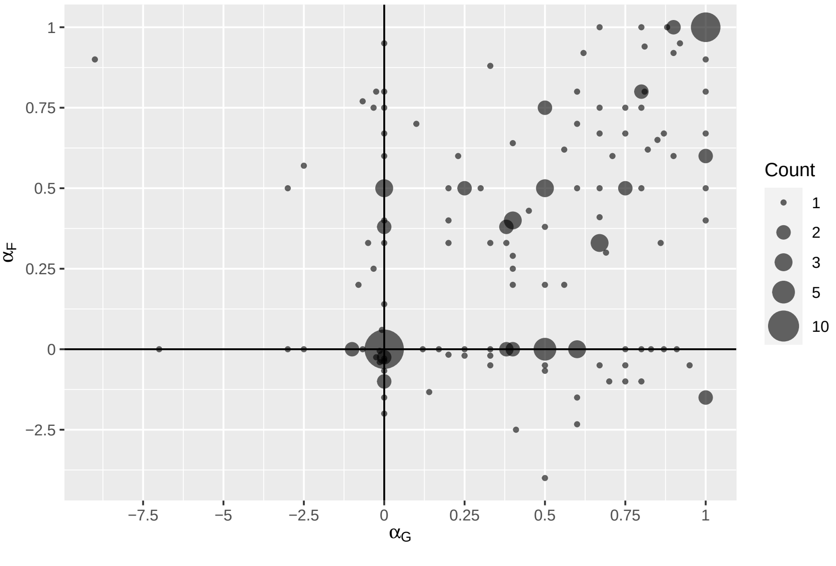


Figure 2: Comparison of $\alpha_F$ and $\alpha_G$.

There are two reasons why we included both the pairs of questions F1, F2 and G1, G2. First, we wanted to find out whether, under the ambiguity, the respondents are willing to bet more in the case that they may determine the winning color themselves than in the case when the winning color is predetermined. It appears that the average bet in situation F2 is 7.66 CZK, while in situation G2, it is 6.87 CZK. The difference is small and thus we consider it negligible. Second, we got two pairs of bets, based on which we can determine the personal coefficient of ambiguity

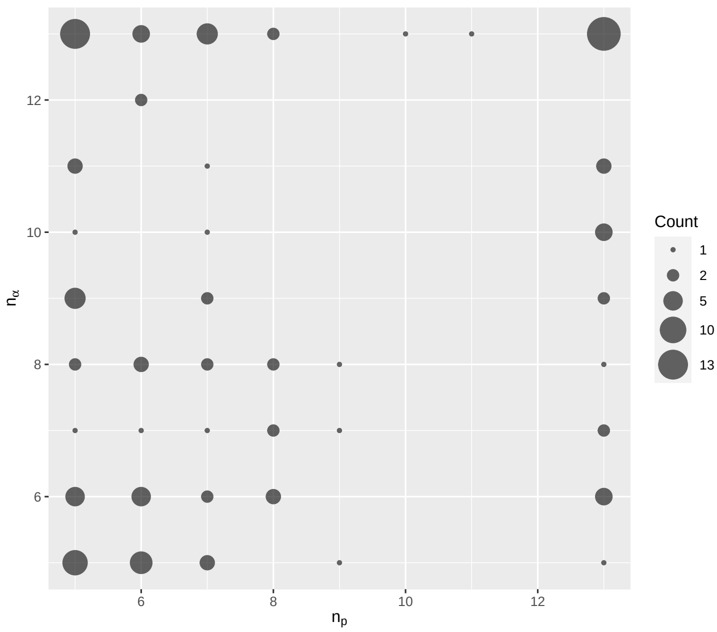


Figure 3: Comparison of $n_\alpha$ and $n_p$

aversion $\alpha$. Thus, the coefficients computed from bets in situations F1 and F2 are denoted by $\alpha_F$, and analogously those computed from bets corresponding to G1 and G2 are denoted by $\alpha_G$. Let us realize that 13 respondents bet 0 CZK on either F1 or G1, so at least one of the coefficients (either $\alpha_F$ or $\alpha_G$, or both) could not be computed for these respondents. In Figure 2 there are 179 points, each with the coordinates corresponding to $\alpha_F$ and $\alpha_G$ of one experimental subject. At first sight, one can see that we cannot claim that the behavior of our subject evinces the stability of the intensity of ambiguity aversion. At this point, let us also remark that the fact that the coefficient $\alpha$ is negative for some of the participants corresponds to the observations of other authors, too. Namely, it is known that there are people seeking ambiguity [18] (see [39] for a nice survey). Some authors also observed that the intensity of ambiguity aversion depends on the type of the underlying problem, and the amount of a possible gain. For example, Crockett et al. [9] found that "typically individuals exhibit an aversion to ambiguity when facing likely events, and a love for ambiguity when facing unlikely events". These observations, in connection with the fact that the amount of 50 CZK is not high enough to make the experiment participants more careful, well explain the instability of the coefficient of ambiguity aversion.

Recall now that in Section 5, two possibilities of measuring the personal coefficient alpha were proposed. To compare their results we proceeded as follows. First, for each experimental person, we considered the average coefficient $\alpha = \frac{\alpha_F + \alpha_G}{2}$ according to which we estimated (using Table 1) the smallest $n$ for which the person should start betting on other colors than red in situations R5, R6, R7, R8, R9, R10, R11, R12. This value is denoted by $n_\alpha$. Thus, for subjects with negative $\alpha$ values, $n_\alpha = 5$, and for subjects with $\alpha \geq 0.545$, $n_\alpha = 13$. It means that we expect a person with a negative $\alpha$ value to bet on other colors than red in all situations R5 – R13, contrary to a person with $\alpha \geq 0.545$, who is expected to bet only on red color in all these situations. Naturally, a person with $\alpha = 0$ is expected to bet on red in situation R5, and on other colors in situations R7 – R13. For $n = 6$, we cannot predict her behavior because her personal expected gain is the same for all the colors. Nevertheless, we define $n_\alpha = 6$ for $\alpha = 0$.

Based on the instability of the ambiguity aversion intensity observed from Figure 2, it would be a great surprise if all the participants in our experiments behaved "rationally" in the sense that, for each of them, one could find $n_p$ such that she would bet on red for all R5, ... , R$n_p$, and bet on other colors in the remaining situations. Nevertheless, 128 (out of those 179 subjects for whom we could compute $\alpha = \frac{\alpha_F + \alpha_G}{2}$) behaved in this "rational" way in our experiments. In Figure 3 we depict the comparison of the following two parameters: $n_\alpha$ is computed in the above-described way, and $n_p$ is observed from the data. To compare them, we again represent each of the 128 participants as a point in

Figure 3. Let us say that only 34 points are directly on the diagonal (i.e., $n_\alpha = n_p$), and for 20 more subjects these two parameters differ just by one, which is considered the full correspondence, because for $\alpha$ close to the breaking points (as e.g., $\alpha$ close to zero) the model does not determine the subject's behavior (the subject can choose an arbitrary color) because of the same values of the personal expected gain. With a small indulgence, we consider the behavior of a subject fitting the model even if $|n_\alpha - n_p| \leq 2$, which holds for 71 out of the above-mentioned 128 subjects.

Let us briefly summarize our observations based on the data. We cannot say that we could estimate a personal coefficient of ambiguity aversion just from one experimental session and that the knowledge of the personal ambiguity aversion coefficient would enable predicting the behavior in decision problems under ambiguity. Nevertheless, the model well describes the collective (general) behavior of human decision-makers. In our experiments, the average coefficient of ambiguity aversion $\bar{\alpha} = 0.2133$, which means that one can expect the subjects to bet on red color in situations R5, R6, R7, and on non-red colors in the remaining situations. This well corresponds with $\overline{n_\alpha} = 7.969$.

## 8. Conclusions

In this paper, we have introduced a belief function model manifesting an ambiguity aversion similar to human decision-makers. The intensity of this aversion is expressed by the personal coefficient ambiguity $\alpha$ with the following semantics: the higher the aversion, the higher the coefficient. As one can see from Figure 2, it may be negative for decision-makers seeking ambiguity. This attitude is observed also by other authors [9, 18, 39]. Surprisingly, the participants in our experiments did not follow the behavior described by Ellsberg in [12]. Namely, almost two-thirds of respondents bet on the red ball even in lottery E2 in our experiments.

We can hardly make any definite conclusions from the analysis of the data gained in our experiments. It may have been influenced by the fact that the reward of 50 CZK is smaller than the rewards paid by other authors. This also explains the fact that uncommonly many respondents exhibited a positive attitude to ambiguity and, perhaps, also the observed instability of this attitude visible from Figure 2.

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## References

[1] D. Aerts, S. Geriente, C. Moreira, and S. Sozzo. Testing ambiguity and machina preferences within a quantum-theoretic framework for decision-making. *Journal of Mathematical Economics*, 78:176–185, 2018.

[2] P. Battigalli, S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci. A note on comparative ambiguity aversion and justifiability. *Econometrica*, 84(5):1903–1916, 2016.

[3] A. Bronevich and G. J. Klir. Measures of uncertainty for imprecise probabilities: an axiomatic approach. *International journal of approximate reasoning*, 51(4):365–390, 2010.

[4] C. Camerer and M. Weber. Recent developments in modeling preferences: Uncertainty and ambiguity. *Journal of risk and uncertainty*, 5(4):325–370, 1992.

[5] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Probabilistic sophistication, second order stochastic dominance and uncertainty aversion. *Journal of Mathematical Economics*, 48(5):271–283, 2012.

[6] G. Choquet. Theory of capacities. In *Annales de l'institut Fourier*, volume 5, pages 131–295, 1954.

[7] B. R. Cobb and P. P. Shenoy. On the plausibility transformation method for translating belief function models to probability models. *International Journal of Approximate Reasoning*, 41(3):314–340, 2006.

[8] G. Coletti, D. Petturiti, and B. Vantaggi. Rationality principles for preferences on belief functions. *Kybernetika*, 51(3):486–507, 2015.

[9] S. Crockett, Y. Izhakian, and J. Jamison. Ellsberg's hidden paradox. *SSRN eLibrary*, 2019. 3423534.

[10] F. Cuzzolin. On the relative belief transform. *International Journal of Approximate Reasoning*, 53(5):786–804, 2012.

[11] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38(2):325–339, 1967.

[12] D. Ellsberg. Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics*, 75(2):643–669, 1961.

[13] D. Ellsberg. *Risk, ambiguity and decision*. Routledge, 2015.

[14] I. Gilboa and M. Marinacci. Ambiguity and the bayesian paradigm. In *Readings in formal epistemology*, pages 385–439. Springer, 2016.

[15] Y. Halevy. Ellsberg revisited: An experimental study. *Econometrica*, 75(2):503–536, 2007.

[16] J. Y. Halpern and R. Fagin. Two views of belief: Belief as generalized probability and belief as evidence. *Artificial Intelligence*, 54(3):275–317, 1992.

[17] L. Hurwicz. The generalized bayes minimax principle: a criterion for decision making under uncertainty. *Cowles Comm. Discuss. Paper Stat*, 335:1950, 1951.
[18] Y. Izhakian. Expected utility with uncertain probabilities theory. *Journal of Mathematical Economics*, 69:91–103, 2017.
[19] J.-Y. Jaffray. Linear utility theory for belief functions. *Operations Research Letters*, 8(2):107–112, 1989.
[20] R. Jiroušek and V. Kratochvíl. On expected utility under ambiguity. In *Symbolic and Quantitative Approaches to Reasoning with Uncertainty: 15th European Conference, ECSQARU 2019, Belgrade, Serbia, September 18-20, 2019, Proceedings*, volume 11726, pages 137–147. Springer Nature, 2019.
[21] R. Jiroušek, V. Kratochvíl, and J. Rauh. A note on the approximation of shenoy's expectation operator using probabilistic transforms. *International Journal of General Systems*, 49(1):48–63, 2020.
[22] R. Jiroušek and P. P. Shenoy. Ambiguity aversion and a decision-theoretic framework using belief functions. In *2017 IEEE Symposium Series on Computational Intelligence (SSCI)*, pages 1–7. IEEE, 2017.
[23] A.-L. Jousselme, C. Liu, D. Grenier, and É. Bossé. Measuring ambiguity in the evidence theory. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, 36(5):890–903, 2006.
[24] B. E. Kahn and R. K. Sarin. Modeling ambiguity in decisions under uncertainty. *Journal of consumer Research*, 15(2):265–272, 1988.
[25] G. J. Klir and H. W. Lewis. Remarks on "measuring ambiguity in the evidence theory". *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, 38(4):995–999, 2008.
[26] S. Kullback and R. A. Leibler. On information and sufficiency. *Annals of Mathematical Statistics*, 22(1):79–86, 1951.
[27] M. Lang. First-order and second-order ambiguity aversion. *Management Science*, 63(4):1254–1269, 2017.
[28] A. Perez. The barycenter concept of a set of probability measures as a tool in statistical decision. In *Probability Theory and Mathematical Statistics*, volume 2, pages 437–450. VNU Science Press, 1986.
[29] L. J. Savage. *The foundations of statistics*. Courier Corporation, 1972.
[30] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.
[31] A. Shahpari and S. A. Seyedin. Using mutual aggregate uncertainty measures in a threat assessment problem constructed by dempster–shafer network. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 45(6):877–886, 2014.
[32] P. P. Shenoy. An expectation operator for belief functions in the dempster–shafer theory. *International Journal of General Systems*, 49(1):112–141, 2020.
[33] P. Smets. The degree of belief in a fuzzy event. *Information sciences*, 25(1):1–19, 1981.
[34] P. Smets. Constructing the pignistic probability function in a context of uncertainty. In *UAI*, volume 89, pages 29–40, 1989.
[35] P. Smets. Decision making in a context where uncertainty is represented by belief functions. In R. P. Srivastava and T. J. Mock, editors, *Belief Functions in Business Decisions*, pages 316–332. Physica-Verlag, Heidelberg, 2002.
[36] P. Smets. Decision making in the tbm: the necessity of the pignistic transformation. *International Journal of Approximate Reasoning*, 38(2):133–147, 2005.
[37] P. Smets and R. Kennes. The transferable belief model. *Artificial Intelligence*, 66(2):191–234, 1994.
[38] R. P. Srivastava. Decision making under ambiguity: A belief-function perspective. *Archives of Control Sciences*, 6(XLII):5–27, 1997.
[39] S. T. Trautmann and G. Van De Kuilen. Ambiguity attitudes. *The Wiley Blackwell handbook of judgment and decision making*, 1:89–116, 2015.