# Performance Bounds for Complex-Valued Independent Vector Analysis

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***Abstract*—Independent Vector Analysis (IVA) is a method for joint Blind Source Separation of multiple datasets with wide area of applications including audio source separation, biomedical data analysis, etc. In this paper, identification conditions and Cramér-Rao Lower Bound (CRLB) on the achievable accuracy are derived for the complex-valued case involving circular and non-circular signals and correlated and uncorrelated datasets. The identification conditions describe when independent sources can be separated from their linear mixture in the statistically consistent manner. The CRLB shows how non-Gaussianty, non-circularity of sources and statistical dependence between datasets influence the attainable separation accuracy. Examples presented in the experimental part confirm the validity of the CRLB. Also, they show certain gap between the attainable accuracy and performance of state-of-the-art algorithms, especially, in case of highly non-circular signals. Hence, there is a room for possible improvements.**

***Index Terms*—Blind source separation, complex-valued signal processing, Cramér-Rao lower bound, independent component/vector analysis, non-circular sources.**

## I. Introduction

INDEPENDENT Vector Analysis (IVA) [1] is a blind source separation (BSS) method of jointly separating multiple signal data sets into independent components. It is a generalization of Independent Component Analysis (ICA) where only one mixture of scalar independent sources is separated [2]. The idea behind IVA is that in a simultaneous decomposition of multiple datasets, signals (components) in different data sets (mixtures) are expected to be statistically dependent, which is the main idea behind a joint multiset data analysis.

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Traditional methods of BSS rely on signal non-Gaussianity, non-stationarity, and/or spectral diversity (sample dependence) [3]. More recently, [4] considers complex-valued sources and their non-circularity as another feature that helps to separate the sources. Thanks to the assumed non-circularity, it is possible to separate Gaussian-distributed sources that cannot be separated otherwise. This paper generalizes results of [4] to the IVA model without the sample dependence.

Multiple data sets with dependence among them need to be jointly processed in many applications [5]. The most common example is the separation of convolutive signal mixtures in the frequency domain, where one frequency bin corresponds to one data set [1], [6]–[8]. Also medical data, such as functional magnetic resonance imaging (fMRI) and electroencephalogram (EEG) data are typically collected from multiple subjects or under different measurement conditions, and analysed jointly [5], [9], [10]. Another example is processing of video sequences in multiple bands [11] and use of IVA for EEG signal enhancement [12]. Complex-valued processing is natural in applications such as fMRI processing. Such data is natively complex and there are advantages for processing it in the complex domain [13], [14]. Also, frequency domain representation is commonly used in the solution of the convolutive problem as well as in processing of signals such as EEG. In addition, in many cases, noncircularity needs to be accounted for, e.g., in the processing of fMRI, radar, and certain communications data [5], [15]. Another practical example showing how non-circularity helps to separate sources is polarization sunglasses [16]. Direct sun-light is represented by circularly polarized (non-polarized) electro magnetic waves. After a reflection, the light becomes partially polarized. The polarization sunglasses allow to filter out (at least partially) the unwanted noncircular waves - the reflections.

Many methods for ICA and IVA have been proposed in the literature; a review can be found in [5]. When evaluating performances of the algorithms, a comparison with theoretical and algorithm-independent performance bounds is very useful. Cramér-Rao lower bound (CRLB), already studied for ICA and IVA in [17]–[24], provides a lower bound on the separation accuracy. The CRLB for extraction of one Source Component Vector (SCV) from IVA mixture where all but one SCVs are Gaussian have been derived in [25]. Recently, the CRLB for complex-valued ICA considering three types of signal diversities, that is, non-Gaussianity, non-whiteness and non-circularity, has been derived in [4]. For IVA, similar study has been provided in [21], however, only for the real-valued case. It means that the bound

has not been yet computed for the complex-valued IVA, which is not a trivial extension of the problem in the real domain. One goal of the paper is to show how non-circularity of the sources can contribute to a better separation performance. Indeed, non-circularity does not have a counterpart in the real domain.

Besides the lower bound on separation accuracy, there is a question whether two (or more) sources are separable in the sense of statistical consistency (no matter how accurately). This poses the problem of finding the identifiability conditions, which can be studied through the CRLB, because the CRLB exists only if the identification conditions are met. This is influenced by all types of signal diversities, including non-circularity in the complex-valued problems. The identification conditions were analyzed in [4] for ICA and in [21] for the real-valued IVA.

The contribution of this paper resides in the full analysis of the complex-valued IVA problem. This includes the induced Cramér-Rao Lower Bound (I-CRLB) for the achievable interference-to-signal ratio (ISR) of the separated signals and the identifibility conditions. The analysis embodies non-circular signals as well as possible nonzero correlations between signals from different data sets. We show how non-circularity and correlatedness can influence the identifiability of signals mixtures that would not be identifiable without these signal features.

The paper is organized as follows. The mixing model is introduced in Section II. The Fisher Information Matrix (FIM) and the CRLB are calculated in Section III. Section IV studies the identification conditions. Section V deals with a special case of uncorrelated mixtures. Numerical examples are presented in Section VI, while the conclusions of Section VII summarize the paper results.

## II. Problem Statement

### A. Mathematical Notations

Throughout the paper, plain, bold lowercase and bold capital letters denote, respectively, scalars, vectors and matrices. Symbols $(\cdot)^T$, $(\cdot)^H$ and $(\cdot)^*$ denote, respectively, transposition, conjugate transpose and complex conjugate. The Matlab convention for matrix/vector concatenation and indexing will be used, e.g., $[1;\, \mathbf{g}] = [1,\, \mathbf{g}^T]^T$, and $(\mathbf{A})_{j,:}$ is the $j$th row of $\mathbf{A}$.

The notation $\mathbf{x} \perp\!\!\!\perp \mathbf{y}$ means that the random vector variables $\mathbf{x}$ and $\mathbf{y}$ are statistically independent. The symbol $\imath$ stands for the imaginary unit. The Kronecker and Hadamard (element-wise) product are denoted by $\otimes$ and $\odot$, respectively. The operator of vectorization $\mathtt{vec}(\mathbf{A})$ denotes an $MN \times 1$ vector stacking the columns of $\mathbf{A} \in \mathbb{C}^{M\times N}$. Let $\boldsymbol{\alpha} = [\alpha_1 \ldots, \alpha_{M_{\boldsymbol{\alpha}}}]$ be a vector listing a subset of rows in $\mathbf{A}$, where $0 < M_{\boldsymbol{\alpha}} < M$, and $\mathbf{E}_{\boldsymbol{\alpha}} = [\mathbf{e}_{\alpha_1}, \ldots, \mathbf{e}_{\alpha_{M_{\boldsymbol{\alpha}}}}]^T \in \mathbb{R}^{M_{\boldsymbol{\alpha}}\times M}$, where $\mathbf{e}_i$ is the $i$th column of the identity matrix, be a corresponding indexing matrix. Then, $\mathbf{E}_{\boldsymbol{\alpha}}\mathbf{A}$ selects the rows in $\mathbf{A}$ indicated by $\boldsymbol{\alpha}$. Next, define $\mathtt{vec}_{\boldsymbol{\alpha}}(\mathbf{A}) \stackrel{\text{def}}{=} \mathtt{vec}(\mathbf{E}_{\boldsymbol{\alpha}}\mathbf{A})$. The symbol $\delta_{m,n}$ is the Kronecker delta, i.e., $\delta_{m,n} = 0$ for $m \neq n$, and $\delta_{m,n} = 1$ for $m = n$. Let $\mathbf{x}$ be a complex-valued multivariate random variable of zero mean. Its covariance matrix will be denoted by $\mathbf{R}_{\mathbf{x}} = \mathrm{E}[\mathbf{x}\mathbf{x}^H]$ and pseudo-covariance matrix by $\boldsymbol{\Omega}_{\mathbf{x}} = \mathrm{E}[\mathbf{x}\mathbf{x}^T]$. The augmented covariance matrix of $\mathbf{x}$ is defined as $\underline{\mathbf{R}}_{\mathbf{x}} = \mathrm{E}[\underline{\mathbf{x}}\underline{\mathbf{x}}^H]$, where $\underline{\mathbf{x}} = [\mathbf{x}; \mathbf{x}^*]$. Let $\mathcal{N}(0, \underline{\mathbf{R}})$ denotes a multivariate complex Gaussian distribution with zero mean and augmented covariance matrix $\underline{\mathbf{R}}$. This covers both circular and non-circular Gaussian variables.

### B. Mixing Model

Let the $k$th mixture (data set), $k = 1, \ldots, K$, where $K$ is the number of mixtures, be described by

$$\mathbf{x}^k = \mathbf{A}^k\mathbf{s}^k, \tag{1}$$

where $\mathbf{s}^k \in \mathbb{C}^d$ is the vector of independent complex-valued source signals (scalar random variables), $\mathbf{A}^k \in \mathbb{C}^{d\times d}$ is the unknown non-singular mixing matrix, and $\mathbf{x}^k \in \mathbb{C}^d$ is the vector of the observed signals. The sources $\mathbf{s}^k$ are assumed to have zero means.

Let $N$ i.i.d. samples of the observed signals be available. The data model for the $n$th sample is thus given by

$$\mathbf{x}^k(n) = \mathbf{A}^k\mathbf{s}^k(n), \tag{2}$$

where $k = 1, \ldots, K$ and $n = 1, \ldots, N$. By defining the augmented matrices $\mathbf{X}$, $\mathbf{A}$ and $\mathbf{S}$, we can describe the entire data as

$$\begin{bmatrix} \mathbf{X^1} \\ \vdots \\ \mathbf{X}^K \end{bmatrix} = \begin{bmatrix} \mathbf{A^1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbf{A}^K \end{bmatrix} \begin{bmatrix} \mathbf{S^1} \\ \vdots \\ \mathbf{S}^K \end{bmatrix}, \tag{3}$$

where $\mathbf{S}^k \in \mathbb{C}^{d\times N}$, $k = 1, \ldots, K$, is the matrix of original sources with elements $(\mathbf{S}^k)_{i,j} = s_i^k(j)$, $i = 1, \ldots, d$; $j = 1 \ldots, N$. The data matrix $\mathbf{X}^k$ represents the mixtures. The joint IVA mixing model can be written as $\mathbf{X} = \mathbf{A}\mathbf{S}$.

We define the $j$th Source Component Vector (SCV) $\mathbf{s}_j$ as $\mathbf{s}_j(n) = [s_j^1(n); \ldots; s_j^K(n)]^T \in \mathbb{C}^K$, $j = 1, \ldots, d$. The goal is to estimate mutually independent SCVs. A special case of the IVA problem where the elements of the vector components are uncorrelated is studied in [26]. Separation of vector components with internal correlations can be based on second-order statistics only; see, e.g., [27]–[30]. In this paper, we admit correlated as well as uncorrelated elements of the SCVs.

The problem to separate the original independent vector components is equivalent with the problem of estimating the block diagonal de-mixing matrix $\mathbf{W}$ such that its $k$th block, denoted by $\mathbf{W}^k$, satisfies $\mathbf{W}^k\mathbf{A}^k = \mathbf{G}^k = \mathbf{D}^k\mathbf{P}^k$, where $\mathbf{D}^k$ and $\mathbf{P}^k$ is a diagonal and a permutation matrix, respectively. The vector involving the free parameters could be then defined as

$$\boldsymbol{\theta} = \left[\mathtt{vec}(\mathbf{W}^1); \ldots; \mathtt{vec}(\mathbf{W}^K)\right]. \tag{4}$$

In general, the separation of each mixture is possible up to the scale and order of the original signals. Since the identification conditions and the I-CRLB do not depend on scale or permutation, we can fix those ambiguities and treat them as known, without any loss in generality. We can assume unpermuted SCVs are scaled to unit variance. Deriving the CRLB with different scaling and permutation would yield the same resulting I-CRLB.

## III. Lower Bound for ISR

According to the scaling ambiguity of the ICA/IVA problem, all signals of interest can be assumed to have unit variance without any loss of generality. Consider an Interference-to-Signal

Ratio (ISR) matrix of $k$th mixture with elements

$$\mathtt{ISR}_{m,n}^{k} = \mathrm{E}\left[|\mathbf{G}_{m,n}^{k}|^2\right], \tag{5}$$

where $\mathbf{G}^k = \hat{\mathbf{W}}^k\mathbf{A}^k$, $m,n,=1,\dots d$. The element $\mathtt{ISR}_{m,n}^{k}$ represents the variance of the contribution of $n$th source to the $m$th estimated source in the $k$th mixture (it is nonzero in general due to imperfect separation).

### A. Equivariance Property

In the following, we exploit a transformation rule saying that the FIM of $\boldsymbol{\varphi}$, denoted as $\mathcal{J}_{\boldsymbol{\varphi}}$, and the FIM of a linearly transformed version $\boldsymbol{\theta} = \mathbf{K}\boldsymbol{\varphi}$, where $\mathbf{K}$ is a non-singular matrix, are related through [31]

$$\mathcal{J}_{\boldsymbol{\theta}} = \mathbf{K}^{-1}\mathcal{J}_{\boldsymbol{\varphi}}\mathbf{K}^{-H}. \tag{6}$$

Let $\mathcal{J}_{\mathbf{I}}$ and $\boldsymbol{\varphi}_{\mathbf{I}} = \mathrm{vec}(\mathbf{I})$ stand for the FIM and the parameter vector, respectively, derived for a case when $\mathbf{A} = \mathbf{I}$. Then, $\boldsymbol{\theta} = (\mathbf{W}^T \otimes \mathbf{I})\boldsymbol{\varphi}_{\mathbf{I}} = \mathrm{vec}(\mathbf{W})$ is a parameter vector of the true demixing matrix. Thus, according to (6) with $\mathbf{K} = \mathbf{W}^T \otimes \mathbf{I} = \mathbf{A}^{-T} \otimes \mathbf{I}$, it holds that

$$\mathcal{J}_{\boldsymbol{\theta}} = (\mathbf{A}^T \otimes \mathbf{I})\mathcal{J}_{\mathbf{I}}(\mathbf{A}^* \otimes \mathbf{I}). \tag{7}$$

When computing the lower bound for ISR, the Induced Cramér-Rao Lower Bound (I-CRLB), the parameter vector is $\boldsymbol{\theta}_{\mathbf{G}} = \mathtt{vec}(\mathbf{G}) = (\mathbf{A}^T \otimes \mathbf{I})\mathtt{vec}(\mathbf{W}) = (\mathbf{A}^T \otimes \mathbf{I})\boldsymbol{\theta}$. (In some other papers, the bound is called the Cramér-Rao-Induced Bound, CRIB, [19].) Similarly, using (6), it also holds that

$$\mathcal{J}_{\mathbf{G}} = (\mathbf{W}^T \otimes \mathbf{I})\mathcal{J}_{\boldsymbol{\theta}}(\mathbf{W}^* \otimes \mathbf{I}), \tag{8}$$

which, together with (7), results in

$$\mathcal{J}_{\mathbf{G}} = \mathcal{J}_{\mathbf{I}}. \tag{9}$$

Due to the invariance of the I-CRLB with respect to $\mathbf{G} = \mathbf{WA}$, the global demixing-mixing matrix, we need to only consider $\mathbf{A} = \mathbf{I}$, i.e., the CRLB of $\mathbf{G}$ depends only on the statistics of the sources [32].

From (9) it follows that the lower bound for ISR does not depend on the mixing matrix. Hence, the I-CRLB can be calculated without any loss of generality by considering the special case that $\mathbf{A} = \mathbf{I}$. This property is related to the *equivariance* of the (J)BSS determined mixing model (1); see, e.g., [2], [32].

The I-CRLB thus provides an algorithm–independent lower bound for achievable separation accuracy. A comparison of the ISR achieved by a given algorithm with the bound shows us its performance, and only statistically efficient algorithms attain the bound.

### B. Cramér-Rao Lower Bound

In the following, we will use the CRLB definition for complex-valued parameters estimation problems from [4], [31], [33]. Let $\boldsymbol{\theta}$ defined in (4) be the parameter vector, and $\underline{\boldsymbol{\theta}} = [\boldsymbol{\theta}; \boldsymbol{\theta}^*]$. For any unbiased estimator of $\underline{\boldsymbol{\theta}}$, it holds that

$$\mathtt{cov}\left(\underline{\boldsymbol{\theta}}\right) \succeq \mathcal{J}^{-1}(\underline{\boldsymbol{\theta}}) = \mathtt{CRLB}(\underline{\boldsymbol{\theta}}), \tag{10}$$

where $\mathbf{C} \succeq \mathbf{D}$ means that $\mathbf{C} - \mathbf{D}$ is positive semi-definite, and $\mathcal{J}(\underline{\boldsymbol{\theta}})$ is the FIM defined, in a block structure, as

$$\mathcal{J}(\underline{\boldsymbol{\theta}}) = \begin{pmatrix} \mathbf{F} & \mathbf{P} \\ \mathbf{P}^* & \mathbf{F}^* \end{pmatrix} = \mathrm{E}\left[\frac{\partial \mathcal{L}}{\partial \underline{\boldsymbol{\theta}}}\left(\frac{\partial \mathcal{L}}{\partial \underline{\boldsymbol{\theta}}}\right)^H\right], \tag{11}$$

where

$$\mathbf{F} = \mathrm{E}\left[\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}^*}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}^*}\right)^H\right], \quad \mathbf{P} = \mathrm{E}\left[\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}^*}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}^*}\right)^T\right]. \tag{12}$$

$\mathcal{L} = \mathcal{L}(\underline{\boldsymbol{\theta}}|\underline{\mathbf{x}})$ stands for the log-likelihood function derived from a statistical model. The derivatives with respect to $\boldsymbol{\theta}^*$ are defined according to the Wirtinger calculus; see, e.g., [20], [30].

### C. Statistical Model

The fundamental assumption of IVA (as well as of ICA) is that $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_d$ are mutually independent, so their joint pdf is a product of the marginal pdfs, $p_{\mathbf{s}}(\mathbf{s}) = \prod_{i=1}^{d} p_{\mathbf{s}_i}(\mathbf{s}_i)$. The pdf of $\mathbf{s}_i$ is not further factorized into products of marginals, which is the way how the dependence within the SCVs is taken into account.

From the probability density transformation theorem,[1] it follows that the joint pdf of one sample of the observation vector $\mathbf{x}$ is [4], [20]

$$p(\mathbf{x}|\boldsymbol{\theta}) = |\det(\mathbf{W})|^2 \prod_{i=1}^{d} p_{\mathbf{s}_i}(\mathbf{s}_i), \tag{13}$$

where $\mathbf{s}_i = [s_i^1; \dots; s_i^K], i = 1, \dots, d$, denotes the $i$th separated SCV. Since $\det(\mathbf{W}) = \prod_{k=1}^{K} \det(\mathbf{W}^k)$, the log-likelihood function is given by

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = \sum_{k=1}^{K} \log|\det(\mathbf{W}^k)|^2 + \sum_{i=1}^{d} \log p_{\mathbf{s}_i}(\mathbf{s}_i). \tag{14}$$

Now, the derivatives in (11) and (12) with respect to the elements of $\mathbf{W}^k$ corresponding to the $m$th row and the $n$th column are, after some computations shown in Appendix A, equal to

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial (\mathbf{W}_{m,n}^k)^*} &= \frac{\partial \log|\det(\mathbf{W}^k)|^2}{\partial (\mathbf{W}_{m,n}^k)^*} + \sum_{i=1}^{d} \frac{\partial \log p_{\mathbf{s}_i}(\mathbf{s}_i)}{\partial (\mathbf{W}_{m,n}^k)^*} \\ &= \left((\mathbf{W}_{m,n}^k)^{-1}\right)^* - \mathbf{x}_n^{k*}\psi_m^k, \end{aligned} \tag{15}$$

where

$$\psi_i^k = \psi_i^k(\mathbf{s}_i) = -\frac{\partial \log p_{\mathbf{s}_i}(\mathbf{s}_i)}{\partial s_i^{k*}} \tag{16}$$

is the score function of the $i$th SCV. Now, by considering the special case $\mathbf{A} = \mathbf{I}$, the derivative simplifies to

$$\frac{\partial \mathcal{L}}{\partial (\mathbf{W}_{m,n}^k)^*} = \delta_{m,n} - \mathbf{s}_n^{k*}\psi_m^k. \tag{17}$$

[1] It is worth pointing out to the square of the determinant in (13) due to the fact that the complex-valued problem is considered; this is a frequently overlooked fact in the signal processing literature.

Dependencies between elements of SCVs are described by the following expressions, where $k, l$ are indices of the mixture and $m$ is the index of the SCV,

$$\kappa_m^{k,l} = \mathrm{E}[\psi_m^k \psi_m^{l\,*}], \tag{18}$$

$$\xi_m^{k,l} = \mathrm{E}[\psi_m^k \psi_m^l], \tag{19}$$

$$\sigma_m^{k,l} = \mathrm{E}[\mathbf{s}_m^k \mathbf{s}_m^{l\,*}], \tag{20}$$

$$\omega_m^{k,l} = \mathrm{E}[\mathbf{s}_m^k \mathbf{s}_m^l], \tag{21}$$

$$\eta_m^{k,l} = \mathrm{E}[\psi_m^k \mathbf{s}_m^{k\,*} \psi_m^{l\,*} \mathbf{s}_m^l], \tag{22}$$

$$\beta_m^{k,l} = \mathrm{E}[\psi_m^k \mathbf{s}_m^{k\,*} \psi_m^l \mathbf{s}_m^{l\,*}]. \tag{23}$$

In addition, the following notation is used for brevity

$$\upsilon_m^{k,k} = \upsilon_m^k, \text{ for } \upsilon \in \{\kappa, \xi, \omega, \eta, \beta\}, \tag{24}$$

$$\sigma_m^{k,k} = \sigma_m^k. \tag{25}$$

### D. Fisher Information Matrix

To compute the FIM, we substitute (17) into the definition (12). The details of the derivation are in Appendix B. There exists a permutation of parameters such that the corresponding $\mathbf{F}$ and $\mathbf{P}$ are block diagonal. The ordering of the parameters is

$$\begin{aligned}\boldsymbol{\theta} = \big[&\mathbf{W}_{1,1}^1, \ldots, \mathbf{W}_{1,1}^K, \mathbf{W}_{2,2}^1, \ldots, \mathbf{W}_{2,2}^K, \ldots \\ &\mathbf{W}_{d,d}^1, \ldots \mathbf{W}_{d,d}^K, \mathbf{W}_{1,2}^1, \ldots, \mathbf{W}_{1,2}^K, \mathbf{W}_{2,1}^1, \ldots, \mathbf{W}_{2,1}^K, \ldots, \\ &\mathbf{W}_{d-1,d}^1, \ldots, \mathbf{W}_{d-1,d}^K, \mathbf{W}_{d,d-1}^1, \ldots, \mathbf{W}_{d,d-1}^K\big].\end{aligned}$$

Then,

$$\mathbf{F} = \begin{pmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}, \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix}, \tag{26}$$

where $\mathbf{N}$ is a block diagonal matrix with $d$ blocks $\mathbf{N}_i$, $i = 1, \ldots, d$, of size $K \times K$, with elements given by $(\mathbf{N}_i)_{k,l} = \eta_i^{k,l} - 1$, for $k, l = 1, \ldots, K$. $\mathbf{H}$ can be handled as a block diagonal matrix with $d(d-1)/2$ blocks, $\mathbf{H}_{i,j}$, of size $2K \times 2\ K$ each of which read

$$\mathbf{H}_{i,j} = \begin{pmatrix} \boldsymbol{\Gamma}_i \odot \mathbf{R}_j & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_j \odot \mathbf{R}_i \end{pmatrix}, \tag{27}$$

for $i = 1, \ldots d,\ j = i+1, \ldots, d$. The elements of $\boldsymbol{\Gamma}_i$ are given by $(\boldsymbol{\Gamma}_i)_{k,l} = \kappa_i^{k,l}$, and the elements of the covariance matrix $\mathbf{R}_j$ are $(\mathbf{R}_j)_{k,l} = \sigma_j^{k,l}, k, l = 1, \ldots, K$. Next, $\mathbf{B}$ is a block diagonal matrix with $d$ blocks $\mathbf{B}_i$, $i = 1, \ldots, d$, of size $K \times K$, with elements given by $(\mathbf{B}_i)_{k,l} = \beta_i^{k,l} - 1$, for $k, l = 1, \ldots, K$. $\mathbf{G}$ is as a block matrix with $d(d-1)/2$ blocks, $\mathbf{G}_{i,j}$, of size $2K \times 2\ K$ that are given by

$$\mathbf{G}_{i,j} = \begin{pmatrix} \mathbf{E_i} \odot \boldsymbol{\Omega_j} & \mathbf{I}_K \\ \mathbf{I}_K & \mathbf{E}_j \odot \boldsymbol{\Omega}_i \end{pmatrix}, \tag{28}$$

for $i = 1, \ldots d,\ j = i+1, \ldots, d$, where the elements of $\boldsymbol{\Omega}_j$ are given by $(\boldsymbol{\Omega}_j)_{k,l} = \omega_j^{k,l}$, the elements of $\mathbf{E}_i$ are $(\mathbf{E}_i)_{k,l} = \xi_i^{k,l}$, $k, l = 1, \ldots, K$.

### E. I-CRLB

According to the CRLB (10), defined as $\mathcal{J}^{-1}$, and using the equivariance property in (9), the I-CRLB, i.e., the lower bound for the ISR, is determined through the sub-block of $\mathcal{J}^{-1}$ corresponding to $\mathbf{F}$ in (11). The permuted matrices $\mathbf{F}$ and $\mathbf{P}$ can be used for the computation with similar results up to the permutation of rows and columns, thus,

$$\mathtt{CRLB}(\boldsymbol{\theta}) = (\mathbf{F} - \mathbf{P}(\mathbf{F}^*)^{-1}\mathbf{P}^*)^{-1}. \tag{29}$$

Since $\mathbf{F}$ and $\mathbf{P}$ are block diagonal having two blocks on the main diagonal, (29) has the same structure. The first block of the $\mathtt{CRLB}(\boldsymbol{\theta})$, corresponding to $\mathbf{N}$ and $\mathbf{B}$, is given by $d$ blocks $\mathbf{D}_i$ of size $K \times K$, $i = 1, \ldots, d$, such that

$$\mathbf{D}_i = \left(\mathbf{N}_i - \mathbf{B}_i(\mathbf{N}_i^*)^{-1}\mathbf{B}_i^*\right)^{-1}. \tag{30}$$

The second block of the $\mathtt{CRLB}(\boldsymbol{\theta})$, corresponding to $\mathbf{H}$ and $\mathbf{G}$ in (26), follows the block structures (27) and (28), and, therefore, it is block diagonal having $d(d-1)/2$ blocks, denoted $\mathbf{M}_{i,j}$, of size $2K \times 2\ K$ that are given by

$$\mathbf{M}_{i,j} = \left(\mathbf{H}_{i,j} - \mathbf{G}_{i,j}(\mathbf{H}_{i,j}{}^*)^{-1}\mathbf{G}_{i,j}{}^*\right)^{-1}, \tag{31}$$

where $i = 1, \ldots d,\ \ j = i+1, \ldots, d$. $\mathbf{M}_{i,j}$ has similar block structure as $\mathbf{H}_{i,j}$ and of $\mathbf{G}_{i,j}$, that is, it consists of four blocks of size $K \times K$, however, it is not possible to compute the inverse matrix in (31) analytically. In general, the lower bound for ISR (5) is given by

$$\mathtt{ISR}_{i,j} = \sum_{k=1}^{K} \mathtt{ISR}_{i,j}^k \geq \mathtt{tr}\left(\mathbf{M}_{i,j}\right)_{1,1}, \tag{32}$$

for $i, j = 1, \ldots d, i \neq j$, where $(\mathbf{M}_{i,j})_{1,1}$ denotes the upper left-corner block of $\mathbf{M}_{i,j}$ of size $K \times K$.

## IV. Identification Conditions

Whenever the FIM is rank deficient, the inverse of the FIM, the CRLB, does not exist, which points to the fact that the achievable variance of the estimation of the model parameters is infinite. Therefore, in this case, the IVA model as a whole cannot be identified, however subspaces that correspond to the non-singular portion can still be identified, see Section VI in [31]. These are identified as $\alpha$-SCV components, subset of rows in an SCV independent of the other rows in the same SCV [5]. We use this fact in order to determine the identification conditions, that is, conditions under which the CRLB exists and is finite [4]. It has been already known that the determined mixtures of i.i.d. sources (1) are blindly separable when none two of them are Gaussian with proportional covariance matrices. This condition holds both for ICA [2] as well as for IVA [21].

The identification conditions for the complex-valued ICA were derived in [4], which also consider signals that may not be i.i.d. In the case of i.i.d signals, as we have in this paper, the result of [4] can be formulated as follows.

*Lemma 1:* Sources $s_m$ and $s_n$ cannot be identified if and only if they are both Gaussian with

$$\omega_m = \exp(\imath\theta)\omega_n, \tag{33}$$

where $\omega_m = \mathrm{E}[s_m^2]$, $\theta \in \mathbb{R}$, $m \neq n$.

From the Lemma 1, it follows that two Gaussian sources with $|\omega_m| = |\omega_n|$ cannot be identified which translates to the requirement that the circularity coefficients of all Gaussian sources must be distinct to satisfy the identification conditions in the complex-valued ICA problem [30], [33]–[35].

To start discussing the identification conditions for the complex-valued IVA, we need to introduce notation similar to the one used in [21], which allows us to denote a particular subset of elements in an SCV: Let $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_{K_{\boldsymbol{\alpha}}}\}$ be a set of $K_{\boldsymbol{\alpha}}$ indices where $0 < K_{\boldsymbol{\alpha}} < K$. Let the complementing subset of $\boldsymbol{\alpha}$ in $\{1, \ldots, K\}$ be denoted by $\boldsymbol{\alpha}^c$.

*Definition 1:* A given SCV $\mathbf{s}_n$ is said to have an $\boldsymbol{\alpha}$-Gaussian component if and only if $\mathtt{vec}_{\boldsymbol{\alpha}}(\mathbf{s}_n) \perp\!\!\!\perp \mathtt{vec}_{\boldsymbol{\alpha}^c}(\mathbf{s}_n)$, and $\mathtt{vec}_{\boldsymbol{\alpha}}(\mathbf{s}_n) \sim \mathcal{N}(0, \underline{\mathbf{R}}_{n;\boldsymbol{\alpha}})$, where $\underline{\mathbf{R}}_{n;\boldsymbol{\alpha}}$ denotes the augmented covariance matrix of $\mathtt{vec}_{\boldsymbol{\alpha}}(\mathbf{s}_n)$ (or the covariance matrix in the real valued case).

In other words, according to Definition 1, the $n$th SCV has an $\boldsymbol{\alpha}$-Gaussian component iff there exists a subset of $\alpha$ rows in the SCV that is independent of all other rows in the same SCV and that the given subset has a multivariate Gaussian distribution with the augmented covariance matrix $\underline{\mathbf{R}}_{n;\boldsymbol{\alpha}}$.

For comparison, in the case of real-valued IVA, there is Lemma 2, proved in [21], that gives the identification conditions.

*Lemma 2:* The real-valued IVA model cannot be identified if $\exists\ \boldsymbol{\alpha} \neq \emptyset$ and $\exists\ m \neq n$ such that $\mathbf{s}_m$ and $\mathbf{s}_n$ have $\boldsymbol{\alpha}$-Gaussian components and $\mathbf{R}_{m;\boldsymbol{\alpha}} = \mathbf{D}\mathbf{R}_{n;\boldsymbol{\alpha}}\mathbf{D}$, where $\mathbf{D}$ is a square diagonal matrix with positive elements on its diagonal.

To determine the identification conditions for the complex-valued IVA, we use the block structure of the FIM (11). The FIM is regular (model is identifiable) if and only if all blocks of the FIM are regular. The FIM for the $(i, j)$th block is given as

$$\mathcal{J}_{i,j} = \begin{pmatrix} \mathbf{H}_{\mathbf{i,j}} & \mathbf{G}_{i,j} \\ \mathbf{G}_{i,j}^* & \mathbf{H}_{i,j}^* \end{pmatrix} \tag{34}$$

where $\mathbf{H}_{i,j}$ and $\mathbf{G}_{i,j}$ are defined in (27) and (28), respectively.

When all elements in the $i$th and $j$th SCVs are dependent, then (34) is not block diagonal. However, if the elements can be partitioned in two groups such that the elements in one group are independent of the other group, then (34) is a block matrix with one block per group. The following theorem gives the identification conditions for complex-valued IVA.

*Theorem 1:* The sources cannot be identified if $\exists\ \boldsymbol{\alpha} \neq \emptyset$ and $\exists\, m \neq n$ such that $\mathbf{s}_m$ and $\mathbf{s}_n$ have $\boldsymbol{\alpha}$-Gaussian components and

$$\mathbf{R}_{i;\boldsymbol{\alpha}} = \mathbf{D}\mathbf{R}_{j;\boldsymbol{\alpha}}\mathbf{D}^H, \tag{35}$$

$$\boldsymbol{\Omega}_{i;\boldsymbol{\alpha}} = \mathbf{D}\boldsymbol{\Omega}_{j;\boldsymbol{\alpha}}\mathbf{D}, \tag{36}$$

where $\mathbf{D}$ is a complex-valued diagonal matrix with nonzero entries on the diagonal, and $\mathbf{R}_{i;\boldsymbol{\alpha}}, \boldsymbol{\Omega}_{i;\boldsymbol{\alpha}}$ are blocks of the augmented covariance matrix given by

$$\underline{\mathbf{R}}_{i;\boldsymbol{\alpha}} = \begin{pmatrix} \mathbf{R}_{\mathbf{i};\boldsymbol{\alpha}} & \boldsymbol{\Omega}_{i;\boldsymbol{\alpha}} \\ \boldsymbol{\Omega}_{i;\boldsymbol{\alpha}}^* & \mathbf{R}_{i;\boldsymbol{\alpha}}^* \end{pmatrix}. \tag{37}$$

*Remark 1:* Following Theorem 1, any non-Gaussian SCVs can always be identified.

*Proof:* Let SCVs $\mathbf{s}_m$ and $\mathbf{s}_n$ have $\boldsymbol{\alpha}$-Gaussian components, and there is an $\boldsymbol{\alpha} \neq \emptyset$. The selected block of the FIM is given by

$$\mathcal{J}_{i,j;\boldsymbol{\alpha}} = \mathtt{cov}\left([\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}; \boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}^*; \boldsymbol{\nabla}_{j,i;\boldsymbol{\alpha}}; \boldsymbol{\nabla}_{j,i;\boldsymbol{\alpha}}^*]\right), \tag{38}$$

where $\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}$ and $\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}^*$ are $2K_{\boldsymbol{\alpha}} \times 1$ vectors given by

$$\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}} = \left(\frac{\partial \mathcal{L}(\mathbf{x}; \mathbf{W})}{\partial \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^*}\right)^T \frac{\partial \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^*}{\partial \mathbf{w}_{i,j}^*}, \tag{39}$$

$$\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}^* = \left(\frac{\partial \mathcal{L}(\mathbf{x}; \mathbf{W})}{\partial \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^*}\right)^T \frac{\partial \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^*}{\partial \mathbf{w}_{i,j}}, \tag{40}$$

where $\mathbf{w}_{i,j} = [w_{i,j}^1, \ldots, w_{i,j}^K]$, $i, j = 1, \ldots, d$, is the estimated parameter vector, i.e., the vector stacking $(i, j)$th elements of the de-mixing matrices $\mathbf{W}^k$, $k = 1, \ldots, K$. When the $\boldsymbol{\alpha}$-component $\underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}$ has complex Gaussian pdfs with an augmented covariance matrix

$$\underline{\mathbf{R}}_{i;\boldsymbol{\alpha}} = \begin{pmatrix} \mathbf{R}_{\mathbf{i};\boldsymbol{\alpha}} & \boldsymbol{\Omega}_{i;\boldsymbol{\alpha}} \\ \boldsymbol{\Omega}_{i;\boldsymbol{\alpha}}^* & \mathbf{R}_{i;\boldsymbol{\alpha}}^* \end{pmatrix}, \tag{41}$$

then, using the equivariance property, see Appendix C, we get

$$\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}} = \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^T \underline{\mathbf{R}}_{i;\boldsymbol{\alpha}}^{-T} \begin{pmatrix} \mathtt{diag}(\underline{\mathbf{s}}_{j;\boldsymbol{\alpha}}^*) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{42}$$

$$\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}^* = \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^T \underline{\mathbf{R}}_{i;\boldsymbol{\alpha}}^{-T} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathtt{diag}(\underline{\mathbf{s}}_{j;\boldsymbol{\alpha}}), \end{pmatrix}. \tag{43}$$

The FIM (38) is singular $\iff$ $(\exists \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \neq \mathbf{0})$ s.t.

$$\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}\boldsymbol{\lambda}_1 + \boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}}^*\boldsymbol{\lambda}_1^* + \boldsymbol{\nabla}_{j,i;\boldsymbol{\alpha}}\boldsymbol{\lambda}_2 + \boldsymbol{\nabla}_{j,i;\boldsymbol{\alpha}}^*\boldsymbol{\lambda}_2{}^* = 0, \tag{44}$$

which after using (42) and (43) translates to

$$\underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^T \underline{\mathbf{R}}_{i;\boldsymbol{\alpha}}^{-T} \mathtt{diag}(\underline{\boldsymbol{\lambda}}_1) \underline{\mathbf{s}}_{j;\boldsymbol{\alpha}}^* + \underline{\mathbf{s}}_{j;\boldsymbol{\alpha}}^T \underline{\mathbf{R}}_{j;\boldsymbol{\alpha}}^{-T} \mathtt{diag}(\underline{\boldsymbol{\lambda}}_2) \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^* = 0, \tag{45}$$

$$\underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^H \underline{\mathbf{R}}_{i;\boldsymbol{\alpha}}^{-1} \mathtt{diag}(\underline{\boldsymbol{\lambda}}_1^*) \underline{\mathbf{s}}_{j;\boldsymbol{\alpha}} + \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^H \mathtt{diag}(\underline{\boldsymbol{\lambda}}_2) \underline{\mathbf{R}}_{j;\boldsymbol{\alpha}}^{-1} \underline{\mathbf{s}}_{j;\boldsymbol{\alpha}} = 0, \tag{46}$$

$$\underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}^H \left(\underline{\mathbf{R}}_{i;\boldsymbol{\alpha}}^{-1} \mathtt{diag}(\underline{\boldsymbol{\lambda}}_1^*) + \mathtt{diag}(\underline{\boldsymbol{\lambda}}_2) \underline{\mathbf{R}}_{j;\boldsymbol{\alpha}}^{-1}\right) \underline{\mathbf{s}}_{j;\boldsymbol{\alpha}} = 0, \tag{47}$$

which holds for $\forall\ \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}, \underline{\mathbf{s}}_{j;\boldsymbol{\alpha}},\ i, j = 1, \ldots, d$. Therefore, the middle term in (47) is zero. Consequently,

$$\underline{\mathbf{R}}_{j;\boldsymbol{\alpha}} = -\mathtt{diag}(\underline{\boldsymbol{\lambda}}_1^*)^{-1} \underline{\mathbf{R}}_{i;\boldsymbol{\alpha}} \mathtt{diag}(\underline{\boldsymbol{\lambda}}_2). \tag{48}$$

Thus, the blocks of $\underline{\mathbf{R}}_{j;\boldsymbol{\alpha}}$ in (41) read

$$\mathbf{R}_{i;\boldsymbol{\alpha}} = -\mathtt{diag}(\boldsymbol{\lambda}_1^*)^{-1} \mathbf{R}_{j;\boldsymbol{\alpha}} \mathtt{diag}(\boldsymbol{\lambda}_2), \tag{49}$$

$$\mathbf{R}_{i;\boldsymbol{\alpha}}^* = -\mathtt{diag}(\boldsymbol{\lambda}_1)^{-1} \mathbf{R}_{j;\boldsymbol{\alpha}}^* \mathtt{diag}(\boldsymbol{\lambda}_2^*), \tag{50}$$

$$\boldsymbol{\Omega}_{i;\boldsymbol{\alpha}} = -\mathtt{diag}(\boldsymbol{\lambda}_1^*)^{-1} \boldsymbol{\Omega}_{j;\boldsymbol{\alpha}} \mathtt{diag}(\boldsymbol{\lambda}_2{}^*), \tag{51}$$

$$\boldsymbol{\Omega}_{i;\boldsymbol{\alpha}}^* = -\mathtt{diag}(\boldsymbol{\lambda}_1)^{-1} \boldsymbol{\Omega}_{j;\boldsymbol{\alpha}}^* \mathtt{diag}(\boldsymbol{\lambda}_2). \tag{52}$$

If there are $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ such that (49)–(52) hold, the identification conditions are violated.

Let $\eta_m^k$ and $\zeta_m^k$ denote absolute value and phase of $\boldsymbol{\lambda}_m$, $m = 1, 2$, respectively, i.e., $\lambda_m^k = \eta_m^k \exp(\imath\zeta_m^k)$, $m = 1, 2$. Since the diagonal elements of $\mathbf{R}_{i;\boldsymbol{\alpha}}$, $\forall i = 1, \ldots, d$, are positive and real-valued, (49) holds only when $-\frac{\eta_2^k}{\eta_1^k} > 0$ and $\zeta_1^k = \zeta_2^k$, for

$k = 1, \ldots, K$. In addition, the covariance matrix $\mathbf{R}_{i;\boldsymbol{\alpha}}$ has to be Hermitian, i.e., $\frac{1}{\lambda_1^k} = \lambda_2^k$, thus, $\frac{1}{\eta_1^k} = \eta_2^k$, $k \in \boldsymbol{\alpha}$. Inserting into (49) and (51), respectively, leads to

$$\mathbf{R}_{i;\boldsymbol{\alpha}} = \mathbf{D}\mathbf{R}_{j;\boldsymbol{\alpha}}\mathbf{D}^H, \tag{53}$$

$$\boldsymbol{\Omega}_{i;\boldsymbol{\alpha}} = \mathbf{D}\boldsymbol{\Omega}_{j;\boldsymbol{\alpha}}\mathbf{D}, \tag{54}$$

where $\mathbf{D}$ is a diagonal matrix with a diagonal elements equal to $d_k = \eta_k \exp(\imath\zeta_k)$, $\eta_k, \zeta_k \in \mathbb{R}$. The proof is done. ■

The identification conditions in Theorem 1 show how the three diversities non-Gaussianity, non-circularity, and dependence influence the separability of data. In contrast to the real-valued IVA, the complex-valued SCVs with proportional covariance matrices (35) can still be separable provided that their pseudo-covariance matrices are not proportional each to the other. This can happen only for non-circular SCVs, because circular SCVs have zero pseudo-covariance matrices (which are trivially proportional).

Note that the condition given by (53) is obviously satisfied in ICA, since the covariance matrices in (53) are replaced by variances. If two sources have the same circularity, then also (54) is satisfied and such sources cannot be identified. By contrast, in IVA only sources with proportional covariance and pseudo-covariance matrices do not meet the identification conditions.

## V. Uncorrelated Elements of the SCV

A special case of IVA is when the elements in SCVs are dependent but uncorrelated, so only higher-order dependencies among them may exist; this was the original statistical model considered in [26] where IVA is deployed for the frequency-domain separation of convolutive mixtures of audio signals. The signals from different frequency bins tend to be uncorrelated, so the signal model is useful here.

For this special case, the computation of the CRLB is significantly simplified, because now the uncorrelatedness means that

$$\sigma_m^{k,l} = \sigma_m^k \delta_{k,l}, \tag{55}$$

$$\omega_m^{k,l} = \omega_m^k \delta_{k,l}. \tag{56}$$

The sub-blocks of $\mathbf{F}$ in (26) are then as follows: $\mathbf{N}$ is a block diagonal matrix with $d$ blocks $\mathbf{N}_i$, $i = 1, \ldots, d$, of size $K \times K$, with elements given by $(\mathbf{N}_i)_{k,l} = \eta_i^{k,l} - 1$, for $k, l = 1 \ldots K$; $\mathbf{H}$ is a block diagonal matrix with $K(d-1)d/2$ blocks of size $2 \times 2$ that read

$$\mathbf{H}_{i,j}^k = \begin{pmatrix} \kappa_i^k \sigma_j^k & 0 \\ 0 & \kappa_j^k \sigma_i^k \end{pmatrix}, \tag{57}$$

for $k = 1, \ldots, K$ and $i = 1, \ldots d$, $j = i+1, \ldots, d$. The sub-blocks of $\mathbf{P}$ in (26) are as follows: $\mathbf{B}$ is a block diagonal matrix with $d$ blocks $\mathbf{B}_i$, $i = 1, \ldots, d$, of size $K \times K$, with elements given by $(\mathbf{B}_i)_{k,l} = \beta_i^{k,l} - 1$, for $k, l = 1 \ldots K$; $\mathbf{G}$ is a block diagonal matrix with $K(d-1)d/2$ blocks of size $2 \times 2$ that read

$$\mathbf{G}_{i,j}^k = \begin{pmatrix} \xi_i^k \omega_j^{k^*} & 1 \\ 1 & \xi_j^k \omega_i^{k^*} \end{pmatrix}, \tag{58}$$

$k = 1, \ldots, K$ and $i = 1, \ldots, d$, $j = i+1, \ldots, d$. Hence, both matrices, $\mathbf{H}$ and $\mathbf{G}$, are block diagonal with $2 \times 2$ blocks when the elements in SCVs are uncorrelated.

### A. I-CRLB for Uncorrelated Elements

Now, since $\mathbf{F}$ and $\mathbf{P}$ are block diagonal matrices with a similar structure, (29) is a block diagonal matrix. Its first $d$ blocks $\mathbf{D}_i$ of size $K \times K$, $i = 1, \ldots, d$, are given by

$$\mathbf{D}_i = \left(\mathbf{N}_i - \mathbf{B}_i(\mathbf{N}_i^*)^{-1}\mathbf{B}_i^*\right)^{-1}, \tag{59}$$

where the elements of $\mathbf{N}_i$ and $\mathbf{B}_i$ are $(\mathbf{N}_i)_{k,l} = \eta_i^{k,l} - 1$ and $(\mathbf{B}_i)_{k,l} = \beta_i^{k,l} - 1$. The other $Kd(d-1)/2$ blocks are given by

$$\begin{aligned} \mathbf{M}_{i,j}^k &= \left(\mathbf{H}_{i,j}^k - \mathbf{G}_{i,j}^k (\mathbf{H}_{i,j}^{k\ *})^{-1}\mathbf{G}_{i,j}^{k\ *}\right)^{-1} \\ &= \begin{pmatrix} \kappa_i^k - \frac{|\xi_i^k|^2|\omega_j^k|^2}{\kappa_i^k} - \frac{1}{\kappa_j^k} & -\frac{\xi_i^k(\omega_j^k)^*}{\kappa_i^k} - \frac{(\xi_j^k)^*\omega_i^k}{\kappa_j^k} \\ -\frac{\xi_i^{k^*}\omega_j^k}{\kappa_i^k} - \frac{\xi_j^k\omega_i^{k^*}}{\kappa_j^k} & \kappa_j^k - \frac{|\xi_j^k|^2|\omega_i^k|^2}{\kappa_j^k} - \frac{1}{\kappa_i^k} \end{pmatrix}^{-1}, \end{aligned} \tag{60}$$

for $k = 1, \ldots, K$ and $i = 1, \ldots d$, $j = i+1, \ldots, d$.

The lower bound for ISR corresponds to

$$\mathtt{ISR}_{i,j}^k = \mathrm{E}\left[(\mathbf{G}_{i,j}^k)^2\right] \geq (\mathbf{M}_{i,j}^k)_{1,1}^{-1}. \tag{61}$$

Thus, after calculation of the inverse in (60) and some further simplifications, the I-CRLB for the $k$th mixture says that

$$\mathtt{ISR}_{i,j}^k \geq \frac{\kappa_i^k(\kappa_i^k\kappa_j^k - 1) - \kappa_j^k|\xi_i^k|^2|\omega_j^k|^2}{u_{i,j}^k - 1}, \tag{62}$$

where

$$\begin{aligned} u_{i,j}^k = (\kappa_i^k\kappa_j^k - 1)^2 - (\kappa_i^k)^2|\xi_j^k|^2|\omega_i^k|^2 - (\kappa_j^k)^2|\xi_i^k|^2|\omega_j^k|^2 \\ + |\xi_i^k\xi_j^k\omega_i^{k^*}\omega_j^{k^*} - 1|^2. \end{aligned} \tag{63}$$

It can be easily verified that when the SCV elements are independent, the bound (62) is equal to that one in [20] ((30) on page 4) for the complex-valued ICA. Following inequalities hold for $\kappa_i^k$, $\eta_i^k$, $\sigma_i^k$ and $\omega_i^k$:

$$\kappa_i^k \geq |\xi_i^k|,\ \kappa_i^k \geq 1, \tag{64}$$

$$|\xi_i^k| \geq |\omega_i^k|,\ \sigma_i^k \geq |\omega_i^k|. \tag{65}$$

In the special case where the $j$th signal is circular, which implies that $\omega_j = \xi_j = 0$, (62) simplifies to

$$\mathtt{ISR}_{i,j}^k \geq \frac{\kappa_i^k}{\kappa_i^k\kappa_j^k - 1}. \tag{66}$$

It is worth noting that the noncircularity of the $i$th signal did not vanish in this case; it influences the value of $\kappa_i^k$.

The following Lemma 3 provides an insight how the dependence between the elements of SCVs affects the bound.

*Lemma 3:* Let $p(s^1, s^{1^*}, \ldots, s^K, s^{K^*}) = p(\mathbf{s}, \mathbf{s}^*)$ denote the joint pdf of $s^1, \ldots, s^K$, and $p_k(s^k, s^{k^*})$ be the marginal pdf of $s^k$, $k = 1, \ldots, K$. Let $\kappa_{\mathrm{IVA}}^k = \mathrm{E}[|\partial \log p_{\mathbf{s}}(\mathbf{s})/\partial s^{k^*}|^2]$ and $\kappa_{\mathrm{ICA}}^k = \mathrm{E}[|\partial \log p_k(s^k)/\partial s^{k^*}|^2]$. Then, $\kappa_{\mathrm{IVA}}^k \geq \kappa_{\mathrm{ICA}}^k$, and the

equality holds when $s^k$ is independent of the other random variables, or, equivalently, when

$$p(\mathbf{s}, \mathbf{s}^*) = p_k(s^k, s^{k*}) p(s^1, s^{1*}, \ldots, s^{k-1*}, s^{k+1}, \ldots, s^{K*}). \tag{67}$$

*Proof:* See Appendix D.

The bound (66) is a monotonously decreasing function of $\kappa_i^k$, since $\kappa_j^k \geq 1$. The Lemma 3 implies that the lower bound for IVA is lower or equal to that one for ICA.

In the special case of Gaussian uncorrelated sources, our analysis confirms an expected result that the sources cannot be separated unless they have different circularity [4].

## VI. Numerical Simulations

The examples in this section illustrate how dependence and non-circularity affect the separation accuracy. Let $K = 2$ mixtures consist of $d = 2$ SCVs.

### A. Gaussian SCVs

In first example, both SCVs are drawn from a Gaussian distribution with density

$$p(\underline{\mathbf{x}}) = \frac{1}{Z} \exp\left\{ -\frac{1}{2} \underline{\mathbf{x}}^H \begin{pmatrix} \mathbf{R} & \mathbf{\Omega} \\ \mathbf{\Omega}^* & \mathbf{R}^* \end{pmatrix}^{-1} \underline{\mathbf{x}} \right\}. \tag{68}$$

The SCVs are scaled to unit variance, which means that diagonal elements of $\mathbf{R}$ and $\mathbf{R}^*$ are set to one. The first SCV is assumed to be circular, which is provided by $\mathbf{\Omega} = \mathbf{\Omega}^* = \mathbf{0}$. Elements of this SCV are uncorrelated (thus, also independent in this case). The other SCV is generated with varying circularity and mutual dependence of elements. The circularity is varied through

$$\mathbf{\Omega} = \gamma \mathbf{R} = \gamma \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \tag{69}$$

where $\gamma \in [0, 1)$ ($\gamma = 0$ implies circularity). The dependence is controlled by $c \in [0, 1)$, where $c$ is an off-diagonal element of $\mathbf{R}$ ($c = 0$ implies uncorrelatedness). Owing to the circularity of one SCV, the bound (32) simplifies to

$$\mathtt{ISR} \geq \left( \mathbf{I} \odot \mathbf{\Gamma} - (\mathbf{I} \odot \mathbf{R})^{-1} \right)^{-1}_{1,1}, \tag{70}$$

where $\mathbf{\Gamma}$ is defined in (27). Since it holds that

$$\mathbf{I} \odot \mathbf{\Gamma} = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix}, \; (\mathbf{I} \odot \mathbf{R})^{-1} = \mathbf{I}, \tag{71}$$

where

$$\kappa = \frac{1}{(1 - c^2)(1 - \gamma^2)}, \tag{72}$$

the resulting bound reads

$$\mathtt{ISR} \geq \frac{(1 - c^2)(1 - \gamma^2)}{1 - (1 - c^2)(1 - \gamma^2)}. \tag{73}$$

Fig. 1 shows the comparison of CRLBs with ISR achieved by the second-order (Gaussian) Independent Vector Analysis (IVA-G) algorithm [36]. The average ISR is computed over 1000 trials for varying values of circularity coefficient $\gamma$ and correlation parameter $c$. The CRLB is given by the right-hand side of (73) for selected values of $c$ as the function of $\gamma$. The bound goes to infinity if and only if both $c$ and $\gamma$ approach zero. This means that Gaussian sources can be separated when other diversities (noncircularity and correlatedness) are taken into account. These results corroborate the identification conditions derived in Sec. IV saying that SCVs cannot be identified if they have proportional covariance and pseudo-covariance matrices, which holds only if $\gamma = 0$ and $c = 0$. As observed in the figure, the performance of IVA-G follows the bound closely providing support for its efficient nature [27] except in the region where the signals are maximally improper/noncircular. As noted in [30], for this case, which is also called "rectilinear," many algorithms developed for noncircular signals are known to fail and hence might require special attention. We note that this is also the case for IVA-G.

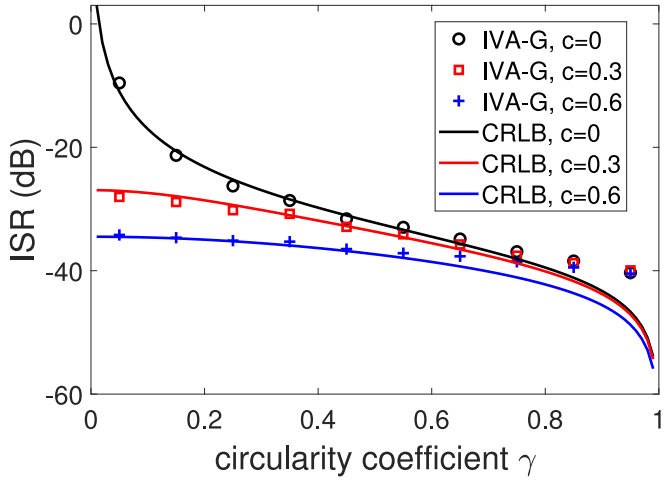


Fig. 1. Achieved ISR and CRLBs for $K = 2$ mixtures, $d = 2$ SCVs, $N = 5000$ i.i.d. samples and varying circularity.

### B. Generalized Gaussian SCVs

In the second example, also the non-Gaussianity is taken into account. The elements of the first SCV are independent circular Gaussian. The second SCV is generated as

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \mathbf{H} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{74}$$

where $x, y$ are drawn from the complex Generalized Gaussian Distribution (GGD) with a shape parameter $\alpha$. The pdf of a GGD variable is [37]

$$p(x, x^*) = \frac{1}{Z} \exp\left( - \left[ \lambda \left( \gamma x^2 + \gamma x^{*2} - 2 x x^* \right) \right]^{\alpha} \right), \tag{75}$$

where $\gamma$ is the circularity coefficient and

$$\lambda = \frac{1}{2} \frac{1}{\gamma^2 - 1} \frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)} \tag{76}$$

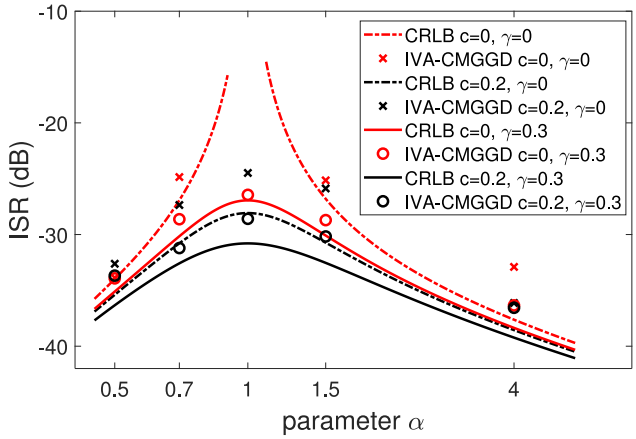


Fig. 2. Achieved ISR and CRLBs for $K = 2$ mixtures, $d = 2$ SCVs, $N = 5000$ i.i.d. samples and varying shape parameter $\alpha$.

is selected to scale $x$ to unit variance. As shown in [20], it holds that

$$\kappa_{\texttt{GGD}} = \mathrm{E}[|\psi(x)|^2] = \frac{\alpha^2\Gamma(2/\alpha)}{(1-\gamma^2)\Gamma^2(1/\alpha)}. \tag{77}$$

Thus, after some computations, the resulting bound (70) is given by

$$\texttt{ISR} \geq \frac{1}{(h_{1,1}^2 + h_{1,2}^2)\kappa_{\texttt{GGD}} - 1}, \tag{78}$$

where $h_{1,j} = (\mathbf{H}^{-1})_{1,j}$, $j = 1, 2$.

Fig. 2 shows the comparison of CRLBs with empirical ISR achieved by the IVA by Complex Multivariate GGD (IVA-CMGGD) algorithm proposed in [38], [39]. The average ISR is computed over 100 trials for varying values of the circularity coefficient $\gamma$, the correlation parameter $c$, and the shape parameter $\alpha$. The CRLB is given by the right-hand side of (78) for selected values of $c$ and $\gamma$ as a function of $\alpha$. The results show how all types of diversity are used to increase the separation accuracy. The gap between the achieved ISR and related CRLB indicates limitations of the algorithm, especially for greater values of $c$ and $\alpha$. Sources cannot be separated if $\alpha = 1$ and $\gamma = c = 0$, which corresponds to the case of independent mixtures of circular Gaussian sources. This is in accord with identifications conditions derived in Sec. IV.

## VII. Conclusion

Different types of diversity discussed in this paper have been shown to affect the separation accuracy and identification conditions related to IVA. In particular, we have proved that, in contrast to ICA, complex-valued IVA can benefit from statistical dependence across mixtures and, in contrast to real-valued IVA, it can benefit from the noncircularity of SCVs. We have determined the most general results of identifiability, which includes the influence of noncircularity, non-Gaussianity (higher-order statistics), and dependence across datasets in one model. In the real-valued IVA, it was known that non-Gaussian SCVs can be always separated while Gaussian SCVs cannot be separated if their covariance matrices are proportional. Now, we have shown that, in the complex-valued IVA, Gaussian SCVs are separable if their pseudo-covariance matrices are not proportional. The numerical simulations support the theory and confirm that the derived bound is a lower bound on achievable accuracy. A possible extension is to also consider sample dependence as an additional type of diversity, which brings the spectral diversity of signals into the equation. This will be an exciting future direction.

## Appendix A

It holds in

$$\begin{aligned}
&\frac{\partial \log|\det(\mathbf{W}^k)|^2}{\partial(\mathbf{W}^k_{m,n})^*} \\
&= \frac{1}{|\det(\mathbf{W}^k)|^2}\frac{\partial \det(\mathbf{W}^k)\left(\det(\mathbf{W}^k)^*\right)}{\partial(\mathbf{W}^k_{m,n})^*} \\
&= \frac{1}{\det(\mathbf{W}^k)^*}\frac{\partial \det(\mathbf{W}^k)^*}{\partial(\mathbf{W}^k_{m,n})^*} \\
&= \texttt{tr}\left((\mathbf{W}^k)^{-1}\frac{\partial \mathbf{W}^{k^*}}{(\mathbf{W}^k_{m,n})^*}\right) = (\mathbf{A}^k_{m,n})^*,
\end{aligned} \tag{79}$$

where the last equality holds when the permutation and scaling ambiguity is avoided, since then $(\mathbf{W}^k)^{-1} = \mathbf{A}^k$. The scaling and permutation ambiguity do not affect the identification conditions and the I-CRLB, thus, we can avoid them by assuming unit variances and unpermuted separated signals. Then, the inverse of the de-mixing matrix equals the mixing matrix.

## Appendix B

The elements of matrices $\mathbf{F}$ and $\mathbf{P}$ in (12) are given by

$$\mathbf{F}^{k,l}_{(m,n),(i,j)} = \mathrm{E}[\psi^k_m(\mathbf{y}^k_n)^*(\psi^l_i)^*\mathbf{y}^l_j] - \delta_{m,n}\delta_{i,j}, \tag{80}$$

$$\mathbf{P}^{k,l}_{(m,n),(i,j)} = \mathrm{E}[\psi^k_m(\mathbf{y}^k_n)^*\psi^l_i(\mathbf{y}^l_j)^*] - \delta_{m,n}\delta_{i,j}. \tag{81}$$

Hence, there are only following possible cases of nonzero values of (80) and (81):

- $m = n = i = j$

$$\mathbf{F}^{k,l}_{(m,m),(m,m)} = \mathrm{E}[\psi^k_m(\mathbf{y}^k_m)^*(\psi^l_m)^*\mathbf{y}^l_m] - 1 = \eta^{k,l}_m - 1, \tag{82}$$

$$\mathbf{P}^{k,l}_{(m,m),(m,m)} = \mathrm{E}[\psi^k_m(\mathbf{y}^k_m)^*\psi^l_m(\mathbf{y}^l_m)^*] - 1 = \beta^{k,l}_m - 1, \tag{83}$$

- $m = i \;\&\; n = j \;\&\; m \neq n$

$$\mathbf{F}^{k,l}_{(m,n),(m,n)} = \kappa^{k,l}_m {\sigma_n}^{k,l}, \tag{84}$$

$$\mathbf{P}^{k,l}_{(m,n),(m,n)} = \xi^{k,l}_m \omega^{k,l^*}_n, \tag{85}$$

where $\kappa^{k,l}_m = \mathrm{E}[\psi^k_m {\psi^l_m}^*]$, ${\sigma_n}^{k,l} = \mathrm{E}[\mathbf{y}^k_n {\mathbf{y}^l_n}^*]$ and $\xi^{k,l}_m = \mathrm{E}[\psi^k_m \psi^l_m]$, $\omega^{k,l}_n = \mathrm{E}[\mathbf{y}^k_n \mathbf{y}^l_n]$,

- $m = j \;\&\; n = i \;\&\; m \neq n$

$$\mathbf{F}^{k,l}_{(m,n),(n,m)} = 0, \tag{86}$$

$$\mathbf{P}^{k,l}_{(m,n),(n,m)} = \delta_{k,l}. \tag{87}$$

If uncorrelated elements of the SCV are assumed, then (84) and (85) are both zeros when $k \neq l$.

## APPENDIX C

The equation (39)

$$\boldsymbol{\nabla}_{i,j;\boldsymbol{\alpha}} = \left( \frac{\partial \mathcal{L}(\mathbf{x};\mathbf{W})}{\partial \underline{\mathbf{s}}^*_{i;\boldsymbol{\alpha}}} \right)^T \frac{\partial \underline{\mathbf{s}}^*_{i;\boldsymbol{\alpha}}}{\partial w^*_{i,j}}, \tag{88}$$

simplifies for Gaussian SCVs as follows:

$$\frac{\partial \mathcal{L}(\mathbf{x};\mathbf{W})}{\partial \underline{\mathbf{s}}^*_{i;\boldsymbol{\alpha}}} = \underline{\mathbf{R}}^{-1}_{i;\boldsymbol{\alpha}} \underline{\mathbf{s}}_{i;\boldsymbol{\alpha}}, \tag{89}$$

$$\frac{\partial \underline{\mathbf{s}}^*_{i;\boldsymbol{\alpha}}}{\partial w^*_{i,j}} = \frac{\partial}{\partial w^*_{i,j}} \begin{pmatrix} \mathbf{W}^* \mathbf{x}^* \\ \mathbf{W}\mathbf{x} \end{pmatrix}. \tag{90}$$

It holds that

$$\frac{\partial}{\partial w^*_{i,j}} \mathbf{W}\mathbf{x} = \mathbf{0} \tag{91}$$

and

$$\frac{\partial}{\partial w^*_{i,j}} \mathbf{W}^* \mathbf{x}^* = \frac{\partial}{\partial w^*_{i,j}} \begin{pmatrix} \mathbf{W}^{1^*} \mathbf{x}^{1^*} \\ \vdots \\ \mathbf{W}^{K^*} \mathbf{x}^{K^*} \end{pmatrix} = \begin{pmatrix} x^{1^*} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & x^{K^*} \end{pmatrix}. \tag{92}$$

Thanks to the equivariance we can write $\mathbf{x} := \mathbf{s}$.

## APPENDIX D

*Lemma 3:* It holds that $\kappa^k_{\mathrm{IVA}} \geq \kappa^k_{\mathrm{ICA}}$.

*Proof:* We start by computing the following auxiliary quantity:

$$\kappa^k_{\mathrm{MI}} = \mathrm{E}\left[ \left| \frac{\partial}{\partial s^k} \left( \log \frac{p(\mathbf{s},\mathbf{s}^*)}{\prod_{i=1}^K p_i(s^i, s^{i^*})} \right) \right|^2 \right]$$

$$= \mathrm{E}\left[ \left| \frac{\partial \log p(\mathbf{s},\mathbf{s}^*)}{\partial s^k} \right|^2 \right] + \mathrm{E}\left[ \left| \frac{\partial \log \left( \prod_{i=1}^K p_i(s^i, s^{i^*}) \right)}{\partial s^k} \right|^2 \right]$$

$$- 2\mathtt{Re}\left( \mathrm{E}\left[ \frac{\partial \log p(\mathbf{s},\mathbf{s}^*)}{\partial s^k} \frac{\partial \log \left( \prod_{i=1}^K p_i(s^i, s^{i^*}) \right)}{\partial s^{k^*}} \right] \right)$$

$$= \kappa^k_{\mathrm{IVA}} + \kappa^k_{\mathrm{ICA}} - 2\mathtt{Re}\left( \mathrm{E}\left[ \frac{\partial \log p(\mathbf{s},\mathbf{s}^*)}{\partial s^k} \frac{\partial \log p_k(s^k, s^{k^*})}{\partial s^{k^*}} \right] \right).$$

By unfolding the last term, we obtain

$$\int_{\mathbf{C}^{2K}} \frac{1}{p(\mathbf{s},\mathbf{s}^*)} \frac{\partial p(\mathbf{s},\mathbf{s}^*)}{\partial s^k} \frac{1}{p_k(s^k, s^{k^*})} \frac{\partial p_k(s^k, s^{k^*})}{\partial s^{k^*}} p(\mathbf{s},\mathbf{s}^*) \mathtt{d}\mathbf{s}\mathtt{d}\mathbf{s}^*$$

$$= \int_{\mathbf{C}^2} \frac{1}{p_k(s^k, s^{k^*})} \frac{\partial p_k(s^k, s^{k^*})}{\partial s^{k^*}} \int_{\mathbf{C}^{2K-2}} \frac{\partial p(\mathbf{s},\mathbf{s}^*)}{\partial s^k} \mathtt{d}\mathbf{s}\mathtt{d}\mathbf{s}^*$$

$$= \int_{\mathbf{C}^2} \frac{1}{p_k(s^k, \mathbf{s}^*)} \left| \frac{\partial p_k(s^k, \mathbf{s}^*)}{\partial s^k} \right|^2 \mathtt{d}s^k \mathtt{d}s^{k^*} = \kappa^k_{\mathrm{ICA}}.$$

The proof is completed as $0 \leq \kappa^k_{\mathrm{MI}} = \kappa^k_{\mathrm{IVA}} - \kappa^k_{\mathrm{ICA}}$. ■

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