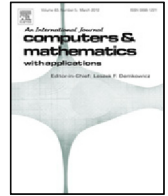




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# Poincaré–Friedrichs type constants for operators involving grad, curl, and div: Theory and numerical experiments

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## ABSTRACT

We give some theoretical as well as computational results on Laplace and Maxwell constants, i.e., on the smallest constants  $c_n > 0$  arising in estimates of the form

$$|u|_{L^2(\Omega)} \leq c_0 |\text{grad } u|_{L^2(\Omega)}, \quad |E|_{L^2(\Omega)} \leq c_1 |\text{curl } E|_{L^2(\Omega)}, \quad |H|_{L^2(\Omega)} \leq c_2 |\text{div } H|_{L^2(\Omega)}.$$

Besides the classical de Rham complex we investigate the complex of elasticity and the complex related to the biharmonic equation and general relativity as well using the general functional analytical concept of Hilbert complexes. We consider mixed boundary conditions and bounded Lipschitz domains of arbitrary topology. Our numerical aspects are presented by examples for the de Rham complex in 2D and 3D which not only confirm our theoretical findings but also indicate some interesting conjectures.

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## 1. Introduction

We present some theoretical results as well as some computations on Laplace and Maxwell constants for bounded Lipschitz domains  $\Omega$  with mixed boundary conditions defined on boundary parts  $\Gamma_\tau$  and  $\Gamma_\nu$  of the boundary  $\Gamma$ . While a lot of our theoretical findings hold for domains  $\Omega$  in arbitrary dimensions, we restrict our numerical experiments to the 2D and 3D cases. Moreover, we verify various theoretical results established in the last few years in [1–4]. There is a recent interest in these eigenvalues, see, e.g., [5–7] and related contributions [8–14], but little results for mixed boundary conditions are known in the literature, except for, e.g., [3,4]. In 3D these constants are the best possible real numbers  $c_{0,\Gamma_\tau}, c_{1,\Gamma_\tau}, c_{2,\Gamma_\tau} > 0$  in the estimates

$$\begin{aligned} \forall u &\in D(\text{grad}_{\Gamma_\tau}) \cap R(\text{div}_{\Gamma_\nu}) & |u|_{L^2(\Omega)} &\leq c_{0,\Gamma_\tau} |\text{grad } u|_{L^2(\Omega)}, \\ \forall E &\in D(\text{curl}_{\Gamma_\tau}) \cap R(\text{curl}_{\Gamma_\nu}) & |E|_{L^2(\Omega)} &\leq c_{1,\Gamma_\tau} |\text{curl } E|_{L^2(\Omega)}, \\ \forall H &\in D(\text{div}_{\Gamma_\tau}) \cap R(\text{grad}_{\Gamma_\nu}) & |H|_{L^2(\Omega)} &\leq c_{2,\Gamma_\tau} |\text{div } H|_{L^2(\Omega)}, \end{aligned}$$

which are often called Poincaré–Friedrichs type constants, cf. Section 2.2 for notations. More precisely, we have

$$\frac{1}{c_{0,\Gamma_\tau}} = \inf_{0 \neq u \in D(\text{grad}_{\Gamma_\tau}) \cap R(\text{div}_{\Gamma_\nu})} \frac{|\text{grad } u|_{L^2(\Omega)}}{|u|_{L^2(\Omega)}} \quad (\text{Friedrichs/Poincaré constants}),$$

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$$\begin{aligned} \frac{1}{c_{1,\Gamma_\tau}} &= \inf_{0 \neq E \in D(\operatorname{curl}_{\Gamma_\tau}) \cap R(\operatorname{curl}_{\Gamma_\nu})} \frac{|\operatorname{curl} E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}} && \text{(rotation/Maxwell constants),} \\ \frac{1}{c_{2,\Gamma_\tau}} &= \inf_{0 \neq H \in D(\operatorname{div}_{\Gamma_\tau}) \cap R(\operatorname{grad}_{\Gamma_\nu})} \frac{|\operatorname{div} H|_{L^2(\Omega)}}{|H|_{L^2(\Omega)}} && \text{(divergence constants).} \end{aligned}$$

We also point out the strong connection to the well known de Rham complex with mixed boundary conditions. It turns out that in the 3D case, cf. [Theorem 2.20](#), the estimates and equations

$$c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}, \quad c_{1,\Gamma} = c_{1,\Gamma_\nu}, \quad c_{0,\Gamma} \leq \min\left\{c_{0,\emptyset}, \frac{\operatorname{diam}(\Omega)}{\pi}\right\}$$

always hold and that in convex domains we even have

$$c_{1,\Gamma} = c_{1,\emptyset} \leq c_{0,\emptyset} \leq \frac{\operatorname{diam}(\Omega)}{\pi}.$$

Here,

$$c_F := c_{0,\Gamma}, \quad c_P := c_{0,\emptyset}$$

are the classical Friedrichs and Poincaré constants, respectively, and the constants

$$c_{M,t} := c_{1,\Gamma} = c_{1,\emptyset} =: c_{M,n}$$

are often called tangential (electric) and normal (magnetic) Maxwell constants, respectively. All these constants relate to minimal positive eigenvalues of certain Laplace and Maxwell operators. More precisely,

$$\lambda_{0,\Gamma_\tau} = \frac{1}{c_{0,\Gamma_\tau}}, \quad \lambda_{1,\Gamma_\tau} = \frac{1}{c_{1,\Gamma_\tau}}, \quad \lambda_{2,\Gamma_\tau} = \frac{1}{c_{2,\Gamma_\tau}}$$

are the smallest positive eigenvalues of the first order matrix operators

$$\begin{bmatrix} 0 & -\operatorname{div}_{\Gamma_\nu} \\ \operatorname{grad}_{\Gamma_\tau} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \operatorname{curl}_{\Gamma_\nu} \\ \operatorname{curl}_{\Gamma_\tau} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\operatorname{grad}_{\Gamma_\nu} \\ \operatorname{div}_{\Gamma_\tau} & 0 \end{bmatrix},$$

respectively, and

$$\lambda_{0,\Gamma_\tau}^2, \quad \lambda_{1,\Gamma_\tau}^2, \quad \lambda_{2,\Gamma_\tau}^2 \tag{1}$$

are the smallest positive eigenvalues of the second order operators

$$-\operatorname{div}_{\Gamma_\nu} \operatorname{grad}_{\Gamma_\tau}, \quad \operatorname{curl}_{\Gamma_\nu} \operatorname{curl}_{\Gamma_\tau}, \quad -\operatorname{grad}_{\Gamma_\nu} \operatorname{div}_{\Gamma_\tau}, \tag{2}$$

respectively. In 2D, cf. [Corollary 2.23](#), we will see that

$$c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{1,\Gamma_\nu} = c_{2,\Gamma_\nu}, \quad c_{0,\Gamma} \leq \min\left\{c_{0,\emptyset}, \frac{\operatorname{diam}(\Omega)}{\pi}\right\}.$$

Generally, in ND, cf. [Theorem 2.25](#), we have

$$c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{N-1,\Gamma_\nu}, \quad c_{q,\Gamma_\tau} = c_{N-q-1,\Gamma_\nu}, \quad c_{0,\Gamma} \leq \min\left\{c_{0,\emptyset}, \frac{\operatorname{diam}(\Omega)}{\pi}\right\}$$

and in convex domains

$$c_{q,\Gamma} = c_{N-q-1,\emptyset} \leq c_{0,\emptyset} \leq \frac{\operatorname{diam}(\Omega)}{\pi}.$$

Here,  $q = 0, \dots, N-1$  and the differential operators  $\operatorname{grad}$ ,  $\operatorname{curl}$ , and  $\operatorname{div}$  are simply replaced by the exterior derivative  $d_q$  acting on the rank  $q$  of the respective differential form. So far, all findings are related to the ND de Rham complex. We will present more examples and results for the 3D elasticity complex as well as for the 3D biharmonic complex.

In a series of numerical tests we discretize the operators (2) by the finite element method and compute upper bounds for the eigenvalues (1) from generalized eigenvalue systems

$$Ku = \lambda^2 Mu$$

with discretized stiffness and mass matrices  $K$  and  $M$ , respectively. There are also recent interests in guaranteed lower bounds, cf. [15–17]. In a search for the smallest positive eigenvalue  $\lambda^2$  we exploit a projection into the range of  $K$  for smaller size problems or the nested iteration technique for large size problems. The latter theoretical results are confirmed by computations in 2D for the unit square and the unit L-shape domain as well as in 3D for the unit cube and the unit Fichera corner domain. Note that all three constants  $c_{\ell,\Gamma_\tau}$ ,  $\ell = 1, 2, 3$ , grow proportionally to the “radius” of the domain. More precisely, e.g., it holds

$$c_\ell(r \cdot \Omega, r \cdot \Gamma_\tau) = r \cdot c_\ell(\Omega, \Gamma_\tau) = r \cdot c_{\ell,\Gamma_\tau},$$

where

$$r \cdot \Omega := \{r \cdot x : x \in \Omega\}, \quad r \cdot \Gamma_r := \{r \cdot x : x \in \Gamma_r\}$$

for some bounded domain  $\Omega \subset \mathbb{R}^3$  being star-shaped with respect to the origin. Moreover, we performed some monotonicity tests (with respect to boundary conditions) which are just partially guaranteed by our theoretical findings. To our surprise we found (numerically) much stronger inequalities in (26), see also the related Fig. 7, giving rise to some interesting conjectures.

## 2. Theoretical results

We shall summarize some basic results from functional analysis and apply those to the classical operators of vector analysis.

### 2.1. Functional analysis toolbox

We start with collecting and citing some results from [3,4,18–20] about the so-called functional analysis toolbox (fa-toolbox).

#### 2.1.1. Preliminaries

Let  $A : D(A) \subset H_0 \rightarrow H_1$  be a densely defined and closed linear operator with domain of definition  $D(A)$  on two Hilbert spaces  $H_0$  and  $H_1$ . Then the adjoint  $A^* : D(A^*) \subset H_1 \rightarrow H_0$  is well defined and characterized by

$$\forall x \in D(A) \quad \forall y \in D(A^*) \quad \langle Ax, y \rangle_{H_1} = \langle x, A^* y \rangle_{H_0}.$$

$A$  and  $A^*$  are both densely defined and closed, but typically unbounded. Often  $(A, A^*)$  is called a dual pair as  $(A^*)^* = \bar{A} = A$ . The projection theorem shows

$$H_0 = N(A) \oplus_{H_0} \overline{R(A^*)}, \quad H_1 = N(A^*) \oplus_{H_1} \overline{R(A)}, \quad (3)$$

often called Helmholtz/Hodge/Weyl decompositions, where we introduce the notation  $N$  for the kernel (or null space) and  $R$  for the range of a linear operator. These orthogonal decompositions reduce the operators  $A$  and  $A^*$ , leading to the injective operators  $\mathcal{A} := A|_{\overline{R(A^*)}}$  and  $\mathcal{A}^* := A^*|_{\overline{R(A)}}$ , i.e.

$$\begin{aligned} \mathcal{A} : D(\mathcal{A}) \subset \overline{R(A^*)} &\rightarrow \overline{R(A)}, & D(\mathcal{A}) &= D(A) \cap \overline{R(A^*)}, \\ \mathcal{A}^* : D(\mathcal{A}^*) \subset \overline{R(A)} &\rightarrow \overline{R(A^*)}, & D(\mathcal{A}^*) &= D(A^*) \cap \overline{R(A)}. \end{aligned}$$

Note that

$$\overline{R(A^*)} = N(A)^{\perp_{H_0}}, \quad \overline{R(A)} = N(A^*)^{\perp_{H_1}}$$

and that  $\mathcal{A}$  and  $\mathcal{A}^*$  are indeed adjoint to each other, i.e.,  $(\mathcal{A}, \mathcal{A}^*)$  is a dual pair as well. Then the inverse operators

$$\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A}), \quad (\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$$

are well defined and bijective, but possibly unbounded. Furthermore, by (3) we have the refined Helmholtz type decompositions

$$D(A) = N(A) \oplus_{H_0} D(\mathcal{A}), \quad D(A^*) = N(A^*) \oplus_{H_1} D(\mathcal{A}^*) \quad (4)$$

and thus we obtain for the ranges

$$R(A) = R(\mathcal{A}), \quad R(A^*) = R(\mathcal{A}^*).$$

#### 2.1.2. Basic results

The following result is a well known and direct consequence of the closed graph theorem and the closed range theorem.

**Lemma 2.1** (fa-toolbox Lemma 1). *The following assertions are equivalent:*

- (i)  $\exists c_A \in (0, \infty) \quad \forall x \in D(\mathcal{A}) \quad |x|_{H_0} \leq c_A |Ax|_{H_1}$
- (i\*)  $\exists c_{A^*} \in (0, \infty) \quad \forall y \in D(\mathcal{A}^*) \quad |y|_{H_1} \leq c_{A^*} |A^*y|_{H_0}$
- (ii)  $R(A) = R(\mathcal{A})$  is closed in  $H_1$ .
- (ii\*)  $R(A^*) = R(\mathcal{A}^*)$  is closed in  $H_0$ .
- (iii)  $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow D(\mathcal{A})$  is bounded by  $c_A$ .
- (iii\*)  $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow D(\mathcal{A}^*)$  is bounded by  $c_{A^*}$ .
- (iv)  $\mathcal{A} : D(\mathcal{A}) \subset R(\mathcal{A}^*) \rightarrow R(A)$  is bijective with continuous inverse.

- (iv\*)  $\mathcal{A}^* : D(\mathcal{A}^*) \subset R(\mathcal{A}) \rightarrow R(\mathcal{A}^*)$  is bijective with continuous inverse.
- (v)  $\mathcal{A} : D(\mathcal{A}) \rightarrow R(\mathcal{A})$  is a topological isomorphism.
- (v\*)  $\mathcal{A}^* : D(\mathcal{A}^*) \rightarrow R(\mathcal{A}^*)$  is a topological isomorphism.

The latter inequalities will be called Poincaré–Friedrichs type estimates. Note that in (iv) and (iv\*) we consider  $\mathcal{A}$  and  $\mathcal{A}^*$  as unbounded linear operators, whereas in (v) and (v\*) we consider  $\mathcal{A}$  and  $\mathcal{A}^*$  as bounded linear operators.

**Lemma 2.2** (fa-toolbox Lemma 2). *The following assertions are equivalent:*

- (i)  $D(\mathcal{A}) \hookrightarrow H_0$  is compact.
- (i\*)  $D(\mathcal{A}^*) \hookrightarrow H_1$  is compact.
- (ii)  $\mathcal{A}^{-1} : R(\mathcal{A}) \rightarrow R(\mathcal{A}^*)$  is compact.
- (ii\*)  $(\mathcal{A}^*)^{-1} : R(\mathcal{A}^*) \rightarrow R(\mathcal{A})$  is compact.

Moreover: Each of these assumptions imply the assertions of [Lemma 2.1](#) (and of [Lemma 2.2](#)).

**Remark 2.3** (Sufficient Assumptions for the Fa-Toolbox).

- (i) If  $R(\mathcal{A})$  is closed, then the assertions of [Lemma 2.1](#) hold.
- (ii) If  $D(\mathcal{A}) \hookrightarrow H_0$  is compact, then the assertions of [Lemmas 2.1](#) and [2.2](#) hold. In particular, the Poincaré–Friedrichs type estimates hold, all ranges are closed and the inverse operators are compact.

### 2.1.3. Constants, Spectra, and Eigenvalues

Let us introduce the “best” constants  $c_A, c_{A^*}$  by utilizing the Rayleigh quotients

$$\frac{1}{c_A} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}}, \quad \frac{1}{c_{A^*}} := \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^*y|_{H_0}}{|y|_{H_1}}. \quad (5)$$

Then  $0 < c_A, c_{A^*} \leq \infty$  and we refer to  $c_A$  and  $c_{A^*}$  as Poincaré–Friedrichs type constants. From now on, we assume that we always deal with these best constants.

**Lemma 2.4** (Constant Lemma). *The Poincaré–Friedrichs type constants coincide, i.e.,  $c_A = c_{A^*}$ .*

In the case that  $R(\mathcal{A})$  is closed, we shall denote

$$\lambda_A := \frac{1}{c_A} = \frac{1}{c_{A^*}} > 0.$$

Let us emphasize that

$$A^*A, \quad AA^*, \quad |A|, \quad |A^*|, \quad \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}, \quad \begin{bmatrix} AA^* & 0 \\ 0 & A^*A \end{bmatrix}, \quad \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \quad (6)$$

are self-adjoint, see [Appendix A](#), and have essentially – except of 0 and taking square roots – the same spectra contained in  $\mathbb{R}$ . Moreover, the first four operators are non-negative. The same holds true for the reduced operators  $\mathcal{A}$  and  $\mathcal{A}^*$ . We will give more details in the next lemma.

**Lemma 2.5** (Constant And Eigenvalue Lemma). *Let  $D(\mathcal{A}) \hookrightarrow H_0$  be compact. Then the operators in (6) have pure and discrete point spectra with no accumulation point in  $\mathbb{R}$ . Moreover:*

- (i)  $\lambda_A$  is the smallest positive eigenvalue of  $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$  and of  $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ .
- (ii)  $\lambda_A^2$  is the smallest positive eigenvalue of  $A^*A$  and of  $AA^*$ .
- (iii)  $\lambda_A^2$  is the smallest positive eigenvalue of  $\begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}$  and of  $\begin{bmatrix} AA^* & 0 \\ 0 & A^*A \end{bmatrix}$ .
- (iv)  $\sigma(A^*A) \setminus \{0\} = \sigma(AA^*) \setminus \{0\} = \sigma\left(\begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}\right) \setminus \{0\} = \sigma\left(\begin{bmatrix} AA^* & 0 \\ 0 & A^*A \end{bmatrix}\right) \setminus \{0\} > 0$
- (v)  $\sigma\left(\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right) \setminus \{0\} = \sigma\left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right) \setminus \{0\} = \pm\sqrt{\sigma(A^*A) \setminus \{0\}}$
- (vi)  $\sigma(A^*A) \setminus \{0\} = \sigma(A^*A)$  and corresponding results hold for all other spectra in (iv).
- (vii)  $\sigma\left(\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right) \setminus \{0\} = \sigma\left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right)$  and corresponding results hold for all other spectra in (v).
- (viii)  $|\mathcal{A}^{-1}|_{R(\mathcal{A}), R(\mathcal{A}^*)} = |(\mathcal{A}^*)^{-1}|_{R(\mathcal{A}^*), R(\mathcal{A})} = c_A$
- (viii\*)  $|\mathcal{A}^{-1}|_{R(\mathcal{A}), D(\mathcal{A})} = |(\mathcal{A}^*)^{-1}|_{R(\mathcal{A}^*), D(\mathcal{A}^*)} = (c_A^2 + 1)^{1/2}$

- (ix)  $|(\mathcal{A}^* \mathcal{A})^{-1}|_{R(\mathcal{A}^*), R(\mathcal{A}^*)} = |(\mathcal{A} \mathcal{A}^*)^{-1}|_{R(\mathcal{A}), R(\mathcal{A})} = c_A^2$   
 (x)  $N(\mathcal{A}) = N(\mathcal{A}^* \mathcal{A}) = N(\mathcal{A} \mathcal{A}^*) = \dots$  and  $N(\mathcal{A}^*) = N(\mathcal{A} \mathcal{A}^*) = N(\mathcal{A}^* \mathcal{A} \mathcal{A}^*) = \dots$   
 (xi)  $R(\mathcal{A}) = R(\mathcal{A} \mathcal{A}^*) = R(\mathcal{A}^* \mathcal{A} \mathcal{A}^*) = \dots$  and  $R(\mathcal{A}^*) = R(\mathcal{A}^* \mathcal{A}) = R(\mathcal{A}^* \mathcal{A} \mathcal{A}^*) = \dots$  and the same holds for the operators  $\mathcal{A}$  and  $\mathcal{A}^*$ .

For a proof see [Appendix A](#).

**Remark 2.6** (Variational Formulations). By [Lemma 2.5](#) the infima in (5) are minima, provided that  $D(\mathcal{A}) \hookrightarrow H_0$  is compact. In particular, the respective minimizers  $x_A$  and  $y_A$  are the eigenvectors to the eigenvalue  $\lambda_A^2$ , i.e.,

$$\inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}} = |Ax_A|_{H_1} = \lambda_A = |A^* y_A|_{H_0} = \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^* y|_{H_0}}{|y|_{H_1}},$$

where we assume without loss of generality  $|x_A|_{H_0} = |y_A|_{H_1} = 1$ . Moreover,

$$\begin{aligned} (A^* A - \lambda_A^2) x_A &= 0, & x_A &\in D(A^* A) \cap R(A^*) = D(\mathcal{A}^* \mathcal{A}) \subset D(\mathcal{A}), \\ (A A^* - \lambda_A^2) y_A &= 0, & y_A &\in D(A A^*) \cap R(A) = D(\mathcal{A} \mathcal{A}^*) \subset D(\mathcal{A}^*), \end{aligned}$$

and the eigenvectors satisfy the variational formulations

$$\begin{aligned} \forall \phi \in D(\mathcal{A}) \quad \langle A x_A, A \phi \rangle_{H_1} &= \lambda_A^2 \langle x_A, \phi \rangle_{H_0}, \\ \forall \psi \in D(\mathcal{A}^*) \quad \langle A^* y_A, A^* \psi \rangle_{H_0} &= \lambda_A^2 \langle y_A, \psi \rangle_{H_1}. \end{aligned}$$

#### 2.1.4. Complex structure results

Now, let

$$A_0 : D(A_0) \subset H_0 \rightarrow H_1, \quad A_1 : D(A_1) \subset H_1 \rightarrow H_2$$

be two densely defined and closed linear operators on three Hilbert spaces  $H_0$ ,  $H_1$ , and  $H_2$  with adjoints

$$A_0^* : D(A_0^*) \subset H_1 \rightarrow H_0, \quad A_1^* : D(A_1^*) \subset H_2 \rightarrow H_1$$

as well as reduced operators  $\mathcal{A}_0$ ,  $\mathcal{A}_0^*$ , and  $\mathcal{A}_1$ ,  $\mathcal{A}_1^*$ . Furthermore, we assume the complex property (also called sequence property) of  $A_0$  and  $A_1$ , that is  $A_1 A_0 = 0$ , i.e.,

$$R(A_0) \subset N(A_1), \tag{7}$$

which is equivalent to  $A_0^* A_1^* = 0$ , i.e.,  $R(A_1^*) \subset N(A_0^*)$ . Recall that

$$R(A_0) = R(\mathcal{A}_0), \quad R(A_0^*) = R(\mathcal{A}_0^*), \quad R(A_1) = R(\mathcal{A}_1), \quad R(A_1^*) = R(\mathcal{A}_1^*).$$

From the Helmholtz type decompositions (3) for  $A = A_0$  and  $A = A_1$  we get in particular

$$H_1 = \overline{R(A_0)} \oplus_{H_1} N(A_0^*), \quad H_1 = \overline{R(A_1^*)} \oplus_{H_1} N(A_1). \tag{8}$$

Introducing the cohomology group

$$N_{0,1} := N(A_1) \cap N(A_0^*),$$

we obtain the refined Helmholtz type decompositions

$$\begin{aligned} N(A_1) &= \overline{R(A_0)} \oplus_{H_1} N_{0,1}, & N(A_0^*) &= \overline{R(A_1^*)} \oplus_{H_1} N_{0,1}, \\ D(A_1) &= \overline{R(A_0)} \oplus_{H_1} (D(A_1) \cap N(A_0^*)), & D(A_0^*) &= \overline{R(A_1^*)} \oplus_{H_1} (D(A_0^*) \cap N(A_1)), \end{aligned} \tag{9}$$

and therefore

$$H_1 = \overline{R(A_0)} \oplus_{H_1} N_{0,1} \oplus_{H_1} \overline{R(A_1^*)}. \tag{10}$$

Let us remark that the first line of (9) can also be written as

$$\overline{R(A_0)} = N(A_1) \cap N_{0,1}^{\perp}, \quad \overline{R(A_1^*)} = N(A_0^*) \cap N_{0,1}^{\perp}.$$

Note that (10) can be further refined and specialized, e.g., to

$$\begin{aligned} D(A_1) &= \overline{R(A_0)} \oplus_{H_1} N_{0,1} \oplus_{H_1} D(\mathcal{A}_1), \\ D(A_0^*) &= D(\mathcal{A}_0^*) \oplus_{H_1} N_{0,1} \oplus_{H_1} \overline{R(A_1^*)}, \\ D(A_1) \cap D(A_0^*) &= D(\mathcal{A}_0^*) \oplus_{H_1} N_{0,1} \oplus_{H_1} D(\mathcal{A}_1). \end{aligned} \tag{11}$$

We observe

$$\begin{aligned} D(\mathcal{A}_1) &= D(A_1) \cap \overline{R(A_1^*)} \subset D(A_1) \cap N(A_0^*) \subset D(A_1) \cap D(A_0^*), \\ D(\mathcal{A}_0^*) &= D(A_0^*) \cap \overline{R(A_0)} \subset D(A_0^*) \cap N(A_1) \subset D(A_0^*) \cap D(A_1), \end{aligned}$$

and using the refined Helmholtz type decompositions (10) and (11) as well as the results of Lemma 2.2 we immediately see:

**Lemma 2.7** (fa-toolbox Lemma 3). *The following assertions are equivalent:*

- (i)  $D(\mathcal{A}_0) \hookrightarrow H_0$ ,  $D(\mathcal{A}_1) \hookrightarrow H_1$ , and  $N_{0,1} \hookrightarrow H_1$  are compact.
- (ii)  $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$  is compact.

In this case, the cohomology group  $N_{0,1}$  has finite dimension.

We summarize:

**Theorem 2.8** (Fa-Toolbox Theorem). *Let the ranges  $R(A_0)$  and  $R(A_1)$  be closed. Then all ranges  $R(A_0)$ ,  $R(A_0^*)$ , and  $R(A_1)$ ,  $R(A_1^*)$  are closed, and the corresponding Poincaré–Friedrichs type estimates hold, i.e. there exists positive constants  $c_{A_0}$ ,  $c_{A_1}$  such that*

$$\begin{aligned} \forall z \in D(\mathcal{A}_0) &= D(A_0) \cap R(A_0^*) & |z|_{H_0} &\leq c_{A_0} |A_0 z|_{H_1}, \\ \forall x \in D(\mathcal{A}_0^*) &= D(A_0^*) \cap R(A_0) = D(A_0^*) \cap N(A_1) \cap N_{0,1}^{\perp_{H_1}} & |x|_{H_1} &\leq c_{A_0} |A_0^* x|_{H_0}, \\ \forall x \in D(\mathcal{A}_1) &= D(A_1) \cap R(A_1^*) = D(A_1) \cap N(A_0^*) \cap N_{0,1}^{\perp_{H_1}} & |x|_{H_1} &\leq c_{A_1} |A_1 x|_{H_2}, \\ \forall y \in D(\mathcal{A}_1^*) &= D(A_1^*) \cap R(A_1) & |y|_{H_2} &\leq c_{A_1} |A_1^* y|_{H_1}, \end{aligned}$$

and

$$\forall x \in D(A_1) \cap D(A_0^*) \cap N_{0,1}^{\perp_{H_1}} \quad |x|_{H_1}^2 \leq c_{A_1}^2 |A_1 x|_{H_2}^2 + c_{A_0}^2 |A_0^* x|_{H_0}^2.$$

Moreover, all refined Helmholtz type decompositions (9)–(11) hold with closed ranges, in particular, e.g.,

$$H_1 = R(A_0) \oplus_{H_1} N_{0,1} \oplus_{H_1} R(A_1^*).$$

**Remark 2.9.** Let us define  $c_{A_0, A_1} > 0$  by

$$\frac{1}{c_{A_0, A_1}^2} := \inf \frac{|A_1 x|_{H_2}^2 + |A_0^* x|_{H_0}^2}{|x|_{H_1}^2},$$

where the infimum is taken over all  $0 \neq x \in D(A_1) \cap D(A_0^*) \cap N_{0,1}^{\perp_{H_1}}$ . Assuming – as mentioned above – that we always take the best constants, we obtain by Theorem 2.8

$$c_{A_0, A_1} = \max\{c_{A_0}, c_{A_1}\}.$$

This can be seen as follows: Theorem 2.8 shows  $c_{A_0, A_1} \leq \max\{c_{A_0}, c_{A_1}\}$ . On the other hand, for

$$x \in D(\mathcal{A}_1) = D(A_1) \cap R(A_1^*) = D(A_1) \cap N(A_0^*) \cap N_{0,1}^{\perp_{H_1}}$$

we have  $|x|_{H_1} \leq c_{A_0, A_1} |A_1 x|_{H_2}$  and hence  $c_{A_1} \leq c_{A_0, A_1}$ . Analogously we get  $c_{A_0} \leq c_{A_0, A_1}$ .

**Remark 2.10.** If  $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$  is compact, then  $D(\mathcal{A}_0) \hookrightarrow H_0$ ,  $D(\mathcal{A}_1) \hookrightarrow H_1$ , and  $D(\mathcal{A}_0^*) \hookrightarrow H_1$ ,  $D(\mathcal{A}_1^*) \hookrightarrow H_2$  are compact, as well as  $\dim N_{0,1} < \infty$ . Hence all ranges are closed, see Remark 2.3, and all assertions of Theorem 2.8 hold.

In other words, the primal and dual complex, i.e.,

$$\begin{array}{ccccc} D(A_0) & \xrightarrow{A_0} & D(A_1) & \xrightarrow{A_1} & H_2, \\ H_0 & \xleftarrow{A_0^*} & D(A_0^*) & \xleftarrow{A_1^*} & D(A_1^*), \end{array} \quad (12)$$

is a Hilbert complex of closed and densely defined linear operators. The additional assumption that the ranges  $R(A_0)$  and  $R(A_1)$  are closed (and so also the ranges  $R(A_0^*)$  and  $R(A_1^*)$ ) is equivalent to the closedness of the Hilbert complex. Moreover, the complex is exact if and only if  $N_{0,1} = \{0\}$ . The complex is called compact, if

$$D(A_1) \cap D(A_0^*) \hookrightarrow H_1 \quad (13)$$

is compact. Remark 2.10 shows that (13) is the crucial assumption for the complex (12).

Finally, we present some results for the (unbounded linear) operator

$$A_0 A_0^* + A_1^* A_1 : D(A_0 A_0^* + A_1^* A_1) \subset H_1 \rightarrow H_1$$

$$\text{with } D(A_0 A_0^* + A_1^* A_1) := \{x \in D(A_1) \cap D(A_0^*) : A_1 x \in D(A_1^*) \wedge A_0^* x \in D(A_0)\}.$$

**Lemma 2.11** (Constant And Eigenvalue Lemma). *Let  $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$  be compact. Then:*

- (i)  $A_0^* A_0$ ,  $A_0 A_0^*$ ,  $A_1^* A_1$ ,  $A_1 A_1^*$ , and  $A_0 A_0^* + A_1^* A_1$  are self-adjoint and have pure and discrete point spectra with no accumulation point in  $\mathbb{R}$ .
  - (ii) The results of [Lemma 2.5](#) hold for  $A_0$  and  $A_1$ , in particular  $\sigma(A_0^* A_0) \setminus \{0\} = \sigma(A_0 A_0^*) \setminus \{0\}$  and  $\sigma(A_1^* A_1) \setminus \{0\} = \sigma(A_1 A_1^*) \setminus \{0\}$  as well as  $N(A_0 A_0^*) = N(A_0^*)$  and  $N(A_1^* A_1) = N(A_1)$ .
  - (iii)  $N(A_0 A_0^* + A_1^* A_1) = N_{0,1}$  and  $R(A_0 A_0^* + A_1^* A_1) = N_{0,1}^{\perp_{H_1}}$ , in particular the range is closed.
  - (iv)  $A_0 A_0^* + A_1^* A_1 : D(A_0 A_0^* + A_1^* A_1) \cap N_{0,1}^{\perp_{H_1}} \subset N_{0,1}^{\perp_{H_1}} \rightarrow N_{0,1}^{\perp_{H_1}}$  is bijective with compact inverse.
  - (iv\*)  $A_0 A_0^* + A_1^* A_1 : D(A_0 A_0^* + A_1^* A_1) \cap N_{0,1}^{\perp_{H_1}} \rightarrow N_{0,1}^{\perp_{H_1}}$  is a topological isomorphism.
- Moreover, the spectrum of  $A_0 A_0^* + A_1^* A_1$  is given by the spectra of  $A_0 A_0^*$  and  $A_1^* A_1$ , i.e.,
- (v)  $\sigma(A_0 A_0^* + A_1^* A_1) \setminus \{0\} = (\sigma(A_0^* A_0) \setminus \{0\}) \cup (\sigma(A_1^* A_1) \setminus \{0\})$ .
  - (v\*) In particular, the smallest positive eigenvalue of  $A_0 A_0^* + A_1^* A_1$  is given by  $\min\{\lambda_{A_0}^2, \lambda_{A_1}^2\}$ .

For a proof see [Appendix A](#).

**Remark 2.12** (Helmholtz Decomposition).  $A_0 A_0^* + A_1^* A_1$  provides the Helmholtz decomposition from [Theorem 2.8](#). To see this, let us denote the orthonormal projector onto the cohomology group  $N_{0,1}$  by  $\pi_{N_{0,1}} : H_1 \rightarrow N_{0,1}$ . Then, for  $x \in H_1$  we have  $(1 - \pi_{N_{0,1}})x \in N_{0,1}^{\perp_{H_1}}$  and

$$\begin{aligned} x &= \pi_{N_{0,1}} x + (1 - \pi_{N_{0,1}})x \\ &= \pi_{N_{0,1}} x + (A_0 A_0^* + A_1^* A_1)(A_0 A_0^* + A_1^* A_1)^{-1}(1 - \pi_{N_{0,1}})x \in N_{0,1} \oplus_{H_1} R(A_0) \oplus_{H_1} R(A_1^*). \end{aligned}$$

## 2.2. Laplace and Maxwell constants in 3D

Now, we specialize to linear acoustics and electromagnetics in 3D, i.e., to the classical operators of the 3D-de Rham complex, cf. [\(12\)](#),

$$\begin{aligned} H_{\Gamma_\tau}(\text{grad}, \Omega) &\xrightarrow{A_0 = \text{grad}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{curl}, \Omega) \xrightarrow{A_1 = \text{curl}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{div}, \Omega) \xrightarrow{A_2 = \text{div}_{\Gamma_\tau}} L^2(\Omega), \\ L^2(\Omega) &\xleftarrow{A_0^* = -\text{div}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{div}, \Omega) \xleftarrow{A_1^* = \text{curl}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{curl}, \Omega) \xleftarrow{A_2^* = -\text{grad}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{grad}, \Omega), \end{aligned} \quad (14)$$

and apply the fa-toolbox to these operators.

More precisely, let  $\Omega \subset \mathbb{R}^3$  be a bounded weak Lipschitz domain, see [\[21, Definition 2.3\]](#) for details, with boundary  $\Gamma := \partial \Omega$ , which is divided into two relatively open weak Lipschitz subsets  $\Gamma_\tau$  and  $\Gamma_\nu := \Gamma \setminus \Gamma_\tau$  (its complement), see [\[21, Definition 2.5\]](#) for details. We shall call  $(\Omega, \Gamma_\tau)$  a bounded weak Lipschitz pair. Moreover, if  $(\Omega, \Gamma_\tau)$  is a bounded weak Lipschitz pair, so is  $(\Omega, \Gamma_\nu)$ . Note that strong Lipschitz (graph of Lipschitz functions) implies weak Lipschitz (Lipschitz manifolds) for the boundary as well as for the interface. We introduce the usual Lebesgue and Sobolev spaces by  $L^2(\Omega)$  and  $H^k(\Omega)$ ,  $k \in \mathbb{N}_0$ . For  $k = 1$  we also write

$$H^1(\Omega) = H(\text{grad}, \Omega) := \{u \in L^2(\Omega) : \text{grad } u \in L^2(\Omega)\}.$$

Homogeneous weak boundary conditions (in the strong sense) are defined by closure of respective test functions, i.e.,

$$H_{\Gamma_\tau}^1(\Omega) = H_{\Gamma_\tau}(\text{grad}, \Omega) := \overline{C_{\Gamma_\tau}^\infty(\Omega)}^{H(\text{grad}, \Omega)}, \quad (15)$$

where

$$C_{\Gamma_\tau}^\infty(\Omega) := \{u|_\Omega : u \in C^\infty(\mathbb{R}^3), \text{ supp } u \text{ compact in } \mathbb{R}^3, \text{ dist}(\text{supp } u, \Gamma_\tau) > 0\}.$$

Analogously we define (using test vector fields)

$$H(\text{curl}, \Omega), \quad H_{\Gamma_\tau}(\text{curl}, \Omega), \quad H(\text{div}, \Omega), \quad H_{\Gamma_\tau}(\text{div}, \Omega).$$

All latter definitions extend to  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , in an obvious way, see [\[22,23\]](#) for details. Throughout this paper and until otherwise stated, we shall assume the latter minimal regularity on  $\Omega$  and  $\Gamma_\tau$ .

**Assumption 2.13.**  $(\Omega, \Gamma_\tau)$  is a bounded weak Lipschitz pair.

As closures of the respective classical operators of vector analysis defined on test functions/vector fields from  $C^\infty_{\Gamma_\tau}(\Omega)$ , we consider the densely defined and closed linear operators

$$\begin{aligned} A_0 &:= \text{grad}_{\Gamma_\tau} : D(\text{grad}_{\Gamma_\tau}) \subset L^2(\Omega) \longrightarrow L^2(\Omega); & u &\mapsto \text{grad } u, \\ A_1 &:= \text{curl}_{\Gamma_\tau} : D(\text{curl}_{\Gamma_\tau}) \subset L^2(\Omega) \longrightarrow L^2(\Omega); & E &\mapsto \text{curl } E, \\ A_2 &:= \text{div}_{\Gamma_\tau} : D(\text{div}_{\Gamma_\tau}) \subset L^2(\Omega) \longrightarrow L^2(\Omega); & H &\mapsto \text{div } H, \end{aligned}$$

together with their adjoints, see [21, Theorem 4.5, Section 5.2] and [22,23, Theorem 4.7, Section 5.2],

$$\begin{aligned} A_0^* &= \text{grad}_{\Gamma_\tau}^* = -\text{div}_{\Gamma_\nu} : D(\text{div}_{\Gamma_\nu}) \subset L^2(\Omega) \longrightarrow L^2(\Omega); & H &\mapsto -\text{div } H, \\ A_1^* &= \text{curl}_{\Gamma_\tau}^* = \text{curl}_{\Gamma_\nu} : D(\text{curl}_{\Gamma_\nu}) \subset L^2(\Omega) \longrightarrow L^2(\Omega); & E &\mapsto \text{curl } E, \\ A_2^* &= \text{div}_{\Gamma_\tau}^* = -\text{grad}_{\Gamma_\nu} : D(\text{grad}_{\Gamma_\nu}) \subset L^2(\Omega) \longrightarrow L^2(\Omega); & u &\mapsto -\text{grad } u. \end{aligned}$$

Note that

$$D(\text{grad}_{\Gamma_\tau}) = H_{\Gamma_\tau}(\text{grad}, \Omega), \quad D(\text{curl}_{\Gamma_\tau}) = H_{\Gamma_\tau}(\text{curl}, \Omega), \quad D(\text{div}_{\Gamma_\tau}) = H_{\Gamma_\tau}(\text{div}, \Omega)$$

and that (14) is indeed a Hilbert complex.

Recently, in [21–23], Weck’s selection theorem, also known as the Maxwell compactness property, has been shown to hold for such bounded weak Lipschitz domains and mixed boundary conditions.

**Theorem 2.14** (Weck’s Selection Theorem). *The embedding*

$$H_{\Gamma_\tau}(\text{curl}, \Omega) \cap H_{\Gamma_\nu}(\text{div}, \Omega) \hookrightarrow L^2(\Omega)$$

is compact.

For a proof see [21–23]. A short historical overview of Weck’s selection theorem is given in the introduction of [21], see also the original paper [24] and [10,25–30].

Now, Theorem 2.14 implies that the crucial assumption (13) holds for the operators  $A_n$  of the de Rham complex (14), cf. the general complex (12). More precisely, by Theorem 2.14

$$\begin{aligned} \text{(a)} \quad & D(A_1) \cap D(A_0^*) = H_{\Gamma_\tau}(\text{curl}, \Omega) \cap H_{\Gamma_\nu}(\text{div}, \Omega) \hookrightarrow L^2(\Omega) = H_1, \\ \text{(b)} \quad & D(A_2) \cap D(A_1^*) = H_{\Gamma_\tau}(\text{div}, \Omega) \cap H_{\Gamma_\nu}(\text{curl}, \Omega) \hookrightarrow L^2(\Omega) = H_2 \end{aligned}$$

are compact and, hence, (14) is a compact Hilbert complex. Thus, by Theorem 2.8 and Remark 2.10, all ranges are closed, all corresponding Poincaré–Friedrichs type estimates hold, and all refined Helmholtz type decompositions (9)–(11) hold with closed ranges. In particular, denoting the corresponding constants by

$$\begin{aligned} \frac{1}{\lambda_{0,\Gamma_\tau}} &:= c_{0,\Gamma_\tau} := c_{\text{grad}_{\Gamma_\tau}} := c_{A_0} = c_{A_0^*} = c_{\text{div}_{\Gamma_\nu}} = c_{2,\Gamma_\nu} = \frac{1}{\lambda_{2,\Gamma_\nu}}, \\ \frac{1}{\lambda_{1,\Gamma_\tau}} &:= c_{1,\Gamma_\tau} := c_{\text{curl}_{\Gamma_\tau}} := c_{A_1} = c_{A_1^*} = c_{\text{curl}_{\Gamma_\nu}} = c_{1,\Gamma_\nu} = \frac{1}{\lambda_{1,\Gamma_\nu}}, \\ \frac{1}{\lambda_{2,\Gamma_\tau}} &:= c_{2,\Gamma_\tau} := c_{\text{div}_{\Gamma_\tau}} := c_{A_2} = c_{A_2^*} = c_{\text{grad}_{\Gamma_\nu}} = c_{0,\Gamma_\nu} = \frac{1}{\lambda_{0,\Gamma_\nu}}, \end{aligned} \tag{16}$$

and introducing the (finite-dimensional) cohomology groups

$$\begin{aligned} \mathcal{H}_1 &:= N_{0,1} := N(A_1) \cap N(A_0^*) = N(\text{curl}_{\Gamma_\tau}) \cap N(\text{div}_{\Gamma_\nu}), \\ \mathcal{H}_2 &:= N_{1,2} := N(A_2) \cap N(A_1^*) = N(\text{div}_{\Gamma_\tau}) \cap N(\text{curl}_{\Gamma_\nu}), \end{aligned}$$

the so-called Dirichlet/Neumann fields, we have by Theorem 2.8 and Remark 2.10 the following inequalities:

**Theorem 2.15** (Poincaré–Friedrichs Type Estimates). *It holds*

$$\begin{aligned} \forall u \in D(\mathcal{A}_0) &= D(\text{grad}_{\Gamma_\tau}) \cap R(\text{div}_{\Gamma_\nu}) & |u|_{L^2(\Omega)} &\leq c_{0,\Gamma_\tau} |\text{grad } u|_{L^2(\Omega)}, \\ \forall E \in D(\mathcal{A}_0^*) &= D(\text{div}_{\Gamma_\nu}) \cap R(\text{grad}_{\Gamma_\tau}) & |E|_{L^2(\Omega)} &\leq c_{0,\Gamma_\tau} |\text{div } E|_{L^2(\Omega)}, \\ \forall E \in D(\mathcal{A}_1) &= D(\text{curl}_{\Gamma_\tau}) \cap R(\text{curl}_{\Gamma_\nu}) & |E|_{L^2(\Omega)} &\leq c_{1,\Gamma_\tau} |\text{curl } E|_{L^2(\Omega)}, \\ \forall H \in D(\mathcal{A}_1^*) &= D(\text{curl}_{\Gamma_\nu}) \cap R(\text{curl}_{\Gamma_\tau}) & |H|_{L^2(\Omega)} &\leq c_{1,\Gamma_\tau} |\text{curl } H|_{L^2(\Omega)}, \end{aligned}$$

and for all  $E \in D(A_1) \cap D(A_0^*) \cap N_{0,1}^\perp = D(\text{curl}_{\Gamma_\tau}) \cap D(\text{div}_{\Gamma_\nu}) \cap \mathcal{H}_1^\perp$

$$|E|_{L^2(\Omega)}^2 \leq c_{1,\Gamma_\tau}^2 |\text{curl } E|_{L^2(\Omega)}^2 + c_{0,\Gamma_\tau}^2 |\text{div } E|_{L^2(\Omega)}^2,$$



where

$$\begin{aligned} R(\text{grad}_{\Gamma_\tau}) &= N(\text{curl}_{\Gamma_\tau}) \cap \mathcal{H}_1^{\perp L^2(\Omega)}, & R(\text{curl}_{\Gamma_\nu}) &= N(\text{div}_{\Gamma_\nu}) \cap \mathcal{H}_1^{\perp L^2(\Omega)}, \\ R(\text{grad}_{\Gamma_\nu}) &= N(\text{curl}_{\Gamma_\nu}) \cap \mathcal{H}_2^{\perp L^2(\Omega)}, & R(\text{curl}_{\Gamma_\tau}) &= N(\text{div}_{\Gamma_\tau}) \cap \mathcal{H}_2^{\perp L^2(\Omega)}. \end{aligned}$$

Let  $c_{0,1,\Gamma_\tau} := c_{\text{grad}_{\Gamma_\tau}, \text{curl}_{\Gamma_\tau}} > 0$  be defined by

$$\frac{1}{c_{0,1,\Gamma_\tau}} := \inf \frac{|\text{curl } E|_{L^2(\Omega)}^2 + |\text{div } E|_{L^2(\Omega)}^2}{|E|_{L^2(\Omega)}^2},$$

where the infimum taken over all  $0 \neq E \in D(\text{curl}_{\Gamma_\tau}) \cap D(\text{div}_{\Gamma_\nu}) \cap \mathcal{H}_1^{\perp L^2(\Omega)}$ .

**Remark 2.16.** By Remark 2.9 it holds  $c_{0,1,\Gamma_\tau} = \max\{c_{0,\Gamma_\tau}, c_{1,\Gamma_\tau}\}$ .

Note that by the symmetry of the de Rham complex the corresponding two estimates for  $\mathcal{A}_2$  and  $\mathcal{A}_2^*$ , i.e.,

$$\begin{aligned} \forall H \in D(\mathcal{A}_2) &= D(\text{div}_{\Gamma_\tau}) \cap R(\text{grad}_{\Gamma_\nu}) & |H|_{L^2(\Omega)} &\leq c_{2,\Gamma_\tau} |\text{div } H|_{L^2(\Omega)}, \\ \forall u \in D(\mathcal{A}_2^*) &= D(\text{grad}_{\Gamma_\nu}) \cap R(\text{div}_{\Gamma_\tau}) & |u|_{L^2(\Omega)} &\leq c_{2,\Gamma_\tau} |\text{grad } u|_{L^2(\Omega)}, \end{aligned}$$

are redundant, as these are already included in the two estimates for  $\mathcal{A}_0$  and  $\mathcal{A}_0^*$  just by interchanging the boundary conditions on  $\Gamma_\tau$  and  $\Gamma_\nu$ . In other words,  $c_{2,\Gamma_\tau} = c_{0,\Gamma_\nu}$ . Furthermore,

$$\begin{aligned} N(\text{grad}_{\Gamma_\tau}) &= \begin{cases} \{0\} & \text{if } \Gamma_\tau \neq \emptyset, \\ \mathbb{R} & \text{if } \Gamma_\tau = \emptyset, \end{cases} \\ R(\text{div}_{\Gamma_\nu}) &= N(\text{grad}_{\Gamma_\tau})^{\perp L^2(\Omega)} = L_{\Gamma_\nu}^2(\Omega) := \begin{cases} L^2(\Omega) & \text{if } \Gamma_\nu \neq \Gamma, \\ L^2(\Omega) \cap \mathbb{R}^{\perp L^2(\Omega)} & \text{if } \Gamma_\nu = \Gamma, \end{cases} \end{aligned}$$

where

$$L^2(\Omega) \cap \mathbb{R}^{\perp L^2(\Omega)} = \{u \in L^2(\Omega) : \langle u, 1 \rangle_{L^2(\Omega)} = 0\} = \{u \in L^2(\Omega) : \int_\Omega u = 0\}.$$

Combinations of the latter operators give the well known operators from acoustics, Maxwell equations, Laplace equations, and the double rotation equations, i.e.,

$$\begin{aligned} M_0 &:= \begin{bmatrix} 0 & A_0^* \\ A_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{grad}_{\Gamma_\tau}^* \\ \text{grad}_{\Gamma_\tau} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\text{div}_{\Gamma_\nu} \\ \text{grad}_{\Gamma_\tau} & 0 \end{bmatrix}, \\ M_1 &:= \begin{bmatrix} 0 & A_1^* \\ A_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{curl}_{\Gamma_\tau}^* \\ \text{curl}_{\Gamma_\tau} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{curl}_{\Gamma_\nu} \\ \text{curl}_{\Gamma_\tau} & 0 \end{bmatrix}, \\ M_2 &:= \begin{bmatrix} 0 & A_2^* \\ A_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{div}_{\Gamma_\tau}^* \\ \text{div}_{\Gamma_\tau} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\text{grad}_{\Gamma_\nu} \\ \text{div}_{\Gamma_\tau} & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} A_0^* A_0 &= \text{grad}_{\Gamma_\tau}^* \text{grad}_{\Gamma_\tau} = -\text{div}_{\Gamma_\nu} \text{grad}_{\Gamma_\tau} =: -\Delta_{\Gamma_\tau}, & A_0 A_0^* &= \text{grad}_{\Gamma_\tau} \text{grad}_{\Gamma_\tau}^* = -\text{grad}_{\Gamma_\tau} \text{div}_{\Gamma_\nu} =: -\diamond_{\Gamma_\nu}, \\ A_1^* A_1 &= \text{curl}_{\Gamma_\tau}^* \text{curl}_{\Gamma_\tau} = \text{curl}_{\Gamma_\nu} \text{curl}_{\Gamma_\tau} =: \square_{\Gamma_\tau}, & A_1 A_1^* &= \text{curl}_{\Gamma_\tau} \text{curl}_{\Gamma_\tau}^* = \text{curl}_{\Gamma_\tau} \text{curl}_{\Gamma_\nu} =: \square_{\Gamma_\nu}, \\ A_2^* A_2 &= \text{div}_{\Gamma_\tau}^* \text{div}_{\Gamma_\tau} = -\text{grad}_{\Gamma_\nu} \text{div}_{\Gamma_\tau} =: -\diamond_{\Gamma_\tau}, & A_2 A_2^* &= \text{div}_{\Gamma_\tau} \text{div}_{\Gamma_\tau}^* = -\text{div}_{\Gamma_\tau} \text{grad}_{\Gamma_\nu} =: -\Delta_{\Gamma_\nu}. \end{aligned}$$

Again,  $M_2$  and the operators involving  $A_2, A_2^*$  are redundant by interchanging the boundary conditions in  $M_0$  and  $A_0, A_0^*$ . Hence, we may focus on  $c_{0,\Gamma_\tau}$  and  $c_{1,\Gamma_\tau}$ . Section 2.1.3 shows the following:

**Theorem 2.17** (Poincaré–Friedrichs Type Constants). *The Poincaré–Friedrichs type constants can be computed by the four Rayleigh quotations*

$$\begin{aligned} \frac{1}{c_{0,\Gamma_\tau}} &= \lambda_{0,\Gamma_\tau} = \inf_{0 \neq u \in D(\text{grad}_{\Gamma_\tau}) \cap L_{\Gamma_\nu}^2(\Omega)} \frac{|\text{grad } u|_{L^2(\Omega)}}{|u|_{L^2(\Omega)}} = \inf_{0 \neq E \in D(\text{div}_{\Gamma_\nu}) \cap R(\text{grad}_{\Gamma_\tau})} \frac{|\text{div } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}}, \\ \frac{1}{c_{1,\Gamma_\tau}} &= \lambda_{1,\Gamma_\tau} = \inf_{0 \neq E \in D(\text{curl}_{\Gamma_\tau}) \cap R(\text{curl}_{\Gamma_\nu})} \frac{|\text{curl } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}} = \inf_{0 \neq H \in D(\text{curl}_{\Gamma_\nu}) \cap R(\text{curl}_{\Gamma_\tau})} \frac{|\text{curl } H|_{L^2(\Omega)}}{|H|_{L^2(\Omega)}}. \end{aligned}$$

Moreover,  $\lambda_{0,\Gamma_\tau}$  is the smallest positive eigenvalue of

$$\begin{bmatrix} 0 & A_0^* \\ A_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\text{div}_{\Gamma_\nu} \\ \text{grad}_{\Gamma_\tau} & 0 \end{bmatrix}$$

and  $\lambda_{0,\Gamma_\tau}^2$  is the smallest (positive) eigenvalue of

$$A_0^* A_0 = -\operatorname{div}_{\Gamma_\tau} \operatorname{grad}_{\Gamma_\tau} = -\Delta_{\Gamma_\tau} \quad \text{and} \quad A_0 A_0^* = -\operatorname{grad}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\tau} = -\diamond_{\Gamma_\tau}.$$

$\lambda_{1,\Gamma_\tau}$  is the smallest positive eigenvalue of

$$\begin{bmatrix} 0 & A_1^* \\ A_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{curl}_{\Gamma_\tau} \\ \operatorname{curl}_{\Gamma_\tau} & 0 \end{bmatrix}$$

and  $\lambda_{1,\Gamma_\tau}^2$  is the smallest (positive) eigenvalue of

$$A_1^* A_1 = \operatorname{curl}_{\Gamma_\tau} \operatorname{curl}_{\Gamma_\tau} = \square_{\Gamma_\tau} \quad \text{and} \quad A_1 A_1^* = \operatorname{curl}_{\Gamma_\tau} \operatorname{curl}_{\Gamma_\tau} = \square_{\Gamma_\tau}.$$

**Remark 2.18** (Variational Formulations). All infima in Theorem 2.17 are minima and the respective minimizers  $u_{0,\Gamma_\tau}$ ,  $E_{0,\Gamma_\tau}$ , and  $E_{1,\Gamma_\tau}$ ,  $H_{1,\Gamma_\tau}$  are the eigenfunctions to the eigenvalues  $\lambda_{0,\Gamma_\tau}^2$  and  $\lambda_{1,\Gamma_\tau}^2$ , i.e.,

$$\lambda_{0,\Gamma_\tau} = \frac{|\operatorname{grad} u_{0,\Gamma_\tau}|_{L^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{L^2(\Omega)}} = \frac{|\operatorname{div} E_{0,\Gamma_\tau}|_{L^2(\Omega)}}{|E_{0,\Gamma_\tau}|_{L^2(\Omega)}}, \quad \lambda_{1,\Gamma_\tau} = \frac{|\operatorname{curl} E_{1,\Gamma_\tau}|_{L^2(\Omega)}}{|E_{1,\Gamma_\tau}|_{L^2(\Omega)}} = \frac{|\operatorname{curl} H_{1,\Gamma_\tau}|_{L^2(\Omega)}}{|H_{1,\Gamma_\tau}|_{L^2(\Omega)}},$$

$$\begin{aligned} (-\Delta_{\Gamma_\tau} - \lambda_{0,\Gamma_\tau}^2) u_{0,\Gamma_\tau} &= 0, & u_{0,\Gamma_\tau} &\in D(\Delta_{\Gamma_\tau}) \cap L_{\Gamma_\tau}^2(\Omega) \subset D(\operatorname{grad}_{\Gamma_\tau}) \cap L_{\Gamma_\tau}^2(\Omega), \\ (-\diamond_{\Gamma_\tau} - \lambda_{0,\Gamma_\tau}^2) E_{0,\Gamma_\tau} &= 0, & E_{0,\Gamma_\tau} &\in D(\diamond_{\Gamma_\tau}) \cap R(\operatorname{grad}_{\Gamma_\tau}) \subset D(\operatorname{div}_{\Gamma_\tau}) \cap R(\operatorname{grad}_{\Gamma_\tau}), \\ (\square_{\Gamma_\tau} - \lambda_{1,\Gamma_\tau}^2) E_{1,\Gamma_\tau} &= 0, & E_{1,\Gamma_\tau} &\in D(\square_{\Gamma_\tau}) \cap R(\operatorname{curl}_{\Gamma_\tau}) \subset D(\operatorname{curl}_{\Gamma_\tau}) \cap R(\operatorname{curl}_{\Gamma_\tau}), \\ (\square_{\Gamma_\tau} - \lambda_{1,\Gamma_\tau}^2) H_{1,\Gamma_\tau} &= 0, & H_{1,\Gamma_\tau} &\in D(\square_{\Gamma_\tau}) \cap R(\operatorname{curl}_{\Gamma_\tau}) \subset D(\operatorname{curl}_{\Gamma_\tau}) \cap R(\operatorname{curl}_{\Gamma_\tau}). \end{aligned}$$

Moreover, the eigenvectors satisfy the variational formulations

$$\begin{aligned} \forall \psi \in D(\operatorname{grad}_{\Gamma_\tau}) & \quad \langle \operatorname{grad} u_{0,\Gamma_\tau}, \operatorname{grad} \psi \rangle_{L^2(\Omega)} = \lambda_{0,\Gamma_\tau}^2 \langle u_{0,\Gamma_\tau}, \psi \rangle_{L^2(\Omega)}, \\ \forall \Psi \in D(\operatorname{div}_{\Gamma_\tau}) & \quad \langle \operatorname{div} E_{0,\Gamma_\tau}, \operatorname{div} \Psi \rangle_{L^2(\Omega)} = \lambda_{0,\Gamma_\tau}^2 \langle E_{0,\Gamma_\tau}, \Psi \rangle_{L^2(\Omega)}, \\ \forall \Phi \in D(\operatorname{curl}_{\Gamma_\tau}) & \quad \langle \operatorname{curl} E_{1,\Gamma_\tau}, \operatorname{curl} \Phi \rangle_{L^2(\Omega)} = \lambda_{1,\Gamma_\tau}^2 \langle E_{1,\Gamma_\tau}, \Phi \rangle_{L^2(\Omega)}, \\ \forall \Theta \in D(\operatorname{curl}_{\Gamma_\tau}) & \quad \langle \operatorname{curl} H_{1,\Gamma_\tau}, \operatorname{curl} \Theta \rangle_{L^2(\Omega)} = \lambda_{1,\Gamma_\tau}^2 \langle H_{1,\Gamma_\tau}, \Theta \rangle_{L^2(\Omega)}. \end{aligned}$$

**Remark 2.19.** We emphasize that Lemma 2.11 provides results for the vector Laplacian

$$A_0 A_0^* + A_1^* A_1 = -\diamond_{\Gamma_\tau} + \square_{\Gamma_\tau} = -\operatorname{grad}_{\Gamma_\tau} \operatorname{div}_{\Gamma_\tau} + \operatorname{curl}_{\Gamma_\tau} \operatorname{curl}_{\Gamma_\tau},$$

which has been recently discussed in, e.g., [31].

### 2.2.1. Known results for the constants in 3D

Let us summarize and cite some recent results from [1–4] about the Poincaré–Friedrichs type constants, i.e., about the Poincaré–Friedrichs constants  $\lambda_{0,\Gamma_\tau}$  and the Maxwell constants  $\lambda_{1,\Gamma_\tau}$ .

**Theorem 2.20** (Poincaré–Friedrichs/maxwell Constants in 3D). For  $c_{\ell,\Gamma_\tau} = 1/\lambda_{\ell,\Gamma_\tau}$  the following holds:

(i) The Poincaré–Friedrichs constants depend monotonically on the boundary conditions, i.e.,

$$\emptyset \neq \tilde{\Gamma}_\tau \subset \Gamma_\tau \Rightarrow c_{0,\Gamma_\tau} \leq c_{0,\tilde{\Gamma}_\tau}.$$

(ii) The Friedrichs constant is always smaller than the Poincaré constant, i.e.,

$$c_{0,\Gamma} \leq c_{0,\emptyset},$$

where  $c_{0,\Gamma}$  is the classical Friedrichs constant and  $c_{0,\emptyset}$  is the classical Poincaré constant. Moreover,  $\lambda_{0,\Gamma}$  is usually called the first Dirichlet–Laplace eigenvalue and  $\lambda_{0,\emptyset}$  is usually called the second Neumann–Laplace eigenvalue.

(iii)  $c_{0,\Gamma} \leq \operatorname{diam}(\Omega)/\pi$

(iv)  $c_{0,\Gamma_\tau} = c_{2,\Gamma_\tau}$

(v)  $c_{1,\Gamma_\tau} = c_{1,\Gamma_\tau}$

(vi)  $c_{0,\Gamma} \leq c_{0,\Gamma_\tau} \leq c_{0,1,\Gamma_\tau} = \max\{c_{0,\Gamma_\tau}, c_{1,\Gamma_\tau}\}$

(vii) If  $\Omega$  is convex, then  $c_{0,\Gamma} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$ .

(viii) If  $\Omega$  is convex, then  $c_{1,\Gamma} = c_{1,\emptyset} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$ .

(ix) If  $\Omega$  is convex, then  $c_{0,\Gamma} \leq c_{0,1,\Gamma} = \max\{c_{0,\Gamma}, c_{1,\Gamma}\} \leq c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$ .

(ix') If  $\Omega$  is convex, then  $c_{0,\Gamma} \leq c_{0,1,\emptyset} = \max\{c_{0,\emptyset}, c_{1,\emptyset}\} = c_{0,\emptyset} \leq \operatorname{diam}(\Omega)/\pi$ .

**Remark 2.21.** To the best of our knowledge, it is an open question whether or not

$$c_{0,\Gamma_\tau} \leq c_{1,\Gamma_\tau} \quad \text{or at least} \quad c_{0,\Gamma} \leq c_{1,\Gamma}$$

holds in general.

### 2.3. Other complexes and constants

So far, we have discussed the de Rham complex (14) in 3D. While in higher dimensions  $N \geq 4$  the situation is very similar to the 3D case (but the adjoint of  $\text{curl}_{\Gamma_\tau}$  is no longer a rotation itself), the situations in 1D and 2D are much simpler. Moreover, similar to the 3D-de Rham complex (14), other important complexes of shape (12) fit nicely into our general fa-toolbox and, therefore, can be handled with our theory, see also [18,19] for details.

#### 2.3.1. 1D-de Rham complex, Laplace and Maxwell Constants in 1D

In 1D the domain  $\Omega$  is an interval and we have just one operator  $A_0 = \text{grad}_{\Gamma_\tau} = (\cdot)'_{\Gamma_\tau}$  with adjoint  $A_0^* = -\text{div}_{\Gamma_\nu} = -(\cdot)'_{\Gamma_\nu}$ , i.e., the complex (12), compare to (14), reads

$$H_{\Gamma_\tau}^1(\Omega) = H_{\Gamma_\tau}(\text{grad}, \Omega) \xrightarrow{A_0 = \text{grad}_{\Gamma_\tau} = (\cdot)'_{\Gamma_\tau}} L^2(\Omega),$$

$$L^2(\Omega) \xleftarrow{A_0^* = -\text{div}_{\Gamma_\nu} = -(\cdot)'_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{div}, \Omega) = H_{\Gamma_\nu}^1(\Omega).$$

Hence, just the Laplacians  $\Delta_{\Gamma_\tau} = \text{div}_{\Gamma_\nu} \text{grad}_{\Gamma_\tau} = (\cdot)''_{\Gamma_\tau}$  and  $\Delta_{\Gamma_\nu} = \text{grad}_{\Gamma_\tau} \text{div}_{\Gamma_\nu} = (\cdot)''_{\Gamma_\nu}$  exist and there are no Maxwell operators. The crucial compact embedding (13) is simply Rellich's selection theorem, compare to Theorem 2.14. Moreover, here in the 1D case we have

$$\begin{aligned} \lambda_{0,\Gamma_\tau} &= \inf_{0 \neq u \in H_{\Gamma_\tau}^1(\Omega) \cap L_{\Gamma_\nu}^2(\Omega)} \frac{|\text{grad } u|_{L^2(\Omega)}}{|u|_{L^2(\Omega)}} = \inf_{0 \neq u \in H_{\Gamma_\tau}^1(\Omega) \cap L_{\Gamma_\nu}^2(\Omega)} \frac{|u'|_{L^2(\Omega)}}{|u|_{L^2(\Omega)}} \\ &= \underbrace{\inf_{0 \neq E \in H_{\Gamma_\nu}(\text{div}, \Omega) \cap \mathcal{R}(\text{grad}_{\Gamma_\tau})} \frac{|\text{div } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}}}_{=\lambda_{2,\Gamma_\nu}} = \inf_{0 \neq E \in H_{\Gamma_\nu}^1(\Omega) \cap L_{\Gamma_\tau}^2(\Omega)} \frac{|E'|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}} = \lambda_{0,\Gamma_\nu}, \end{aligned}$$

i.e., it is sufficient to compute the eigenvalues  $\lambda_{0,\Gamma_\tau}$ , and we can also give a meaning to  $\lambda_{2,\Gamma_\nu}$ . Thus

$$\lambda_{0,\Gamma_\tau} = \lambda_{0,\Gamma_\nu} = \lambda_{2,\Gamma_\nu} = \frac{1}{c_{2,\Gamma_\nu}} = \frac{1}{c_{0,\Gamma_\nu}} = \frac{1}{c_{0,\Gamma_\tau}}.$$

Note that

$$\lambda_{0,\Gamma_\tau} = \frac{|\text{grad } u_{0,\Gamma_\tau}|_{L^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{L^2(\Omega)}} = \frac{|u'_{0,\Gamma_\tau}|_{L^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{L^2(\Omega)}} = \frac{|\text{div } E_{0,\Gamma_\nu}|_{L^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{L^2(\Omega)}} = \frac{|E'_{0,\Gamma_\nu}|_{L^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{L^2(\Omega)}} = \lambda_{0,\Gamma_\nu}.$$

Theorem 2.20 turns to:

**Corollary 2.22** (Poincaré–Friedrichs/Maxwell constants in 1D). For  $c_{\ell,\Gamma_\tau} = 1/\lambda_{\ell,\Gamma_\tau}$  the following holds:

- (i)  $\emptyset \neq \tilde{\Gamma}_\tau \subset \Gamma_\tau \Rightarrow c_{0,\Gamma_\tau} \leq c_{0,\tilde{\Gamma}_\tau}$
- (ii)  $c_{0,\Gamma} = c_{0,\emptyset} \leq \text{diam}(\Omega)/\pi$
- (iii)  $c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{0,\Gamma_\nu}$
- (iv) There is no  $c_{1,\Gamma_\tau}$ , but  $c_{2,\Gamma_\nu} = c_{0,\Gamma_\nu} = c_{0,\Gamma_\tau}$ .

#### 2.3.2. 2D-de Rham complex, Laplace and Maxwell Constants in 2D

In 2D there are just the two operators  $A_0 = \text{grad}_{\Gamma_\tau}$  and  $A_1 = \text{curl}_{\Gamma_\tau} = \text{div}_{\Gamma_\tau} R$  with adjoints  $A_0^* = -\text{div}_{\Gamma_\nu}$  and  $A_1^* = \text{curl}_{\Gamma_\nu} = R \text{grad}_{\Gamma_\nu}$ , where

$$\text{curl } E = \text{div } RE = \partial_1 E_2 - \partial_2 E_1, \quad \vec{\text{curl}} u = R \text{grad } u = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the complex (12), compare to (14), reads

$$H_{\Gamma_\tau}^1(\Omega) = H_{\Gamma_\tau}(\text{grad}, \Omega) \xrightarrow{A_0 = \text{grad}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{curl}, \Omega) = H_{\Gamma_\tau}(\text{div}, \Omega) R \xrightarrow{A_1 = \text{curl}_{\Gamma_\tau}} L^2(\Omega),$$

$$L^2(\Omega) \xleftarrow{A_0^* = -\text{div}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{div}, \Omega) \xleftarrow{A_1^* = \text{curl}_{\Gamma_\nu}} H_{\Gamma_\nu}(\vec{\text{curl}}, \Omega) = H_{\Gamma_\nu}^1(\Omega).$$

Hence, we have the Laplacian  $\Delta_{\Gamma_\tau} = \text{div}_{\Gamma_\tau} \text{grad}_{\Gamma_\tau}$  and  $\Delta_{\Gamma_\nu} = \text{grad}_{\Gamma_\tau} \text{div}_{\Gamma_\nu}$ , as well as the second order Maxwell operators (related to the 3D notations)

$$\begin{aligned}\square_{\Gamma_\tau} &= \vec{\text{curl}}_{\Gamma_\nu} \text{curl}_{\Gamma_\tau} = R \text{grad}_{\Gamma_\nu} \text{div}_{\Gamma_\tau} R = R \Delta_{\Gamma_\tau} R, \\ \square_{\Gamma_\nu} &= \text{curl}_{\Gamma_\tau} \vec{\text{curl}}_{\Gamma_\nu} = \text{div}_{\Gamma_\tau} R R \text{grad}_{\Gamma_\nu} = -\text{div}_{\Gamma_\tau} \text{grad}_{\Gamma_\nu} = -\Delta_{\Gamma_\nu}.\end{aligned}$$

By Lemma 2.7 the crucial compact embedding (13) is just Rellich's selection theorem, compare to Theorem 2.14. Moreover, here in the 2D case we have

$$\begin{aligned}\lambda_{0,\Gamma_\tau} &= \inf_{0 \neq u \in H_{\Gamma_\tau}^1(\Omega) \cap L_{\Gamma_\nu}^2(\Omega)} \frac{|\text{grad } u|_{L^2(\Omega)}}{|u|_{L^2(\Omega)}} = \inf_{0 \neq u \in H_{\Gamma_\tau}^1(\Omega) \cap L_{\Gamma_\nu}^2(\Omega)} \frac{|\vec{\text{curl}} u|_{L^2(\Omega)}}{|u|_{L^2(\Omega)}} \\ &= \underbrace{\inf_{0 \neq E \in H_{\Gamma_\nu}(\text{div}, \Omega) \cap R(\text{grad}_{\Gamma_\tau})} \frac{|\text{div } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}}}_{=\lambda_{2,\Gamma_\nu}} \stackrel{E \sim RE}{=} \inf_{0 \neq E \in H_{\Gamma_\nu}(\text{curl}, \Omega) \cap R(\vec{\text{curl}}_{\Gamma_\tau})} \frac{|\text{curl } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}} = \lambda_{1,\Gamma_\nu},\end{aligned}$$

i.e., it is sufficient to compute the eigenvalues  $\lambda_{0,\Gamma_\tau}$ , and we can also give a meaning to  $\lambda_{2,\Gamma_\nu}$ . Thus

$$\lambda_{0,\Gamma_\tau} = \lambda_{1,\Gamma_\nu} = \lambda_{2,\Gamma_\nu} = \frac{1}{c_{2,\Gamma_\nu}} = \frac{1}{c_{1,\Gamma_\nu}} = \frac{1}{c_{0,\Gamma_\tau}}.$$

Note that

$$\lambda_{0,\Gamma_\tau} = \frac{|\text{grad } u_{0,\Gamma_\tau}|_{L^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{L^2(\Omega)}} = \frac{|\vec{\text{curl}} u_{0,\Gamma_\tau}|_{L^2(\Omega)}}{|u_{0,\Gamma_\tau}|_{L^2(\Omega)}} = \frac{|\text{div } E_{0,\Gamma_\nu}|_{L^2(\Omega)}}{|E_{0,\Gamma_\nu}|_{L^2(\Omega)}} = \frac{|\text{curl } E_{1,\Gamma_\nu}|_{L^2(\Omega)}}{|E_{1,\Gamma_\nu}|_{L^2(\Omega)}} = \lambda_{1,\Gamma_\nu},$$

i.e., in our 3D-notation  $H_{1,\Gamma_\tau} = u_{0,\Gamma_\tau}$  and  $E_{0,\Gamma_\nu} = RE_{1,\Gamma_\nu}$ . Theorem 2.20 turns to:

**Corollary 2.23** (Poincaré–Friedrichs/Maxwell constants in 2D). For  $c_{\ell,\Gamma_\tau} = 1/\lambda_{\ell,\Gamma_\tau}$  the following holds:

(i) The Poincaré–Friedrichs constants depend monotonically on the boundary conditions, i.e.,

$$\emptyset \neq \tilde{\Gamma}_\tau \subset \Gamma_\tau \Rightarrow c_{0,\Gamma_\tau} \leq c_{0,\tilde{\Gamma}_\tau}.$$

(ii) The Friedrichs constant is always smaller than the Poincaré constant, i.e.,  $c_{0,\Gamma} \leq c_{0,\emptyset}$ .

(iii)  $c_{0,\Gamma} \leq \text{diam}(\Omega)/\pi$

(iv)  $c_{0,\Gamma} \leq c_{0,\Gamma_\tau} = c_{1,\Gamma_\nu} = c_{2,\Gamma_\nu} \leq c_{0,1,\Gamma_\tau} = \max\{c_{0,\Gamma_\tau}, c_{1,\Gamma_\tau}\} = \max\{c_{0,\Gamma_\tau}, c_{0,\Gamma_\nu}\}$

(v) If  $\Omega$  is convex, then  $c_{0,\Gamma} \leq c_{0,\emptyset} \leq \text{diam}(\Omega)/\pi$ .

### 2.3.3. ND-de Rham complex, Laplace and Maxwell constants in ND

In ND differential forms generalize suitable functions and vector fields used for  $N = 1, 2, 3$ . The de Rham complex of (12) in ND, compare to (14), consists of  $N$  differential operators  $A_q := A_{q,\Gamma_\tau} := d_{q,\Gamma_\tau}$ ,  $q = 0, \dots, N-1$ , with adjoints  $A_q^* = A_{q,\Gamma_\tau}^* = -\iota_{q+1,\Gamma_\nu}$  acting on alternating  $q$  resp.  $(q+1)$ -forms, i.e.,

$$\begin{aligned}\dots &\xrightarrow{\dots} H_{\Gamma_\tau}(d_q, \Omega) \xrightarrow{A_q=d_{q,\Gamma_\tau}} H_{\Gamma_\tau}(d_{q+1}, \Omega) \xrightarrow{A_{q+1}=d_{q+1,\Gamma_\tau}} \dots, \\ \dots &\xleftarrow{A_{q-1}^*=-\iota_{q,\Gamma_\nu}} H_{\Gamma_\nu}(\iota_q, \Omega) \xleftarrow{A_q^*=-\iota_{q+1,\Gamma_\nu}} H_{\Gamma_\nu}(\iota_{q+1}, \Omega) \xleftarrow{\dots} \dots,\end{aligned}$$

see, e.g., [32–39] for details about the complex and numerical applications. Hence, the second order “Laplace” and “Maxwell” operators are simply

$$A_q^* A_q = -\iota_{q+1,\Gamma_\nu} d_{q,\Gamma_\tau}, \quad A_q A_q^* = -d_{q,\Gamma_\tau} \iota_{q+1,\Gamma_\nu},$$

and for the constants and eigenvalues  $c_{q,\Gamma_\tau} = 1/\lambda_{q,\Gamma_\tau}$  we have

$$\begin{aligned}\forall E \in D(A_q) = D(d_{q,\Gamma_\tau}) \cap R(\iota_{q+1,\Gamma_\nu}) &\quad |E|_{L^{2,q}(\Omega)} \leq c_{q,\Gamma_\tau} |d_q E|_{L^{2,q+1}(\Omega)}, \\ \forall H \in D(A_q^*) = D(\iota_{q+1,\Gamma_\nu}) \cap R(d_{q,\Gamma_\tau}) &\quad |H|_{L^{2,q+1}(\Omega)} \leq c_{q,\Gamma_\tau} |\iota_{q+1} H|_{L^{2,q}(\Omega)}.\end{aligned}$$

The crucial compact embeddings (13) are given by the following theorem from [23, Theorem 4.9] or [22, Theorem 4.8].

**Theorem 2.24** (Weck's Selection Theorem). The embeddings

$$D(A_q) \cap D(A_{q-1}^*) = H_{\Gamma_\tau}(d_q, \Omega) \cap H_{\Gamma_\nu}(\iota_q, \Omega) \hookrightarrow L^{2,q}(\Omega)$$

are compact.

The general theory, the definition  $\iota_{q+1, \Gamma_\tau} = \pm \star_{N-q} d_{N-q-1, \Gamma_\tau} \star_{q+1}$ , where  $\star_q$  is the Hodge star-operator, and the substitution  $E = \star_{q+1} H$  shows again a symmetry for the eigenvalues, i.e.,

$$\begin{aligned} \lambda_{q, \Gamma_\tau} &= \inf_{0 \neq E \in D(\mathcal{A}_{q, \Gamma_\tau}) = D(d_{q, \Gamma_\tau}) \cap R(\iota_{q+1, \Gamma_\tau})} \frac{|d_q E|_{L^{2, q+1}(\Omega)}}{|E|_{L^{2, q}(\Omega)}} \\ &= \inf_{0 \neq H \in D(\mathcal{A}_{q, \Gamma_\tau}^*) = D(\iota_{q+1, \Gamma_\tau}) \cap R(d_{q, \Gamma_\tau})} \frac{|\iota_{q+1} H|_{L^{2, q}(\Omega)}}{|H|_{L^{2, q+1}(\Omega)}} \\ &= \inf_{0 \neq E \in D(\mathcal{A}_{N-q-1, \Gamma_\tau}) = D(d_{N-q-1, \Gamma_\tau}) \cap R(\iota_{N-q, \Gamma_\tau})} \frac{|d_{N-q-1} E|_{L^{2, N-q}(\Omega)}}{|E|_{L^{2, N-q-1}(\Omega)}} = \lambda_{N-q-1, \Gamma_\tau}. \end{aligned}$$

Therefore, we obtain the relations

$$\frac{1}{c_{q, \Gamma_\tau}} = \lambda_{q, \Gamma_\tau} = \lambda_{N-q-1, \Gamma_\tau} = \frac{1}{c_{N-q-1, \Gamma_\tau}},$$

which also confirm (for  $N = 1, 2$ ) the results of Sections 2.3.1 and 2.3.2. Using the notations from the 3D case we define

$$\frac{1}{c_{q-1, q, \Gamma_\tau}^2} := \lambda_{q-1, q, \Gamma_\tau}^2 := \inf \frac{|d_q E|_{L^{2, q+1}(\Omega)}^2 + |\iota_q E|_{L^{2, q-1}(\Omega)}^2}{|E|_{L^{2, q}(\Omega)}^2},$$

where the infimum is taken over all  $0 \neq E \in D(\mathcal{A}_{q, \Gamma_\tau}) \cap D(\mathcal{A}_{q-1, \Gamma_\tau}^*) = D(d_{q, \Gamma_\tau}) \cap D(\iota_{q, \Gamma_\tau})$  being perpendicular to the respective generalized Dirichlet–Neumann forms  $N(\mathcal{A}_{q, \Gamma_\tau}) \cap N(\mathcal{A}_{q-1, \Gamma_\tau}^*)$ . Theorem 2.20 turns to:

**Theorem 2.25** (Poincaré–Friedrichs/maxwell Constants in ND). For  $c_{q, \Gamma_\tau} = 1/\lambda_{q, \Gamma_\tau}$  the following holds:

(i) The Poincaré–Friedrichs constants depend monotonically on the boundary conditions, i.e.,

$$\emptyset \neq \tilde{\Gamma}_\tau \subset \Gamma_\tau \Rightarrow c_{0, \Gamma_\tau} \leq c_{0, \tilde{\Gamma}_\tau}.$$

(ii) The Friedrichs constant is always smaller than the Poincaré constant, i.e.,  $c_{0, \Gamma} \leq c_{0, \emptyset}$ .

(iii)  $c_{0, \Gamma} \leq \text{diam}(\Omega)/\pi$

(iv)  $c_{q, \Gamma_\tau} = c_{N-q-1, \Gamma_\tau}$

(v)  $c_{q-1, q, \Gamma_\tau} = \max\{c_{q-1, \Gamma_\tau}, c_{q, \Gamma_\tau}\}$

(vi) If  $\Omega$  is topologically trivial, then  $c_{0, \Gamma} \leq c_{q-1, q, \Gamma_\tau}$ .

(vii) If  $\Omega$  is convex, then  $c_{0, \Gamma} \leq c_{0, \emptyset} \leq \text{diam}(\Omega)/\pi$ .

(viii) If  $\Omega$  is convex, then  $c_{q, \Gamma}, c_{q, \emptyset} \leq c_{0, \emptyset} \leq \text{diam}(\Omega)/\pi$ .

(ix) If  $\Omega$  is convex, then  $c_{0, \Gamma} \leq c_{q-1, q, \Gamma} = \max\{c_{q-1, \Gamma}, c_{q, \Gamma}\} \leq c_{0, \emptyset} \leq \text{diam}(\Omega)/\pi$ .

(ix') If  $\Omega$  is convex, then  $c_{0, \Gamma} \leq c_{q-1, q, \emptyset} = \max\{c_{q-1, \emptyset}, c_{q, \emptyset}\} \leq c_{0, \emptyset} \leq \text{diam}(\Omega)/\pi$ .

For proofs and details see [4]. To show (vi), for which an argument is missing in [4], let  $I$  be a multi-index of order  $q$  and let  $u \in H_I^1(\Omega) = H_I(\text{grad}, \Omega)$ . Then

$$E := u \, dx^I \in H_{I_\tau}^{1, q}(\Omega) \subset H_{I_\tau}(d_q, \Omega) \cap H_{I_\tau}(\iota_q, \Omega)$$

and we have by approximation and the triviality of Dirichlet–Neumann forms

$$\begin{aligned} |u|_{L^2(\Omega)} &= |E|_{L^{2, q}(\Omega)} \leq c_{q-1, q, \Gamma_\tau} (|d_q E|_{L^{2, q+1}(\Omega)}^2 + |\iota_q E|_{L^{2, q-1}(\Omega)}^2)^{1/2} \\ &= c_{q-1, q, \Gamma_\tau} \left( \sum_{\ell=1}^N |\partial_\ell E|_{L^{2, q}(\Omega)}^2 \right)^{1/2} = c_{q-1, q, \Gamma_\tau} |\text{grad } u|_{L^2(\Omega)}, \end{aligned}$$

showing  $c_{0, \Gamma} \leq c_{q-1, q, \Gamma_\tau}$ .

### 2.3.4. 3D-elasticity complex

The complex (involving vector as well as symmetric tensor fields)

$$H_{\Gamma_\tau}(\text{Grad}, \Omega) \xrightarrow{A_0 = \text{sym Grad}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{Curl Curl}^\top, \mathbb{S}, \Omega) \xrightarrow{A_1 = \text{Curl Curl}_{\Gamma_\tau}^\top} H_{\Gamma_\tau}(\text{Div}, \mathbb{S}, \Omega) \xrightarrow{A_2 = \text{Div}_{\Gamma_\tau}} L^2(\Omega),$$

$$L^2(\Omega) \xleftarrow{A_0^* = -\text{Div}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{Div}, \mathbb{S}, \Omega) \xleftarrow{A_1^* = \text{Curl Curl}_{\Gamma_\tau}^\top} H_{\Gamma_\tau}(\text{Curl Curl}^\top, \mathbb{S}, \Omega) \xleftarrow{A_2^* = \text{sym Grad}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{Grad}, \Omega),$$

is related to elasticity, see, e.g., [36,40–47] for details about the complex and numerical applications. Note that, indeed, by Korn's inequality the regularity

$$D(A_0) = D(\text{sym Grad}_{\Gamma_\tau}) = H_{\Gamma_\tau}(\text{sym Grad}, \Omega) = H_{\Gamma_\tau}(\text{Grad}, \Omega) = H_{\Gamma_\tau}^1(\Omega)$$

holds. The “second order Laplace and Maxwell” operators are given by

$$\begin{aligned} A_0^* A_0 &= -\text{Div}_{\Gamma_v} \text{sym Grad}_{\Gamma_\tau}, & A_0 A_0^* &= -\text{sym Grad}_{\Gamma_\tau} \text{Div}_{\Gamma_v}, \\ A_1^* A_1 &= \text{Curl Curl}_{\Gamma_v}^\top \text{Curl Curl}_{\Gamma_\tau}^\top, & A_1 A_1^* &= \text{Curl Curl}_{\Gamma_\tau}^\top \text{Curl Curl}_{\Gamma_v}^\top, \end{aligned}$$

and for the constants and eigenvalues  $c_{\ell, \Gamma_\tau}^{\text{ela}} = 1/\lambda_{\ell, \Gamma_\tau}^{\text{ela}}$  we have

$$\begin{aligned} \forall v \in D(\mathcal{A}_0) &= D(\text{sym Grad}_{\Gamma_\tau}) \cap R(\text{Div}_{\Gamma_v}) & |v|_{L^2(\Omega)} &\leq c_{0, \Gamma_\tau}^{\text{ela}} |\text{sym Grad } v|_{L^2(\mathbb{S}, \Omega)}, \\ \forall S \in D(\mathcal{A}_0^*) &= D(\text{Div}_{\Gamma_v}) \cap R(\text{sym Grad}_{\Gamma_\tau}) & |S|_{L^2(\mathbb{S}, \Omega)} &\leq c_{0, \Gamma_\tau}^{\text{ela}} |\text{Div } S|_{L^2(\Omega)}, \\ \forall S \in D(\mathcal{A}_1) &= D(\text{Curl Curl}_{\Gamma_\tau}^\top) \cap R(\text{Curl Curl}_{\Gamma_v}^\top) & |S|_{L^2(\mathbb{S}, \Omega)} &\leq c_{1, \Gamma_\tau}^{\text{ela}} |\text{Curl Curl}^\top S|_{L^2(\mathbb{S}, \Omega)}, \\ \forall T \in D(\mathcal{A}_1^*) &= D(\text{Curl Curl}_{\Gamma_v}^\top) \cap R(\text{Curl Curl}_{\Gamma_\tau}^\top) & |T|_{L^2(\mathbb{S}, \Omega)} &\leq c_{1, \Gamma_\tau}^{\text{ela}} |\text{Curl Curl}^\top T|_{L^2(\mathbb{S}, \Omega)}, \\ \forall T \in D(\mathcal{A}_2) &= D(\text{Div}_{\Gamma_\tau}) \cap R(\text{sym Grad}_{\Gamma_v}) & |T|_{L^2(\mathbb{S}, \Omega)} &\leq c_{2, \Gamma_\tau}^{\text{ela}} |\text{Div } T|_{L^2(\Omega)}, \\ \forall v \in D(\mathcal{A}_2^*) &= D(\text{sym Grad}_{\Gamma_v}) \cap R(\text{Div}_{\Gamma_\tau}) & |v|_{L^2(\Omega)} &\leq c_{2, \Gamma_\tau}^{\text{ela}} |\text{sym Grad } v|_{L^2(\mathbb{S}, \Omega)}. \end{aligned}$$

As in the 3D Maxwell case the last two inequalities are already given by the first two. Note that

$$\begin{aligned} N(\text{sym Grad}_{\Gamma_\tau}) &= \begin{cases} \{0\} & \text{if } \Gamma_\tau \neq \emptyset, \\ \text{RM} & \text{if } \Gamma_\tau = \emptyset, \end{cases} \\ R(\text{Div}_{\Gamma_v}) = N(\text{sym Grad}_{\Gamma_\tau})^{\perp L^2(\Omega)} &= \begin{cases} L^2(\Omega) & \text{if } \Gamma_v \neq \Gamma, \\ L^2(\Omega) \cap \text{RM}^{\perp L^2(\Omega)} & \text{if } \Gamma_v = \Gamma, \end{cases} \end{aligned}$$

where RM denotes the space of global rigid motions. The crucial compact embeddings (13) have recently been proved in [48].

**Theorem 2.26** (Selection Theorems for Elasticity). *The embedding*

$$D(\mathcal{A}_1) \cap D(\mathcal{A}_0^*) = H_{\Gamma_\tau}(\text{Curl Curl}^\top, \mathbb{S}, \Omega) \cap H_{\Gamma_v}(\text{Div}, \mathbb{S}, \Omega) \hookrightarrow L^2(\mathbb{S}, \Omega)$$

is compact.

Note that by the latter theorem the embedding

$$D(\mathcal{A}_2) \cap D(\mathcal{A}_1^*) = H_{\Gamma_\tau}(\text{Div}, \mathbb{S}, \Omega) \cap H_{\Gamma_v}(\text{Curl Curl}^\top, \mathbb{S}, \Omega) \hookrightarrow L^2(\mathbb{S}, \Omega)$$

is compact as well by interchanging  $\Gamma_\tau$  and  $\Gamma_v$ .

Similar to the 3D Maxwell case we get the following theorem, cf. Theorem 2.20.

**Theorem 2.27** (Poincaré–Friedrichs Type Constants for Elasticity). *For  $c_{\ell, \Gamma_\tau}^{\text{ela}} = 1/\lambda_{\ell, \Gamma_\tau}^{\text{ela}}$  the following holds:*

(i) *The Poincaré–Friedrichs type constants depend monotonically on the boundary conditions, i.e.,*

$$\emptyset \neq \tilde{\Gamma}_\tau \subset \Gamma_\tau \Rightarrow c_{0, \Gamma_\tau}^{\text{ela}} \leq c_{0, \tilde{\Gamma}_\tau}^{\text{ela}}.$$

(ii)  $c_{0, \Gamma_\tau}^{\text{ela}} = c_{2, \Gamma_v}^{\text{ela}}$  and  $c_{1, \Gamma_\tau}^{\text{ela}} = c_{1, \Gamma_v}^{\text{ela}}$ .

**Remark 2.28** (Poincaré–Friedrichs Type Constants for Elasticity). The Poincaré–Friedrichs type constants of the elasticity complex  $c_{0, \Gamma_\tau}^{\text{ela}} = c_{2, \Gamma_v}^{\text{ela}}$  are related to the classical Poincaré–Friedrichs constants  $c_{0, \Gamma_\tau} = c_{2, \Gamma_v}$  by Korn’s inequality, i.e.,

$$\begin{aligned} \forall v \in D(\mathcal{A}_0) &= \underbrace{D(\text{sym Grad}_{\Gamma_\tau}) \cap R(\text{Div}_{\Gamma_v})}_{=H_{\Gamma_\tau}^1(\Omega)} & |\text{Grad } v|_{L^2(\Omega)} &\leq c_{k, \Gamma_\tau} |\text{sym Grad } v|_{L^2(\mathbb{S}, \Omega)}. \end{aligned}$$

More precisely,

$$c_{2, \Gamma_v}^{\text{ela}} = c_{0, \Gamma_\tau}^{\text{ela}} \leq c_{k, \Gamma_\tau} c_{0, \Gamma_\tau} = c_{k, \Gamma_\tau} c_{2, \Gamma_v}$$

holds, as for all  $v \in D(\mathcal{A}_0)$

$$|v|_{L^2(\Omega)} \leq c_{0, \Gamma_\tau} |\text{Grad } v|_{L^2(\Omega)} \leq c_{k, \Gamma_\tau} c_{0, \Gamma_\tau} |\text{sym Grad } v|_{L^2(\mathbb{S}, \Omega)}.$$

In particular, for  $\Gamma_\tau = \Gamma$  we know  $c_{k, \Gamma} \leq \sqrt{2}$ , see [49,50], which shows by Theorem 2.20

$$c_{2, \emptyset}^{\text{ela}} = c_{0, \Gamma}^{\text{ela}} \leq \sqrt{2} c_{0, \Gamma} \leq \sqrt{2} \min\{c_{0, \Gamma_\tau}, c_{0, \emptyset}, \frac{\text{diam}(\Omega)}{\pi}\} \leq \frac{\sqrt{2}}{\pi} \text{diam}(\Omega).$$

### 2.3.5. 3D-Biharmonic Complex (div Div-complex)

The complex (involving scalar as well as symmetric and deviatoric tensor fields)

$$\begin{aligned} H_{\Gamma_\tau}^2(\Omega) &= H_{\Gamma_\tau}(\text{Grad grad}, \Omega) \xrightarrow{A_0 = \text{Grad grad}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{Curl}, \mathbb{S}, \Omega) \xrightarrow{A_1 = \text{Curl}_{\Gamma_\tau}} H_{\Gamma_\tau}(\text{Div}, \mathbb{T}, \Omega) \xrightarrow{A_2 = \text{Div}_{\Gamma_\tau}} L^2(\Omega), \\ L^2(\Omega) &\xleftarrow{A_0^* = \text{div Div}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{div Div}, \mathbb{S}, \Omega) \xleftarrow{A_1^* = \text{sym Curl}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{sym Curl}, \mathbb{T}, \Omega) \xleftarrow{A_2^* = -\text{dev Grad}_{\Gamma_\nu}} H_{\Gamma_\nu}(\text{Grad}, \Omega), \end{aligned}$$

arises in general relativity and for the biharmonic equation, see, e.g., [20] for details and, e.g., [51–55] for numerical applications. Note that, indeed, similar to using Korn's inequality in the latter section, the regularity

$$D(A_2^*) = D(\text{dev Grad}_{\Gamma_\nu}) = H_{\Gamma_\nu}(\text{dev Grad}, \Omega) = H_{\Gamma_\nu}(\text{Grad}, \Omega) = H_{\Gamma_\nu}^1(\Omega)$$

holds, cf. [20, Lemma 3.2]. The “second order Laplace and Maxwell” operators are given by

$$\begin{aligned} A_0^* A_0 &= \text{div Div}_{\Gamma_\nu} \text{Grad grad}_{\Gamma_\tau}, & A_0 A_0^* &= \text{Grad grad}_{\Gamma_\tau} \text{div Div}_{\Gamma_\nu}, \\ A_1^* A_1 &= \text{sym Curl}_{\Gamma_\nu} \text{Curl}_{\Gamma_\tau}, & A_1 A_1^* &= \text{Curl}_{\Gamma_\tau} \text{sym Curl}_{\Gamma_\nu}, \\ A_2^* A_2 &= -\text{dev Grad}_{\Gamma_\nu} \text{Div}_{\Gamma_\tau}, & A_2 A_2^* &= -\text{Div}_{\Gamma_\tau} \text{dev Grad}_{\Gamma_\nu}, \end{aligned}$$

and for the constants and eigenvalues  $c_{\ell, \Gamma_\tau}^{\text{bih}} = 1/\lambda_{\ell, \Gamma_\tau}^{\text{bih}}$  we have

$$\begin{aligned} \forall u \in D(A_0) &= D(\text{Grad grad}_{\Gamma_\tau}) \cap R(\text{div Div}_{\Gamma_\nu}) & |u|_{L^2(\Omega)} &\leq c_{0, \Gamma_\tau}^{\text{bih}} |\text{Grad grad } u|_{L^2(\mathbb{S}, \Omega)}, \\ \forall S \in D(A_0^*) &= D(\text{div Div}_{\Gamma_\nu}) \cap R(\text{Grad grad}_{\Gamma_\tau}) & |S|_{L^2(\mathbb{S}, \Omega)} &\leq c_{0, \Gamma_\tau}^{\text{bih}} |\text{div Div } S|_{L^2(\Omega)}, \\ \forall S \in D(A_1) &= D(\text{Curl}_{\Gamma_\tau}) \cap R(\text{sym Curl}_{\Gamma_\nu}) & |S|_{L^2(\mathbb{S}, \Omega)} &\leq c_{1, \Gamma_\tau}^{\text{bih}} |\text{Curl } S|_{L^2(\mathbb{T}, \Omega)}, \\ \forall T \in D(A_1^*) &= D(\text{sym Curl}_{\Gamma_\nu}) \cap R(\text{Curl}_{\Gamma_\tau}) & |T|_{L^2(\mathbb{T}, \Omega)} &\leq c_{1, \Gamma_\tau}^{\text{bih}} |\text{sym Curl } T|_{L^2(\mathbb{S}, \Omega)}, \\ \forall T \in D(A_2) &= D(\text{Div}_{\Gamma_\tau}) \cap R(\text{dev Grad}_{\Gamma_\nu}) & |T|_{L^2(\mathbb{T}, \Omega)} &\leq c_{2, \Gamma_\tau}^{\text{bih}} |\text{Div } T|_{L^2(\Omega)}, \\ \forall v \in D(A_2^*) &= D(\text{dev Grad}_{\Gamma_\nu}) \cap R(\text{Div}_{\Gamma_\tau}) & |v|_{L^2(\Omega)} &\leq c_{2, \Gamma_\tau}^{\text{bih}} |\text{dev Grad } v|_{L^2(\mathbb{T}, \Omega)}. \end{aligned}$$

We emphasize that this complex is the first non-symmetric one and we get additional results for the operators involving  $A_2$ . Note that

$$\begin{aligned} N(\text{Grad grad}_{\Gamma_\tau}) &= \begin{cases} \{0\} & \text{if } \Gamma_\tau \neq \emptyset, \\ \mathbf{P}^1 & \text{if } \Gamma_\tau = \emptyset, \end{cases} \\ R(\text{div Div}_{\Gamma_\nu}) = N(\text{Grad grad}_{\Gamma_\tau})^{\perp L^2(\Omega)} &= \begin{cases} L^2(\Omega) & \text{if } \Gamma_\nu \neq \Gamma, \\ L^2(\Omega) \cap (\mathbf{P}^1)^{\perp L^2(\Omega)} & \text{if } \Gamma_\nu = \Gamma, \end{cases} \\ N(\text{dev Grad}_{\Gamma_\nu}) &= \begin{cases} \{0\} & \text{if } \Gamma_\nu \neq \emptyset, \\ \text{RT} & \text{if } \Gamma_\nu = \emptyset, \end{cases} \\ R(\text{Div}_{\Gamma_\tau}) = N(\text{dev Grad}_{\Gamma_\nu})^{\perp L^2(\Omega)} &= \begin{cases} L^2(\Omega) & \text{if } \Gamma_\tau \neq \Gamma, \\ L^2(\Omega) \cap \text{RT}^{\perp L^2(\Omega)} & \text{if } \Gamma_\tau = \Gamma, \end{cases} \end{aligned}$$

where  $\mathbf{P}^1$  denotes the polynomials of order less than 1 and RT the space of global Raviart–Thomas vector fields. The crucial compact embeddings (13) have recently been proved in [20, Lemma 3.22].

**Theorem 2.29** (Selection Theorems for the Biharmonic Complex). *The embeddings*

$$\begin{aligned} D(A_1) \cap D(A_0^*) &= H_{\Gamma_\tau}(\text{Curl}, \mathbb{S}, \Omega) \cap H_{\Gamma_\nu}(\text{div Div}, \mathbb{S}, \Omega) \hookrightarrow L^2(\mathbb{S}, \Omega), \\ D(A_2) \cap D(A_1^*) &= H_{\Gamma_\tau}(\text{Div}, \mathbb{T}, \Omega) \cap H_{\Gamma_\nu}(\text{sym Curl}, \mathbb{T}, \Omega) \hookrightarrow L^2(\mathbb{T}, \Omega) \end{aligned}$$

are compact.

Similar to the 3D Maxwell case and the 3D elasticity case we get the following result, cf. Theorems 2.20 and 2.27, and Remark 2.28.

**Remark 2.30** (Poincaré–Friedrichs Type Constants for the Biharmonic Complex). For  $c_{\ell, \Gamma_\tau}^{\text{bit}} = 1/\lambda_{\ell, \Gamma_\tau}^{\text{bit}}$  the following holds:

(i) The Poincaré–Friedrichs type constants depend monotonically on the boundary conditions, i.e.,

$$\emptyset \neq \tilde{\Gamma}_\tau \subset \Gamma_\tau \Rightarrow c_{0, \tilde{\Gamma}_\tau}^{\text{bih}} \leq c_{0, \Gamma_\tau}^{\text{bih}}.$$

(ii) Due to the lack a symmetry in the biharmonic complex there are no further formulas relating  $c_{0, \Gamma_\tau}^{\text{bih}}$  to  $c_{2, \Gamma_\nu}^{\text{bih}}$  or  $c_{1, \Gamma_\tau}^{\text{bih}}$  to  $c_{1, \Gamma_\nu}^{\text{bih}}$ .

(iii) As pointed out in [Remark 2.28](#) for the elasticity complex, there is a similar relation between the Poincaré–Friedrichs type constants of the biharmonic complex  $c_{0,\Gamma_\tau}^{\text{bih}}$  and  $c_{2,\Gamma_\tau}^{\text{bih}}$  and the classical Poincaré–Friedrichs constants  $c_{0,\Gamma_\tau} = c_{2,\Gamma_\nu}$  by the classical Poincaré–Friedrichs estimate and a Korn like inequality, i.e.,

$$\forall v \in D(\mathcal{A}_2^*) = \underbrace{D(\text{dev Grad}_{\Gamma_\nu})}_{=\mathcal{H}_{\Gamma_\nu}^1(\Omega)} \cap R(\text{Div}_{\Gamma_\tau}) \quad |\text{Grad } v|_{L^2(\Omega)} \leq c_{\text{dev},\Gamma_\nu} |\text{dev Grad } v|_{L^2(\mathbb{T},\Omega)},$$

cf. [\[20, Lemma 3.2\]](#). More precisely,  $c_{0,\Gamma_\tau}^{\text{bih}} \leq c_{0,\Gamma_\tau}^2$  and  $c_{2,\Gamma_\tau}^{\text{bih}} \leq c_{\text{dev},\Gamma_\nu} c_{0,\Gamma_\nu}$  hold, as

$$\forall v \in D(\mathcal{A}_2^*) \quad |v|_{L^2(\Omega)} \leq c_{0,\Gamma_\nu} |\text{Grad } v|_{L^2(\Omega)} \leq c_{\text{dev},\Gamma_\nu} c_{0,\Gamma_\nu} |\text{dev Grad } v|_{L^2(\mathbb{T},\Omega)}.$$

### 3. Analytical examples

In the sequel we will compute all Poincaré–Friedrichs and Maxwell eigenvalues for the unit cube in 1D, 2D, and 3D with mixed boundary conditions on canonical boundary parts. We emphasize that the completeness of the respective eigensystems can be shown as in [\[7\]](#).

#### 3.1. 1D

Let  $\Omega := I := (0, 1)$ ,  $\Gamma = \{0, 1\}$ , and  $\Gamma_\tau \in P(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \Gamma\}$ , and recall [Section 2.3.1](#). From [Appendix B.1](#) we see

$$c_{0,\Gamma} = c_{0,\emptyset} = \frac{1}{\pi}, \quad c_{0,\{0\}} = c_{0,\{1\}} = \frac{2}{\pi}. \quad (17)$$

Note that from  $c_{0,\Gamma_\tau} = c_{0,\Gamma_\nu}$ , see [Corollary 2.22](#), we already know  $c_{0,\Gamma} = c_{0,\emptyset}$  and  $c_{0,\{0\}} = c_{0,\{1\}}$ .

**Remark 3.1.** [Corollary 2.22](#) may be verified by this example.

- (i)  $\emptyset \neq \{0\}, \{1\} \subset \Gamma \Rightarrow c_{0,\Gamma} = \frac{1}{\pi} \leq \frac{2}{\pi} = c_{0,\{0\}} = c_{0,\{1\}}$
- (ii)  $c_{0,\Gamma} = c_{0,\emptyset} = \frac{1}{\pi} = \frac{\text{diam}(\Omega)}{\pi}$

#### 3.2. 2D

Let  $\Omega := I^2$ ,  $I := (0, 1)$ ,  $\Gamma = \overline{\Gamma_b \cup \Gamma_t \cup \Gamma_l \cup \Gamma_r}$ , where  $\Gamma_b$ ,  $\Gamma_t$ ,  $\Gamma_l$ ,  $\Gamma_r$  are the open bottom, top, left, and right boundary parts of  $\Gamma$ , respectively, and  $\Gamma_\tau \in P(\{\Gamma_b, \Gamma_t, \Gamma_l, \Gamma_r\})$ , and recall [Section 2.3.2](#). We shall use canonical index notations such as

$$\Gamma_{b,l} := \text{int}(\overline{\Gamma_b \cup \Gamma_l}), \quad \Gamma_{b,l,t} := \text{int}(\overline{\Gamma_b \cup \Gamma_l \cup \Gamma_t}).$$

From [Appendix B.2](#) we see

$$\begin{aligned} c_{0,\emptyset} &= \frac{1}{\pi}, & c_{0,\Gamma_{b,l}} &= c_{0,\Gamma_{b,r}} = c_{0,\Gamma_{t,l}} = c_{0,\Gamma_{t,r}} = \frac{\sqrt{2}}{\pi}, \\ c_{0,\Gamma_b} &= c_{0,\Gamma_t} = c_{0,\Gamma_l} = c_{0,\Gamma_r} = \frac{2}{\pi}, & c_{0,\Gamma_{b,l,r}} &= c_{0,\Gamma_{t,l,r}} = c_{0,\Gamma_{b,t,l}} = c_{0,\Gamma_{b,t,r}} = \frac{2}{\sqrt{5}\pi}, \\ c_{0,\Gamma_{b,t}} &= c_{0,\Gamma_{l,r}} = \frac{1}{\pi}, & c_{0,\Gamma} &= \frac{1}{\sqrt{2}\pi}. \end{aligned} \quad (18)$$

**Remark 3.2.** [Corollary 2.23](#) may be verified by this example.

- (i)  $\emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma \Rightarrow c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_b} = \frac{2}{\pi}$
- (i')  $\emptyset \neq \Gamma_l \subset \Gamma_{l,r} \subset \Gamma_{b,l,r} \subset \Gamma \Rightarrow c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{l,r}} = \frac{1}{\pi} \leq c_{0,\Gamma_l} = \frac{2}{\pi}$
- (ii)  $c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset}$
- (iii)  $c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq \frac{\sqrt{2}}{\pi} = \frac{\text{diam}(\Omega)}{\pi}$
- (iv)  $\Omega$  is convex and  $c_{0,\Gamma} = \frac{1}{\sqrt{2}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{2}}{\pi} = \frac{\text{diam}(\Omega)}{\pi}$ .



### 3.3. 3D

Let  $\Omega := \widehat{\Omega} \times I = I^3$ ,  $\widehat{\Omega} := I^2$ ,  $I := (0, 1)$ ,  $\Gamma = \overline{\Gamma_b \cup \Gamma_t \cup \Gamma_l \cup \Gamma_r \cup \Gamma_f \cup \Gamma_k}$ , where  $\Gamma_b, \Gamma_t, \Gamma_l, \Gamma_r, \Gamma_f, \Gamma_k$  are the open bottom, top, left, right, front, and back boundary parts of  $\Gamma$ , respectively, and  $\Gamma_\tau \in P(\{\Gamma_b, \Gamma_t, \Gamma_l, \Gamma_r, \Gamma_f, \Gamma_k\})$ , and recall Section 2.2 as well as Theorem 2.20. Again, we use canonical index notations such as

$$\Gamma_{b,r} := \text{int}(\overline{\Gamma_b \cup \Gamma_r}), \quad \Gamma_{b,r,k,f} := \text{int}(\overline{\Gamma_b \cup \Gamma_r \cup \Gamma_k \cup \Gamma_f}).$$

From Appendix B.3 we see for  $c_{0,\Gamma_\tau}$

$$\begin{aligned} c_{0,\emptyset} &= \frac{1}{\pi}, \\ c_{0,\Gamma_b} &= c_{0,\Gamma_t} = c_{0,\Gamma_l} = c_{0,\Gamma_r} = c_{0,\Gamma_f} = c_{0,\Gamma_k} = \frac{2}{\pi}, \\ c_{0,\Gamma_{b,t}} &= c_{0,\Gamma_{l,r}} = c_{0,\Gamma_{f,k}} = \frac{1}{\pi}, \\ c_{0,\Gamma_{b,l}} &= c_{0,\Gamma_{b,r}} = c_{0,\Gamma_{b,f}} = c_{0,\Gamma_{b,k}} \\ &= c_{0,\Gamma_{t,l}} = c_{0,\Gamma_{t,r}} = c_{0,\Gamma_{t,f}} = c_{0,\Gamma_{t,k}} = c_{0,\Gamma_{f,l}} = c_{0,\Gamma_{f,r}} = c_{0,\Gamma_{k,l}} = c_{0,\Gamma_{k,r}} = \frac{\sqrt{2}}{\pi}, \\ c_{0,\Gamma_{b,t,l}} &= c_{0,\Gamma_{b,t,r}} = c_{0,\Gamma_{b,t,f}} = c_{0,\Gamma_{b,t,k}} = c_{0,\Gamma_{l,r,b}} \\ &= c_{0,\Gamma_{l,r,t}} = c_{0,\Gamma_{l,r,f}} = c_{0,\Gamma_{l,r,k}} = c_{0,\Gamma_{f,k,l}} = c_{0,\Gamma_{f,k,r}} = c_{0,\Gamma_{f,k,b}} = c_{0,\Gamma_{f,k,t}} = \frac{2}{\sqrt{5\pi}}, \\ c_{0,\Gamma_{b,k,l}} &= c_{0,\Gamma_{b,l,f}} = c_{0,\Gamma_{b,f,r}} = c_{0,\Gamma_{b,r,k}} = c_{0,\Gamma_{t,k,l}} = c_{0,\Gamma_{t,l,f}} = c_{0,\Gamma_{t,f,r}} = c_{0,\Gamma_{t,r,k}} = \frac{2}{\sqrt{3\pi}}, \\ c_{0,\Gamma_{b,t,l,r}} &= c_{0,\Gamma_{b,t,f,k}} = c_{0,\Gamma_{l,r,f,k}} = \frac{1}{\sqrt{2\pi}}, \\ c_{0,\Gamma_{b,t,l,k}} &= c_{0,\Gamma_{b,t,f,l}} = c_{0,\Gamma_{b,t,r,f}} = c_{0,\Gamma_{b,t,r,k}} = c_{0,\Gamma_{l,r,f,t}} = c_{0,\Gamma_{l,r,f,b}} \\ &= c_{0,\Gamma_{l,r,t,k}} = c_{0,\Gamma_{l,r,b,k}} = c_{0,\Gamma_{f,k,b,l}} = c_{0,\Gamma_{f,k,t,l}} = c_{0,\Gamma_{f,k,b,r}} = c_{0,\Gamma_{f,k,r,r}} = \frac{2}{\sqrt{6\pi}}, \\ c_{0,\Gamma_{b,t,l,r,k}} &= c_{0,\Gamma_{b,t,l,r,f}} = c_{0,\Gamma_{b,t,l,f,k}} = c_{0,\Gamma_{b,t,r,f,k}} = c_{0,\Gamma_{b,l,r,f,k}} = c_{0,\Gamma_{t,l,r,f,k}} = \frac{2}{3\pi}, \\ c_{0,\Gamma} &= \frac{1}{\sqrt{3\pi}}, \end{aligned} \tag{19}$$

and for  $c_{1,\Gamma_\tau}$

$$\begin{aligned} c_{1,\emptyset} &= c_{1,\Gamma} = \frac{1}{\sqrt{2\pi}}, \\ c_{1,\Gamma_b} &= c_{1,\Gamma_t} = c_{1,\Gamma_l} = c_{1,\Gamma_r} = c_{1,\Gamma_f} = c_{1,\Gamma_k} = \frac{2}{\sqrt{5\pi}}, \\ c_{1,\Gamma_{l,r}} &= c_{1,\Gamma_{b,t}} = c_{1,\Gamma_{f,k}} = \frac{1}{\pi}, \\ c_{1,\Gamma_{b,l}} &= c_{1,\Gamma_{b,r}} = c_{1,\Gamma_{b,f}} = c_{1,\Gamma_{b,k}} \\ &= c_{1,\Gamma_{t,l}} = c_{1,\Gamma_{t,r}} = c_{1,\Gamma_{t,f}} = c_{1,\Gamma_{t,k}} = c_{1,\Gamma_{f,l}} = c_{1,\Gamma_{l,k}} = c_{1,\Gamma_{k,r}} = c_{1,\Gamma_{f,r}} = \frac{\sqrt{2}}{\pi}, \\ c_{1,\Gamma_{b,t,l}} &= c_{1,\Gamma_{b,t,r}} = c_{1,\Gamma_{b,t,f}} = c_{1,\Gamma_{b,t,k}} = c_{1,\Gamma_{l,r,t}} \\ &= c_{1,\Gamma_{l,r,b}} = c_{1,\Gamma_{l,r,f}} = c_{1,\Gamma_{l,r,k}} = c_{1,\Gamma_{f,k,l}} = c_{1,\Gamma_{f,k,r}} = c_{1,\Gamma_{f,k,t}} = c_{1,\Gamma_{f,k,b}} = \frac{2}{\pi}, \\ c_{1,\Gamma_{b,k,l}} &= c_{1,\Gamma_{b,r,k}} = c_{1,\Gamma_{b,l,f}} = c_{1,\Gamma_{b,r,f}} = c_{1,\Gamma_{t,k,l}} = c_{1,\Gamma_{t,r,k}} = c_{1,\Gamma_{t,l,f}} = c_{1,\Gamma_{t,r,f}} = \frac{2}{\sqrt{3\pi}}, \end{aligned} \tag{20}$$

and all the other remaining cases follow by  $c_{1,\Gamma_\nu} = c_{1,\Gamma_\tau}$  as well as symmetry.

**Remark 3.3.** Theorem 2.20 may be verified by these examples. E.g.:

$$(i) \emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,t,l,r} \subset \Gamma_{b,t,l,r,f} \subset \Gamma \Rightarrow$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3\pi}} \leq c_{0,\Gamma_{b,t,l,r,f}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,t,l,r}} = \frac{1}{\sqrt{2\pi}} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5\pi}} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2\pi}} \leq c_{0,\Gamma_b} = \frac{2}{\pi}$$

$$(i') \emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma \Rightarrow$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,f,l,r,t}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,f,l,r}} = \frac{2}{\sqrt{6}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_b} = \frac{2}{\pi}$$

$$(i'') \emptyset \neq \Gamma_l \subset \Gamma_{l,r} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma \Rightarrow$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,f,l,r,t}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,f,l,r}} = \frac{2}{\sqrt{6}\pi} \leq c_{0,\Gamma_{b,l,r}} = \frac{2}{\sqrt{5}\pi} \leq c_{0,\Gamma_{l,r}} = \frac{1}{\pi} \leq c_{0,\Gamma_l} = \frac{2}{\pi}$$

$$(i''') \emptyset \neq \Gamma_l \subset \Gamma_{b,l} \subset \Gamma_{b,f,l} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma \Rightarrow$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,f,l,r,t}} = \frac{2}{3\pi} \leq c_{0,\Gamma_{b,f,l,r}} = \frac{2}{\sqrt{6}\pi} \leq c_{0,\Gamma_{b,f,l}} = \frac{2}{\sqrt{3}\pi} \leq c_{0,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \leq c_{0,\Gamma_l} = \frac{2}{\pi}$$

$$(ii) c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset}$$

$$(iii) c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{\sqrt{3}}{\pi} = \frac{\text{diam}(\Omega)}{\pi}$$

$$(iv) \Omega \text{ is convex and } c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\text{diam}(\Omega)}{\pi}.$$

$$(v) \Omega \text{ is convex and } c_{1,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\text{diam}(\Omega)}{\pi}.$$

$$(vi) \Omega \text{ is convex and}$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\sqrt{2}\pi} = c_{0,1,\Gamma} = \max\{c_{0,\Gamma}, c_{1,\Gamma}\} = \max\left\{\frac{1}{\sqrt{3}\pi}, \frac{1}{\sqrt{2}\pi}\right\} \leq \frac{1}{\pi} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\text{diam}(\Omega)}{\pi},$$

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \leq \frac{1}{\pi} = c_{0,1,\emptyset} = \max\{c_{0,\emptyset}, c_{1,\emptyset}\} = \max\left\{\frac{1}{\pi}, \frac{1}{\sqrt{2}\pi}\right\} = c_{0,\emptyset} \leq \frac{\sqrt{3}}{\pi} = \frac{\text{diam}(\Omega)}{\pi}.$$

**Remark 3.4.** In general, the Maxwell constants do not have monotonicity properties, which can also be verified by the latter examples. In fact, in our examples, the Maxwell constants are monotone increasing up to a certain situation in the ‘middle’, where the tangential and the normal boundary condition are equally strong, and from there on the Maxwell constants are monotone decreasing. E.g.:

$$\bullet \emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,t,l,r} \subset \Gamma_{b,t,l,r,f} \subset \Gamma, \text{ but}$$

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_k} = \frac{2}{\sqrt{5}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r}} = c_{1,\Gamma_{f,k}} = \frac{1}{\pi} \\ \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,l,r}} = \frac{2}{\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_b} = \frac{2}{\sqrt{5}\pi}.$$

$$\bullet \emptyset \neq \Gamma_b \subset \Gamma_{b,l} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma, \text{ but}$$

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_k} = \frac{2}{\sqrt{5}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,f,l,r}} = c_{1,\Gamma_{t,k}} = \frac{2}{\sqrt{2}\pi} \\ \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,l,r}} = \frac{2}{\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_{b,l}} = \frac{2}{\sqrt{2}\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_b} = \frac{2}{\sqrt{5}\pi}.$$

$$\bullet \emptyset \neq \Gamma_l \subset \Gamma_{l,r} \subset \Gamma_{b,l,r} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma, \text{ but}$$

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_k} = \frac{2}{\sqrt{5}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,f,l,r}} = c_{1,\Gamma_{t,k}} = \frac{2}{\sqrt{2}\pi} \\ \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,l,r}} = \frac{2}{\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_{l,r}} = \frac{1}{\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_l} = \frac{2}{\sqrt{5}\pi}.$$

$$\bullet \emptyset \neq \Gamma_l \subset \Gamma_{b,l} \subset \Gamma_{b,f,l} \subset \Gamma_{b,f,l,r} \subset \Gamma_{b,f,l,r,t} \subset \Gamma, \text{ but}$$

$$c_{1,\Gamma} = \frac{1}{\sqrt{2}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,t,l,r,f}} = c_{1,\Gamma_k} = \frac{2}{\sqrt{5}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{b,f,l,r}} = c_{1,\Gamma_{t,k}} = \frac{2}{\sqrt{2}\pi} \\ \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_{b,l,f}} = \frac{2}{\sqrt{3}\pi} \stackrel{\text{ok}}{\leq} c_{1,\Gamma_{l,b}} = \frac{2}{\sqrt{2}\pi} \stackrel{\text{not ok}}{\not\leq} c_{1,\Gamma_l} = \frac{2}{\sqrt{5}\pi}.$$

#### 4. Numerical examples

The finite element method (FEM) is applied for evaluation of the Rayleigh quotients on finite dimensional subspaces. Constants are therefore approximated and convergence to their exact values is expected for higher dimensions. Assuming

that  $\Omega$  is discretized by a triangular (2D) or a tetrahedral (3D) mesh  $\mathcal{T}$ , we use only the lowest order finite elements available:

- Linear Lagrange (P1) nodal elements  $\theta_i^{P1}$  for approximations of  $H_{\Gamma_\tau}(\text{grad}, \Omega)$  spaces,
- Linear Nédélec (N) edge elements  $\Theta_i^N$  for approximations of  $H_{\Gamma_\tau}(\text{curl}, \Omega)$  spaces,
- Linear Raviart–Thomas (RT) face elements  $\Theta_i^{RT}$  for approximations of  $H_{\Gamma_\tau}(\text{div}, \Omega)$  spaces.

We assemble the mass matrices  $M^{P1}$ ,  $M^N$ ,  $M^{RT}$  and the stiffness matrices  $K^{P1}$ ,  $K^N$ ,  $K^{RT}$  defined by

$$\begin{aligned} M_{ij}^{P1} &= \langle \theta_i^{P1}, \theta_j^{P1} \rangle_{L^2(\Omega)}, & K_{ij}^{P1} &= \langle \text{grad } \theta_i^{P1}, \text{grad } \theta_j^{P1} \rangle_{L^2(\Omega)}, \\ M_{ij}^N &= \langle \Theta_i^N, \Theta_j^N \rangle_{L^2(\Omega)}, & K_{ij}^N &= \langle \text{curl } \Theta_i^N, \text{curl } \Theta_j^N \rangle_{L^2(\Omega)}, \\ M_{ij}^{RT} &= \langle \Theta_i^{RT}, \Theta_j^{RT} \rangle_{L^2(\Omega)}, & K_{ij}^{RT} &= \langle \text{div } \Theta_i^{RT}, \text{div } \Theta_j^{RT} \rangle_{L^2(\Omega)}, \end{aligned}$$

where the indices  $i, j$  are the global numbers of the corresponding degrees of freedom, i.e., they are related to mesh nodes (for P1 elements) or mesh edges or faces (for N and RT elements). By using the affine mappings (for P1 elements) or Piola mappings (for N and RT elements) from reference elements we can assemble the local matrices. Detailed implementation of finite element assemblies is explained in [56,57]. Squares of terms from Theorem 2.17 are easy to evaluate as quadratic forms with mass and stiffness matrices:

$$\begin{aligned} |u|_{L^2(\Omega)}^2 &= M^{P1} u^{P1} \cdot u^{P1}, & |\text{grad } u|_{L^2(\Omega)}^2 &= K^{P1} u^{P1} \cdot u^{P1}, \\ |E|_{L^2(\Omega)}^2 &= M^N E^N \cdot E^N, & |\text{curl } E|_{L^2(\Omega)}^2 &= K^N E^N \cdot E^N, \\ |H|_{L^2(\Omega)}^2 &= M^{RT} H^{RT} \cdot H^{RT}, & |\text{div } H|_{L^2(\Omega)}^2 &= K^{RT} H^{RT} \cdot H^{RT}, \end{aligned}$$

where  $u^{P1}$ ,  $E^N$ , and  $H^{RT}$  represent (column) vectors of coefficients with respect to their finite element bases of P1, N, and RT, respectively.

#### 4.1. Poincaré–Friedrichs and Divergence Constants

The classical Friedrichs constant  $c_{0,\Gamma}$  is approximated as

$$\frac{1}{c_{0,\Gamma,P1}^2} = \lambda_{0,\Gamma,P1}^2 = \min_{\substack{0 \neq u^{P1}, \\ u_{\Gamma}^{P1} = 0}} \frac{K^{P1} u^{P1} \cdot u^{P1}}{M^{P1} u^{P1} \cdot u^{P1}},$$

where  $u_{\Gamma}^{P1}$  denotes a subvector of  $u^{P1}$  in indices corresponding to boundary nodes.  $\lambda_{0,\Gamma,P1}^2$  is the minimal (positive) eigenvalue of the generalized eigenvalue problem

$$K^{P1} u^{P1} = \lambda^2 M^{P1} u^{P1}, \quad u_{\Gamma}^{P1} = 0,$$

and may also be found by computing the minimal (positive) eigenvalue of

$$K_{\text{int}}^{P1} u_{\text{int}}^{P1} = \lambda^2 M_{\text{int}}^{P1} u_{\text{int}}^{P1},$$

where  $K_{\text{int}}^{P1}$ ,  $M_{\text{int}}^{P1}$ , and  $u_{\text{int}}^{P1}$  are restrictions of the matrices  $K^{P1}$ ,  $M^{P1}$ , and the vector  $u^{P1}$ , respectively, to indices corresponding to internal mesh nodes only. Note that  $K_{\text{int}}^{P1}$  is regular.

The classical Poincaré constant  $c_{0,\emptyset}$  is approximated as

$$\frac{1}{c_{0,\emptyset,P1}^2} = \lambda_{0,\emptyset,P1}^2 = \min_{\substack{0 \neq u^{P1}, \\ u^{P1} \cdot \mathbf{1}^{P1} = 0}} \frac{K^{P1} u^{P1} \cdot u^{P1}}{M^{P1} u^{P1} \cdot u^{P1}},$$

where the constraint  $u^{P1} \cdot \mathbf{1}^{P1} = 0$  means that the vector  $u^{P1}$  has to be perpendicular to the constant vector of ones.  $\lambda_{0,\emptyset,P1}^2$  is the minimal positive eigenvalue of the generalized eigenvalue problem

$$K^{P1} u^{P1} = \lambda^2 M^{P1} u^{P1}.$$

The minimal eigenvalue here is  $\lambda^2 = 0$  and the corresponding eigenvector is the constant vector of ones. Analogously, the Poincaré–Friedrichs (Laplace) constants for mixed boundary conditions  $c_{0,\Gamma_\tau}$  is approximated by using the same techniques and finite elements P1. More precisely, for  $\Gamma_\tau \neq \emptyset$  we have

$$\frac{1}{c_{0,\Gamma_\tau,P1}^2} = \lambda_{0,\Gamma_\tau,P1}^2 = \min_{\substack{0 \neq u^{P1}, \\ u_{\Gamma_\tau}^{P1} = 0}} \frac{K^{P1} u^{P1} \cdot u^{P1}}{M^{P1} u^{P1} \cdot u^{P1}}, \quad (21)$$

where  $u_{\Gamma_\tau}^{P1}$  denotes a subvector of  $u^{P1}$  in indices corresponding to boundary nodes of  $\Gamma_\tau$ .  $\lambda_{0,\Gamma_\tau,P1}^2$  is the minimal (positive) eigenvalue of the generalized eigenvalue problem

$$K^{P1} u^{P1} = \lambda^2 M^{P1} u^{P1}, \quad u_{\Gamma_\tau}^{P1} = 0,$$

and may be computed again by solving a restricted problem (to internal nodes and some boundary nodes) with a regular stiffness matrix  $K_{\text{int}, \Gamma_\tau}^{P1}$ , i.e.,

$$K_{\text{int}, \Gamma_\tau}^{P1} u_{\text{int}, \Gamma_\tau}^{P1} = \lambda^2 M_{\text{int}, \Gamma_\tau}^{P1} u_{\text{int}, \Gamma_\tau}^{P1}.$$

As in any dimension the Poincaré–Friedrichs constants can be computed either as a gradient or as a divergence constant, see [Theorem 2.17](#), we can approximate

$$c_{0, \Gamma_\tau} = c_{2, \Gamma_v}$$

either by (21) or by

$$\frac{1}{c_{2, \Gamma_v, \text{RT}}^2} = \lambda_{2, \Gamma_v, \text{RT}}^2 = \min_{\substack{0 \neq H^{\text{RT}}, \\ H_{\Gamma_v}^{\text{RT}} = 0, \\ H^{\text{RT}} \perp N(K^{\text{RT}})}} \frac{K^{\text{RT}} H^{\text{RT}} \cdot H^{\text{RT}}}{M^{\text{RT}} H^{\text{RT}} \cdot H^{\text{RT}}},$$

where  $H_{\Gamma_v}^{\text{RT}}$  denotes a subvector of  $H^{\text{RT}}$  in indices corresponding to boundary faces of  $\Gamma_v$  (boundary edges in 2D).  $\lambda_{2, \Gamma_v, \text{RT}}^2$  is the minimal positive eigenvalue of the generalized eigenvalue problem

$$K^{\text{RT}} H^{\text{RT}} = \lambda^2 M^{\text{RT}} H^{\text{RT}}, \quad H_{\Gamma_v}^{\text{RT}} = 0,$$

respectively,

$$K_{\text{int}, \Gamma_v}^{\text{RT}} H_{\text{int}, \Gamma_v}^{\text{RT}} = \lambda^2 M_{\text{int}, \Gamma_v}^{\text{RT}} H_{\text{int}, \Gamma_v}^{\text{RT}},$$

where  $K_{\text{int}, \Gamma_v}^{\text{RT}}$ ,  $M_{\text{int}, \Gamma_v}^{\text{RT}}$ , and  $H_{\text{int}, \Gamma_v}^{\text{RT}}$  are restrictions of the matrices  $K^{\text{RT}}$ ,  $M^{\text{RT}}$ , and the vector  $H^{\text{RT}}$  to indices corresponding to ‘free’ mesh faces (edges in 2D) only. Note that there are a lot of first zero eigenvalues  $\lambda^2 = 0$  as neither  $K^{\text{RT}}$  nor  $K_{\text{int}, \Gamma_v}^{\text{RT}}$  are regular due to the existence of large kernels  $N(K^{\text{RT}})$  and  $N(K_{\text{int}, \Gamma_v}^{\text{RT}})$  since all rotations belong to the kernel of the divergence.

#### 4.2. Maxwell constants

While the computation of the Poincaré–Friedrichs constants  $c_{0, \Gamma_\tau} = c_{2, \Gamma_v}$  is more or less independent of the dimension, the computation of the Maxwell constants is different in 2D and 3D, or generally, in ND. By [Remark 3.2\(v\)](#) we have in 2D

$$c_{1, \Gamma_v} = c_{0, \Gamma_\tau},$$

and thus the Maxwell constants can simply be computed by the corresponding Poincaré–Friedrichs (Laplace) constants. In particular, for the tangential (electric) and normal (magnetic) Maxwell constants it holds

$$c_{1, \Gamma} = c_{0, \emptyset}, \quad c_{1, \emptyset} = c_{0, \Gamma}.$$

By [Remark 3.3\(vii\)](#) we have in 3D

$$c_{1, \Gamma_\tau} = c_{1, \Gamma_v},$$

and thus this Maxwell constant has to be calculated separately, since the simple link to the Poincaré–Friedrichs (Laplace) constants is lost in higher dimensions. In particular, for the tangential (electric) and normal (magnetic) Maxwell constants it holds

$$c_{1, \Gamma} = c_{1, \emptyset}.$$

The Maxwell constants are approximated as

$$\frac{1}{c_{1, \Gamma_\tau, N}^2} = \lambda_{1, \Gamma_\tau, N}^2 = \min_{\substack{0 \neq E^N, \\ E_{\Gamma_\tau}^N = 0, \\ E^N \perp N(K^N)}} \frac{K^N E^N \cdot E^N}{M^N E^N \cdot E^N},$$

where  $E_{\Gamma_\tau}^N$  denotes a subvector of  $E^N$  in indices corresponding to boundary edges of  $\Gamma_\tau$ .  $\lambda_{1, \Gamma_\tau, N}^2$  is the minimal positive eigenvalue of the generalized eigenvalue problem

$$K^N E^N = \lambda^2 M^N E^N, \quad E_{\Gamma_\tau}^N = 0,$$

respectively,

$$K_{\text{int}, \Gamma_\tau}^N E_{\text{int}, \Gamma_\tau}^N = \lambda^2 M_{\text{int}, \Gamma_\tau}^N E_{\text{int}, \Gamma_\tau}^N,$$

where  $K_{\text{int}, \Gamma_\tau}^N$ ,  $M_{\text{int}, \Gamma_\tau}^N$ , and  $E_{\text{int}, \Gamma_\tau}^N$  are restrictions of the matrices  $K^N$ ,  $M^N$ , and the vector  $E^N$  to indices corresponding to ‘free’ mesh edges only. Note that similar to the computation of the divergence constants there are a lot of first zero eigenvalues  $\lambda^2 = 0$  as neither  $K^N$  nor  $K_{\text{int}, \Gamma_\tau}^N$  are regular due to the existence of large kernels  $N(K^N)$  and  $N(K_{\text{int}, \Gamma_\tau}^N)$  since now all gradients belong to the kernel of the rotation.

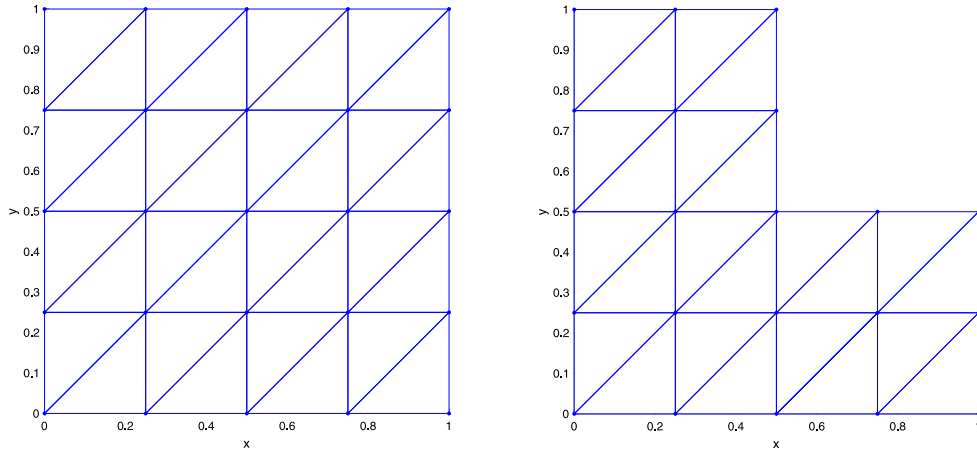


Fig. 1. Coarse (level 1) triangular meshes for the unit square and the L-shape domains.

Table 1

Constants computed for the unit square domain and full boundary conditions.

Mesh level	$c_{0,\emptyset,P1}$	$c_{2,\Gamma,RT}$	$c_{1,\Gamma,N}$	$c_{1,\emptyset,N}$	$c_{2,\emptyset,RT}$	$c_{0,\Gamma,P1}$
1	0.31072999	0.32316745	0.32316745	0.22328039	0.22328039	0.20912552
2	0.31631302	0.31953907	0.31953907	0.22460517	0.22460517	0.22083319
3	0.31780225	0.31861815	0.31861815	0.22495907	0.22495907	0.22400032
4	0.31818232	0.31838701	0.31838701	0.22504898	0.22504898	0.22480828
5	0.31827795	0.31832917	0.31832917	0.22507155	0.22507155	0.22501131
6	0.31830190	0.31831471	0.31831471	0.22507720	0.22507720	0.22506213
7	0.31830789	0.31831109	0.31831109	0.22507861	0.22507861	0.22507484
$\infty$	0.31830988	0.31830988	0.31830988	0.22507907	0.22507907	0.22507907

We emphasize that the Maxwell constants are also approximated by

$$\frac{1}{c_{1,\Gamma_b,N}^2} = \lambda_{1,\Gamma_b,N}^2 = \min_{\substack{0 \neq E^N, \\ E_{\Gamma_b}^N = 0, \\ E^N \perp N(\mathbb{K}^N)}} \frac{\mathbb{K}^N E^N \cdot E^N}{\mathbb{M}^N E^N \cdot E^N}.$$

#### 4.3. 2D computations

We demonstrate two benchmarks with the unit square and the L-shape domain, the first with known and the second with unknown values of the constants. Their coarse (level 1) meshes are displayed in Fig. 1. For the unit square we have by Remark 3.2 exact values

$$c_{0,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \approx 0.22507907, \quad c_{0,\emptyset} = c_{1,\Gamma} = \frac{1}{\pi} \approx 0.31830988,$$

and our approximative values converge to them, see Table 1. This extends results of [58, Table 1]. For one case of mixed boundary conditions (with missing boundary part  $\Gamma_b$ ) we have by Remark 3.2 and (18) exact values

$$c_{0,\Gamma_{t,l,r}} = c_{1,\Gamma_b} = \frac{2}{\sqrt{5}\pi} \approx 0.28470501, \quad c_{0,\Gamma_b} = c_{1,\Gamma_{t,l,r}} = \frac{2}{\pi} \approx 0.63661977,$$

and our approximative values converge again to them, see Table 2. Approximative values for the L-shape domain are provided in Tables 3 and 4. We notice a quadratic convergence of all constants with respect to the mesh size  $h$ . It means that an absolute error of any considered constant approximation is reduced by a factor of 4 after each uniform mesh refinement. Geometrical parameters of triangular meshes used for the unit square domain are given in Table 9.

#### 4.4. 3D computations

We present two benchmarks with the unit cube and the Fichera corner domain, the first with known and the second with unknown values of the constants. Their coarse (level 1) meshes are displayed in Fig. 2.

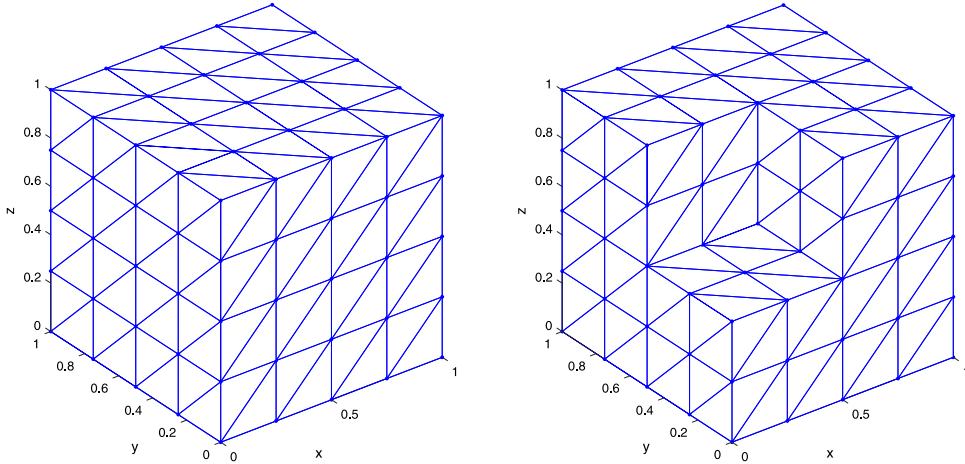


Fig. 2. Coarse (level 1) tetrahedral meshes for the unit cube and the Fichera corner domains.

Table 2

Constants computed for the unit square domain and mixed boundary conditions.

Mesh level	$C_{0,\Gamma_b,P1}$	$C_{2,\Gamma_{t,L,R},RT}$	$C_{1,\Gamma_{t,L,R},N}$	$C_{1,\Gamma_b,N}$	$C_{2,\Gamma_b,RT}$	$C_{0,\Gamma_{t,L,R},P1}$
1	0.63267458	0.63798842	0.63798842	0.28486798	0.28486798	0.27318834
2	0.63560893	0.63696095	0.63696095	0.28473767	0.28473767	0.28172459
3	0.63636500	0.63670501	0.63670501	0.28471277	0.28471277	0.28395286
4	0.63655592	0.63664108	0.63664108	0.28470693	0.28470693	0.28451652
5	0.63660380	0.63662510	0.63662510	0.28470549	0.28470549	0.28465786
6	0.63661578	0.63662110	0.63662110	0.28470514	0.28470514	0.28469323
7	0.63661877	0.63662011	0.63662011	0.28470505	0.28470505	0.28470207
$\infty$	0.63661977	0.63661977	0.63661977	0.28470501	0.28470501	0.28470501

Table 3

Constants computed for the L-shape domain and full boundary conditions.

Mesh level	$C_{0,\emptyset,P1}$	$C_{2,\Gamma,RT}$	$C_{1,\Gamma,N}$	$C_{1,\emptyset,N}$	$C_{2,\emptyset,RT}$	$C_{0,\Gamma,P1}$
1	0.39156654	0.43611331	0.43611331	0.16795692	0.16795692	0.13325394
2	0.40370423	0.42045050	0.42045050	0.16377267	0.16377267	0.15232573
3	0.40850306	0.41492017	0.41492017	0.16214127	0.16214127	0.15838355
4	0.41038725	0.41287500	0.41287500	0.16148392	0.16148392	0.16020361
5	0.41112643	0.41209870	0.41209870	0.16121879	0.16121879	0.16076463
6	0.41141712	0.41179918	0.41179918	0.16111230	0.16111230	0.16094566
7	0.41153175	0.41168242	0.41168242	0.16106970	0.16106970	0.16100698

Table 4

Constants computed for the L-shape domain and mixed boundary conditions.

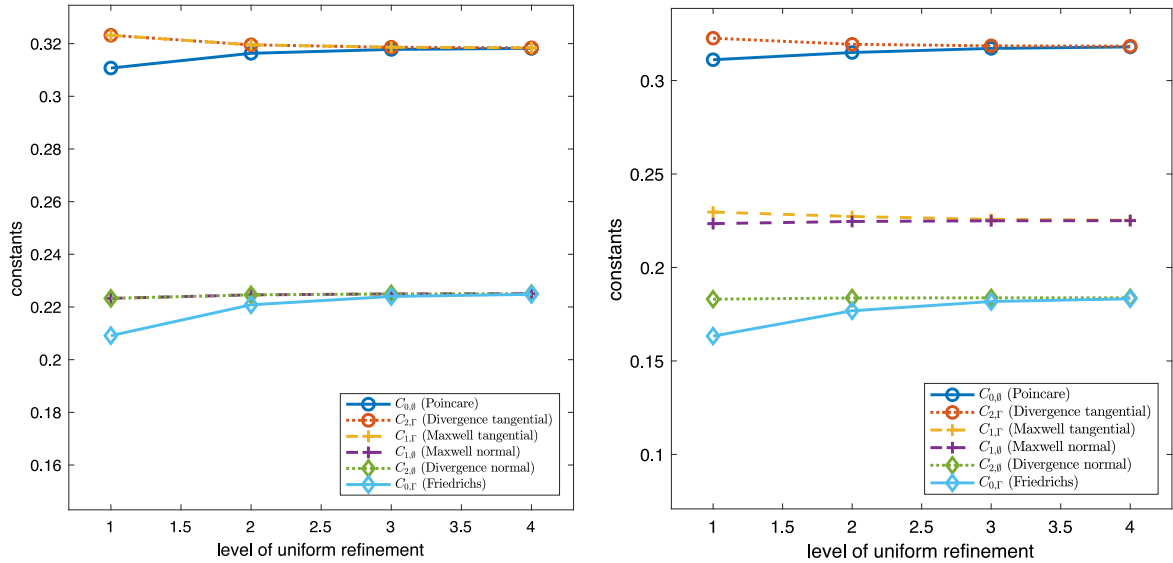
Mesh level	$C_{0,\Gamma_b,P1}$	$C_{2,\Gamma_{t,L,R},RT}$	$C_{1,\Gamma_{t,L,R},N}$	$C_{1,\Gamma_b,N}$	$C_{2,\Gamma_b,RT}$	$C_{0,\Gamma_{t,L,R},P1}$
1	0.55287499	0.58356116	0.58356116	0.24038804	0.24038804	0.21444362
2	0.56332946	0.57483917	0.57483917	0.23765352	0.23765352	0.22916286
3	0.56716139	0.57152377	0.57152377	0.23648111	0.23648111	0.23363643
4	0.56857589	0.57025101	0.57025101	0.23597974	0.23597974	0.23498908
5	0.56910703	0.56975715	0.56975715	0.23577100	0.23577100	0.23541318
6	0.56930976	0.56956402	0.56956402	0.23568569	0.23568569	0.23555262
7	0.56938813	0.56948808	0.56948808	0.23565122	0.23565122	0.23560066

For the unit cube we have by Remark 3.3 exact values

$$c_{0,\Gamma} = \frac{1}{\sqrt{3}\pi} \approx 0.18377629, \quad c_{1,\Gamma} = c_{1,\emptyset} = \frac{1}{\sqrt{2}\pi} \approx 0.22507907, \quad c_{0,\emptyset} = \frac{1}{\pi} \approx 0.31830988,$$

and our approximative values converge to them, see Table 5. For one case of mixed boundary conditions (with missing boundary part  $\Gamma_b$ ) we have by Remark 3.3 and (19), (20) exact values

$$c_{0,\Gamma_{t,L,R},f,k} = \frac{2}{3\pi} \approx 0.21220659, \quad c_{1,\Gamma_{t,L,R},f,k} = c_{1,\Gamma_b} = \frac{2}{\sqrt{5}\pi} \approx 0.28470501, \quad c_{0,\Gamma_b} = \frac{2}{\pi} \approx 0.63661977,$$



**Fig. 3.** Constants computed for the unit square domain (left) and the unit cube domain (right) with full boundary conditions. Theoretically, it holds in 2D (left)

$$c_{0,\Gamma} = c_{2,\theta} = c_{1,\theta} = \frac{1}{\sqrt{2}\pi} \approx 0.225 < c_{0,\theta} = c_{2,\Gamma} = c_{1,\Gamma} = \frac{1}{\pi} \approx 0.318$$

and in 3D (right)

$$c_{0,\Gamma} = c_{2,\theta} = \frac{1}{\sqrt{3}\pi} \approx 0.184 < c_{1,\Gamma} = c_{1,\theta} = \frac{1}{\sqrt{2}\pi} \approx 0.225 < c_{0,\theta} = c_{2,\Gamma} = \frac{1}{\pi} \approx 0.318.$$

**Table 5**

Constants computed for the unit cube domain and full boundary conditions.

Mesh level	$C_{0,\theta,P1}$	$C_{2,\Gamma,RT}$	$C_{1,\Gamma,N}$	$C_{1,\theta,N}$	$C_{2,\theta,RT}$	$C_{0,\Gamma,P1}$
1	0.31114284	0.32265677	0.22964649	0.22346361	0.18305860	0.16330104
2	0.31500347	0.31939334	0.22727295	0.22461307	0.18369611	0.17685247
3	0.31720303	0.31857551	0.22577016	0.22497862	0.18375776	0.18178558
4	0.31799426	0.31837527	0.22526682	0.22505528	0.18377095	0.18324991
$\infty$	0.31830988	0.31830988	0.22507907	0.22507907	0.18377629	0.18377629

**Table 6**

Constants computed for the unit cube domain and mixed boundary conditions.

Mesh level	$C_{0,\Gamma_b,P1}$	$C_{2,\Gamma_{t,l,r,f,k},RT}$	$C_{1,\Gamma_{t,l,r,f,k},N}$	$C_{1,\Gamma_b,N}$	$C_{2,\Gamma_b,RT}$	$C_{0,\Gamma_{t,l,r,f,k},P1}$
1	0.63279353	0.63799454	0.28810408	0.28568645	0.21207495	0.19466267
2	0.63563506	0.63694323	0.28621730	0.28506833	0.21221199	0.20590030
3	0.63636820	0.63669754	0.28518535	0.28483451	0.21220585	0.21033840
4	0.63655623	0.63663874	0.28483637	0.28474355	0.21220553	0.21170560
$\infty$	0.63661977	0.63661977	0.28470501	0.28470501	0.21220659	0.21220659

**Table 7**

Constants computed for the Fichera corner domain and full boundary conditions.

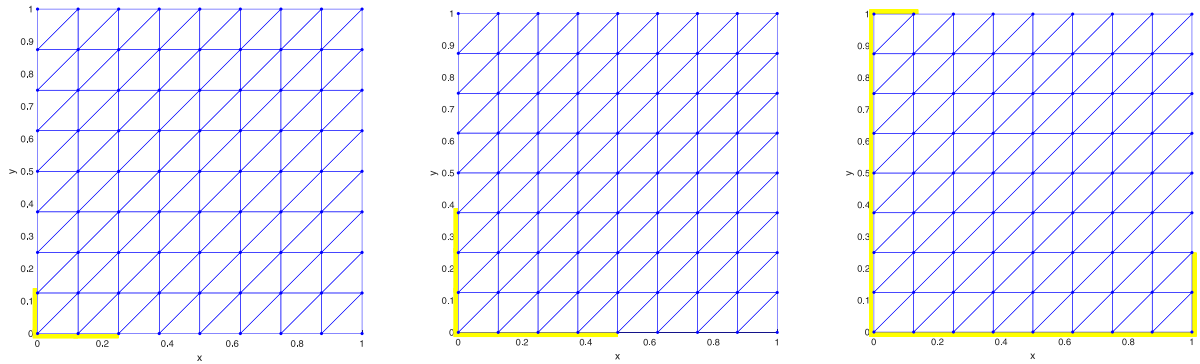
Mesh level	$C_{0,\theta,P1}$	$C_{2,\Gamma,RT}$	$C_{1,\Gamma,N}$	$C_{1,\theta,N}$	$C_{2,\theta,RT}$	$C_{0,\Gamma,P1}$
1	0.34328060	0.37118723	0.30375245	0.26905796	0.15938388	0.12490491
2	0.35193318	0.36341148	0.28961049	0.27500043	0.15638922	0.14329827
3	0.35628919	0.36072519	0.28329415	0.27728510	0.15500593	0.15042074
4	0.35808207	0.35976508	0.28054899	0.27811443	0.15444291	0.15286199

and our approximative values converge again to them, see Table 6. Approximative values for the Fichera corner domain are provided in Tables 7 and 8. We notice a slightly lower than quadratic convergence of all constants with respect to the mesh size  $h$ . Geometrical parameters of tetrahedral meshes used for the unit cube domain are given in Table 10 (see Fig. 3).

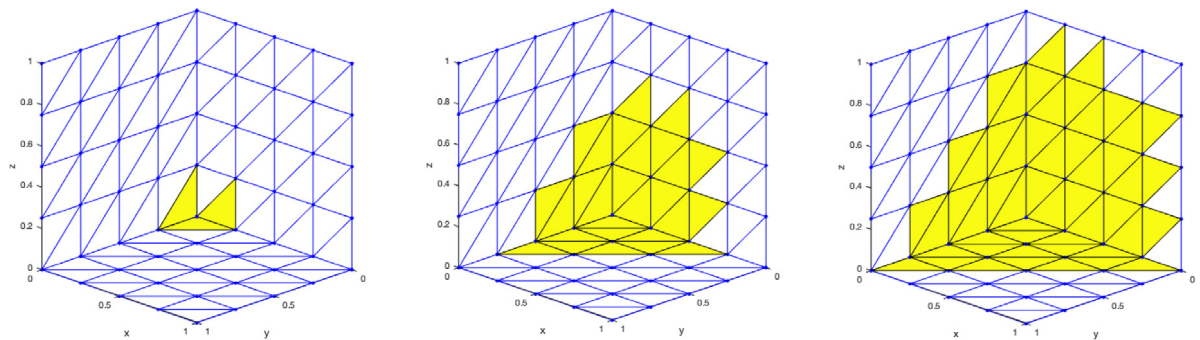
**Table 8**

Constants computed for the Fichera corner domain and mixed boundary conditions.

Mesh level	$C_0, \Gamma_b, P1$	$C_2, \Gamma_{t,l,r,f,k}, RT$	$C_1, \Gamma_{t,l,r,f,k}, N$	$C_1, \Gamma_b, N$	$C_2, \Gamma_b, RT$	$C_0, \Gamma_{t,l,r,f,k}, P1$
1	0.59192242	0.60790729	0.32388929	0.30017867	0.19741476	0.17047720
2	0.59806507	0.60397104	0.31363043	0.30335884	0.19584527	0.18566517
3	0.60032617	0.60255178	0.30884202	0.30465706	0.19504508	0.19157456
4	0.60117716	0.60202588	0.30682949	0.30515364	0.19471006	0.19355739



**Fig. 4.** Elements of a monotone increasing sequence of Neumann edges  $\Gamma_t$  with 3 (left), 7 (middle) and 19 (right) Neumann edges marked in yellow. A full boundary  $\Gamma$  of the considered level 2 mesh of the unit square domain consists of 32 edges. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)



**Fig. 5.** Elements of a monotone increasing sequence of Neumann faces  $\Gamma_v$  with 3 (left), 27 (middle) and 51 (right) Neumann faces marked in yellow. A full boundary  $\Gamma$  of the considered level 1 mesh of the unit cube domain consists of 192 faces. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

#### 4.5. Testing of the monotonicity properties

We perform some monotonicity tests on the constants depending on the respective boundary conditions, i.e., we display the mapping

$$\Gamma_v \mapsto (c_0, \Gamma_t, c_0, \Gamma_v, c_1, \Gamma_t, c_1, \Gamma_v, c_2, \Gamma_t, c_2, \Gamma_v)$$

for a monotone increasing sequence of  $\Gamma_v$ . Figs. 4 and 5 depict examples of such sequences in 2D/3D. The boundary part  $\Gamma_v$  is represented discretely as a set of Neumann edges in 2D or a set of Neumann faces in 3D. Boundary faces or edges are checked for their connectivity and a breadth-first search (BFS) algorithm is applied to order them in a sequence. All constants are then evaluated for every element of the sequence and the results are displayed in Figs. 6 and 7.

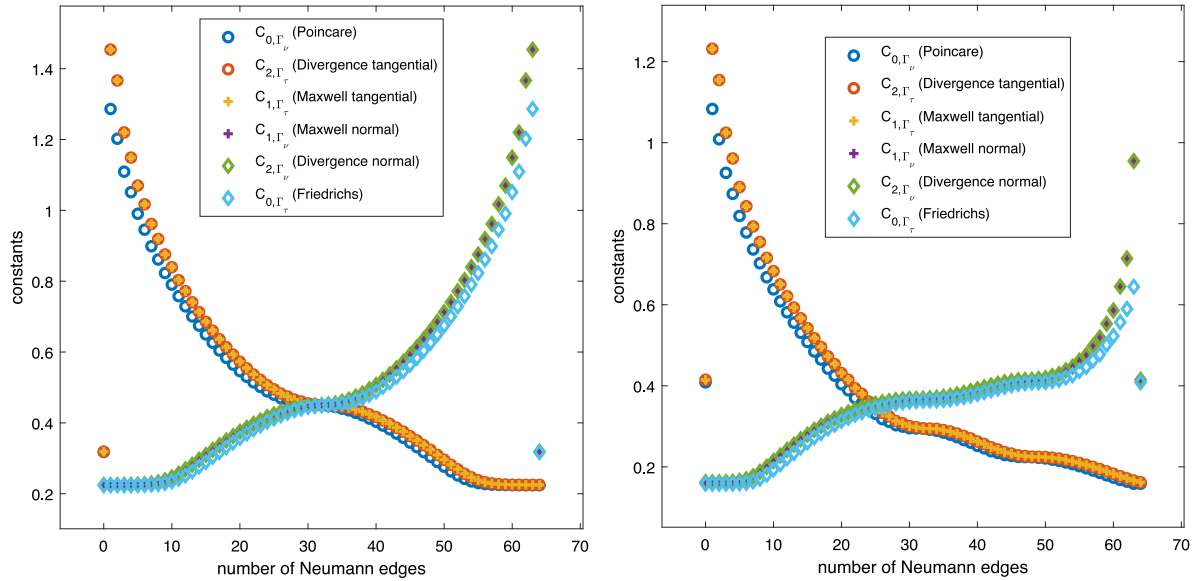
#### 4.6. Computational details and MATLAB code

It is more computationally demanding to evaluate divergence and Maxwell constants than Laplace constants, since the numbers of faces (in 3D) and edges are higher than the number of nodes, cf. e.g. Tables 9 and 10.

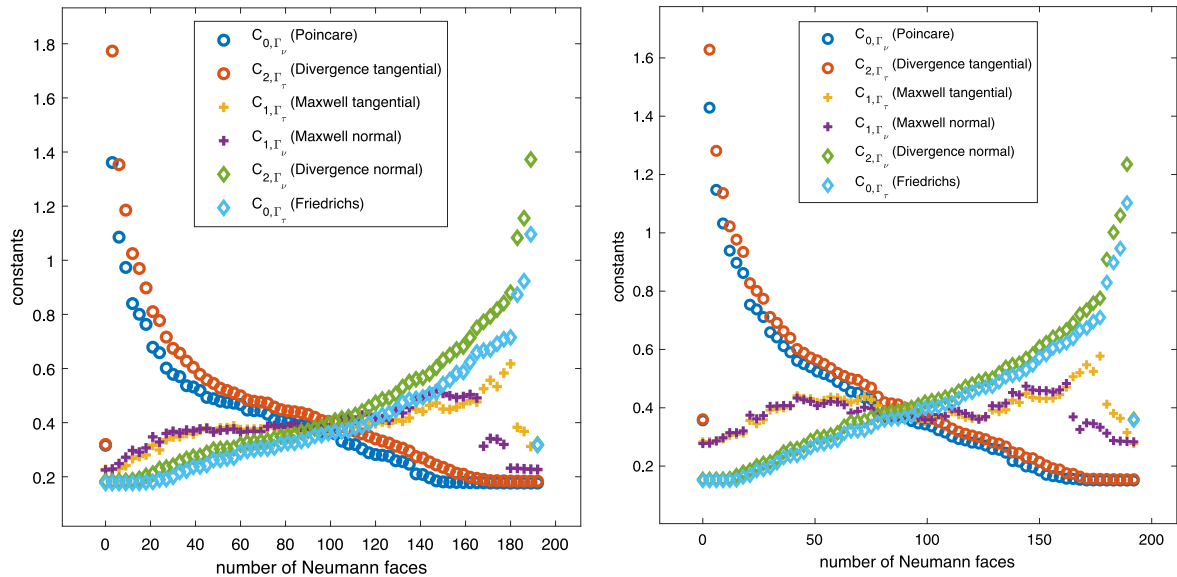
A generalized eigenvalue system

$$Kv = \lambda^2 Mv \quad (22)$$





**Fig. 6.** Constants computed for the level 3 meshes of the unit square domain (left) and of the L-shape domain (right) - monotonicity test for a monotone increasing sequence of Neumann edges  $\Gamma_\nu$ .



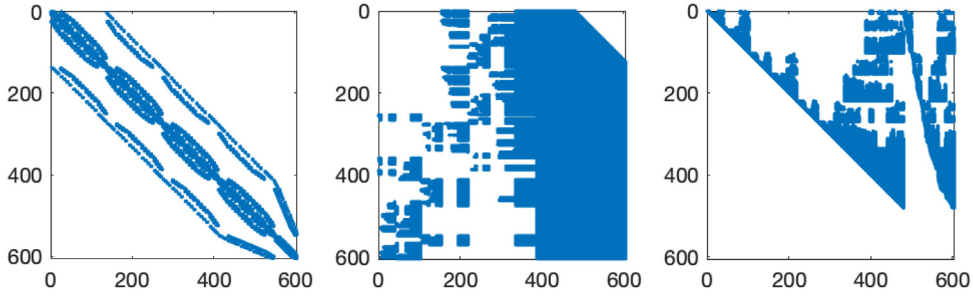
**Fig. 7.** Constants computed for the level 1 meshes of unit cube domain (left) and of the Fichera corner domain (right) - monotonicity test for a monotone increasing sequence of Neumann faces  $\Gamma_\nu$ .

with a positive semidefinite and symmetric matrix  $K \in \mathbb{R}^{n \times n}$  and a positive definite and symmetric matrix  $M \in \mathbb{R}^{n \times n}$  is solved for a smallest positive eigenvalue  $\lambda^2 > 0$ . We apply two computational techniques.

#### 4.6.1. A nested iteration technique

An eigenvalue evaluated on a coarser mesh (e.g. by the second technique explained below) is used as initial guess on a finer (uniformly refined) mesh, where an inbuilt MATLAB function `eigs` is applied for the search of the closest eigenvalue. Without additional preconditioning (multigrid, domain decompositions) of eigenvalue solvers we can efficiently find smallest positive eigenvalues for all considered meshes.

However, it was noticed this technique did not converge for some cases of mixed boundary conditions in 3D because a sequence of corresponding Laplace constants did not form a monotone sequence in the monotonicity test. Then, since the



**Fig. 8.** An example of a positive semidefinite matrix  $K \in \mathbb{R}^{604 \times 604}$  (left) taken as a  $K^N$  matrix from level 1 mesh discretization of the unit cube domain and matrices  $\tilde{Q}$  (middle) and  $\tilde{R}$  (right) from the QR decomposition (23). The number of last zero rows of  $\tilde{R}$  is 124 and it determines the dimension of the kernel of  $K$ . Therefore, the dimension of the range of  $K$  is 480.

**Table 9**

Discretization of the unit square domain by uniform triangular meshes.

Mesh level	Elements	Nodes	Edges	Boundary edges
1	32	25	56	16
2	128	81	208	32
3	512	289	800	64
4	2.028	1.089	3.136	128
5	8.192	4.225	12.416	256
6	32.768	16.641	49.408	512
7	131.072	66.049	197.120	1.024

dimension of the kernel of a corresponding stiffness matrix is known (0 or 1), we simply compute the smallest eigenvalue or two smallest eigenvalues with 0 being the smallest eigenvalue.

#### 4.6.2. A projection to the range of $K$

We apply the QR-decomposition of  $K$  in the form

$$KE = \tilde{Q}\tilde{R}, \quad (23)$$

where  $E \in \mathbb{R}^{n \times n}$  is a permutation matrix,  $\tilde{Q} \in \mathbb{R}^{n \times n}$  is an orthogonal matrix and  $\tilde{R} \in \mathbb{R}^{n \times n}$  is an upper triangular matrix with diagonal entries ordered in decreasing order as

$$|\tilde{R}_{1,1}| \geq \dots \geq |\tilde{R}_{r,r}| \geq \dots \geq |\tilde{R}_{n,n}|.$$

The number  $r \leq n$  of nonzero entries of the sequence above then determines the range  $K$  and all rows of  $\tilde{R}$  with indices larger than  $r$  are zero rows, cf. Figs. 8 and 9. Therefore, we also have

$$KE = QR \quad (24)$$

where  $Q \in \mathbb{R}^{n \times r}$  is a restriction of  $\tilde{Q}$  to its first  $r$  columns and  $R \in \mathbb{R}^{r \times n}$  a restriction of  $\tilde{R}$  to its first  $r$  rows. Then, a mapping  $v = Qz$  projects a (column) vector  $z \in \mathbb{R}^r$  to the range of  $K$  and the generalized eigenvalue system (22) to

$$KQz = \lambda^2 MQz.$$

The multiplication of both sides by  $Q^T M^{-1}$  transforms the above relation (since  $Q^T Q$  is an identity matrix of size  $r \times r$ ) to a standard eigenvalue problem

$$Q^T M^{-1} KQz = \lambda^2 z. \quad (25)$$

In view of (24), the symmetry of  $K$  and the orthogonality of the permutation matrix  $E$ , it holds

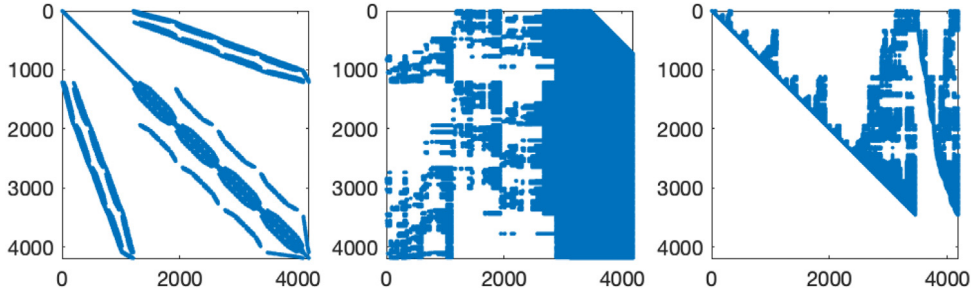
$$KQ = (QRE^T)Q = (QRE^T)^T Q = ER^T Q^T Q = ER^T$$

and this formula is applied in our practical computations. A matrix  $M^{-1}$  is full and expensive to compute, its memory storage is large and the multiplication with  $Q^T M^{-1}$  is costly. Therefore, this projection technique can only be applied for coarser meshes.

#### 4.6.3. A MATLAB code

Numerical evaluations are based on finite element assemblies from [56,57] and also utilizes a 3D cube mesh and mesh visualizations from [59]. The code is freely available for download and testing at:

<https://www.mathworks.com/matlabcentral/fileexchange/23991>



**Fig. 9.** An example of a positive semidefinite matrix  $K \in \mathbb{R}^{4184 \times 4184}$  (left) taken as a  $K^N$  matrix from level 2 mesh discretization of the unit cube domain and matrices  $\bar{Q}$  (middle) and  $\bar{R}$  (right) from the QR decomposition (23). The number of last zero rows of  $\bar{R}$  is 728 and it determines the dimension of the kernel of  $K$ . Therefore, the dimension of the range of  $K$  is 3456.

**Table 10**

Discretization of the unit cube domain by uniform tetrahedral meshes.

Mesh level	Elements	Nodes	Edges	Faces	Boundary faces
1	384	125	604	864	192
2	3.072	729	4.184	6.528	768
3	24.576	4.913	31.024	50.688	3.072
4	196.608	35.937	238.688	399.360	12.288

It can be easily modified to other domains and boundary conditions. The starting scripts for testing are `start_2D` and `start_3D`. To a given mesh, it automatically determines its boundary. In 2D, the code can also visualize eigenfunctions, see Fig. 10 for the case of the L-shape domain.

## 5. Discussion of the numerical results and conclusions

Our numerical results, especially in 3D, did verify all the theoretical assertions of Theorem 2.20, see also Remarks 3.3 and 3.2, in particular,

- the monotone dependence of the Poincaré–Friedrichs and divergence constants on the boundary conditions, i.e., the monotonicity of the mapping

$$\Gamma_v \mapsto c_{0,\Gamma_\tau} = c_{2,\Gamma_v},$$

- the ‘independence’ of the Maxwell constants on the boundary conditions, i.e.,

$$\forall \Gamma_\tau \subset \Gamma \quad c_{1,\Gamma_\tau} = c_{1,\Gamma_v},$$

- as well as the boundedness of the full tangential and normal Maxwell constants by the Poincaré constant for convex  $\Omega$ , i.e.,

$$c_{1,\Gamma} = c_{1,\emptyset} \leq c_{0,\emptyset} = c_{2,\Gamma}.$$

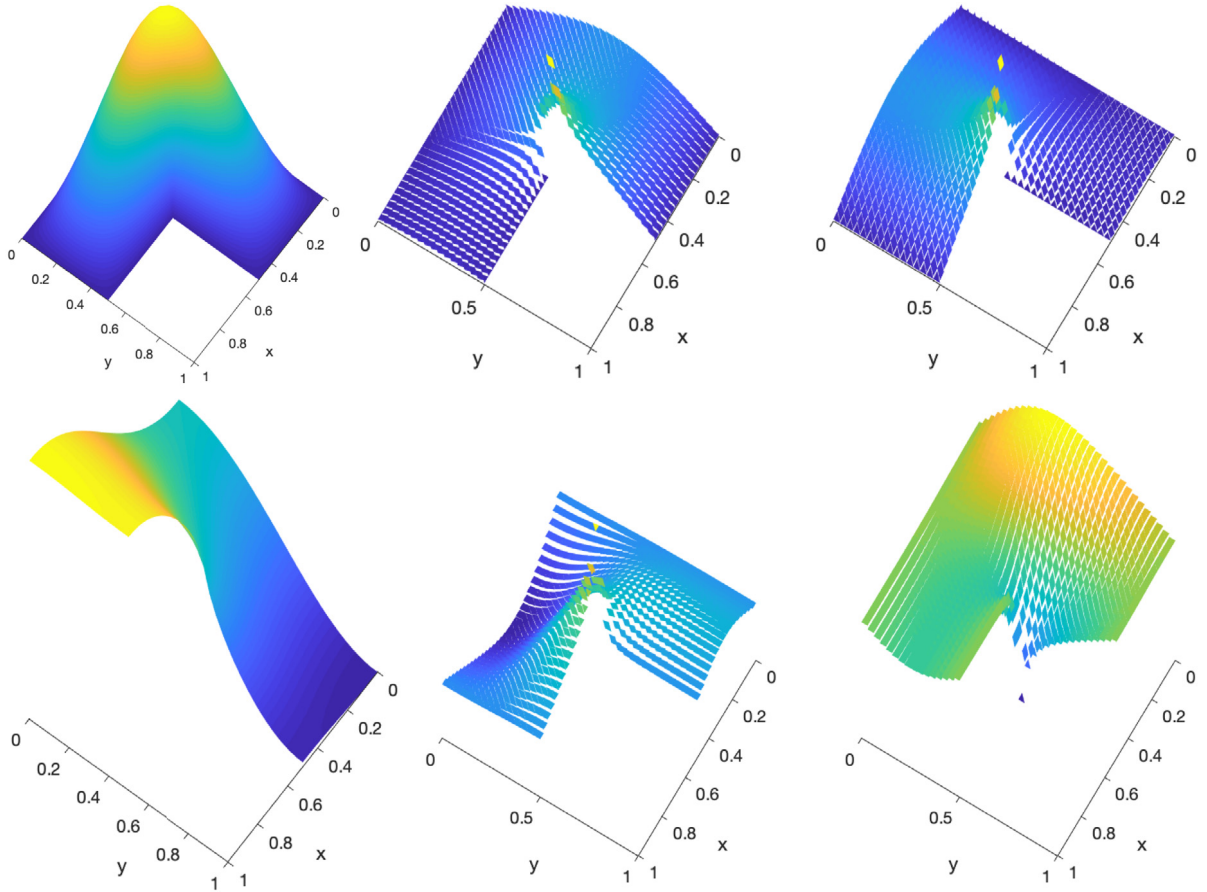
While the first two assertions hold for general bounded Lipschitz domains and Lipschitz interfaces, the third assertion is analytically proved only for convex domains and the full boundary conditions. In our numerical experiments, the unit cube served as a prototype for a convex domain, and we picked the Fichera corner domain as a typical example of a non-convex domain, see Fig. 2 for both initial meshes.

### 5.1. Extended inequalities

To our surprise, even for mixed boundary conditions and for non-convex geometries, the *extended inequalities*

$$c_{0,\Gamma} \leq \min\{c_{0,\Gamma_\tau}, c_{0,\Gamma_v}\} \leq c_{1,\Gamma_\tau} = c_{1,\Gamma_v} \leq \max\{c_{0,\Gamma_\tau}, c_{0,\Gamma_v}\} \leq \sup_{\Gamma_\tau \neq \emptyset} c_{0,\Gamma_\tau} = \sup_{\Gamma_v \neq \Gamma} c_{2,\Gamma_v} \quad (26)$$

seem to hold for our examples, see Fig. 7. In these special cases the Maxwell constants are always in between the Poincaré–Friedrichs (Laplace) constants. We emphasize that our examples possess (piecewise) vanishing curvature. It remains an open question if (26) is true – at least partially – in general or, e.g., for polyhedra. Moreover, if  $\Gamma_\tau$  approaches  $\emptyset$ , the Poincaré–Friedrichs constants  $c_{0,\Gamma_\tau}$  seem to be bounded, i.e., the suprema in (26) appear to be bounded, although a kernel of dimension 1 (constants) is approximated.



**Fig. 10.** Eigenfunctions - Friedrichs (top left), Poincaré (bottom left), tangential Maxwell (top middle and right), normal Maxwell (bottom middle and right) - for the L-shape domain with full boundary conditions.

### 5.1.1. Hints for the extended inequalities

We note the well known integration by parts formula

$$|\text{grad } E|_{L^2(\Omega)}^2 = |\text{curl } E|_{L^2(\Omega)}^2 + |\text{div } E|_{L^2(\Omega)}^2, \quad (27)$$

being valid for all vector fields  $E \in H_{\Gamma}(\text{grad}, \Omega)$ , the closure of  $\Omega$ -compactly supported test fields, see (15). Using a more sophisticated integration by parts formula from [49, Corollary 6], which has been proved already in, e.g., [12, Theorem 2.3] for the case of full boundary conditions, we see that (27) remains true for *polyhedral* domains  $\Omega$  and for vector fields

$$E \in H_{\Gamma_{\tau}, \Gamma_{\nu}}(\text{grad}, \Omega) := \overline{C_{\Gamma_{\tau}, \Gamma_{\nu}}^{\infty}(\Omega)}^{H(\text{grad}, \Omega)} \subset H(\text{grad}, \Omega) \cap H_{\Gamma_{\tau}}(\text{curl}, \Omega) \cap H_{\Gamma_{\nu}}(\text{div}, \Omega), \quad (28)$$

where

$$C_{\Gamma_{\tau}, \Gamma_{\nu}}^{\infty}(\Omega) := \{\Phi|_{\Omega} : \Phi \in C^{\infty}(\mathbb{R}^3), \text{ supp } \Phi \text{ compact in } \mathbb{R}^3, n \times \Phi|_{\Gamma_{\tau}} = 0, n \cdot \Phi|_{\Gamma_{\nu}} = 0\}.$$

Note that these results at least go back to the book of Grisvard [60, Theorem 3.1.1.2], see also the book of Leis [29, p. 156–157].

A first hint for a possible explanation of (26) is then the following observation: Let  $E_{1, \Gamma_{\tau}}$  be the minimizer from Remark 2.18. Then

$$E_{1, \Gamma_{\tau}} \in D(\text{curl}_{\Gamma_{\tau}}) \cap R(\text{curl}_{\Gamma_{\nu}}) \subset H_{\Gamma_{\tau}}(\text{curl}, \Omega) \cap H_{\Gamma_{\nu}}(\text{div}, \Omega), \quad \text{div } E_{1, \Gamma_{\tau}} = 0.$$

Hence, if  $\Omega$  is a *polyhedron* and if  $E_{1, \Gamma_{\tau}}$  is regular<sup>1</sup> enough, i.e.,  $E_{1, \Gamma_{\tau}} \in H_{\Gamma_{\tau}, \Gamma_{\nu}}(\text{grad}, \Omega)$ , then by (27) and (28)

$$\lambda_{1, \Gamma_{\tau}} = \frac{|\text{curl } E_{1, \Gamma_{\tau}}|_{L^2(\Omega)}}{|E_{1, \Gamma_{\tau}}|_{L^2(\Omega)}} = \frac{|\text{grad } E_{1, \Gamma_{\tau}}|_{L^2(\Omega)}}{|E_{1, \Gamma_{\tau}}|_{L^2(\Omega)}}.$$

<sup>1</sup> The additional regularity of the minimizer  $E_{1, \Gamma_{\tau}}$  is not realistic.

Moreover, if  $E_{1,\Gamma_\tau}$  admits the additional regularity  $E_{1,\Gamma_\tau} \in H_{\Gamma_\tau}(\text{grad}, \Omega)$ , then

$$\lambda_{1,\Gamma_\tau} \geq \inf_{0 \neq E \in H_{\Gamma_\tau}(\text{grad}, \Omega)} \frac{|\text{grad } E|_{L^2(\Omega)}}{|E|_{L^2(\Omega)}} = \lambda_{0,\Gamma_\tau}.$$

## Acknowledgements

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## Appendix A. Some proofs

**Proof of (6).** To show that, e.g.,  $A^*A$  is self-adjoint, we first observe that  $A^*A$  is symmetric. Hence, so is  $A^*A + 1$ . By Riesz' representation theorem, for any  $f \in H_0$  there exists a unique  $x \in D(A)$  such that

$$\forall \varphi \in D(A) \quad \langle Ax, A\varphi \rangle_{H_1} + \langle x, \varphi \rangle_{H_0} = \langle f, \varphi \rangle_{H_0}.$$

Thus,  $Ax \in D(A^*)$  and  $A^*Ax = f - x$ , i.e.,  $x \in D(A^*A)$  and  $(A^*A + 1)x = f$ . In other words,  $A^*A + 1$  is onto. Therefore,  $A^*A + 1$  is self-adjoint and so is  $A^*A$ . Note that we did not need the additional assumption that  $R(A)$  is closed or that  $A$  resp.  $A^*$  is onto.

We also present an alternative proof of the self-adjointness of  $A^*A$  in the case that  $R(A)$  is closed. For this, let  $y \in H_0$  such that there exists  $z \in H_0$  with

$$\forall x \in D(A^*A) \quad \langle A^*Ax, y \rangle_{H_0} = \langle x, z \rangle_{H_0}. \quad (29)$$

Picking  $x \in N(A)$  shows that  $z \perp_{H_0} N(A)$ , i.e., we have  $z \in R(A^*)$  by (3). For  $\varphi \in D(A^*)$  we note  $A^*\varphi \in R(A^*) = R(\mathcal{A}^*)$ . Thus there is

$$\psi_\varphi := (\mathcal{A}^*)^{-1}A^*\varphi \in D(\mathcal{A}^*) \subset R(A) = R(\mathcal{A}) \quad \text{with} \quad A^*\psi_\varphi = A^*\varphi.$$

Moreover, there exists  $x_\varphi := A^{-1}\psi_\varphi \in D(\mathcal{A})$  with  $Ax_\varphi = \psi_\varphi$  and thus  $x_\varphi \in D(\mathcal{A}^*\mathcal{A})$ . By (29) we see

$$\begin{aligned} \langle A^*\varphi, y \rangle_{H_0} &= \langle A^*Ax_\varphi, y \rangle_{H_0} = \langle x_\varphi, z \rangle_{H_0} = \langle x_\varphi, A^*(\mathcal{A}^*)^{-1}z \rangle_{H_0} = \langle Ax_\varphi, (\mathcal{A}^*)^{-1}z \rangle_{H_1} \\ &= \langle \varphi, (\mathcal{A}^*)^{-1}z \rangle_{H_1} + \underbrace{\langle \psi_\varphi - \varphi, (\mathcal{A}^*)^{-1}z \rangle_{H_1}}_{=0}, \end{aligned}$$

as  $\psi_\varphi - \varphi \in N(A^*) \perp_{H_1} R(A) \supset D(\mathcal{A}^*) \ni (\mathcal{A}^*)^{-1}z$ . Therefore,  $y \in D(A)$  and  $Ay = (\mathcal{A}^*)^{-1}z \in D(A^*)$ , showing  $y \in D(A^*A)$  and  $A^*Ay = z$ . This proves  $(A^*A)^* = A^*A$ .  $\square$

**Proof of Lemma 2.5.** We show a few selected assertions of Lemma 2.5.

- For an eigenvalue  $\lambda > 0$  and an eigenvector  $(x, y)$  of  $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$  it holds  $A^*y = \lambda x$  and  $Ax = \lambda y$ . Note that  $x = 0$  implies  $y = 0$ . Thus  $0 \neq x \in D(A^*A)$  and  $A^*Ax = \lambda A^*y = \lambda^2 x$ , i.e.,  $x$  is an eigenvector and  $\lambda^2$  is an eigenvalue of  $A^*A$ .
- If  $\lambda^2 > 0$  is an eigenvalue and  $x$  is an eigenvector of  $A^*A$ , then  $y_\pm := \pm \lambda^{-1}Ax \in D(A^*)$  and  $A^*y_\pm = \pm \lambda^{-1}A^*Ax = \pm \lambda x$ , i.e.,  $(x, y_\pm)$  is an eigenvector and  $\pm \lambda$  is an eigenvalue of  $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ . Note that  $y_\pm \neq 0$  as  $y_\pm = 0$  implies  $x = 0$ .
- If  $\lambda^2 > 0$  is an eigenvalue and  $x$  is an eigenvector of  $A^*A$ , then  $y := Ax \in D(A^*)$  and we have  $A^*y = A^*Ax = \lambda^2 x \in D(A)$ . Hence  $y \in D(AA^*)$  and  $AA^*y = \lambda^2 Ax = \lambda^2 y$ , i.e.,  $\lambda^2$  is an eigenvalue and  $y$  is an eigenvector of  $AA^*$ . Note that  $y \neq 0$  as  $y = 0$  implies  $x = 0$ .
- To show that indeed, e.g.,  $\lambda_A^2$  is the smallest positive eigenvalue of  $A^*A$ , let us consider a sequence  $(\tilde{x}_n)$  in  $D(\mathcal{A}) \setminus \{0\}$  with

$$\frac{|A\tilde{x}_n|_{H_1}}{|\tilde{x}_n|_{H_0}} \rightarrow \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}} = \lambda_A > 0.$$

Then  $(x_n) := (\tilde{x}_n/|\tilde{x}_n|_{H_0}) \subset D(\mathcal{A})$  with  $|x_n|_{H_0} = 1$  and

$$\lambda_A \leq |Ax_n|_{H_1} \rightarrow \lambda_A.$$

Hence  $(x_n)$  is bounded in  $D(\mathcal{A})$ , yielding a subsequence – again denoted by  $(x_n)$  – as well as  $x_A \in H_0$  and  $y_A \in H_1$  with  $x_n \rightharpoonup x_A$  in  $H_0$ ,  $Ax_n \rightharpoonup y_A$  in  $H_1$ , and  $x_n \rightarrow x_A$  in  $H_0$ . Then  $x_A \in D(A)$  and  $Ax_A = y_A$  as for all  $\psi \in D(A^*)$

$$\langle y_A, \psi \rangle_{H_1} \leftarrow \langle Ax_n, \psi \rangle_{H_1} = \langle x_n, A^*\psi \rangle_{H_0} \rightarrow \langle x_A, A^*\psi \rangle_{H_0}.$$

Note that  $x_A \in R(A^*)$  as  $x_n \in R(A^*) = N(A)^{\perp_{H_0}}$ , especially,  $x_A \in D(\mathcal{A})$ . Moreover,  $|x_A|_{H_0} = 1$ .

By elementary calculations<sup>2</sup> we obtain for all  $\varphi, \phi \in D(\mathcal{A})$

$$|\langle A\varphi, A\phi \rangle_{H_1} - \lambda_A^2 \langle \varphi, \phi \rangle_{H_0}|^2 \leq 2(|A\varphi|_{H_1}^2 - \lambda_A^2 |\varphi|_{H_0}^2)(|A\phi|_{H_1}^2 - \lambda_A^2 |\phi|_{H_0}^2).$$

In particular, for  $\varphi := x_n$  we get for all  $\phi \in D(\mathcal{A})$

$$|\langle Ax_n, A\phi \rangle_{H_1} - \lambda_A^2 \langle x_n, \phi \rangle_{H_0}|^2 \leq 2(|Ax_n|_{H_1}^2 - \lambda_A^2)(|A\phi|_{H_1}^2 - \lambda_A^2 |\phi|_{H_0}^2) \rightarrow 0 \quad (30)$$

and thus

$$\begin{aligned} & |\langle Ax_A, A\phi \rangle_{H_1} - \lambda_A^2 \langle x_A, \phi \rangle_{H_0}| \\ & \leq |\langle Ax_A - x_n, A\phi \rangle_{H_1}| + \lambda_A^2 |\langle x_A - x_n, \phi \rangle_{H_0}| + |\langle Ax_n, A\phi \rangle_{H_1} - \lambda_A^2 \langle x_n, \phi \rangle_{H_0}| \rightarrow 0. \end{aligned}$$

Hence, for all  $\phi \in D(\mathcal{A})$

$$\langle Ax_A, A\phi \rangle_{H_1} = \lambda_A^2 \langle x_A, \phi \rangle_{H_0}. \quad (31)$$

For  $\phi \in D(A) = N(A) \oplus_{H_0} D(\mathcal{A})$ , see the Helmholtz type decomposition (4), we decompose

$$\phi = \phi_N + \phi_{\mathcal{A}} \in N(A) \oplus_{H_0} D(\mathcal{A})$$

and compute by using (31),  $A\phi = A\phi_{\mathcal{A}}$ , and  $x_A \in R(A^*) \perp_{H_0} N(A)$

$$\langle Ax_A, A\phi \rangle_{H_1} = \langle Ax_A, A\phi_{\mathcal{A}} \rangle_{H_1} = \lambda_A^2 \langle x_A, \phi_{\mathcal{A}} \rangle_{H_0} = \lambda_A^2 \langle x_A, \phi \rangle_{H_0}.$$

Therefore, (31) holds for all  $\phi \in D(A)$ , i.e.,

$$\forall \phi \in D(A) \quad \langle Ax_A, A\phi \rangle_{H_1} = \lambda_A^2 \langle x_A, \phi \rangle_{H_0}. \quad (32)$$

This implies  $Ax_A \in D(A^*)$ , i.e.,  $x_A \in D(A^*A)$  and  $A^*Ax_A = \lambda_A^2 x_A$ . We even have  $x_A \in D(\mathcal{A}^*\mathcal{A})$ . Thus,  $\lambda_A^2$  is an eigenvalue and  $x_A$  is an eigenvector of  $A^*A$ . Note that (31) or (32) implies (for  $\phi = x_A$ )  $|Ax_A|_{H_1}^2 = \lambda_A^2 |x_A|_{H_0}^2$ , i.e.,  $|Ax_A|_{H_1} = \lambda_A$ .

Finally, we show that  $(x_n)$  even converges strongly in  $D(\mathcal{A})$ , i.e.,  $(Ax_n)$  converges strongly in  $H_1$  respectively in  $R(A)$ . For this, we get for all  $\phi \in D(\mathcal{A})$  by (30) and (31)

$$\begin{aligned} & |\langle Ax_n - x_A, A\phi \rangle_{H_1} - \lambda_A^2 \langle x_n - x_A, \phi \rangle_{H_0}|^2 \\ & = |\langle Ax_n, A\phi \rangle_{H_1} - \lambda_A^2 \langle x_n, \phi \rangle_{H_0}|^2 \leq 2(|Ax_n|_{H_1}^2 - \lambda_A^2)(|A\phi|_{H_1}^2 - \lambda_A^2 |\phi|_{H_0}^2). \end{aligned}$$

In particular, for  $\phi = x_n - x_A$  we see

$$| |Ax_n - x_A|_{H_1}^2 - \lambda_A^2 |x_n - x_A|_{H_0}^2 | \leq c(|Ax_n|_{H_1}^2 - \lambda_A^2) \rightarrow 0,$$

and hence

$$|Ax_n - x_A|_{H_1}^2 \leq \lambda_A^2 |x_n - x_A|_{H_0}^2 + | |Ax_n - x_A|_{H_1}^2 - \lambda_A^2 |x_n - x_A|_{H_0}^2 | \rightarrow 0.$$

• For  $0 \neq \lambda \in \sigma(A^*A)$  we have  $A^*Ax = \lambda x$  for some  $0 \neq x \in D(A^*A) = D(\mathcal{A}^*A)$ . Hence  $x \in R(A^*)$  and thus  $x \in D(\mathcal{A})$ , showing  $0 \neq x \in D(\mathcal{A}^*A)$ . So  $\lambda \in \sigma(\mathcal{A}^*\mathcal{A})$ .

• For  $0 \neq \lambda \in \sigma\left(\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right) \setminus \{0\}$  we have  $A^*y = \lambda x$  and  $Ax = \lambda y$  for some  $0 \neq (x, y) \in D(A) \times D(A^*)$ . Hence  $(x, y) \in R(A^*) \times R(A)$  and thus  $(x, y) \in D(\mathcal{A}) \times D(\mathcal{A}^*)$ , showing  $0 \neq (x, y) \in D(\mathcal{A}) \times D(\mathcal{A}^*)$ . So  $\lambda \in \sigma\left(\begin{bmatrix} 0 & \mathcal{A}^* \\ \mathcal{A} & 0 \end{bmatrix}\right)$ .

• It holds

$$|\mathcal{A}^{-1}|_{R(A), R(A^*)} = \sup_{0 \neq y \in D(\mathcal{A}^{-1})} \frac{|\mathcal{A}^{-1}y|_{H_0}}{|y|_{H_1}} = \sup_{0 \neq x \in D(\mathcal{A})} \frac{|x|_{H_0}}{|Ax|_{H_1}} = \left( \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_1}}{|x|_{H_0}} \right)^{-1} = c_A.$$

<sup>2</sup> For all  $\varphi, \phi \in D(\mathcal{A})$  and for all  $\varepsilon \in \mathbb{R}$  it holds  $\lambda_A |\varphi + \varepsilon \phi|_{H_0} \leq |A(\varphi + \varepsilon \phi)|_{H_1}$ , i.e.,

$$0 \leq \underbrace{(|A\varphi|_{H_1}^2 - \lambda_A^2 |\varphi|_{H_0}^2)}_{=: \alpha \geq 0} + 2\varepsilon \underbrace{\Re(\langle A\varphi, A\phi \rangle_{H_1} - \lambda_A^2 \langle \varphi, \phi \rangle_{H_0})}_{=: \delta} + \varepsilon^2 \underbrace{(|A\phi|_{H_1}^2 - \lambda_A^2 |\phi|_{H_0}^2)}_{=: \gamma \geq 0}.$$

Let  $\beta := \Re \delta$  and  $0 \leq f(\varepsilon) := \alpha + 2\beta\varepsilon + \gamma\varepsilon^2$ . If  $\gamma = 0$  then  $\beta = 0$ . For  $\gamma > 0$  the minimum of  $f$  is attained at  $\varepsilon = -\beta/\gamma$  and thus  $0 \leq f(-\beta/\gamma) = \alpha - 2\beta^2/\gamma + \beta^2/\gamma = \alpha - \beta^2/\gamma$  yielding  $\beta^2 \leq \alpha\gamma$ . Replacing  $\varepsilon$  by  $-i\varepsilon$  shows the same inequality  $\beta^2 \leq \alpha\gamma$  for  $\beta := \Im \delta$ . Hence  $|\delta|^2 \leq 2\alpha\gamma$ .



• Let  $x_{A^*A}$  with  $|x_{A^*A}|_{H_0} = 1$  be an eigenvector of  $A^*A$  to the eigenvalue  $\lambda_A^2$ . Then  $x_{A^*A} \in R(A^*)$  and  $\lambda_A^2(\mathcal{A}^*\mathcal{A})^{-1}x_{A^*A} = x_{A^*A}$ . Thus

$$\begin{aligned} |(\mathcal{A}^*\mathcal{A})^{-1}|_{R(A^*),R(A^*)} &\leq |\mathcal{A}^{-1}|_{R(A),R(A^*)} |(\mathcal{A}^*)^{-1}|_{R(A^*),R(A)} = c_A^2 \\ &= \sup_{0 \neq x \in D((\mathcal{A}^*\mathcal{A})^{-1})} \frac{|(\mathcal{A}^*\mathcal{A})^{-1}x|_{H_0}}{|x|_{H_0}} \geq |(\mathcal{A}^*\mathcal{A})^{-1}x_{A^*A}|_{H_0} = \frac{1}{\lambda_A^2} = c_A^2. \end{aligned}$$

• For  $x \in N(A^*A)$  we have  $0 = \langle A^*Ax, x \rangle_{H_0} = |Ax|_{H_1}^2$ , i.e.,  $x \in N(A)$ . Analogously, we see  $N(AA^*) = N(A^*)$ . For  $x \in N(AA^*)$  we have  $Ax \in N(AA^*) = N(A^*)$ , i.e.,  $x \in N(A^*A) = N(A)$ . The latter arguments can be repeated for any higher power.

• For  $y \in R(A)$  we see  $x := \mathcal{A}^{-1}y \in D(\mathcal{A}) \subset R(A^*)$  and  $z := (\mathcal{A}^*)^{-1}x \in D(\mathcal{A}^*) \subset R(A)$ . Thus  $z \in D(\mathcal{A}\mathcal{A}^*)$  and  $A^*z = x$  and  $Ax = y$  as well as  $AA^*z = Ax = y \in R(\mathcal{A}\mathcal{A}^*) = R(AA^*)$ . The latter arguments can be repeated for any higher power, completing the proof.  $\square$

**Proof of Lemma 2.11.** (i) By (6) we just have to show that  $A_0A_0^* + A_1^*A_1$  is self-adjoint. For this, let  $y \in H_1$  such that there exists  $z \in H_1$  with

$$\forall x \in D(A_0A_0^* + A_1^*A_1) \quad \langle (A_0A_0^* + A_1^*A_1)x, y \rangle_{H_1} = \langle x, z \rangle_{H_1}. \quad (33)$$

Picking  $x \in N_{0,1}$  shows that  $z \perp_{H_1} N_{0,1}$  and hence, according to Theorem 2.8,  $y$  and  $z$  can be orthogonally decomposed into

$$\begin{aligned} y &= y_{R(A_0)} + y_{R(A_1^*)} + y_{N_{0,1}} \in R(A_0) \oplus_{H_1} R(A_1^*) \oplus_{H_1} N_{0,1}, \\ z &= z_{R(A_0)} + z_{R(A_1^*)} \in R(A_0) \oplus_{H_1} R(A_1^*). \end{aligned}$$

(33) implies for all  $x \in D(A_0A_0^* + A_1^*A_1)$

$$\begin{aligned} \langle A_0A_0^*x, y_{R(A_0)} \rangle_{H_1} + \langle A_1^*A_1x, y_{R(A_1^*)} \rangle_{H_1} &= \langle (A_0A_0^* + A_1^*A_1)x, y \rangle_{H_1} \\ &= \langle x, z \rangle_{H_1} = \langle x, z_{R(A_0)} \rangle_{H_1} + \langle x, z_{R(A_1^*)} \rangle_{H_1}. \end{aligned} \quad (34)$$

For  $x \in D(\mathcal{A}_1^*\mathcal{A}_1) \subset R(A_1^*) \subset N(A_0^*)$  we see by (34) that  $\langle A_1^*A_1x, y_{R(A_1^*)} \rangle_{H_1} = \langle x, z_{R(A_1^*)} \rangle_{H_1}$  holds, yielding by (6), i.e.,  $\mathcal{A}_1^*\mathcal{A}_1$  is self-adjoint, that  $y_{R(A_1^*)} \in D(\mathcal{A}_1^*\mathcal{A}_1) \subset N(A_0^*)$  with  $A_1^*A_1y_{R(A_1^*)} = z_{R(A_1^*)}$ . Analogously we see by using  $x \in D(\mathcal{A}_0\mathcal{A}_0^*)$  that  $y_{R(A_0)} \in D(\mathcal{A}_0\mathcal{A}_0^*) \subset N(A_1)$  with  $A_0A_0^*y_{R(A_0)} = z_{R(A_0)}$ . Thus  $y \in D(A_0A_0^* + A_1^*A_1)$  with  $(A_0A_0^* + A_1^*A_1)y = z_{R(A_0)} + z_{R(A_1^*)} = z$ , i.e., we have shown  $(A_0A_0^* + A_1^*A_1)^* = A_0A_0^* + A_1^*A_1$ .

(v) Let  $0 \neq \lambda \in \sigma(A_0A_0^* + A_1^*A_1)$  and let  $0 \neq x \in D(A_0A_0^* + A_1^*A_1)$  be an eigenvector to the eigenvalue  $\lambda$ . Then  $y := A_1^*A_1x = \lambda x - A_0A_0^*x \in D(A_1^*A_1)$  and

$$A_1^*A_1y = \lambda A_1^*A_1x = \lambda y.$$

Thus, as long as  $y \neq 0$ ,  $\lambda$  is an eigenvalue of  $A_1^*A_1$  with eigenvector  $y$ . On the other hand, if  $y = 0$ , then  $z := A_0A_0^*x = \lambda x \in D(A_0A_0^*) \setminus \{0\}$  and  $A_0A_0^*z = \lambda A_0A_0^*x = \lambda z$ . Hence  $\lambda$  is an eigenvalue of  $A_0A_0^*$  with eigenvector  $z$ . This shows

$$\sigma(A_0A_0^* + A_1^*A_1) \setminus \{0\} \subset (\sigma(A_0A_0^*) \setminus \{0\}) \cup (\sigma(A_1^*A_1) \setminus \{0\}).$$

For the other inclusion, let, e.g.,  $0 \neq \lambda \in \sigma(A_1^*A_1)$  and let  $0 \neq x \in D(A_1^*A_1)$  be an eigenvector to the eigenvalue  $\lambda$ . Then  $x \in R(A_1^*) \subset N(A_0^*)$  and thus  $(A_0A_0^* + A_1^*A_1)x = A_1^*A_1x = \lambda x$ , i.e.,  $\lambda$  is an eigenvalue of  $A_0A_0^* + A_1^*A_1$  with eigenvector  $x$ . Thus

$$\sigma(A_1^*A_1) \setminus \{0\} \subset \sigma(A_0A_0^* + A_1^*A_1) \setminus \{0\},$$

and analogously we show  $\sigma(A_0A_0^*) \setminus \{0\} \subset \sigma(A_0A_0^* + A_1^*A_1) \setminus \{0\}$ .

(iii) Let  $x \in N(A_0A_0^* + A_1^*A_1)$ . Then

$$0 = \langle (A_0A_0^* + A_1^*A_1)x, x \rangle_{H_1} = |A_0^*x|_{H_0}^2 + |A_1x|_{H_2}^2,$$

showing  $x \in N_{0,1}$ . As  $D(A_1) \cap D(A_0^*) \hookrightarrow H_1$  is compact, so is  $D(A_0A_0^* + A_1^*A_1) \hookrightarrow H_1$ , showing that the range  $R(A_0A_0^* + A_1^*A_1)$  is closed by Remark 2.3(ii). Thus

$$R(A_0A_0^* + A_1^*A_1) = N(A_0A_0^* + A_1^*A_1)^{\perp_{H_1}} = N_{0,1}^{\perp_{H_1}},$$

finishing the proof.  $\square$

## Appendix B. Analytical calculations

We compute the exact eigenvalues and eigenfunctions of Section 3 in detail.

### B.1. 1D

Recall the situation and notations from Sections 2.3.1 and 3.1. In particular,

$$\frac{1}{c_{0,\Gamma_\tau}} = \lambda_{0,\Gamma_\tau} = \lambda_{0,\Gamma_\nu} = \frac{1}{c_{0,\Gamma_\nu}}.$$

Let  $u = u_{0,\Gamma_\tau}$  be the first eigenfunction for the eigenvalue  $\lambda^2$  with  $\lambda = \lambda_{0,\Gamma_\tau} > 0$  of  $-\Delta_{\Gamma_\tau}$ . Hence, we have  $E_{0,\Gamma_\nu} = \text{grad } u_{0,\Gamma_\tau}$  and

$$u \in D(\Delta_{\Gamma_\tau}) \cap L^2_{\Gamma_\nu}(\Omega) \subset H^1_{\Gamma_\tau}(\Omega) \cap L^2_{\Gamma_\nu}(\Omega), \quad \text{grad } u = u' \in D(\text{div}_{\Gamma_\nu}) = H^1_{\Gamma_\nu}(\Omega),$$

as well as

$$(-\Delta - \lambda^2)u = -u'' - \lambda^2 u = 0.$$

Then

$$u = \alpha \sin(\lambda x) + \beta \cos(\lambda x), \quad u'(x) = \alpha \lambda \cos(\lambda x) - \beta \lambda \sin(\lambda x).$$

For the different boundary conditions we get:

- $\Gamma_\tau = \emptyset$  and  $\Gamma_\nu = \Gamma$ , i.e.,  $u'(0) = u'(1) = 0$ :  $\alpha = 0$ ,  $\lambda = n\pi$ ,  $n \in \mathbb{N}_0$ , i.e.,

$$\lambda_{0,\emptyset} = \pi, \quad u_{0,\emptyset}(x) = \beta \cos(\pi x).$$

Note that in this case the first eigenvalue is  $\lambda = 0$ .

- $\Gamma_\tau = \{0\}$  and  $\Gamma_\nu = \{1\}$ , i.e.,  $u(0) = u'(1) = 0$ :  $\beta = 0$ ,  $\lambda = (n - 1/2)\pi$ ,  $n \in \mathbb{N}$ , i.e.,

$$\lambda_{0,\{0\}} = \frac{\pi}{2}, \quad u_{0,\{0\}}(x) = \alpha \sin\left(\frac{\pi}{2}x\right).$$

- $\Gamma_\tau = \{1\}$  and  $\Gamma_\nu = \{0\}$ , i.e.,  $u'(0) = u(1) = 0$ :  $\alpha = 0$ ,  $\lambda = (n - 1/2)\pi$ ,  $n \in \mathbb{N}$ , i.e.,

$$\lambda_{0,\{1\}} = \frac{\pi}{2}, \quad u_{0,\{1\}}(x) = \beta \cos\left(\frac{\pi}{2}x\right).$$

- $\Gamma_\tau = \Gamma$  and  $\Gamma_\nu = \emptyset$ , i.e.,  $u(0) = u(1) = 0$ :  $\beta = 0$ ,  $\lambda = n\pi$ ,  $n \in \mathbb{N}$ , i.e.,

$$\lambda_{0,\Gamma} = \pi, \quad u_{0,\Gamma}(x) = \alpha \sin(\pi x).$$

Note that from  $\lambda_{0,\Gamma_\tau} = \lambda_{0,\Gamma_\nu}$  we already know  $\lambda_{0,\Gamma} = \lambda_{0,\emptyset}$  and  $\lambda_{0,\{0\}} = \lambda_{0,\{1\}}$ , i.e.,

$$\lambda_{0,\Gamma} = \lambda_{0,\emptyset} = \pi, \quad \lambda_{0,\{0\}} = \lambda_{0,\{1\}} = \frac{\pi}{2}.$$

### B.2. 2D

Recall the situation and notations from Sections 2.3.2 and 3.2. In particular,

$$\frac{1}{c_{0,\Gamma_\tau}} = \lambda_{0,\Gamma_\tau} = \lambda_{1,\Gamma_\nu} = \frac{1}{c_{1,\Gamma_\nu}}.$$

Let  $u = u_{0,\Gamma_\tau}$  be the first eigenfunction for the eigenvalue  $\lambda^2$  with  $\lambda = \lambda_{0,\Gamma_\tau} > 0$  of  $-\Delta_{\Gamma_\tau}$ . Hence, we have  $E_{0,\Gamma_\nu} = \text{grad } u_{0,\Gamma_\tau}$  and<sup>3</sup>

$$u \in D(\Delta_{\Gamma_\tau}) \cap L^2_{\Gamma_\nu}(\Omega) \subset H^1_{\Gamma_\tau}(\Omega) \cap L^2_{\Gamma_\nu}(\Omega), \quad \text{grad } u \in D(\text{div}_{\Gamma_\nu}) = H_{\Gamma_\nu}(\text{div}, \Omega),$$

as well as

$$(-\Delta - \lambda^2)u = 0.$$

Separation of variables shows with  $u(x) = u_1(x_1)u_2(x_2)$  and  $\text{grad } u(x) = \begin{bmatrix} u'_1(x_1)u_2(x_2) \\ u_1(x_1)u'_2(x_2) \end{bmatrix}$

$$0 = (-\Delta - \lambda^2)u(x) = -u''_1(x_1)u_2(x_2) - u_1(x_1)u''_2(x_2) - \lambda^2 u_1(x_1)u_2(x_2).$$

<sup>3</sup> Note that

$$\begin{aligned} E_{1,\Gamma_\tau} &\in D(\square_{\Gamma_\tau}) \cap R(\vec{\text{curl}}_{\Gamma_\nu}) \subset H_{\Gamma_\tau}(\text{curl}, \Omega) \cap R(\vec{\text{curl}}_{\Gamma_\nu}), \\ H_{1,\Gamma_\nu} &= \text{curl } E_{1,\Gamma_\tau} \in D(\vec{\text{curl}}_{\Gamma_\nu}) \cap R(\text{curl}_{\Gamma_\tau}) = H_{\Gamma_\nu}(\vec{\text{curl}}, \Omega) \cap L^2_{\Gamma_\tau}(\Omega) = H^1_{\Gamma_\nu}(\Omega) \cap L^2_{\Gamma_\tau}(\Omega). \end{aligned}$$



For fixed  $x_2$  with  $u_2(x_2) \neq 0$  we get

$$-u_1''(x_1) - \mu_1^2 u_1(x_1) = 0, \quad \mu_1^2 = \frac{u_2''(x_2)}{u_2(x_2)} + \lambda^2,$$

i.e.,

$$-u_1''(x_1) - \mu_1^2 u_1(x_1) = 0, \quad -u_2''(x_2) - \mu_2^2 u_2(x_2) = 0, \quad \lambda^2 = \mu_1^2 + \mu_2^2.$$

The Dirichlet boundary conditions, i.e.,

$$u = 0 \quad \text{on} \quad \Gamma_\tau,$$

reduce to Dirichlet boundary conditions for  $u_1$  and  $u_2$ , respectively, and the Neumann boundary conditions, i.e.,

$$n \cdot \text{grad} u = 0 \quad \text{on} \quad \Gamma_\nu,$$

reduce to Dirichlet boundary conditions for  $u_1'$  and  $u_2'$ , respectively. More precisely, we have:

- $\Gamma_l, n = -e^1, x_1 = 0$ :

$$\begin{aligned} 0 = u|_{\Gamma_l} = u_1 u_2|_{\Gamma_l} &\Rightarrow u_1(0) = 0, \\ 0 = n \cdot \text{grad} u|_{\Gamma_l} = -u_1' u_2|_{\Gamma_l} &\Rightarrow u_1'(0) = 0. \end{aligned}$$

- $\Gamma_r, n = e^1, x_1 = 1$ :

$$\begin{aligned} 0 = u|_{\Gamma_r} = u_1 u_2|_{\Gamma_r} &\Rightarrow u_1(1) = 0, \\ 0 = n \cdot \text{grad} u|_{\Gamma_r} = u_1' u_2|_{\Gamma_r} &\Rightarrow u_1'(1) = 0. \end{aligned}$$

- $\Gamma_b, n = -e^2, x_2 = 0$ :

$$\begin{aligned} 0 = u|_{\Gamma_b} = u_1 u_2|_{\Gamma_b} &\Rightarrow u_2(0) = 0, \\ 0 = n \cdot \text{grad} u|_{\Gamma_b} = -u_1 u_2'|_{\Gamma_b} &\Rightarrow u_2'(0) = 0. \end{aligned}$$

- $\Gamma_t, n = e^2, x_2 = 1$ :

$$\begin{aligned} 0 = u|_{\Gamma_t} = u_1 u_2|_{\Gamma_t} &\Rightarrow u_2(1) = 0, \\ 0 = n \cdot \text{grad} u|_{\Gamma_t} = u_1 u_2'|_{\Gamma_t} &\Rightarrow u_2'(1) = 0. \end{aligned}$$

The 1D case shows for the different boundary conditions the following:

- $\Gamma_\tau = \emptyset$  and  $\Gamma_\nu = \Gamma$ , i.e.,  $u_1'(0) = u_1'(1) = u_2'(0) = u_2'(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2}\pi$ ,  $n, m \in \mathbb{N}_0$ , and

$$\lambda_{0,\emptyset} = \pi, \quad u_{0,\emptyset}(x) = \alpha \cos(\pi x_1) + \beta \cos(\pi x_2).$$

Note that in this case the first eigenvalue is  $\lambda = 0$ .

- $\Gamma_\tau = \Gamma_b$  and  $\Gamma_\nu = \Gamma_{t,l,r}$ , i.e.,  $u_1'(0) = u_1'(1) = u_2(0) = u_2'(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ , i.e.,  $\lambda = \sqrt{n^2 + (m - 1/2)^2}\pi$ ,  $n \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_b} = \frac{1}{2}\pi, \quad u_{0,\Gamma_b}(x) = \alpha \sin\left(\frac{\pi}{2}x_2\right).$$

- $\Gamma_\tau = \Gamma_{b,t}$  and  $\Gamma_\nu = \Gamma_{l,r}$ , i.e.,  $u_1'(0) = u_1'(1) = u_2(0) = u_2(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2}\pi$ ,  $n \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,t}} = \pi, \quad u_{0,\Gamma_{b,t}}(x) = \alpha \sin(\pi x_2).$$

- $\Gamma_\tau = \Gamma_{b,l}$  and  $\Gamma_\nu = \Gamma_{t,r}$ , i.e.,  $u_1(0) = u_1'(1) = u_2(0) = u_2'(1) = 0$ :  $\mu_1 = (n - 1/2)\pi$ ,  $\mu_2 = (m - 1/2)\pi$ , i.e.,  $\lambda = \sqrt{(n - 1/2)^2 + (m - 1/2)^2}\pi$ ,  $n, m \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,l}} = \frac{\sqrt{2}}{2}\pi, \quad u_{0,\Gamma_{b,l}}(x) = \alpha \sin\left(\frac{\pi}{2}x_1\right) \sin\left(\frac{\pi}{2}x_2\right).$$

- $\Gamma_\tau = \Gamma_{b,l,r}$  and  $\Gamma_\nu = \Gamma_t$ , i.e.,  $u_1(0) = u_1(1) = u_2(0) = u_2'(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ , i.e.,  $\lambda = \sqrt{n^2 + (m - 1/2)^2}\pi$ ,  $n, m \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,l,r}} = \frac{\sqrt{5}}{2}\pi, \quad u_{0,\Gamma_{b,l,r}}(x) = \alpha \sin(\pi x_1) \sin\left(\frac{\pi}{2}x_2\right).$$

- $\Gamma_\tau = \Gamma$  and  $\Gamma_\nu = \emptyset$ , i.e.,  $u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2}\pi$ ,  $n, m \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma} = \sqrt{2}\pi, \quad u_{0,\Gamma}(x) = \alpha \sin(\pi x_1) \sin(\pi x_2).$$

All other cases follow by symmetry, i.e.,

$$\begin{aligned}\lambda_{0,\emptyset} &= \pi, & \lambda_{0,\Gamma_{b,l}} &= \lambda_{0,\Gamma_{b,r}} = \lambda_{0,\Gamma_{t,l}} = \lambda_{0,\Gamma_{t,r}} = \frac{\sqrt{2}}{2}\pi, \\ \lambda_{0,\Gamma_b} &= \lambda_{0,\Gamma_t} = \lambda_{0,\Gamma_l} = \lambda_{0,\Gamma_r} = \frac{1}{2}\pi, & \lambda_{0,\Gamma_{b,l,r}} &= \lambda_{0,\Gamma_{t,l,r}} = \lambda_{0,\Gamma_{b,t,l}} = \lambda_{0,\Gamma_{b,t,r}} = \frac{\sqrt{5}}{2}\pi, \\ \lambda_{0,\Gamma_{b,t}} &= \lambda_{0,\Gamma_{l,r}} = \pi, & \lambda_{0,\Gamma} &= \sqrt{2}\pi.\end{aligned}$$

### B.3. 3D

Recall the situation and notations from Section 2.2, Theorem 2.20, and Section 3.3. In particular,

$$\frac{1}{c_{0,\Gamma_t}} = \lambda_{0,\Gamma_t} = \lambda_{2,\Gamma_v} = \frac{1}{c_{2,\Gamma_v}}, \quad \frac{1}{c_{1,\Gamma_t}} = \lambda_{1,\Gamma_t} = \lambda_{1,\Gamma_v} = \frac{1}{c_{1,\Gamma_v}}.$$

Let  $u = u_{0,\Gamma_t}$  be the first eigenfunction for the eigenvalue  $\lambda^2$  with  $\lambda = \lambda_{0,\Gamma_t} > 0$  of  $-\Delta_{\Gamma_t}$ . Analogously, let  $E = E_{1,\Gamma_t}$  be the first eigenfunction for the eigenvalue  $\tilde{\lambda}^2$  with  $\tilde{\lambda} = \lambda_{1,\Gamma_t} > 0$  of  $\square_{\Gamma_t}$ . Hence,

$$\begin{aligned}u &\in D(\Delta_{\Gamma_t}) \cap L^2_{\Gamma_v}(\Omega) \subset H^1_{\Gamma_t}(\Omega) \cap L^2_{\Gamma_v}(\Omega), & E &\in D(\square_{\Gamma_t}) \cap R(\text{curl}_{\Gamma_v}) \subset H_{\Gamma_t}(\text{curl}, \Omega) \cap R(\text{curl}_{\Gamma_v}), \\ \text{grad } u &\in D(\text{div}_{\Gamma_v}) = H_{\Gamma_v}(\text{div}, \Omega), & \text{curl } E &\in D(\text{curl}_{\Gamma_v}) \cap R(\text{curl}_{\Gamma_t}) = H_{\Gamma_v}(\text{curl}, \Omega) \cap R(\text{curl}_{\Gamma_t}),\end{aligned}$$

and we have by  $-\Delta = \text{curl curl} - \text{grad div} = \square - \text{grad div}$

$$(-\Delta - \lambda^2)u = 0, \quad (-\Delta - \tilde{\lambda}^2)E = (\square - \tilde{\lambda}^2)E = 0,$$

as  $\text{div } E = 0$ . Let us first discuss  $u$ . Separation of variables shows with

$$u(x) = \widehat{u}(\widehat{x})u_3(x_3) = u_1(x_1)u_2(x_2)u_3(x_3), \quad \widehat{u}(\widehat{x}) = u_1(x_1)u_2(x_2), \quad x = \begin{bmatrix} \widehat{x} \\ x_3 \end{bmatrix}, \quad \widehat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\text{grad } u(x) = \begin{bmatrix} u_3(x_3) \text{grad } \widehat{u}(\widehat{x}) \\ u'_3(x_3) \widehat{u}(\widehat{x}) \end{bmatrix} = \begin{bmatrix} u'_1(x_1)u_2(x_2)u_3(x_3) \\ u_1(x_1)u'_2(x_2)u_3(x_3) \\ u_1(x_1)u_2(x_2)u'_3(x_3) \end{bmatrix}, \quad \text{grad } \widehat{u}(\widehat{x}) = \begin{bmatrix} u'_1(x_1)u_2(x_2) \\ u_1(x_1)u'_2(x_2) \end{bmatrix}$$

that

$$0 = (-\Delta - \lambda^2)u(x) = -\Delta \widehat{u}(\widehat{x})u_3(x_3) - \widehat{u}(\widehat{x})u''_3(x_3) - \lambda^2 \widehat{u}(\widehat{x})u_3(x_3).$$

For fixed  $x_3$  with  $u_3(x_3) \neq 0$  we get

$$-\Delta \widehat{u}(\widehat{x}) - \widehat{\mu}^2 \widehat{u}(\widehat{x}) = 0, \quad \widehat{\mu}^2 = \frac{u''_3(x_3)}{u_3(x_3)} + \lambda^2,$$

i.e.,

$$-\Delta \widehat{u}(\widehat{x}) - \widehat{\mu}^2 \widehat{u}(\widehat{x}) = 0, \quad -u''_3(x_3) - \mu_3^2 u_3(x_3) = 0, \quad \lambda^2 = \widehat{\mu}^2 + \mu_3^2.$$

From the 2D case we already know  $\widehat{\mu}^2 = \mu_1^2 + \mu_2^2$  and the splitting of  $\widehat{u}$ , i.e.,

$$\lambda^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$$

and

$$-u''_1(x_1) - \mu_1^2 u_1(x_1) = 0, \quad -u''_2(x_2) - \mu_2^2 u_2(x_2) = 0, \quad -u''_3(x_3) - \mu_3^2 u_3(x_3) = 0.$$

The Dirichlet boundary conditions, i.e.,

$$u = 0 \quad \text{on } \Gamma_t,$$

reduce to Dirichlet boundary conditions for  $u_1$ ,  $u_2$ , and  $u_3$ , respectively, and the Neumann boundary conditions, i.e.,

$$n \cdot \text{grad } u = 0 \quad \text{on } \Gamma_v,$$

reduce to Neumann boundary conditions for  $\widehat{u}$  and  $u_3$  and hence to Dirichlet boundary conditions for  $u'_1$ ,  $u'_2$ , and  $u'_3$ , respectively.

- $\Gamma_k$ ,  $n = -e^1$ ,  $x_1 = 0$ :

$$\begin{aligned}0 &= u|_{\Gamma_k} = u_1 u_2 u_3|_{\Gamma_k} & \Rightarrow & & u_1(0) &= 0, \\ 0 &= n \cdot \text{grad } u|_{\Gamma_k} = -u'_1 u_2 u_3|_{\Gamma_k} & \Rightarrow & & u'_1(0) &= 0.\end{aligned}$$

- $\Gamma_f, n = e^1, x_1 = 1$ :

$$0 = u|_{\Gamma_f} = u_1 u_2 u_3|_{\Gamma_f} \Rightarrow u_1(1) = 0,$$

$$0 = n \cdot \text{grad } u|_{\Gamma_f} = u'_1 u_2 u_3|_{\Gamma_f} \Rightarrow u'_1(1) = 0.$$

- $\Gamma_l, n = -e^2, x_2 = 0$ :

$$0 = u|_{\Gamma_l} = u_1 u_2 u_3|_{\Gamma_l} \Rightarrow u_2(0) = 0,$$

$$0 = n \cdot \text{grad } u|_{\Gamma_l} = -u_1 u'_2 u_3|_{\Gamma_l} \Rightarrow u'_2(0) = 0.$$

- $\Gamma_r, n = e^2, x_2 = 1$ :

$$0 = u|_{\Gamma_r} = u_1 u_2 u_3|_{\Gamma_r} \Rightarrow u_2(1) = 0,$$

$$0 = n \cdot \text{grad } u|_{\Gamma_r} = u_1 u'_2 u_3|_{\Gamma_r} \Rightarrow u'_2(1) = 0.$$

- $\Gamma_b, n = -e^3, x_3 = 0$ :

$$0 = u|_{\Gamma_b} = u_1 u_2 u_3|_{\Gamma_b} \Rightarrow u_3(0) = 0,$$

$$0 = n \cdot \text{grad } u|_{\Gamma_b} = -u_1 u_2 u'_3|_{\Gamma_b} \Rightarrow u'_3(0) = 0.$$

- $\Gamma_t, n = e^3, x_3 = 1$ :

$$0 = u|_{\Gamma_t} = u_1 u_2 u_3|_{\Gamma_t} \Rightarrow u_3(1) = 0,$$

$$0 = n \cdot \text{grad } u|_{\Gamma_t} = u_1 u_2 u'_3|_{\Gamma_t} \Rightarrow u'_3(1) = 0.$$

The 1D case shows for the different boundary conditions the following:

- $\Gamma_\tau = \emptyset$  and  $\Gamma_v = \Gamma$ , i.e.,  $u'_1(0) = u'_1(1) = u'_2(0) = u'_2(1) = u'_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n, m, k \in \mathbb{N}_0$ , and

$$\lambda_{0,\emptyset} = \pi, \quad u_{0,\emptyset}(x) = \alpha \cos(\pi x_1) + \beta \cos(\pi x_2) + \gamma \cos(\pi x_3).$$

Note that in this case the first eigenvalue is  $\lambda = 0$ .

- $\Gamma_\tau = \Gamma_b$  and  $\Gamma_v = \Gamma_{t,l,r,f,k}$ , i.e.,  $u'_1(0) = u'_1(1) = u'_2(0) = u'_2(1) = u_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = (k - 1/2)\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2 + (k - 1/2)^2}\pi$ ,  $n, m \in \mathbb{N}_0$ ,  $k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_b} = \frac{1}{2}\pi, \quad u_{0,\Gamma_b}(x) = \alpha \sin\left(\frac{\pi}{2}x_3\right).$$

- $\Gamma_\tau = \Gamma_{b,t}$  and  $\Gamma_v = \Gamma_{l,r,f,k}$ , i.e.,  $u'_1(0) = u'_1(1) = u'_2(0) = u'_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n, m \in \mathbb{N}_0$ ,  $k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,t}} = \pi, \quad u_{0,\Gamma_{b,t}}(x) = \alpha \sin(\pi x_3).$$

- $\Gamma_\tau = \Gamma_{b,l}$  and  $\Gamma_v = \Gamma_{t,r,f,k}$ , i.e.,  $u'_1(0) = u'_1(1) = u_2(0) = u'_2(1) = u_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = (k - 1/2)\pi$ , i.e.,  $\lambda = \sqrt{n^2 + (m - 1/2)^2 + (k - 1/2)^2}\pi$ ,  $n \in \mathbb{N}_0$ ,  $m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,l}} = \frac{\sqrt{2}}{2}\pi, \quad u_{0,\Gamma_{b,l}}(x) = \alpha \sin\left(\frac{\pi}{2}x_2\right) \sin\left(\frac{\pi}{2}x_3\right).$$

- $\Gamma_\tau = \Gamma_{b,t,l}$  and  $\Gamma_v = \Gamma_{r,f,k}$ , i.e.,  $u'_1(0) = u'_1(1) = u_2(0) = u'_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{n^2 + (m - 1/2)^2 + k^2}\pi$ ,  $n \in \mathbb{N}_0$ ,  $m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,t,l}} = \frac{\sqrt{5}}{2}\pi, \quad u_{0,\Gamma_{b,t,l}}(x) = \alpha \sin\left(\frac{\pi}{2}x_2\right) \sin(\pi x_3).$$

- $\Gamma_\tau = \Gamma_{b,l,k}$  and  $\Gamma_v = \Gamma_{r,f,t}$ , i.e.,  $u_1(0) = u'_1(1) = u_2(0) = u'_2(1) = u_3(0) = u'_3(1) = 0$ :  $\mu_1 = (n - 1/2)\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = (k - 1/2)\pi$ , i.e.,  $\lambda = \sqrt{(n - 1/2)^2 + (m - 1/2)^2 + (k - 1/2)^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,l,k}} = \frac{\sqrt{3}}{2}\pi, \quad u_{0,\Gamma_{b,l,k}}(x) = \alpha \sin\left(\frac{\pi}{2}x_1\right) \sin\left(\frac{\pi}{2}x_2\right) \sin\left(\frac{\pi}{2}x_3\right).$$

- $\Gamma_\tau = \Gamma_{b,t,l,r}$  and  $\Gamma_v = \Gamma_{f,k}$ , i.e.,  $u'_1(0) = u'_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n \in \mathbb{N}_0$ ,  $m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,t,l,r}} = \sqrt{2}\pi, \quad u_{0,\Gamma_{b,t,l,r}}(x) = \alpha \sin(\pi x_2) \sin(\pi x_3).$$

- $\Gamma_\tau = \Gamma_{b,t,l,k}$  and  $\Gamma_v = \Gamma_{f,r}$ , i.e.,  $u_1(0) = u'_1(1) = u_2(0) = u'_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = (n - 1/2)\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{(n - 1/2)^2 + (m - 1/2)^2 + k^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,t,l,k}} = \frac{\sqrt{6}}{2}\pi, \quad u_{0,\Gamma_{b,t,l,k}}(x) = \alpha \sin\left(\frac{\pi}{2}x_1\right) \sin\left(\frac{\pi}{2}x_2\right) \sin(\pi x_3).$$

•  $\Gamma_\tau = \Gamma_{b,t,l,r,k}$  and  $\Gamma_v = \Gamma_f$ , i.e.,  $u_1(0) = u'_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = (n - 1/2)\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{(n - 1/2)^2 + m^2 + k^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma_{b,t,l,r,k}} = \frac{3}{2}\pi, \quad u_{0,\Gamma_{b,t,l,r,k}}(x) = \alpha \sin\left(\frac{\pi}{2}x_1\right) \sin(\pi x_2) \sin(\pi x_3).$$

•  $\Gamma_\tau = \Gamma$  and  $\Gamma_v = \emptyset$ , i.e.,  $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\lambda = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and

$$\lambda_{0,\Gamma} = \sqrt{3}\pi, \quad u_{0,\Gamma}(x) = \alpha \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3).$$

All other cases follow by symmetry, i.e.,

$$\begin{aligned} \lambda_{0,\emptyset} &= \pi, \\ \lambda_{0,\Gamma_b} &= \lambda_{0,\Gamma_t} = \lambda_{0,\Gamma_l} = \lambda_{0,\Gamma_r} = \lambda_{0,\Gamma_f} = \lambda_{0,\Gamma_k} = \frac{1}{2}\pi, \\ \lambda_{0,\Gamma_{b,t}} &= \lambda_{0,\Gamma_{l,r}} = \lambda_{0,\Gamma_{f,k}} = \pi, \\ \lambda_{0,\Gamma_{b,l}} &= \lambda_{0,\Gamma_{b,r}} = \lambda_{0,\Gamma_{b,f}} = \lambda_{0,\Gamma_{b,k}} \\ &= \lambda_{0,\Gamma_{t,l}} = \lambda_{0,\Gamma_{t,r}} = \lambda_{0,\Gamma_{t,f}} = \lambda_{0,\Gamma_{t,k}} = \lambda_{0,\Gamma_{f,l}} = \lambda_{0,\Gamma_{f,r}} = \lambda_{0,\Gamma_{k,l}} = \lambda_{0,\Gamma_{k,r}} = \frac{\sqrt{2}}{2}\pi, \\ \lambda_{0,\Gamma_{b,t,l}} &= \lambda_{0,\Gamma_{b,t,r}} = \lambda_{0,\Gamma_{b,t,f}} = \lambda_{0,\Gamma_{b,t,k}} = \lambda_{0,\Gamma_{l,r,b}} \\ &= \lambda_{0,\Gamma_{l,r,t}} = \lambda_{0,\Gamma_{l,r,f}} = \lambda_{0,\Gamma_{l,r,k}} = \lambda_{0,\Gamma_{f,k,l}} = \lambda_{0,\Gamma_{f,k,r}} = \lambda_{0,\Gamma_{f,k,b}} = \lambda_{0,\Gamma_{f,k,t}} = \frac{\sqrt{5}}{2}\pi, \\ \lambda_{0,\Gamma_{b,k,l}} &= \lambda_{0,\Gamma_{b,l,f}} = \lambda_{0,\Gamma_{b,f,r}} = \lambda_{0,\Gamma_{b,r,k}} = \lambda_{0,\Gamma_{t,k,l}} = \lambda_{0,\Gamma_{t,l,f}} = \lambda_{0,\Gamma_{t,f,r}} = \lambda_{0,\Gamma_{t,r,k}} = \frac{\sqrt{3}}{2}\pi, \\ \lambda_{0,\Gamma_{b,t,l,r}} &= \lambda_{0,\Gamma_{b,t,f,k}} = \lambda_{0,\Gamma_{l,r,f,k}} = \sqrt{2}\pi, \\ \lambda_{0,\Gamma_{b,t,l,k}} &= \lambda_{0,\Gamma_{b,t,f,l}} = \lambda_{0,\Gamma_{b,t,r,f}} = \lambda_{0,\Gamma_{b,t,r,k}} = \lambda_{0,\Gamma_{l,r,f,t}} = \lambda_{0,\Gamma_{l,r,f,b}} \\ &= \lambda_{0,\Gamma_{l,r,t,k}} = \lambda_{0,\Gamma_{l,r,b,k}} = \lambda_{0,\Gamma_{f,k,b,l}} = \lambda_{0,\Gamma_{f,k,t,l}} = \lambda_{0,\Gamma_{f,k,b,r}} = \lambda_{0,\Gamma_{f,k,r,r}} = \frac{\sqrt{6}}{2}\pi, \\ \lambda_{0,\Gamma_{b,t,l,r,k}} &= \lambda_{0,\Gamma_{b,t,l,r,f}} = \lambda_{0,\Gamma_{b,t,l,f,k}} = \lambda_{0,\Gamma_{b,t,r,f,k}} = \lambda_{0,\Gamma_{b,l,r,f,k}} = \lambda_{0,\Gamma_{t,l,r,f,k}} = \frac{3}{2}\pi, \\ \lambda_{0,\Gamma} &= \sqrt{3}\pi. \end{aligned}$$

Now, we take care of  $E$ . As  $\operatorname{div} E = 0$  and  $(-\Delta - \tilde{\lambda}^2)E = 0$ , a simple ansatz is given by, e.g.,

$$E := \operatorname{curl} U = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \\ 0 \end{bmatrix}, \quad U(x) := u(x)e^3 = u(x) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where  $u$  is a solution of  $(-\Delta - \tilde{\lambda}^2)u = 0$ , i.e.,  $(-\Delta - \tilde{\lambda}^2)U = 0$ . Then  $\operatorname{div} E = 0$  and

$$-\Delta E = \operatorname{curl} \operatorname{curl} E = \operatorname{curl} \operatorname{curl} \operatorname{curl} U = -\operatorname{curl} \Delta U = \tilde{\lambda}^2 \operatorname{curl} U = \tilde{\lambda}^2 E.$$

As  $u$  solves  $(-\Delta - \tilde{\lambda}^2)u = 0$  we have again by separation of variables

$$u(x) = u_1(x_1)u_2(x_2)u_3(x_3), \quad \operatorname{grad} u(x) = \begin{bmatrix} u'_1(x_1)u_2(x_2)u_3(x_3) \\ u_1(x_1)u'_2(x_2)u_3(x_3) \\ u_1(x_1)u_2(x_2)u'_3(x_3) \end{bmatrix}$$

as well as

$$\tilde{\lambda}^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$$

and

$$-u''_1(x_1) - \mu_1^2 u_1(x_1) = 0, \quad -u''_2(x_2) - \mu_2^2 u_2(x_2) = 0, \quad -u''_3(x_3) - \mu_3^2 u_3(x_3) = 0.$$

Moreover, by the complex property  $R(\operatorname{curl}_{\Gamma_v}) \subset N(\operatorname{div}_{\Gamma_v})$ ,  $E$  must satisfy

$$\begin{aligned} E &\in D(\operatorname{curl}_{\Gamma_\tau}) \cap R(\operatorname{curl}_{\Gamma_v}) \subset D(\operatorname{curl}_{\Gamma_\tau}) \cap N(\operatorname{div}_{\Gamma_v}), \\ \operatorname{curl} E &\in D(\operatorname{curl}_{\Gamma_v}) \cap R(\operatorname{curl}_{\Gamma_\tau}) \subset D(\operatorname{curl}_{\Gamma_v}) \cap N(\operatorname{div}_{\Gamma_\tau}), \end{aligned}$$

i.e., in classical terms

$$n \times E|_{\Gamma_\tau} = 0, \quad n \cdot E|_{\Gamma_v} = 0, \quad n \times \operatorname{curl} E|_{\Gamma_v} = 0, \quad n \cdot \operatorname{curl} E|_{\Gamma_\tau} = 0.$$

As the fourth boundary condition is implied by the first one and the second boundary condition is implied by the third one ( $n \times \text{curl } E|_{\Gamma_v} = 0 \Rightarrow 0 = n \cdot \text{curl curl } E|_{\Gamma_v} = \tilde{\lambda}^2 n \cdot E|_{\Gamma_v}$ ), the third and fourth ones are (almost) redundant, and we are (almost) left with the simple boundary conditions

$$n \times E|_{\Gamma_t} = 0, \quad n \cdot E|_{\Gamma_v} = 0,$$

except for some special cases, where also the third one

$$n \times \text{curl } E|_{\Gamma_v} = 0$$

is needed. For the computations of the boundary conditions we note<sup>4</sup>

$$\text{curl } E = \begin{bmatrix} -\partial_3 E_2 \\ \partial_3 E_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix} = \begin{bmatrix} \partial_1 \partial_3 u \\ \partial_2 \partial_3 u \\ -\partial_1^2 u - \partial_2^2 u \end{bmatrix},$$

and thus

$$E = \begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \\ 0 \end{bmatrix}, \quad e^1 \times E = \begin{bmatrix} 0 \\ 0 \\ E_2 \end{bmatrix}, \quad e^2 \times E = \begin{bmatrix} 0 \\ 0 \\ -E_1 \end{bmatrix}, \quad e^3 \times E = \begin{bmatrix} -E_2 \\ E_1 \\ 0 \end{bmatrix},$$

$$e^3 \times \text{curl } E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\partial_3 E_2 \\ \partial_3 E_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix} = -\begin{bmatrix} \partial_3 E_1 \\ \partial_3 E_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\partial_2 \partial_3 u \\ \partial_1 \partial_3 u \\ 0 \end{bmatrix}.$$

As an alternative we can also set boundary conditions for  $U$  directly. Since

$$E = \text{curl } U \in R(\text{curl}_{\Gamma_v}),$$

we get  $n \times U|_{\Gamma_v} = u n \times e^3|_{\Gamma_v} = 0$ .

- $\Gamma_k, n = -e^1, x_1 = 0$ :

$$\begin{aligned} 0 = n \times E|_{\Gamma_k} &= -E_2 e^3|_{\Gamma_k} = \partial_1 u e^3|_{\Gamma_k} = u'_1 u_2 u_3 e^3|_{\Gamma_k} &\Rightarrow u'_1(0) &= 0, \\ 0 = n \cdot E|_{\Gamma_k} &= -E_1|_{\Gamma_k} = -\partial_2 u|_{\Gamma_k} = -u_1 u'_2 u_3|_{\Gamma_k} &\Rightarrow u_1(0) &= 0. \end{aligned}$$

Alternatively,

$$0 = u n \times e^3|_{\Gamma_k} = u e^2|_{\Gamma_k} = u_1 u_2 u_3 e^2|_{\Gamma_k} \Rightarrow u_1(0) = 0.$$

- $\Gamma_f, n = e^1, x_1 = 1$ :

$$\begin{aligned} 0 = n \times E|_{\Gamma_f} &= E_2 e^3|_{\Gamma_f} = -\partial_1 u e^3|_{\Gamma_f} = -u'_1 u_2 u_3 e^3|_{\Gamma_f} &\Rightarrow u'_1(1) &= 0, \\ 0 = n \cdot E|_{\Gamma_f} &= E_1|_{\Gamma_f} = \partial_2 u|_{\Gamma_f} = u_1 u'_2 u_3|_{\Gamma_f} &\Rightarrow u_1(1) &= 0. \end{aligned}$$

Alternatively,

$$0 = u n \times e^3|_{\Gamma_f} = -u e^2|_{\Gamma_f} = -u_1 u_2 u_3 e^2|_{\Gamma_f} \Rightarrow u_1(1) = 0.$$

- $\Gamma_l, n = -e^2, x_2 = 0$ :

$$\begin{aligned} 0 = n \times E|_{\Gamma_l} &= E_1 e^3|_{\Gamma_l} = \partial_2 u e^3|_{\Gamma_l} = u_1 u'_2 u_3 e^3|_{\Gamma_l} &\Rightarrow u'_2(0) &= 0, \\ 0 = n \cdot E|_{\Gamma_l} &= -E_2|_{\Gamma_l} = \partial_1 u|_{\Gamma_l} = u'_1 u_2 u_3|_{\Gamma_l} &\Rightarrow u_2(0) &= 0. \end{aligned}$$

Alternatively,

$$0 = u n \times e^3|_{\Gamma_l} = -u e^1|_{\Gamma_l} = -u_1 u_2 u_3 e^1|_{\Gamma_l} \Rightarrow u_2(0) = 0.$$

- $\Gamma_r, n = e^2, x_2 = 1$ :

$$\begin{aligned} 0 = n \times E|_{\Gamma_r} &= -E_1 e^3|_{\Gamma_r} = -\partial_2 u e^3|_{\Gamma_r} = -u_1 u'_2 u_3 e^3|_{\Gamma_r} &\Rightarrow u'_2(1) &= 0, \\ 0 = n \cdot E|_{\Gamma_r} &= E_2|_{\Gamma_r} = -\partial_1 u|_{\Gamma_r} = -u'_1 u_2 u_3|_{\Gamma_r} &\Rightarrow u_2(1) &= 0. \end{aligned}$$

Alternatively,

$$0 = u n \times e^3|_{\Gamma_r} = u e^1|_{\Gamma_r} = u_1 u_2 u_3 e^1|_{\Gamma_r} \Rightarrow u_2(1) = 0.$$

<sup>4</sup> Alternatively,  $\text{curl } E = \text{curl curl } U = -\Delta U + \text{grad div } U = -\Delta u e^3 + \text{grad } \partial_3 u = \begin{bmatrix} \partial_1 \partial_3 u \\ \partial_2 \partial_3 u \\ -\partial_1^2 u - \partial_2^2 u \end{bmatrix} = \begin{bmatrix} -\partial_3 E_2 \\ \partial_3 E_1 \\ \partial_1 E_2 - \partial_2 E_1 \end{bmatrix}.$

- $\Gamma_b, n = -e^3, x_3 = 0$ :

$$0 = n \times E|_{\Gamma_b} = \begin{bmatrix} E_2 \\ -E_1 \\ 0 \end{bmatrix}|_{\Gamma_b} = -\begin{bmatrix} \partial_1 u \\ \partial_2 u \\ 0 \end{bmatrix}|_{\Gamma_b} = -\begin{bmatrix} u'_1 u_2 u_3 \\ u_1 u'_2 u_3 \\ 0 \end{bmatrix}|_{\Gamma_b} \Rightarrow u_3(0) = 0,$$

$$0 = n \cdot E|_{\Gamma_b} = 0 \quad (\text{no condition}),$$

$$0 = n \times \text{curl } E|_{\Gamma_b} = \begin{bmatrix} \partial_3 E_1 \\ \partial_3 E_2 \\ 0 \end{bmatrix}|_{\Gamma_b} = \begin{bmatrix} \partial_2 \partial_3 u \\ -\partial_1 \partial_3 u \\ 0 \end{bmatrix}|_{\Gamma_b} = \begin{bmatrix} u_1 u'_2 u'_3 \\ -u'_1 u_2 u'_3 \\ 0 \end{bmatrix}|_{\Gamma_b} \Rightarrow u'_3(0) = 0.$$

Alternatively,

$$0 = u n \times e^3|_{\Gamma_b} = 0 \quad (\text{no condition}).$$

- $\Gamma_t, n = e^3, x_3 = 1$ :

$$0 = n \times E|_{\Gamma_t} = \begin{bmatrix} -E_2 \\ E_1 \\ 0 \end{bmatrix}|_{\Gamma_t} = \begin{bmatrix} \partial_1 u \\ \partial_2 u \\ 0 \end{bmatrix}|_{\Gamma_t} = \begin{bmatrix} u'_1 u_2 u_3 \\ u_1 u'_2 u_3 \\ 0 \end{bmatrix}|_{\Gamma_t} \Rightarrow u_3(1) = 0,$$

$$0 = n \cdot E|_{\Gamma_t} = 0 \quad (\text{no condition}),$$

$$0 = n \times \text{curl } E|_{\Gamma_t} = -\begin{bmatrix} \partial_3 E_1 \\ \partial_3 E_2 \\ 0 \end{bmatrix}|_{\Gamma_t} = \begin{bmatrix} -\partial_2 \partial_3 u \\ \partial_1 \partial_3 u \\ 0 \end{bmatrix}|_{\Gamma_t} = \begin{bmatrix} -u_1 u'_2 u'_3 \\ u'_1 u_2 u'_3 \\ 0 \end{bmatrix}|_{\Gamma_t} \Rightarrow u'_3(1) = 0.$$

Alternatively,

$$0 = u n \times e^3|_{\Gamma_t} = 0 \quad (\text{no condition}).$$

By construction, i.e.,

$$E = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \\ 0 \end{bmatrix},$$

$u$  can be constant in one variable  $x_n$  and simultaneously in two variables  $x_1, x_3$  and  $x_2, x_3$ , respectively, but not simultaneously in the two variables  $x_1, x_2$  since this implies  $E = 0$ . The 1D case shows for the different boundary conditions the following:

- $\Gamma_\tau = \emptyset$  and  $\Gamma_v = \Gamma$ , i.e.,  $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u'_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n, m \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$ , and

$$\lambda_{1,\emptyset} = \sqrt{2}\pi, \quad u_{1,\emptyset}(x) = \alpha \sin(\pi x_1) \sin(\pi x_2), \quad E_{1,\emptyset}(x) = \alpha \pi \begin{bmatrix} \sin(\pi x_1) \cos(\pi x_2) \\ -\cos(\pi x_1) \sin(\pi x_2) \\ 0 \end{bmatrix}.$$

Note that we already know from the theory that

$$\lambda_{1,\Gamma} = \lambda_{1,\emptyset} = \sqrt{2}\pi.$$

In the particular computation we get  $u'_1(0) = u'_1(1) = u'_2(0) = u'_2(1) = u_3(0) = u_3(1) = 0$  for  $\Gamma_\tau = \Gamma$  and  $\Gamma_v = \emptyset$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n, k \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$ , or  $m, k \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$ . We emphasize that here the actual case  $n = m = 0$  is not allowed as this would imply  $E = 0$ , see our discussion above. The eigenvectors are

$$u_{1,\Gamma}(x) = \alpha \cos(\pi x_1) \sin(\pi x_3) + \beta \cos(\pi x_2) \sin(\pi x_3),$$

$$E_{1,\Gamma}(x) = \alpha \pi \sin(\pi x_1) \sin(\pi x_3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \beta \pi \sin(\pi x_2) \sin(\pi x_3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- $\Gamma_\tau = \Gamma_b$  and  $\Gamma_v = \Gamma_{t,b,r,f,k}$ , i.e.,  $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = (k - 1/2)\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + m^2 + (k - 1/2)^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and the minimum and eigenvectors are

$$\tilde{\lambda} = \frac{3}{2}\pi, \quad u(x) = \alpha \sin(\pi x_1) \sin(\pi x_2) \sin\left(\frac{\pi}{2} x_3\right), \quad E(x) = \alpha \pi \sin\left(\frac{\pi}{2} x_3\right) \begin{bmatrix} \sin(\pi x_1) \cos(\pi x_2) \\ -\cos(\pi x_1) \sin(\pi x_2) \\ 0 \end{bmatrix}.$$

If  $\Gamma_\tau = \Gamma_t$  and  $\Gamma_v = \Gamma_{t,b,r,f,k}$ , i.e.,  $u_1(0) = u_1(1) = u'_2(0) = u_2(1) = u'_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + k^2}\pi$ ,  $n, m \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$ , and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_t} = \frac{\sqrt{5}}{2}\pi, \quad u_{1,\Gamma_t}(x) = \alpha \sin(\pi x_1) \cos\left(\frac{\pi}{2} x_2\right), \quad E_{1,\Gamma_t}(x) = \alpha \frac{\pi}{2} \begin{bmatrix} -\sin(\pi x_1) \sin\left(\frac{\pi}{2} x_2\right) \\ 2 \cos(\pi x_1) \cos\left(\frac{\pi}{2} x_2\right) \\ 0 \end{bmatrix}.$$

This shows that by replacing the ansatz for  $E$  by, e.g.,

$$E := \operatorname{curl} U = \begin{bmatrix} 0 \\ \partial_3 u \\ -\partial_2 u \end{bmatrix}, \quad U(x) := u(x) e^1 = u(x) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

we get also the smaller eigenvalue  $\tilde{\lambda} = (\sqrt{5}/2)\pi$  in the case  $\Gamma_\tau = \Gamma_b$ . Hence, by symmetry

$$\lambda_{1,\Gamma_b} = \lambda_{1,\Gamma_t} = \lambda_{1,\Gamma_l} = \lambda_{1,\Gamma_r} = \lambda_{1,\Gamma_f} = \lambda_{1,\Gamma_k} = \frac{\sqrt{5}}{2}\pi.$$

•  $\Gamma_\tau = \Gamma_{b,t}$  and  $\Gamma_v = \Gamma_{l,r,f,k}$ , i.e.,  $u_1(0) = u_1(1) = u_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and the minimum and eigenvectors are

$$\tilde{\lambda} = \sqrt{3}\pi, \quad u(x) = \alpha \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \quad E(x) = \alpha \pi \sin(\pi x_3) \begin{bmatrix} \sin(\pi x_1) \cos(\pi x_2) \\ -\cos(\pi x_1) \sin(\pi x_2) \\ 0 \end{bmatrix}.$$

If  $\Gamma_\tau = \Gamma_{l,r}$  and  $\Gamma_v = \Gamma_{b,t,f,k}$ , i.e.,  $u_1(0) = u_1(1) = u'_2(0) = u'_2(1) = u'_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = m\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + m^2 + k^2}\pi$ ,  $n \in \mathbb{N}$ ,  $m, k \in \mathbb{N}_0$ , and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{l,r}} = \pi, \quad u_{1,\Gamma_{l,r}}(x) = \alpha \sin(\pi x_1), \quad E_{1,\Gamma_{l,r}}(x) = -\alpha \pi \cos(\pi x_1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, again by changing the ansatz, we get also the smaller eigenvalue  $\tilde{\lambda} = \pi$  in the case  $\Gamma_\tau = \Gamma_{b,t}$ . Thus, by symmetry

$$\lambda_{1,\Gamma_{l,r}} = \lambda_{1,\Gamma_{b,t}} = \lambda_{1,\Gamma_{f,k}} = \pi.$$

•  $\Gamma_\tau = \Gamma_{b,l}$  and  $\Gamma_v = \Gamma_{t,r,f,k}$ , i.e.,  $u_1(0) = u_1(1) = u'_2(0) = u'_2(1) = u_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = (k - 1/2)\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + (k - 1/2)^2}\pi$  with  $n, m, k \in \mathbb{N}$ , and the minimum and eigenvectors are

$$\tilde{\lambda} = \frac{\sqrt{6}}{2}\pi, \quad u(x) = \alpha \sin(\pi x_1) \cos(\frac{\pi}{2}x_2) \sin(\frac{\pi}{2}x_3), \quad E(x) = -\alpha \frac{\pi}{2} \sin(\frac{\pi}{2}x_3) \begin{bmatrix} \sin(\pi x_1) \sin(\frac{\pi}{2}x_2) \\ 2 \cos(\pi x_1) \cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}.$$

If  $\Gamma_\tau = \Gamma_{f,l}$  and  $\Gamma_v = \Gamma_{b,t,r,k}$ , i.e.,  $u_1(0) = u'_1(1) = u'_2(0) = u_2(1) = u'_3(0) = u'_3(1) = 0$ :  $\mu_1 = (n - 1/2)\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{(n - 1/2)^2 + (m - 1/2)^2 + k^2}\pi$ ,  $n, m \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$ , and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{f,l}} = \frac{\sqrt{2}}{2}\pi, \quad u_{1,\Gamma_{f,l}}(x) = \alpha \sin(\frac{\pi}{2}x_1) \cos(\frac{\pi}{2}x_2), \quad E_{1,\Gamma_{f,l}}(x) = -\alpha \frac{\pi}{2} \begin{bmatrix} \sin(\frac{\pi}{2}x_1) \sin(\frac{\pi}{2}x_2) \\ \cos(\frac{\pi}{2}x_1) \cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}.$$

Hence, again by changing the ansatz, we obtain also the smaller eigenvalue  $\tilde{\lambda} = (\sqrt{2}/2)\pi$  in the case  $\Gamma_\tau = \Gamma_{b,l}$ . Thus, by symmetry

$$\lambda_{1,\Gamma_{b,l}} = \lambda_{1,\Gamma_{b,r}} = \lambda_{1,\Gamma_{b,f}} = \lambda_{1,\Gamma_{b,k}} \\ = \lambda_{1,\Gamma_{t,l}} = \lambda_{1,\Gamma_{t,r}} = \lambda_{1,\Gamma_{t,f}} = \lambda_{1,\Gamma_{t,k}} = \lambda_{1,\Gamma_{f,l}} = \lambda_{1,\Gamma_{l,k}} = \lambda_{1,\Gamma_{k,r}} = \lambda_{1,\Gamma_{f,r}} = \frac{\sqrt{2}}{2}\pi.$$

•  $\Gamma_\tau = \Gamma_{b,l,t}$  and  $\Gamma_v = \Gamma_{r,f,k}$ , i.e.,  $u_1(0) = u_1(1) = u'_2(0) = u_2(1) = u_3(0) = u_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + k^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and the minimum and eigenvectors are

$$\tilde{\lambda} = \frac{3}{2}\pi, \quad u(x) = \alpha \sin(\pi x_1) \cos(\frac{\pi}{2}x_2) \sin(\pi x_3), \quad E(x) = -\alpha \frac{\pi}{2} \sin(\pi x_3) \begin{bmatrix} \sin(\pi x_1) \sin(\frac{\pi}{2}x_2) \\ 2 \cos(\pi x_1) \cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}.$$

If  $\Gamma_\tau = \Gamma_{f,l,k}$  and  $\Gamma_v = \Gamma_{t,r,b}$ , i.e.,  $u'_1(0) = u'_1(1) = u'_2(0) = u_2(1) = u'_3(0) = u'_3(1) = 0$ :  $\mu_1 = n\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = k\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{n^2 + (m - 1/2)^2 + k^2}\pi$ ,  $n, k \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ , and the minimum and eigenvectors are

$$\lambda_{1,\Gamma_{f,l,k}} = \frac{1}{2}\pi, \quad u_{1,\Gamma_{f,l,k}}(x) = \alpha \cos(\frac{\pi}{2}x_2), \quad E_{1,\Gamma_{f,l,k}}(x) = -\alpha \frac{\pi}{2} \sin(\frac{\pi}{2}x_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, again by changing the ansatz, we obtain also the smaller eigenvalue  $\tilde{\lambda} = (1/2)\pi$  in the case  $\Gamma_\tau = \Gamma_{b,l,t}$ . Thus, by symmetry

$$\begin{aligned} \lambda_{1,\Gamma_{b,l,t}} &= \lambda_{1,\Gamma_{b,r,t}} = \lambda_{1,\Gamma_{b,f,t}} = \lambda_{1,\Gamma_{b,k,t}} = \lambda_{1,\Gamma_{r,l,t}} = \lambda_{1,\Gamma_{r,l,b}} \\ &= \lambda_{1,\Gamma_{r,l,f}} = \lambda_{1,\Gamma_{r,l,k}} = \lambda_{1,\Gamma_{f,k,l}} = \lambda_{1,\Gamma_{f,k,r}} = \lambda_{1,\Gamma_{f,k,t}} = \lambda_{1,\Gamma_{f,k,b}} = \frac{1}{2}\pi. \end{aligned}$$

•  $\Gamma_\tau = \Gamma_{b,l,k}$  and  $\Gamma_\nu = \Gamma_{t,r,f}$ , i.e.,  $u'_1(0) = u_1(1) = u'_2(0) = u_2(1) = u_3(0) = u'_3(1) = 0$ ;  $\mu_1 = (n - 1/2)\pi$ ,  $\mu_2 = (m - 1/2)\pi$ ,  $\mu_3 = (k - 1/2)\pi$ , i.e.,  $\tilde{\lambda} = \sqrt{(n - 1/2)^2 + (m - 1/2)^2 + (k - 1/2)^2}\pi$ ,  $n, m, k \in \mathbb{N}$ , and the minimum and eigenvectors are

$$\begin{aligned} \lambda_{1,\Gamma_{b,l,k}} &= \frac{\sqrt{3}}{2}\pi, \\ u_{1,\Gamma_{b,l,k}}(x) &= \alpha \cos\left(\frac{\pi}{2}x_1\right) \cos\left(\frac{\pi}{2}x_2\right) \sin\left(\frac{\pi}{2}x_3\right), \quad E_{1,\Gamma_{b,l,k}}(x) = \alpha \frac{\pi}{2} \sin\left(\frac{\pi}{2}x_3\right) \begin{bmatrix} -\cos(\frac{\pi}{2}x_1) \sin(\frac{\pi}{2}x_2) \\ \sin(\frac{\pi}{2}x_1) \cos(\frac{\pi}{2}x_2) \\ 0 \end{bmatrix}. \end{aligned}$$

By symmetry

$$\lambda_{1,\Gamma_{b,l,k}} = \lambda_{1,\Gamma_{b,r,k}} = \lambda_{1,\Gamma_{b,l,f}} = \lambda_{1,\Gamma_{b,r,f}} = \lambda_{1,\Gamma_{t,l,k}} = \lambda_{1,\Gamma_{t,r,k}} = \lambda_{1,\Gamma_{t,l,f}} = \lambda_{1,\Gamma_{t,r,f}} = \frac{\sqrt{3}}{2}\pi.$$

We summarize

$$\begin{aligned} \lambda_{1,\emptyset} &= \lambda_{1,\Gamma} = \sqrt{2}\pi, \\ \lambda_{1,\Gamma_b} &= \lambda_{1,\Gamma_t} = \lambda_{1,\Gamma_l} = \lambda_{1,\Gamma_r} = \lambda_{1,\Gamma_f} = \lambda_{1,\Gamma_k} = \frac{\sqrt{5}}{2}\pi, \\ \lambda_{1,\Gamma_{l,r}} &= \lambda_{1,\Gamma_{b,t}} = \lambda_{1,\Gamma_{f,k}} = \pi, \\ \lambda_{1,\Gamma_{b,l}} &= \lambda_{1,\Gamma_{b,r}} = \lambda_{1,\Gamma_{b,f}} = \lambda_{1,\Gamma_{b,k}} \\ &= \lambda_{1,\Gamma_{t,l}} = \lambda_{1,\Gamma_{t,r}} = \lambda_{1,\Gamma_{t,f}} = \lambda_{1,\Gamma_{t,k}} = \lambda_{1,\Gamma_{f,l}} = \lambda_{1,\Gamma_{l,k}} = \lambda_{1,\Gamma_{k,r}} = \lambda_{1,\Gamma_{f,r}} = \frac{\sqrt{2}}{2}\pi, \\ \lambda_{1,\Gamma_{b,l,t}} &= \lambda_{1,\Gamma_{b,r,t}} = \lambda_{1,\Gamma_{b,f,t}} = \lambda_{1,\Gamma_{b,k,t}} = \lambda_{1,\Gamma_{t,l,t}} \\ &= \lambda_{1,\Gamma_{r,l,b}} = \lambda_{1,\Gamma_{r,l,f}} = \lambda_{1,\Gamma_{r,l,k}} = \lambda_{1,\Gamma_{f,k,l}} = \lambda_{1,\Gamma_{f,k,r}} = \lambda_{1,\Gamma_{f,k,t}} = \lambda_{1,\Gamma_{f,k,b}} = \frac{1}{2}\pi, \\ \lambda_{1,\Gamma_{b,l,k}} &= \lambda_{1,\Gamma_{b,r,k}} = \lambda_{1,\Gamma_{b,l,f}} = \lambda_{1,\Gamma_{b,r,f}} = \lambda_{1,\Gamma_{t,l,k}} = \lambda_{1,\Gamma_{t,r,k}} = \lambda_{1,\Gamma_{t,l,f}} = \lambda_{1,\Gamma_{t,r,f}} = \frac{\sqrt{3}}{2}\pi, \end{aligned}$$

and all other cases follow by  $\lambda_{1,\Gamma_\nu} = \lambda_{1,\Gamma_\tau}$  as well as symmetry.

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