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ScienceDirect

Fuzzy Sets and Systems 415 (2021) 1–26

FUZZY sets and systems

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The impact on the properties of the EFGM copulas when extending this family

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Received 25 May 2020; received in revised form 21 October 2020; accepted 4 November 2020
Available online 17 November 2020

Abstract

Several extensions of the family of (bivariate) Eyraud-Farlie-Gumbel-Morgenstern copulas (EFGM copulas) are considered. Some of them are well-known from the literature, others have recently been suggested (copulas based on quadratic constructions, based on some forms of convexity, and polynomial copulas). For each of these extensions we analyze which properties of EFGM copulas are preserved (or even improved) and which are (partly) lost. Such properties can be structural (order theoretical or topological) in nature, or algebraic (symmetry or being a polynomial) or analytic (absolute continuity). Other examples are forms of convexity, quadrant dependence, and symmetry with respect to copula transformations. The last group of properties considered here is related to some dependence parameters.

Keywords: Eyraud-Farlie-Gumbel-Morgenstern copula; Dependence parameter; Perturbation; Polynomial copula; Schur concavity; Ultramodularity

1. Introduction

The class of bivariate distributions we now call Eyraud-Farlie-Gumbel-Morgenstern distributions was studied first by Eyraud [29], and then by Morgenstern [65] (for Cauchy marginals) and by Gumbel [38] (for exponential marginals), and it was further generalized by Farlie [30]. This important and efficient class of bivariate distributions with given marginals induces, in the case of uniform marginals, the family of Eyraud-Farlie-Gumbel-Morgenstern

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https://doi.org/10.1016/j.fss.2020.11.001

copulas (EFGM copulas for short) which is used quite often when rather weak dependences of exchangeable random variables are modeled.

Note that in most papers the EFGM copulas were simply called Farlie-Gumbel-Morgenstern copulas (FGM copulas for short), since the results of [29] had been forgotten for many years. Their quotation in the monograph [28] (see also [14,15,64]) contributed to a movement to rename this family of copulas and to pay proper credit to the early achievements of Eyraud. We will join this movement and consistently speak about Eyraud-Farlie-Gumbel-Morgenstern (EFGM) copulas.

In this paper we exploit some recent results concerning quadratic constructions of copulas [53] as well as copula constructions based on ultramodularity and Schur concavity [75] in order to propose new parametric families of copulas extending the original family of EFGM copulas (see also [55]), and we analyze which of the properties of EFGM copulas are preserved (or even improved) by these extensions, and which are (partly) lost.

The paper is organized as follows: in Section 2 we recall some preliminaries about copulas, followed by a short description of EFGM copulas in Section 3. In Section 4 we review some known extensions of EFGM copulas introduced in [43] and in [26,56,57,61] (the latter being based on perturbations of copulas). Extensions of EFGM copulas based on quadratic constructions on copulas (see [53]) are discussed in Section 5, and extensions based on some forms of convexity (such as ultramodularity and Schur concavity, see [75]) are the topic of Section 6. Finally we present polynomial copulas (in particular of degree 5, compare [81]) which by construction are natural extensions of EFGM copulas (Section 7).

2. Preliminaries

In this contribution we will work only with two-dimensional (or bivariate) copulas, and we will simply call them copulas. Recall that a copula is a bivariate cumulative distribution function (restricted to $[0,1]^2$) with uniform marginals on $[0,1]$, which captures the whole dependence structure of the random pair [86].

The name "copula" for functions linking an n-dimensional distribution and its one-dimensional marginals goes back to Sklar [86], but such functions have been studied before, e.g., by Hoeffding [39,40], Fréchet [34], Dall'Aglio [20–22], and Féron [33].

Here is the formal definition of a copula: a function $C\colon [0,1]^2 \to [0,1]$ is called a copula if the following two conditions are satisfied:

(i) C satisfies, for each $x \in [0,1]$, the boundary conditions

$$C(x,0) = C(0,x) = 0 \quad \text{and} \quad C(x,1) = C(1,x) = x; \tag{2.1}$$

(ii) C is 2-increasing, i.e., for all $x_1, x_2, y_1, y_2 \in [0,1]$ satisfying $x_1 \le x_2$ and $y_1 \le y_2$ the following inequality holds:

$$C(x_2,y_2) - C(x_1,y_2) + C(x_1,y_1) - C(x_2,y_1) \ge 0. \tag{2.2}$$

Three distinguished examples of copulas are the Fréchet-Hoeffding lower bound W, the product (or independence) copula Π, and the Fréchet-Hoeffding upper bound M defined by, respectively,

$$W(x,y) = \max(x+y-1,0), \qquad \Pi(x,y) = xy, \qquad M(x,y) = \min(x,y).$$

In the context of statistical dependence these three copulas have some clear interpretations: Π models the independence of random variables, while the Fréchet-Hoeffding bounds W and M model their counter- and comonotone dependence, respectively (for more information, see, e.g., [71, Remark 2.2, Theorem 3.1]).

For the rest of the paper we will use the shortcut $\mathscr{C}$ for the set of all copulas. The set $\mathscr{C}$ has many nice properties (for instance, it is a closed convex subset of the set of all functions from the unit square $[0,1]^2$ into the unit interval $[0,1]$, and the Fréchet-Hoeffding bounds W and M are its smallest and greatest element), but it does not form a lattice (compare [4,27,31,69]).

Here we will recall only a few basic and important properties of copulas. For more details we recommend the monographs [28,44,67].

Each copula $C\colon [0,1]^2 \to [0,1]$ is 1-Lipschitz and, therefore continuous. A copula $C \in \mathscr{C}$ is called absolutely continuous if there exists a function $\varphi_C\colon [0,1]^2 \to [0,\infty[$ such that

$$C(x, y) = \int_0^x \int_0^y \varphi_C(u, v)\, dv du.$$

Clearly, we may put $\varphi_C(x, y) = \frac{\partial^2 C(x,y)}{\partial y \partial x}$ for all points $(x, y) \in [0, 1]^2$ where this cross derivative of C exists.

Several methods for fitting copulas to real data (in particular, maximal likelihood methods) require the absolute continuity of the copulas under consideration. The fact that each EFGM copula is absolutely continuous is one of the most important reasons for their use in applications [70,85,91].

If $C: [0, 1]^2 \to [0, 1]$ is an arbitrary copula then several related copulas can be considered, among them the x-flipping $C^{x\text{flip}}: [0, 1]^2 \to [0, 1]$ and the y-flipping $C^{y\text{flip}}: [0, 1]^2 \to [0, 1]$ of C, and the survival copula $C^{\text{surv}}: [0, 1]^2 \to [0, 1]$, which are defined (see [23,67]) by, respectively,

$$\begin{aligned} C^{x\text{flip}}(x, y) &= y - C(1 - x, y), \\ C^{y\text{flip}}(x, y) &= x - C(x, 1 - y), \\ C^{\text{surv}}(x, y) &= x + y - 1 + C(1 - x, 1 - y). \end{aligned} \tag{2.3}$$

A copula $C: [0, 1]^2 \to [0, 1]$ which is invariant with respect to the construction of survival copulas, i.e., which satisfies $C^{\text{surv}} = C$, is also called radially symmetric [36] (see also [5,6,18]). Obviously, for each copula $C: [0, 1]^2 \to [0, 1]$ we have

$$C^{\text{surv}} = (C^{x\text{flip}})^{y\text{flip}} = (C^{y\text{flip}})^{x\text{flip}}.$$

Note, however, that the dual $C^*: [0, 1]^2 \to [0, 1]$ of a copula C given below is never a copula:

$$C^*(x, y) = x + y - C(x, y). \tag{2.4}$$

We shall be concerned with several dependence parameters of a copula $C: [0, 1]^2 \to [0, 1]$, in particular with Spearman's rho [87], Kendall's tau [48], Blomqvist's beta [12], and Gini's gamma [37] which can be defined for each copula $C \in \mathscr{C}$ and assume their values in the interval $[-1, 1]$. The corresponding functions $\varrho, \tau, \beta, \gamma: \mathscr{C} \to [-1, 1]$ are given by (see, e.g., [67]), respectively:

$$\begin{aligned} \varrho(C) &= 12 \iint_{[0,1]^2} C(x, y)\, dx\, dy - 3, & \tau(C) &= 4 \iint_{[0,1]^2} C(x, y)\, dC(x, y) - 1, \\ \beta(C) &= 4\, C(\tfrac{1}{2}, \tfrac{1}{2}) - 1, & \gamma(C) &= 4 \int_0^1 (C(x, x) + C(x, 1 - x))\, dx - 2. \end{aligned} \tag{2.5}$$

For the three basic copulas W, Π and M we obtain the following special values:

$$\begin{aligned} \varrho(W) &= \tau(W) = \beta(W) = \gamma(W) = -1, \\ \varrho(\Pi) &= \tau(\Pi) = \beta(\Pi) = \gamma(\Pi) = 0, \\ \varrho(M) &= \tau(M) = \beta(M) = \gamma(M) = 1. \end{aligned} \tag{2.6}$$

Recall that the set $\mathscr{C}^{\mathbb{N}}$ is just the set of all functions from $\mathbb{N}$ to $\mathscr{C}$ or, equivalently, the set of all sequences $(C_n)_{n\in\mathbb{N}}$ of copulas. Then we know that the four dependence parameters given in (2.5) are continuous: whenever a sequence $(C_n)_{n\in\mathbb{N}} \in \mathscr{C}^{\mathbb{N}}$ converges pointwise (and, subsequently, uniformly) to some copula $C: [0, 1]^2 \to [0, 1]$ then, as shown in [77] (or in [28, Lemma 2.4.8]), also the corresponding sequences of dependence parameters converge, i.e.,

$$\begin{aligned} \lim_{n\to\infty} \varrho(C_n) &= \varrho(C), & \lim_{n\to\infty} \tau(C_n) &= \tau(C), \\ \lim_{n\to\infty} \beta(C_n) &= \beta(C), & \lim_{n\to\infty} \gamma(C_n) &= \gamma(C). \end{aligned} \tag{2.7}$$

We shall consider (mainly in Section 6) also some other properties of real functions, such as ultramodularity [60] and Schur concavity [79]. In the case of copulas, they are defined as follows [51,52]:

Definition 2.1. Let $C\colon [0,1]^2 \to [0,1]$ be a copula. Then C is called

(i) ultramodular if for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^2$ satisfying $\mathbf{x}+\mathbf{y}+\mathbf{z} \in [0,1]^2$ the following inequality holds:

$$C(\mathbf{x}+\mathbf{y}+\mathbf{z}) + C(\mathbf{x}) \geq C(\mathbf{x}+\mathbf{y}) + C(\mathbf{x}+\mathbf{z}).$$

(ii) Schur concave if for all $(x, y), (u, v) \in [0,1]^2$ satisfying $x + y = u + v$ and $\min(x, y) \leq \min(u, v)$ the following inequality holds:

$$C(x, y) \leq C(u, v).$$

Finally let us mention that a copula $C\colon [0,1]^2 \to [0,1]$ (see [58]) is called

$$\begin{aligned} &\text{positively quadrant dependent (PQD for short)} && \text{if } C \geq \Pi, \\ &\text{negatively quadrant dependent (NQD for short)} && \text{if } C \leq \Pi. \end{aligned} \tag{2.8}$$

3. Eyraud-Farlie-Gumbel-Morgenstern copulas

The family of Eyraud-Farlie-Gumbel-Morgenstern copulas (see [29,30,38,65,67]) is one of the most important and most frequently used families of copulas (for some recent applications of EFGM copulas see, e.g., [70,85,91]).

Definition 3.1. Consider the parameter set $[-1,1] \subset \mathbb{R}$ and the function $\xi^{\mathbf{EFGM}}\colon [-1,1] \to \mathscr{C}$ given by $\xi^{\mathbf{EFGM}}(\theta) = C_\theta^{\mathbf{EFGM}}$, where each Eyraud-Farlie-Gumbel-Morgenstern copula (EFGM copula for short) $C_\theta^{\mathbf{EFGM}}\colon [0,1]^2 \to [0,1]$ is defined by

$$C_\theta^{\mathbf{EFGM}}(x, y) = xy + \theta x(1-x)y(1-y). \tag{3.1}$$

Let us denote by $\mathscr{C}^{\mathbf{EFGM}} \subset \mathscr{C}$ the set of EFGM copulas, i.e., the function $\xi^{\mathbf{EFGM}}\colon [-1,1] \to \mathscr{C}$ assigns to each parameter $\theta \in [-1,1]$ the corresponding EFGM copula $C_\theta^{\mathbf{EFGM}}$. We only recall that the parameter set $[-1,1]$ of the family $(C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}$ of EFGM copulas is a closed interval and, therefore, a convex, linearly ordered subset of the real line and, subsequently, a complete bounded chain with smallest element -1 and greatest element 1.

Remark 3.2. The set of EFGM copulas $\mathscr{C}^{\mathbf{EFGM}}$, the function $\xi^{\mathbf{EFGM}}\colon [-1,1] \to \mathscr{C}$ and the family $(C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}$ of EFGM copulas have a number of remarkable properties:

[EFGM1] The family $(C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}$ of EFGM copulas is a one-parametric family of copulas, and the function $\xi^{\mathbf{EFGM}}$ is continuous and strictly increasing. The set $\mathscr{C}^{\mathbf{EFGM}}$ of EFGM copulas is a closed convex subset of $\mathscr{C}$, and the function $\xi^{\mathbf{EFGM}}$ preserves convex combinations. The set $\mathscr{C}^{\mathbf{EFGM}}$ of EFGM copulas is a complete bounded chain with smallest element $C_{-1}^{\mathbf{EFGM}}$ and greatest element $C_1^{\mathbf{EFGM}}$, and the function $\xi^{\mathbf{EFGM}}$ preserves arbitrary joins and meets.

[EFGM2] As can be seen directly from (3.1), each EFGM copula $C_\theta^{\mathbf{EFGM}}$ is symmetric, and it is determined by a polynomial (of degree 4, and only $C_0^{\mathbf{EFGM}} = \Pi$ is of degree 2). As an immediate consequence, each $C_\theta^{\mathbf{EFGM}}$ is absolutely continuous, and its density function $\varphi_\theta^{\mathbf{EFGM}}\colon [0,1]^2 \to \mathbb{R}$ is given by

$$\varphi_\theta^{\mathbf{EFGM}}(x, y) = \frac{\partial^2 C_\theta^{\mathbf{EFGM}}(x, y)}{\partial x\, \partial y} = 1 + \theta(1-2x)(1-2y). \tag{3.2}$$

[EFGM3] Each EFGM copula $C_\theta^{\mathbf{EFGM}}$ is Schur concave [50,79], and an EFGM copula $C_\theta^{\mathbf{EFGM}}$ is ultramodular if and only if $\theta \in [-1, 0]$. An EFGM copula $C_\theta^{\mathbf{EFGM}}$ is NQD if and only if $\theta \leq 0$, and PQD if and only if $\theta \geq 0$ (see [58,67]). The family $(C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}$ is symmetric with respect to both x- and y-flipping in the following sense and radially symmetric: for each $\theta \in [-1,1]$ we have

$$C_\theta^{\mathbf{EFGM}} = (C_{-\theta}^{\mathbf{EFGM}})^{x\,\mathrm{flip}} = (C_{-\theta}^{\mathbf{EFGM}})^{y\,\mathrm{flip}} = (C_\theta^{\mathbf{EFGM}})^{\mathrm{surv}}. \tag{3.3}$$

[EFGM4] For each $\theta \in [-1, 1]$, the four dependence parameters $\varrho(C_\theta^{\mathbf{EFGM}})$, $\tau(C_\theta^{\mathbf{EFGM}})$, $\beta(C_\theta^{\mathbf{EFGM}})$ and $\gamma(C_\theta^{\mathbf{EFGM}})$ of the EFGM copulas are linear functions of the parameter θ

$$\varrho(C_\theta^{\mathbf{EFGM}}) = \frac{1}{3}\theta, \qquad \tau(C_\theta^{\mathbf{EFGM}}) = \frac{2}{9}\theta, \qquad \beta(C_\theta^{\mathbf{EFGM}}) = \frac{1}{4}\theta, \qquad \gamma(C_\theta^{\mathbf{EFGM}}) = \frac{4}{15}\theta \tag{3.4}$$

and, subsequently, proportional to each other in a very simple way:

$$\begin{gathered}\varrho(C_\theta^{\mathbf{EFGM}}) : \tau(C_\theta^{\mathbf{EFGM}}) = 3:2, \qquad \varrho(C_\theta^{\mathbf{EFGM}}) : \beta(C_\theta^{\mathbf{EFGM}}) = 4:3, \\ \varrho(C_\theta^{\mathbf{EFGM}}) : \gamma(C_\theta^{\mathbf{EFGM}}) = 5:4.\end{gathered} \tag{3.5}$$

As an immediate consequence of property [EFGM4] in Remark 3.2, we obtain for each $\theta \in [-1, 1]$

$$\varrho(C_\theta^{\mathbf{EFGM}}) \in \left[-\tfrac{1}{3}, \tfrac{1}{3}\right], \quad \tau(C_\theta^{\mathbf{EFGM}}) \in \left[-\tfrac{2}{9}, \tfrac{2}{9}\right], \quad \beta(C_\theta^{\mathbf{EFGM}}) \in \left[-\tfrac{1}{4}, \tfrac{1}{4}\right], \quad \gamma(C_\theta^{\mathbf{EFGM}}) \in \left[-\tfrac{4}{15}, \tfrac{4}{15}\right], \tag{3.6}$$

meaning that only weak dependencies can be modeled by means of EFGM copulas.

This fact is one of the main reasons to look for suitable extensions of the family of EFGM copulas preserving (most of) the positive properties of the EFGM copulas and where the corresponding dependence parameters cover greater subintervals of $[-1, 1]$.

Definition 3.3. A family $(C_\theta)_{\theta\in\Theta}$ of copulas satisfying

$$\{\varrho(C_\theta) \mid \theta \in \Theta\} = \{\tau(C_\theta) \mid \theta \in \Theta\} = \{\beta(C_\theta) \mid \theta \in \Theta\} = \{\gamma(C_\theta) \mid \theta \in \Theta\} = [-1, 1] \tag{3.7}$$

is called dependence-complete.

4. Some known extensions of EFGM copulas

To keep, as much as possible, the main advantages of the family of EFGM copulas as listed in Remark 3.2, and to reduce (some of) its limitations (in particular, the fact that the values of the dependence parameters given in (3.5) are contained in (subintervals of) $\left[-\frac{1}{3}, \frac{1}{3}\right]$), in a number of papers extensions of the set and/or the family of EFGM copulas have been proposed, e.g., in [3,10,11,19,43,57,83,84].

It is obvious that there are infinitely many trivial extensions of the set $\mathscr{C}^{\mathbf{EFGM}}$ of EFGM copulas, e.g., each set of copulas $\mathscr{C}_{[\mathrm{ext}]}$ satisfying $\mathscr{C}^{\mathbf{EFGM}} \subseteq \mathscr{C}_{[\mathrm{ext}]} \subseteq \mathscr{C}$. Clearly, such non-specific "extensions" of the set $\mathscr{C}^{\mathbf{EFGM}}$ of EFGM copulas almost never will satisfy any of the properties [EFGM1]–[EFGM4] in Remark 3.2.

We therefore will focus on more specific extensions of the family of EFGM copulas of the following type: consider a non-empty parameter set J and a function $\xi : J \to \mathscr{C}$ defined by $\xi(j) = C_j$, giving rise to a set of copulas $\mathscr{C}^J = \{C_j \mid j \in J\} \subseteq \mathscr{C}$ and to a family of copulas $(C_j)_{j\in J}$ (which, in some sense, extends the family of EFGM copulas).

In such a case, we shall provide a summary of the properties the function ξ, the set of copulas $\mathscr{C}^J$ and the family of copulas $(C_j)_{j\in J}$, in full analogy to the properties [EFGM1]–[EFGM4] in Remark 3.2. We will denote the corresponding properties by [EX]–[DP] and, wherever applicable, present them according to the following scheme:

- [EX] **Extension:** relation between the family $(C_j)_{j\in J}$ and the family of EFGM copulas.
- [SP] **Structural Properties:** order theoretical, topological and geometrical properties of the parameter set J, the function ξ and the set $\mathscr{C}^J$.
- [AP] **Algebraic and Analytical Properties:** symmetry of the copulas C_j, being a polynomial, being absolutely continuous or singular.
- [CP] **Copula Properties** of the copulas C_j such as ultramodularity, Schur concavity, positive or negative quadrant dependence, some symmetry with respect to x-flipping and y-flipping, and radial symmetry (invariance with respect to the construction of survival copulas).
- [DP] **Dependence Parameters:** properties of Spearman's rho, Kendall's tau, Blomqvist's beta and Gini's gamma of the copulas C_j.

Based on these considerations let us now turn to the analysis of some known extensions of the family of EFGM copulas $(C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}$. The first of these extensions was introduced in [43]:

Definition 4.1. Consider the parameter set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and the function $\xi^{[\mathrm{H}]} \colon \overline{\mathbb{R}} \to \mathscr{C}$ given by $\xi^{[\mathrm{H}]}(\theta) = C_\theta^{[\mathrm{H}]}$, where each copula $C_\theta^{[\mathrm{H}]} \colon [0,1]^2 \to [0,1]$ is defined by

$$C_\theta^{[\mathrm{H}]}(x,y) = \begin{cases} W(x,y) & \text{if } \theta = -\infty, \\ \max(W(x,y), xy + \theta xy(1-x)(1-y)) & \text{if } \theta \in \,]-\infty, 0[\,, \\ \min(M(x,y), xy + \theta xy(1-x)(1-y)) & \text{if } \theta \in [0, \infty[\,, \\ M(x,y) & \text{if } \theta = \infty. \end{cases} \tag{4.1}$$

Obviously, the parameter set $\overline{\mathbb{R}}$ satisfies $[-1,1] \subset \overline{\mathbb{R}}$, it is a closed set and, therefore, a complete bounded chain with smallest element $-\infty$ and greatest element ∞.

It is immediately seen that, for each $\theta \in [-1,1]$, the EFGM copula $C_\theta^{\mathbf{EFGM}}$ defined in (3.1) and the copula $C_\theta^{[\mathrm{H}]}$ given in (4.1) coincide. Therefore the family $\big(C_\theta^{\mathbf{EFGM}}\big)_{\theta\in[-1,1]}$ of EFGM copulas is a subfamily of $\big(C_\theta^{[\mathrm{H}]}\big)_{\theta\in\overline{\mathbb{R}}}$ or, equivalently, $\big(C_\theta^{[\mathrm{H}]}\big)_{\theta\in\overline{\mathbb{R}}}$ is an extension of the family of EFGM copulas.

We denote by $\mathscr{C}^{[\mathrm{H}]} \subset \mathscr{C}$ the set of copulas defined by (4.1), i.e., the function $\xi^{[\mathrm{H}]} \colon \overline{\mathbb{R}} \to \mathscr{C}$ assigns to each $\theta \in \overline{\mathbb{R}}$ the corresponding copula $C_\theta^{[\mathrm{H}]} \in \mathscr{C}^{[\mathrm{H}]}$.

Remark 4.2. Some of the most relevant properties of the function $\xi^{[\mathrm{H}]} \colon \overline{\mathbb{R}} \to \mathscr{C}$, the set of copulas $\mathscr{C}^{[\mathrm{H}]}$ and the family of copulas $\big(C_\theta^{[\mathrm{H}]}\big)_{\theta\in\overline{\mathbb{R}}}$ are:

[EX] Because of $C_\theta^{[\mathrm{H}]} = C_\theta^{\mathbf{EFGM}}$ for each $\theta \in [-1,1]$, the one-parametric family $\big(C_\theta^{[\mathrm{H}]}\big)_{\theta\in\overline{\mathbb{R}}}$ is an extension of the family of EFGM copulas $\big(C_\theta^{\mathbf{EFGM}}\big)_{\theta\in[-1,1]}$. Moreover, the family $\big(C_\theta^{[\mathrm{H}]}\big)_{\theta\in\overline{\mathbb{R}}}$ is comprehensive, i.e., it contains all three basic copulas W, Π and M: $C_{-\infty}^{[\mathrm{H}]} = W$, $C_0^{[\mathrm{H}]} = \Pi$ and $C_\infty^{[\mathrm{H}]} = M$.

[SP] The function $\xi^{[\mathrm{H}]}$ is continuous and strictly increasing. The set $\mathscr{C}^{[\mathrm{H}]}$ is a closed, but not a convex, subset of $\mathscr{C}$, and it is a complete bounded chain with smallest element W and greatest element M.

[AP] A copula $C_\theta^{[\mathrm{H}]} \in \mathscr{C}^{[\mathrm{H}]}$ is absolutely continuous if and only if $\theta \in [-1,1]$.

[CP] From [EFGM3] in Remark 3.2 and [EX] it follows that a copula $C_\theta^{[\mathrm{H}]} \in \mathscr{C}^{[\mathrm{H}]}$ is Schur concave if $\theta \in [-1,1] \cup \{-\infty, \infty\}$, ultramodular if $\theta \in [-\infty, 0]$, NQD if $\theta \in [-\infty, 0]$, and PQD if $\theta \in [0, \infty]$. The family $(C_\theta^{[\mathrm{H}]})_{\theta\in\overline{\mathbb{R}}}$ is symmetric with respect to x- and y-flipping in the following sense and radially symmetric: for each $\theta \in \overline{\mathbb{R}}$ we have

$$C_\theta^{[\mathrm{H}]} = (C_{-\theta}^{[\mathrm{H}]})^{x\text{flip}} = (C_{-\theta}^{[\mathrm{H}]})^{y\text{flip}} = (C_\theta^{[\mathrm{H}]})^{\text{surv}}. \tag{4.2}$$

[DP] Taking into account the continuity and monotonicity of $\xi^{[\mathrm{H}]}$ (see [SP]), the family $(C_\theta^{[\mathrm{H}]})_{\theta\in\overline{\mathbb{R}}}$ is dependence-complete as a consequence of (2.6)–(2.7).

Example 4.3. Let $C \colon [0,1]^2 \to [0,1]$ be some copula and recall the perturbation of copulas (i.e., a distortion of the original copula $C \in \mathscr{C}$ by means of some function which depends on the variables x, y and $C(x,y)$) as introduced and discussed in [26,56,57,61] (see also [32,63,82]).

(i) Consider the parameter set $[-1,1]$ and the function $\xi_C^{[\mathrm{pert}]} \colon [-1,1] \to \mathscr{C}$ given by $\xi_C^{[\mathrm{pert}]}(\theta) = C_\theta^{[\mathrm{pert}]}$, where each $C_\theta^{[\mathrm{pert}]} \colon [0,1]^2 \to [0,1]$ is defined by

$$C_\theta^{[\mathrm{pert}]}(x,y) = \begin{cases} C(x,y) + \theta C(x,y)(C(x,y) + 1 - x - y) & \text{if } \theta < 0, \\ C(x,y) + \theta(x - C(x,y))(y - C(x,y)) & \text{if } \theta \geq 0. \end{cases} \tag{4.3}$$

(ii) In the special case $C = \Pi$ we obtain $\xi_\Pi^{[\mathrm{pert}]} = \xi^{\mathbf{EFGM}}$ or, equivalently, $\Pi_\theta^{[\mathrm{pert}]} = C_\theta^{\mathbf{EFGM}}$ for each $\theta \in [-1,1]$, i.e.,

$$(\Pi_\theta^{[\mathrm{pert}]})_{\theta\in[-1,1]} = (C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}.$$

Note that for each $\theta \in [-1, 1]$ we have

$$C_\theta^{[\mathrm{pert}]} = |\theta| \cdot C_{\mathrm{sign}(\theta)}^{[\mathrm{pert}]} + (1 - |\theta|) \cdot C,$$

i.e., the restrictions of the function $\xi_C^{[\mathrm{pert}]} \colon [-1, 1] \to \mathscr{C}$ to the intervals $[-1, 0]$ and $[0, 1]$ are both linear. Together with Lemma 4.4 below, this allows us to show some nice properties of the four dependence parameters ϱ, τ, β, $\gamma \colon \mathscr{C} \to [-1, 1]$ given in (2.5).

Lemma 4.4. *Suppose that the parameter set is an interval $[a, b] \subset \mathbb{R}$ with $a < b$ and that $\xi \colon [a, b] \to \mathscr{C}$ given by $\xi(j) = C_j$ is a linear function and, therefore, $C_{(1-\lambda)a+\lambda b} = (1 - \lambda)C_a + \lambda C_b$ for each $\lambda \in [0, 1]$. Then we have:*

(i) *For each function $\delta \colon \mathscr{C} \to [-1, 1]$ such that $\delta \in \{\varrho, \beta, \gamma\}$ the function $\delta^{[\xi]} \colon [0, 1] \to [-1, 1]$ defined by $\delta^{[\xi]}(\lambda) = \delta\big((1 - \lambda)C_a + \lambda C_b\big)$ is a linear function.*
(ii) *The function $\tau^{[\xi]} \colon [0, 1] \to [-1, 1]$ defined by $\tau^{[\xi]}(\lambda) = \tau\big((1 - \lambda)C_a + \lambda C_b\big)$ is a quadratic function.*

Proof. The validity of (i) follows directly from (2.5) and the linearity of the Lebesgue integral.

Concerning (ii), note that for any two copulas $C, D \in \{C_j \mid j \in [a, b]\}$ we have $\frac{1}{2}C + \frac{1}{2}D \in \{C_j \mid j \in [a, b]\}$ and

$$\begin{aligned}\tau\Big(\frac{1}{2}C + \frac{1}{2}D\Big) &= 4 \iint\limits_{[0,1]^2} \Big(\frac{1}{2}C + \frac{1}{2}D\Big)(x, y) \Big(\frac{1}{2}dC + \frac{1}{2}dD\Big)(x, y) - 1 \\ &= \frac{1}{4}\tau(C) + \Bigg(\iint\limits_{[0,1]^2} C(x, y)\, dD(x, y) - \frac{1}{4}\Bigg) + \Bigg(\iint\limits_{[0,1]^2} D(x, y)\, dC(x, y) - \frac{1}{4}\Bigg) + \frac{1}{4}\tau(D).\end{aligned} \tag{4.4}$$

Using (4.4) we obtain for each $\lambda \in [0, 1]$

$$\begin{aligned}\tau^{[\xi]}(\lambda) &= (1 - \lambda)^2\tau\big(C_a\big) + 4\lambda(1 - \lambda)\Bigg(\iint\limits_{[0,1]^2} C_b(x, y)\, dC_a(x, y) - \frac{1}{4}\Bigg) \\ &\quad + 4\lambda(1 - \lambda)\Bigg(\iint\limits_{[0,1]^2} C_a(x, y)\, dC_b(x, y) - \frac{1}{4}\Bigg) + \lambda^2\tau\big(C_b\big) \\ &= (1 - \lambda)^2\tau^{[\xi]}(0) + \lambda(1 - \lambda)\big((4\tau^{[\xi]}\big(\tfrac{1}{2}\big) - \tau^{[\xi]}(0) - \tau^{[\xi]}(1)\big) + \lambda^2\tau^{[\xi]}(1) \\ &= A_0\lambda^2 + A_1\lambda + A_2,\end{aligned}$$

where $A_0 = 2\tau^{[\xi]}(0) - 4\tau^{[\xi]}\big(\tfrac{1}{2}\big) + 2\tau^{[\xi]}(1)$, $A_1 = -3\tau^{[\xi]}(0) + 4\tau^{[\xi]}\big(\tfrac{1}{2}\big) - \tau^{[\xi]}(1)$ and $A_2 = \tau^{[\xi]}(0)$. This shows that $\tau^{[\xi]}$ is a polynomial of degree 2 in λ. □

Remark 4.5. Here are some properties of the family $(C_\theta^{[\mathrm{pert}]})_{(C,\theta)\in\mathscr{C}\times[-1,1]}$ considered in Example 4.3:

[EX] As a consequence of Example 4.3(ii), for each subset $\mathscr{C}_\Pi$ of the set of copulas $\mathscr{C}$ satisfying $\Pi \in \mathscr{C}_\Pi$ the two-parameter family $(C_\theta^{[\mathrm{pert}]})_{(C,\theta)\in\mathscr{C}_\Pi\times[-1,1]}$ is an extension of the family $(C_\theta^{\mathbf{EFGM}})_{\theta\in[-1,1]}$ of EFGM copulas.
[SP] For each $C \in \mathscr{C}$ the function $\xi_C^{[\mathrm{pert}]} \colon [-1, 1] \to \mathscr{C}$ considered in Example 4.3(i) is continuous and monotone non-decreasing.
[AP] If $C \in \mathscr{C}$ is symmetric or absolutely continuous or polynomial then for each $\theta \in [-1, 1]$ also $C_\theta^{[\mathrm{pert}]}$ has the corresponding property.
[CP] The perturbation preserves the invariance of $C \in \mathscr{C}$ with respect to x- and y-flipping and the radial symmetry as follows:

$$\begin{aligned} C^{x\mathrm{flip}} = C \quad &\Longrightarrow \quad \big(C_\theta^{[\mathrm{pert}]}\big)^{x\mathrm{flip}} = C_{-\theta}^{[\mathrm{pert}]} \text{ for each } \theta \in [-1, 1]; \\ C^{y\mathrm{flip}} = C \quad &\Longrightarrow \quad \big(C_\theta^{[\mathrm{pert}]}\big)^{y\mathrm{flip}} = C_{-\theta}^{[\mathrm{pert}]} \text{ for each } \theta \in [-1, 1]; \\ C^{\mathrm{surv}} = C \quad &\Longrightarrow \quad \big(C_\theta^{[\mathrm{pert}]}\big)^{\mathrm{surv}} = C_\theta^{[\mathrm{pert}]} \text{ for each } \theta \in [-1, 1]\end{aligned}$$

If $C \in \mathscr{C}$ is NQD then for each $\theta \in [-1, 0]$ also $C_\theta^{[\text{pert}]}$ is NQD. Analogously, if $C \in \mathscr{C}$ is PQD then for each $\theta \in [0, 1]$ also $C_\theta^{[\text{pert}]}$ is PQD.

[DP] As a consequence of Lemma 4.4, for each copula $C \in \mathscr{C}$ the dependence parameters $\varrho(C_\theta^{[\text{pert}]})$, $\beta(C_\theta^{[\text{pert}]})$ and $\gamma(C_\theta^{[\text{pert}]})$ depend linearly on the parameter θ, and the dependence parameter $\tau(C_\theta^{[\text{pert}]})$ depends quadratically on the parameter θ in each of the intervals $[-1, 0]$ and $[0, 1]$.

Example 4.6. Another approach related to the family of EFGM copulas has recently been proposed and exemplified in [18].

(i) Let $\varphi, \psi \colon [0, 1] \to [0, \infty]$ be two functions which satisfy the boundary conditions $\varphi(0) = \psi(0) = 0$ and $\varphi(1) = \psi(1) = \infty$, and also some regularity conditions. Consider the family of copulas $\big(C_\theta^{[\varphi,\psi]}\big)_{\theta\in[-1,0]}$, where each $C_\theta^{[\varphi,\psi]} \colon [0, 1]^2 \to [0, 1]$ is given by (using the convention $0 \cdot \infty = 0$)

$$C_\theta^{[\varphi,\psi]}(x, y) = xy + \theta \frac{xy}{1 + \varphi(x)\psi(y)}.$$

Both families of copulas $\big(C_\theta^{\mathbf{EFGM}}\big)_{\theta\in[-1,1]}$ and $\big(C_\theta^{[\varphi,\psi]}\big)_{\theta\in[-1,0]}$ can be seen as particular cases of copulas $C_{h,\theta} \colon [0, 1]^2 \to [0, 1]$ where $h \colon [0, 1]^2 \to [0, 1]$ is a suitable function, $\theta \in [-1, 1]$ and

$$C_{h,\theta}(x, y) = xy + \theta \frac{xy}{1 + h(x, y)}.$$

(ii) If the functions $\varphi, \psi \colon [0, 1] \to [0, \infty]$ are defined by $\varphi(x) = \psi(x) = \frac{\sqrt{2x-x^2}}{1-x}$, then for each $\theta \in [-1, 0]$ the copulas $C_\theta^{[\varphi,\psi]}$ and $C_\theta^{\mathbf{EFGM}}$ as given by (3.1) coincide for $x = y \in [0, 1]$, i.e., in this particular case the diagonal sections of the negatively quadrant dependent EFGM copulas are recovered.

(iii) As a special case, consider the functions $\varphi_0, \psi_0 \colon [0, 1] \to [0, \infty]$ defined by $\varphi_0(x) = \psi_0(x) = \frac{x}{1-x}$. For the parameter set $[-1, 0]$ and the function $\xi^{[\text{N}]} \colon [-1, 0] \to \mathscr{C}$ given by $\xi^{[\text{N}]}(\theta) = C_\theta^{[\varphi_0,\psi_0]} = C_\theta^{[\text{N}]}$ we obtain the following formula for the copula $C_\theta^{[\text{N}]} \colon [0, 1]^2 \to [0, 1]$ which is sometimes called copula N (see [18, (2.2)])

$$C_\theta^{[\text{N}]}(x, y) = xy\left(1 + \theta \frac{(1-x)(1-y)}{xy + (1-x)(1-y)}\right), \tag{4.5}$$

and which gives rise to the family of copulas $\big(C_\theta^{[\text{N}]}\big)_{\theta\in[-1,0]}$ and to the set of copulas $\mathscr{C}^{[\text{N}]} = \big\{C_\theta^{[\text{N}]} \mid \theta \in [-1, 0]\big\}$.

Remark 4.7. We summarize some relevant properties of the function $\xi^{[\text{N}]}$, the set of copulas $\mathscr{C}^{[\text{N}]}$ and the family of copulas $\big(C_\theta^{[\text{N}]}\big)_{\theta\in[-1,0]}$ given in Example 4.6(iii) (for proofs and additional properties see [18, Subsection 2.2]):

[EX] The formula for each $C_\theta^{[\text{N}]}$ is just a fraction of polynomials occurring in the definition of the EFGM copula $C_\theta^{\mathbf{EFGM}}$, but the family $\big(C_\theta^{[\text{N}]}\big)_{\theta\in[-1,0]}$ is no extension of the subfamily $\big(C_\theta^{\mathbf{EFGM}}\big)_{\theta\in[-1,0]}$ of EFGM copulas.

[SP] The function $\xi^{[\text{N}]}$ is continuous and strictly increasing. The set of copulas $\mathscr{C}^{[\text{N}]}$ is a closed convex subset of $\mathscr{C}$, and the function $\xi^{[\text{N}]}$ preserves convex combinations. The set $\mathscr{C}^{[\text{N}]}$ is a complete bounded chain with smallest element $C_{-1}^{[\text{N}]}$ and greatest element $C_0^{[\text{N}]} = \Pi$.

[AP] For each $\theta \in [-1, 0]$ the copula $C_\theta^{[\text{N}]}$ is absolutely continuous.

[CP] For each $\theta \in [-1, 0]$ the copula $C_\theta^{[\text{N}]}$ is ultramodular, NQD, and radial symmetric, i.e., we have $\big(C_\theta^{[\text{N}]}\big)^{\text{surv}} = C_\theta^{[\text{N}]}$.

[DP] For each $\theta \in [-1, 0]$ the dependence parameters $\varrho(C_\theta^{[\text{N}]})$, $\tau(C_\theta^{[\text{N}]})$, $\beta(C_\theta^{[\text{N}]})$ and $\gamma(C_\theta^{[\text{N}]})$ are determined by the parameter θ as follows:

$$\varrho(C_\theta^{[\text{N}]}) = \frac{3}{8}(\pi^2 - 8)\theta, \qquad \tau(C_\theta^{[\text{N}]}) = \frac{1}{4}(\pi^2 - 8)\theta + \frac{1}{48}(3\pi^2 - 32)\theta^2,$$

$$\beta(C_\theta^{[\text{N}]}) = \frac{1}{2}\theta, \qquad \gamma(C_\theta^{[\text{N}]}) = \frac{1}{2}(\pi - 2)\theta.$$

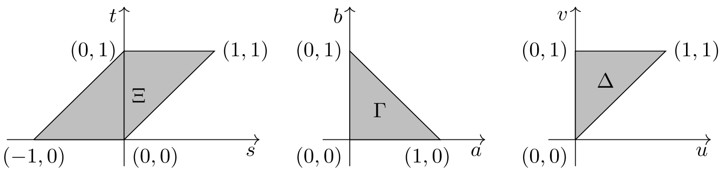


Fig. 1. The parallelogram $\Xi = \{(s,t) \in \mathbb{R}^2 \mid 0 \le t \le 1, 0 \le t - s \le 1\}$ (left), and the triangles $\Gamma = \{(a,b) \in [0,1]^2 \mid a + b \le 1\}$ (center) and $\Delta = \{(u,v) \in [0,1]^2 \mid u \le v\}$ (right).

Remark 4.7 shows that there exist interesting families of copulas which (partially) satisfy the properties [SP]–[DP] without being an extension of (a subfamily of) the family of EFGM copulas.

5. Generalizations of the family of EFGM copulas based on quadratic constructions of copulas

Let us consider the parallelogram $\Xi \subset \mathbb{R}^2$ with the four vertices $(-1,0)$, $(0,0)$, $(1,1)$, and $(0,1)$ visualized in Fig. 1 (left) which will be useful for several constructions in this section. It is easy to see that the set Ξ can also be described as follows:

$$\Xi = \{(s,t) \in \mathbb{R}^2 \mid \{t, t-s\} \in [0,1]\}. \tag{5.1}$$

The set Ξ turns out to be closely related to the subset $\mathscr{G}$ of the set of all functions from $[-1,1]$ to $[0,1]$ given by

$$\mathscr{G} = \{f \colon [-1,1] \to [0,1] \mid f(x) - x \in [0,1] \text{ for all } x \in [-1,1]\}. \tag{5.2}$$

Obviously, the set $\mathscr{G}$ is convex and, equipped with the pointwise order $\le$, a bounded lattice whose bottom and top elements $f^{\perp}, f_{\top} \colon [-1,1] \to [0,1]$ are given by, respectively,

$$f_{\perp}(x) = \max(x,0), \qquad f^{\top}(x) = \min(x+1,1). \tag{5.3}$$

The relation between the sets Ξ and $\mathscr{G}$ is established by the following equivalence which holds for each function $f \colon [-1,1] \to [0,1]$:

$$f \in \mathscr{G} \quad \text{if and only if} \quad (x, f(x)) \in \Xi \text{ for all } x \in [-1,1]. \tag{5.4}$$

Recently, a construction for copulas based on ternary quadratic polynomials [53] has been introduced, and the following result using the set Ξ defined above in (5.1) will be crucial for our investigations:

Theorem 5.1 *([53, Theorem 1]). For each $(s,t) \in \Xi$ and for each copula $C \colon [0,1]^2 \to [0,1]$ the function $C^{[Q]}_{(s,t)} \colon [0,1]^2 \to [0,1]$ given by*

$$C^{[Q]}_{(s,t)}(x,y) = s(C(x,y))^2 + (s(1-x-y) + 1 - t)C(x,y) + txy \tag{5.5}$$

is a copula.

Considering the parameter set $\mathscr{C} \times \Xi$ and the function $\xi^{[Q]} \colon \mathscr{C} \times \Xi \to \mathscr{C}$ given by $\xi^{[Q]}(C,(s,t)) = C^{[Q]}_{(s,t)}$, we obtain the family of copulas

$$\left(C^{[Q]}_{(s,t)}\right)_{(C,(s,t)) \in \mathscr{C} \times \Xi}. \tag{5.6}$$

In the following example we consider the triangle $\Gamma \subset \mathbb{R}^2$ with the vertices $(0,0)$, $(1,0)$ and $(0,1)$ as visualized in Fig. 1 (center) and observe that the set Γ can equivalently be defined as

$$\Gamma = \{(a,b) \in [0,1]^2 \mid a + b \le 1\}. \tag{5.7}$$

Example 5.2. Several families of copulas known from the literature turn out to be subfamilies of the family $\big(C^{[Q]}_{(s,t)}\big)_{(s,t)\in\Xi}$ for appropriate copulas $C \in \mathscr{C}$.

(i) Consider the parameter set $\Gamma \subset \mathbb{R}^2$ and the function $\xi^{[F]}\colon \Gamma \to \mathscr{C}$ given by $\xi^{[F]}(a,b) = C^{[F]}_{(a,b)}$, where each $C^{[F]}_{(a,b)}\colon [0,1]^2 \to [0,1]$ is defined by (see [35])

$$C^{[F]}_{(a,b)} = aW + bM + (1-a-b)\Pi. \tag{5.8}$$

The two-parameter family of copulas $\big(C^{[F]}_{(a,b)}\big)_{(a,b)\in\Gamma}$ as defined by (5.8) is called the family of Fréchet copulas, and we shall write briefly $\mathscr{C}^{[F]}$ for the set of all Fréchet copulas.

Recalling the definitions of the sets Γ in (5.7) and Ξ in (5.1) we see that for each $(s,t) \in \Xi$ we obtain

$$C^{[F]}_{(0,0)} = \Pi^{[Q]}_{(0,1)} = \Pi, \qquad C^{[F]}_{(1-t,0)} = W^{[Q]}_{(s,t)}, \qquad C^{[F]}_{(0,1+s-t)} = M^{[Q]}_{(s,t)}. \tag{5.9}$$

As a consequence, we have the following close relationship between the Fréchet copulas $C^{[F]}_{(a,b)}$ and the copulas $C^{[Q]}_{(s,t)}$ given by (5.5), showing that the family $\big(C^{[F]}_{(a,b)}\big)_{(a,b)\in\Gamma}$ is a subfamily of the family of copulas $\big(C^{[Q]}_{(s,t)}\big)_{(C,(s,t))\in\mathscr{C}\times\Xi}$ given by (5.6): for each $(a,b) \in \Gamma$ we get

$$C^{[F]}_{(a,b)} = \begin{cases} \Pi^{[Q]}_{(0,1)} & \text{if } a=b=0,\\ \frac{a}{a+b} W^{[Q]}_{(0,1-a-b)} + \frac{b}{a+b} M^{[Q]}_{(0,1-a-b)} & \text{otherwise.} \end{cases}$$

(ii) It is easy to check that for all $(\theta,t_1), (\theta,t_2) \in \Xi$ we have

$$\Pi^{[Q]}_{(\theta,t_1)} = \Pi^{[Q]}_{(\theta,t_2)} = C^{\mathbf{EFGM}}_{\theta}. \tag{5.10}$$

Taking into account (5.4), it follows that for each function $f \in \mathscr{G}$ we have

$$\big(\Pi^{[Q]}_{(\theta,f(\theta))}(x,y)\big)_{\theta\in[-1,1]} = \big(C^{\mathbf{EFGM}}_{\theta}\big)_{\theta\in[-1,1]}. \tag{5.11}$$

As a consequence, for each subset $\mathscr{C}_\Pi \subseteq \mathscr{C}$ satisfying $\Pi \in \mathscr{C}_\Pi$ and for each $f \in \mathscr{G}$ the family

$$\big(C^{[Q]}_{(\theta,f(\theta))}\big)_{(C,\theta)\in\mathscr{C}_\Pi\times[-1,1]} \tag{5.12}$$

is an extension of the family of EFGM copulas $\big(C^{\mathbf{EFGM}}_{\theta}\big)_{\theta\in[-1,1]}$.

Remark 5.3. Fix the function $f_0 \in \mathscr{G}$ given by $f_0(x) = \frac{x+1}{2}$ and consider the parameter set $\mathscr{C} \times [-1,1]$ and the function $\xi^{[Q]}_{f_0}\colon \mathscr{C}\times[-1,1] \to \mathscr{C}$ given by $\xi^{[Q]}_{f_0}(C,\theta) = C^{[Q]}_{(\theta,f_0(\theta))}$. Here are some properties of the family of copulas $\big(C^{[Q]}_{(\theta,f_0(\theta))}\big)_{(C,\theta)\in\mathscr{C}\times[-1,1]}$ and some subfamilies thereof:

[EX] Because of Example 5.2(ii) the family of copulas $\big(C^{[Q]}_{(\theta,f_0(\theta))}\big)_{(C,\theta)\in\mathscr{C}\times[-1,1]}$ is an extension of the family of EFGM copulas.

[SP] For each copula $C \in \mathscr{C}$ the section $\xi^{[Q]}_{f_0}(C,\cdot)\colon [-1,1] \to \mathscr{C}$ of the function $\xi^{[Q]}_{f_0}$ is a continuous and strictly increasing function, and the set $\big\{C^{[Q]}_{(\theta,f_0(\theta))} \,\big|\, \theta \in [-1,1]\big\}$ is the convex hull of the set $\big\{C^{[Q]}_{(-1,0)}, C^{[Q]}_{(1,1)}\big\}$.

[AP] If a copula $C \in \mathscr{C}$ is symmetric then for each $\theta \in [-1,1]$ also the copula $C^{[Q]}_{(\theta,f_0(\theta))}$ is symmetric. Similarly, for each polynomial or absolutely continuous copula $C \in \mathscr{C}$ and for each $\theta \in [-1,1]$ also the copula $C^{[Q]}_{(\theta,f_0(\theta))}$ is a polynomial or absolutely continuous copula, respectively.

[CP] If a copula $C \in \mathscr{C}$ is NQD then for each $\theta \in [-1,0]$ also the copula $C^{[Q]}_{(\theta,f_0(\theta))}$ is NQD. Analogously, if $\theta \in [0,1]$ then each PQD copula $C \in \mathscr{C}$ induces a PQD copula $C^{[Q]}_{(\theta,f_0(\theta))}$.

[DP] As a consequence of Lemma 5.5 below, there is no copula $C \in \mathscr{C}$ such that the subfamily $\big(C^{[Q]}_{(\theta,f_0(\theta))}\big)_{\theta\in[-1,1]}$ is dependence-complete.

With the help of the bottom element $f_\perp$ (see (5.3)) of the bounded lattice $\mathscr{G}$ we can, for each $C \in \mathscr{C}$, recover the family of perturbated copulas $\big(C^{[\mathrm{pert}]}_{\theta}\big)_{\theta\in[-1,1]}$ mentioned in Example 4.3(i).

Example 5.4. For an arbitrary copula $C \in \mathscr{C}$ and a parameter $\theta \in [-1,1]$, the copula $C^{[\mathrm{Q}]}_{(\theta, f_\perp(\theta))}$ is given by

$$C^{[\mathrm{Q}]}_{(\theta, f_\perp(\theta))} = \begin{cases} C^{[\mathrm{Q}]}_{(\theta,0)} & \text{if } \theta \le 0, \\ C^{[\mathrm{Q}]}_{(\theta,\theta)} & \text{if } \theta > 0. \end{cases}$$

Therefore, for each $C \in \mathscr{C}$ the family $\big(C^{[\mathrm{Q}]}_{(\theta, f_\perp(\theta))}\big)_{\theta\in[-1,1]}$ coincides with the family $\big(C^{[\mathrm{pert}]}_{\theta}\big)_{\theta\in[-1,1]}$ given by (4.3).

Lemma 5.5. *For all pairs $(C, f) \in \mathscr{C} \times \mathscr{G}$ each of the following four sets*

$$\big\{\varrho\big(C^{[\mathrm{Q}]}_{(\theta, f(\theta))}\big) \mid \theta \in [-1,1]\big\}, \qquad \big\{\tau\big(C^{[\mathrm{Q}]}_{(\theta, f(\theta))}\big) \mid \theta \in [-1,1]\big\},$$
$$\big\{\beta\big(C^{[\mathrm{Q}]}_{(\theta, f(\theta))}\big) \mid \theta \in [-1,1]\big\}, \qquad \big\{\gamma\big(C^{[\mathrm{Q}]}_{(\theta, f(\theta))}\big) \mid \theta \in [-1,1]\big\}$$

is a proper subset of the interval $[-1,1]$.

Proof. Fix an arbitrary pair $(C, f) \in \mathscr{C} \times \mathscr{G}$ and note that for each $(s,t) \in \Xi$ the copula $C^{[\mathrm{Q}]}_{(s,t)}$ is a convex combination of the copulas $C^{[\mathrm{Q}]}_{(-1,0)}$, $C^{[\mathrm{Q}]}_{(0,0)} (= C)$, $C^{[\mathrm{Q}]}_{(1,1)}$, and $C^{[\mathrm{Q}]}_{(0,1)} (= \Pi)$.

(i) If $C = W$ then we obtain $C^{[\mathrm{Q}]}_{(1,1)} = \Pi$ and, subsequently, $C^{[\mathrm{Q}]}_{(s,t)} \le \Pi$, i.e., $\varrho\big(C^{[\mathrm{Q}]}_{(s,t)}\big) < 0$ for each $(s,t) \in \Xi$.
(ii) If $C = M$ then we get $C^{[\mathrm{Q}]}_{(-1,0)} = \Pi$ and, as a consequence, $C^{[\mathrm{Q}]}_{(s,t)} \ge \Pi$, i.e., $\varrho\big(C^{[\mathrm{Q}]}_{(s,t)}\big) > 0$ for each $(s,t) \in \Xi$.
(iii) If $C \notin \{W, M\}$ then we have $\big\{C^{[\mathrm{Q}]}_{(-1,0)}, C^{[\mathrm{Q}]}_{(1,1)}\big\} \cap \{W, M\} = \emptyset$ and, therefore, $C^{[\mathrm{Q}]}_{(s,t)} \notin \{W, M\}$, i.e., $-1 < \varrho\big(C^{[\mathrm{Q}]}_{(s,t)}\big) < 1$ for each $(s,t) \in \Xi$.

The corresponding assertions for the other dependence parameters τ, β, and γ are shown in complete analogy. □

Remark 5.6. Let $C \in \mathscr{C}$ be an arbitrary copula.

(i) If $C \ne W$ then $C^{[\mathrm{Q}]}_{(-1,0)} < C$.
(ii) If $C \ne M$ then $C < C^{[\mathrm{Q}]}_{(1,1)}$.
(iii) As a consequence of (i) and (ii) we have $C^{[\mathrm{Q}]}_{(-1,0)} < C^{[\mathrm{Q}]}_{(1,1)}$.
(iv) If $f \in \mathscr{G}$ is a continuous, strictly increasing function then also the function $\xi^{[\mathrm{Q}]}_{(C,f)} \colon [-1,1] \to \mathscr{C}$ given by $\xi^{[\mathrm{Q}]}_{(C,f)}(\theta) = C^{[\mathrm{Q}]}_{(\theta, f(\theta))}$ is continuous and strictly increasing.

Remark 5.7. Consider an arbitrary pair of continuous functions $(f, g) \in \mathscr{G} \times \mathscr{G}$. Then we have:

(i) $M^{[\mathrm{Q}]}_{(-1, f(-1))} = W^{[\mathrm{Q}]}_{(1, f(1))} = \Pi$;
(ii) the functions $\xi^{[\mathrm{Q}]}_{(M,f)} \colon [-1,1] \to \mathscr{C}$ and $\xi^{[\mathrm{Q}]}_{(W,f)} \colon [-1,1] \to \mathscr{C}$ are continuous;
(iii) Consider the family of copulas $\big(K^{[f,g]}_\theta\big)_{\theta\in[-1,1]}$ defined by

$$K^{[f,g]}_\theta = \begin{cases} W^{[\mathrm{Q}]}_{(2\theta+1, g(2\theta+1))} & \text{if } \theta \le 0, \\ M^{[\mathrm{Q}]}_{(2\theta-1, f(2\theta-1))} & \text{if } \theta > 0, \end{cases} \tag{5.13}$$

and the function $\xi^{[f,g]} \colon [-1,1] \to \mathscr{C}$ defined by $\xi^{[f,g]}(\theta) = K^{[f,g]}_\theta$. The function $\xi^{[f,g]}$ is continuous, and we have $K^{[f,g]}_{-1} = W$, $K^{[f,g]}_0 = \Pi$ and $K^{[f,g]}_1 = M$, i.e., the family $\big(K^{[f,g]}_\theta\big)_{\theta\in[-1,1]}$ is comprehensive;
(iv) the family of copulas $\big(K^{[f,g]}_\theta\big)_{\theta\in[-1,1]}$ turns out to be a subfamily of the family of Fréchet copulas $\big(C^{[\mathrm{F}]}_{(a,b)}\big)_{(a,b)\in\Gamma}$ as given in (5.8) — indeed,

$$K^{[f,g]}_\theta = \begin{cases} g(2\theta+1)\Pi + (1 - g(2\theta+1))W & \text{if } \theta \le 0, \\ (f(1 - 2\theta + f(2\theta-1))\Pi + (2\theta - f(2\theta-1))M & \text{if } \theta > 0. \end{cases}$$

Example 5.8. For the special functions $f, g \in \mathscr{G}$ given by $f(x) = g(x) = \frac{x+1}{2}$ we obtain some special Fréchet copulas as given in (5.8):

$$K_\theta^{[f,g]} = \max(-\theta, 0)W + \max(\theta, 0)M + (1 - |\theta|)\Pi = C^{[\mathrm{F}]}_{(\max(-\theta,0),\max(\theta,0))}.$$

In Remark 4.2 we have seen that the family of copulas $\big(C_\theta^{[\mathrm{H}]}\big)_{\theta\in\overline{\mathbb{R}}}$ is a dependence-complete extension of the family of EFGM copulas (see (3.7)). Based on the family of copulas $\big(C^{[\mathrm{Q}]}_{(\theta,f(\theta))}\big)_{(C,f,\theta)\in\mathscr{C}\times\mathscr{G}\times[-1,1]}$ considered in Remark 5.3 (which is an extension of the family of EFGM copulas), we can introduce some other dependence-complete families of copulas.

Example 5.9. For an arbitrary and fixed copula C we put

$$C^{[\mathrm{Q}]}_{\langle 0\rangle} = C^{[\mathrm{Q}]}_{(0,0)} = C, \qquad C^{[\mathrm{Q}]}_{\langle 1\rangle} = C^{[\mathrm{Q}]}_{(1,1)}, \qquad C^{[\mathrm{Q}]}_{\langle -1\rangle} = C^{[\mathrm{Q}]}_{(-1,0)} \tag{5.14}$$

and, for each $n \in \mathbb{N}$,

$$C^{[\mathrm{Q}]}_{\langle n+1\rangle} = \Big(C^{[\mathrm{Q}]}_{\langle n\rangle}\Big)^{[\mathrm{Q}]}_{\langle 1\rangle} \qquad \text{and} \qquad C_{\langle -(n+1)\rangle} = \Big(C^{[\mathrm{Q}]}_{\langle -n\rangle}\Big)^{[\mathrm{Q}]}_{\langle -1\rangle}. \tag{5.15}$$

Denoting, as usual, the set of integers by $\mathbb{Z}$, the sequence $(C^{[\mathrm{Q}]}_{\langle n\rangle})_{n\in\mathbb{Z}}$ of copulas is monotone non-decreasing, and we have

$$C^{[\mathrm{Q}]}_{\langle -\infty\rangle} = \lim_{n\to\infty} C^{[\mathrm{Q}]}_{\langle -n\rangle} = W \qquad \text{and} \qquad C^{[\mathrm{Q}]}_{\langle \infty\rangle} = \lim_{n\to\infty} C^{[\mathrm{Q}]}_{\langle n\rangle} = M.$$

For each $r \in \mathbb{R} \setminus \mathbb{Z}$ we put

$$C^{[\mathrm{Q}]}_{\langle r\rangle} = \begin{cases} \Big(C^{[\mathrm{Q}]}_{\langle\lfloor r\rfloor\rangle}\Big)^{[\mathrm{Q}]}_{(r-\lfloor r\rfloor, f_\perp(r-\lfloor r\rfloor))} = \Big(C^{[\mathrm{Q}]}_{\langle\lfloor r\rfloor\rangle}\Big)^{[\mathrm{Q}]}_{(r-\lfloor r\rfloor, r-\lfloor r\rfloor)} & \text{if } r > 0, \\ \Big(C^{[\mathrm{Q}]}_{\langle\lceil r\rceil\rangle}\Big)^{[\mathrm{Q}]}_{(r-\lceil r\rceil, f_\perp(r-\lceil r\rceil))} = \Big(C^{[\mathrm{Q}]}_{\langle\lceil r\rceil\rangle}\Big)^{[\mathrm{Q}]}_{(r-\lceil r\rceil, 0)} & \text{if } r < 0, \end{cases} \tag{5.16}$$

where $\lfloor x\rfloor$ denotes the floor and $\lceil x\rceil$ the ceiling of $x \in \mathbb{R}$. Obviously, for each $n \in \mathbb{Z}$ we have $\lim\limits_{r\to n} C^{[\mathrm{Q}]}_{\langle r\rangle} = C^{[\mathrm{Q}]}_{\langle n\rangle}$ (as defined in (5.15)).

Remark 5.10. Following the notations of Example 5.9, for each copula $C \in \mathscr{C}$ consider the parameter set $\overline{\mathbb{R}}$ and the function $\xi_C^{[\mathrm{Q}]} : \overline{\mathbb{R}} \to \mathscr{C}$ given by $\xi_C^{[\mathrm{Q}]}(r) = C^{[\mathrm{Q}]}_{\langle r\rangle}$. Then we have:

[EX] For each $\theta \in [-1, 1]$ we have $\Pi^{[\mathrm{Q}]}_{\langle\theta\rangle} = C_\theta^{\mathbf{EFGM}}$. Therefore, the family $\big(\Pi^{[\mathrm{Q}]}_{\langle r\rangle}\big)_{r\in\overline{\mathbb{R}}}$ of copulas is a comprehensive extension of the family of EFGM copulas.

[SP] The function $\xi_C^{[\mathrm{Q}]}$ is continuous and monotone non-decreasing and, if $C \notin \{W, M\}$, even strictly increasing.

[AP] If the copula C is symmetric then for each $r \in \mathbb{R}$ the copula $C^{[\mathrm{Q}]}_{\langle r\rangle}$ is symmetric. Similarly, if the copula C is polynomial or absolutely continuous then for each $r \in \mathbb{R}$ the copula $C^{[\mathrm{Q}]}_{\langle r\rangle}$ is polynomial or absolutely continuous, respectively (see [81]).

[CP] If the copula C is NQD then for each $r \in [-\infty, 0]$ also the copula $C^{[\mathrm{Q}]}_{\langle r\rangle}$ is NQD. Analogously, if C is PQD then for each $r \in [0, \infty]$ also the copula $C^{[\mathrm{Q}]}_{\langle r\rangle}$ is PQD.

[DP] The family of copulas $\big(C^{[\mathrm{Q}]}_{\langle r\rangle}\big)_{r\in\overline{\mathbb{R}}}$ is dependence-complete. Moreover, since for each $n \in \mathbb{Z}$ the restriction $\xi_C^{[\mathrm{Q}]} \restriction_{[n-1,n]}$ of the function $\xi_C^{[\mathrm{Q}]}$ to the interval $[n-1, n]$ is linear, it follows from (2.5) and the linearity of the Lebesgue integral that for each $n \in \mathbb{Z}$ the restrictions of the functions $\varrho^{\left(\xi_C^{[\mathrm{Q}]},\overline{\mathbb{R}}\right)}$, $\beta^{\left(\xi_C^{[\mathrm{Q}]},\overline{\mathbb{R}}\right)}$ and $\gamma^{\left(\xi_C^{[\mathrm{Q}]},\overline{\mathbb{R}}\right)}$ (see (5.18)) to the interval $[n-1, n]$ are linear functions, and Lemma 4.4 implies that for each $n \in \mathbb{Z}$ the restriction of the function $\tau^{\left(\xi_C^{[\mathrm{Q}]},\overline{\mathbb{R}}\right)}$ to the interval $[n-1, n]$ is a quadratic function.

Remark 5.11. As an immediate consequence of Remark 5.10 we obtain:

(i) Since we have $M^{[Q]}_{\langle -\infty\rangle} = W$, $M^{[Q]}_{\langle -1\rangle} = \Pi$, and $M^{[Q]}_{\langle r\rangle} = M$ for all $r \geq 0$, the family of copulas $\big(M^{[Q]}_{\langle r\rangle}\big)_{r\in[-\infty,0]}$ is comprehensive and dependence-complete.
(ii) Similarly, because of $W^{[Q]}_{\langle \infty\rangle} = M$, $W^{[Q]}_{\langle 1\rangle} = \Pi$, and $W^{[Q]}_{\langle r\rangle} = W$ for all $r \leq 0$, the family of copulas $\big(W^{[Q]}_{\langle r\rangle}\big)_{r\in[0,\infty]}$ is comprehensive and dependence-complete.
(iii) Since we have $M^{[Q]}_{\langle r\rangle} = \Pi^{[Q]}_{\langle r+1\rangle}$ for each $r \leq -1$ and $W^{[Q]}_{\langle r\rangle} = \Pi^{[Q]}_{\langle r-1\rangle}$ for each $r \geq 1$, the family $\big(M^{[Q]}_{\langle r\rangle}\big)_{r\in[-\infty,0]}$ contains all negatively quadrant dependent EFGM copulas, while $\big(W^{[Q]}_{\langle r\rangle}\big)_{r\in[0,\infty]}$ contains all positively quadrant dependent EFGM copulas (see [58,67]).

Example 5.12. If we fix a continuous function $f \in \mathscr{G}$ satisfying $f(0) = 0$ and the parameter set $\overline{\mathbb{R}}$ then we can introduce another family of copulas which turns out to be a dependence-complete extension of the family of EFGM copulas: for $r \in \overline{\mathbb{R}}$ we put

$$\Pi^{[Q,f]}_r = \begin{cases} W & \text{if } r = -\infty, \\ \Pi^{[Q]}_{\langle r\rangle} & \text{if } r \in \mathbb{Z}, \\ M & \text{if } r = \infty, \\ \Big(\Pi^{[Q]}_{\langle \lfloor r\rfloor\rangle}\Big)^{[Q]}_{(r-\lfloor r\rfloor, f(r-\lfloor r\rfloor))} & \text{if } r \in \,]0,\infty[\,\setminus \mathbb{Z}, \\ \Big(\Pi^{[Q]}_{\langle \lceil r\rceil\rangle}\Big)^{[Q]}_{(r-\lceil r\rceil, f(r-\lceil r\rceil))} & \text{if } r \in \,]-\infty,0[\,\setminus \mathbb{Z}. \end{cases} \tag{5.17}$$

Using the function $\xi^{[Q,f]}_{\Pi}\colon \overline{\mathbb{R}} \to \mathscr{C}$ given by $\xi^{[Q,f]}_{\Pi}(r) = \Pi^{[Q,f]}_r$, we obtain the family of copulas $\big(\Pi^{[Q,f]}_r\big)_{r\in\overline{\mathbb{R}}}$. Note that independently of the choice of f, the subfamily $\big(\Pi^{[Q,f]}_\theta\big)_{\theta\in[-1,1]}$ of $\big(\Pi^{[Q,f]}_r\big)_{r\in\overline{\mathbb{R}}}$ coincides with the family of EFGM copulas $\big(C^{\mathbf{EFGM}}_\theta\big)_{\theta\in[-1,1]}$.

Remark 5.13. Fix a continuous function $f \in \mathscr{G}$ satisfying $f(0) = 0$. Here are some properties of the family of copulas $\big(\Pi^{[Q,f]}_r\big)_{r\in\overline{\mathbb{R}}}$ considered in Example 5.12:

[EX] As mentioned in Example 5.12, the family $\big(\Pi^{[Q,f]}_r\big)_{r\in\overline{\mathbb{R}}}$ is an extension of the family of EFGM copulas.
[SP] The function $\xi^{[Q,f]}_{\Pi}$ is continuous but not necessarily monotone.
[AP] For each $r \in \overline{\mathbb{R}}$ the copula $\Pi^{[Q,f]}_r$ is symmetric. Similarly, for each $r \in \mathbb{R}$ the copula $\Pi^{[Q,f]}_r$ is polynomial and, therefore, absolutely continuous.
[CP] The copula $\Pi^{[Q,f]}_r$ is NQD if and only if $r \in [-\infty, 0]$, and it is PQD if and only if $r \in [0,\infty]$. As a consequence of [75, Proposition 3.8], for each $r \in [-\infty, 0]$ the copula $\Pi^{[Q,f]}_r$ is ultramodular on Δ and Schur concave on Δ (and, taking into account its symmetry, even Schur concave on the whole unit square).
[DP] The family of copulas $\big(\Pi^{[Q,f]}_r\big)_{r\in\overline{\mathbb{R}}}$ is dependence-complete.

Let us come back to the copulas $\Pi^{[Q]}_{\langle r\rangle}$ which are a special case of those considered in Example 5.9.

Remark 5.14. We summarize some properties of the function $\xi^{[Q]}_{\Pi}$ given in Remark 5.10, the set of copulas $\mathscr{C}^{\langle Q,\Pi\rangle} = \{\Pi^{[Q]}_{\langle r\rangle} \mid r \in \overline{\mathbb{R}}\}$ and the family of copulas $\big(\Pi^{[Q]}_{\langle r\rangle}\big)_{r\in\overline{\mathbb{R}}}$:

[EX] As mentioned in Remark 5.10, the family of copulas $\big(\Pi^{[Q]}_{\langle r\rangle}\big)_{r\in\overline{\mathbb{R}}}$ is a comprehensive extension of the family of EFGM copulas $\big(C^{\mathbf{EFGM}}_\theta\big)_{\theta\in[-1,1]}$.
[SP] As mentioned in Remark 5.10, the function $\xi^{[Q]}_{\Pi}$ is continuous and strictly increasing. The set $\mathscr{C}^{\langle Q,\Pi\rangle}$ is a closed subset of $\mathscr{C}$, but it is not a convex set. The set $\mathscr{C}^{\langle Q,\Pi\rangle}$ is a complete bounded chain whose smallest element is $\Pi^{[Q]}_{\langle -\infty\rangle} = W$ and whose greatest element is $\Pi^{[Q]}_{\langle \infty\rangle} = M$, and the function $\xi^{[Q]}_{\Pi}$ preserves arbitrary meets and joins.

Table 1
Values of Spearman's rho, Blomqvist's beta, Gini's gamma and Kendall's tau for some members of the family of copulas $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$.

r	$\varrho(\Pi^{[Q]}_{\langle r\rangle})$	$\beta(\Pi^{[Q]}_{\langle r\rangle})$	$\gamma(\Pi^{[Q]}_{\langle r\rangle})$	$\tau(\Pi^{[Q]}_{\langle r\rangle})$	r	$\tau(\Pi^{[Q]}_{\langle r\rangle})$
1	0.333333	0.266667	0.250000	0.222222	**0.5**	0.111111
2	0.513333	0.412698	0.390625	0.349252	**1.5**	0.282823
3	0.625268	0.506863	0.483459	0.435357	**2.5**	0.389530
4	0.700777	0.573274	0.550163	0.498837	**3.5**	0.464729
5	0.754591	0.622910	0.600751	0.548105	**4.5**	0.521473
6	0.794524	0.661565	0.640601	0.587714	**5.5**	0.566213
7	0.825091	0.692604	0.672893	0.620392	**6.5**	0.602598
8	0.849078	0.718130	0.699643	0.647896	**7.5**	0.632883
9	0.868288	0.739526	0.722196	0.671418	**8.5**	0.658554
10	0.883939	0.757743	0.741490	0.691797	**9.5**	0.680634

[AP] For each $r\in\overline{\mathbb{R}}$ the copula $\Pi^{[Q]}_{\langle r\rangle}$ is a polynomial and, as a consequence, absolutely continuous.

[CP] For each $r\in[-\infty,0]$ the copula $\Pi^{[Q]}_{\langle r\rangle}$ is ultramodular and NQD, and for each $r\in[0,\infty]$ the copula $\Pi^{[Q]}_{\langle r\rangle}$ is PQD. Moreover, $\Pi^{[Q]}_{\langle r\rangle}$ is Schur concave for each $r\in\overline{\mathbb{R}}$. The family $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ is also symmetric with respect to x- and y-flipping in the following sense and radially symmetric:

$$\Pi^{[Q]}_{\langle r\rangle}=\left(\Pi^{[Q]}_{\langle -r\rangle}\right)^{x\text{flip}}=\left(\Pi^{[Q]}_{\langle -r\rangle}\right)^{y\text{flip}}=\left(\Pi^{[Q]}_{\langle r\rangle}\right)^{\text{surv}}.$$

[DP] As mentioned in Remark 5.10, the family of copulas $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ is dependence-complete. For some selected members of the family $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ the values of Spearman's rho, Gini's gamma, Blomqvist's beta and Kendall's tau are listed in Table 1 and visualized in Fig. 2.

Consider for each copula $C\in\mathscr{C}$ the function $\varrho^{\left(\xi^{[Q]}_C,\overline{\mathbb{R}}\right)}\colon\overline{\mathbb{R}}\to[-1,1]$ defined by

$$\varrho^{\left(\xi^{[Q]}_C,\overline{\mathbb{R}}\right)}(r)=\varrho\big(C^{[Q]}_{\langle r\rangle}\big) \tag{5.18}$$

and the analogously defined functions $\tau^{\left(\xi^{[Q]}_C,\overline{\mathbb{R}}\right)}$, $\beta^{\left(\xi^{[Q]}_C,\overline{\mathbb{R}}\right)}$, and $\gamma^{\left(\xi^{[Q]}_C,\overline{\mathbb{R}}\right)}$.

For $C=\Pi$ we see that all these functions describing the different dependence parameters of the family of copulas $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ are strictly increasing bijections between $\overline{\mathbb{R}}$ and $[-1,1]$. These functions are also odd, i.e., their graphs are symmetric with respect to the origin $(0,0)$, i.e., we have $\varrho\big(\Pi^{[Q]}_{\langle -r\rangle}\big)=-\varrho\big(\Pi^{[Q]}_{\langle r\rangle}\big)$ for all $r\in\overline{\mathbb{R}}$ (and analogously for τ, β and γ).

Although we have already observed in Remark 5.11(i) that the family of copulas $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ is dependence-complete, we show the values of the dependence parameters $\varrho\big(\Pi^{[Q]}_{\langle r\rangle}\big)$, $\gamma\big(\Pi^{[Q]}_{\langle r\rangle}\big)$, $\beta\big(\Pi^{[Q]}_{\langle r\rangle}\big)$, and $\tau\big(\Pi^{[Q]}_{\langle r\rangle}\big)$ for some members of the family of copulas $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ in Table 1, and partial graphs of the functions $\varrho^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$, $\gamma^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$, $\beta^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$, and $\tau^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$ in Fig. 2.

It follows from the definition of the family of copulas $(\Pi^{[Q]}_{\langle r\rangle})_{r\in\overline{\mathbb{R}}}$ given in Example 5.9 and from the formulas for the parameters $\varrho\big(\Pi^{[Q]}_{\langle r\rangle}\big)$, $\beta\big(\Pi^{[Q]}_{\langle r\rangle}\big)$ and $\gamma\big(\Pi^{[Q]}_{\langle r\rangle}\big)$ according to (2.5) that the functions $\varrho^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$, $\beta^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$ and $\gamma^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$ are piecewise linear or, more precisely, they are linear on each interval $[r-1,r]$ with $r\in\mathbb{Z}$.

On the other hand, for the same reasons the function $\tau^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$ is linear on the interval $[-1,1]$ only, i.e., for the original family of EFGM copulas, in which case $\tau^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}(r)=\frac{2}{9}r$. On any other interval $[r-1,r]$ with $r\in\mathbb{Z}\setminus\{0,1\}$, the function $\tau^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$ is quadratic. To determine these quadratic functions, the values of $\tau^{\left(\xi^{[Q]}_\Pi,\overline{\mathbb{R}}\right)}$ have also been computed in the midpoints of the respective intervals.

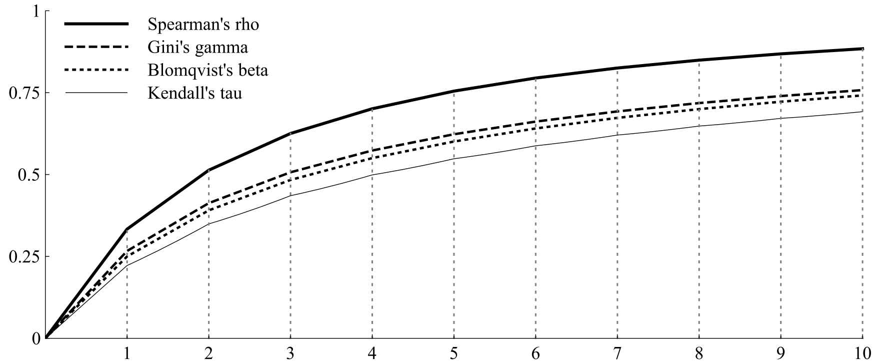


Fig. 2. Spearman's rho, Gini's gamma, Blomqvist's beta, and Kendall's tau for the subfamily $\big(\Pi^{[Q]}_{\langle r\rangle}\big)_{r\in[0,10]}$ of the family of copulas $\big(\Pi^{[Q]}_{\langle r\rangle}\big)_{r\in\overline{\mathbb{R}}}$.

6. Constructions based on some forms of convexity

Another construction method for copulas based on ultramodular and Schur concave copulas (as given in Definition 2.1) and an appropriate transformation function have recently been introduced in [75], based on some earlier results in [50].

Let us denote by Δ the upper left triangle of the unit square, i.e., $\Delta = \{(u, v) \in [0, 1]^2 \mid u \le v\}$ (see Fig. 1 (right)), and put

$$\mathscr{F} = \{f : [0, 1] \to [0, 1] \mid f \ge \mathrm{id}_{[0,1]} \text{ and } f \text{ is monotone non-decreasing and convex}\}. \tag{6.1}$$

Considering for each copula $C : [0, 1]^2 \to [0, 1]$ its dual $C^* : [0, 1]^2 \to [0, 1]$ given in (2.4), the following result is of great importance for our further investigations (see also some related work in [24,50,54,59]).

Theorem 6.1 *([75, Theorem 3.5]). Let $f : [0, 1] \to [0, 1]$ be a function and $D : [0, 1]^2 \to [0, 1]$ be a copula. If $f \in \mathscr{F}$ and D is both ultramodular and Schur concave on Δ then, for each copula $C : [0, 1]^2 \to [0, 1]$, the composite function $D(C, f(C^*)) : [0, 1]^2 \to [0, 1]$ given by*

$$D(C, f(C^*))(x, y) = D(C(x, y), f(C^*(x, y))), \tag{6.2}$$

is a copula.

Observe first that, if both copulas C and D are absolutely continuous and the function f is twice differentiable, Theorem 6.1 tells us that the composite function $D(C, f(C^*))$ given by (6.2) is also an absolutely continuous copula.

Furthermore, for the special case $f = \mathrm{id}_{[0,1]}$ we can show the following result:

Proposition 6.2. *Let $C, D : [0, 1]^2 \to [0, 1]$ be two copulas, and assume that D is ultramodular and Schur concave on Δ. Then the equality $D(C, C^*) = C$ holds if and only if $C = W$ or $D = M$.*

Proof. The sufficiency is obvious because of $W^*(x, y) = 1$ whenever $W(x, y) \ne 0$ and $C \le C^*$ for each copula C.

To show the necessity, suppose that $D(C, C^*) = C$. For each $w \in [0, 1]$ there is a pair $(x_w, y_w) \in [0, 1]^2$ such that $C(x_w, y_w) = w$, implying that

$$D(w, x_w + y_w - w) = D(C(x_w, y_w), C^*(x_w, y_w)) = C(x_w, y_w) = w.$$

Put $v_w = \inf\{x + y - w \mid C(x, y) = w\}$ and notice that v_w exists because of $C(x_w, y_w) = w$ and the continuity of C. As a consequence, for each $w \in]0, 1[$, the function $D(w, \cdot) : [w, 1] \to [0, w]$ is constant on the interval $[v_w, 1]$. Next, since D is ultramodular on Δ, the function $D(w, \cdot)$ is convex which is possible only if $v_w = 1$ or $v_w = w$. Then, because of the continuity of C, we necessarily have either $v_w = 1$ for all $w \in]0, 1[$ or $v_w = w$ for all $w \in]0, 1[$. In the latter case, $D(w, w) = w$ for all $w \in]0, 1[$ and, therefore, $D = M$. If $v_w = 1$ for all $w \in]0, 1[$ then $C(x, y) = w$ implies $x + y - w = 1$ and, subsequently, $C(x, y) = x + y - 1$ for all $w \in]0, 1[$, i.e., $C = W$, thus completing the proof. $\square$

Recall the set of functions $\mathscr{F}$ defined in (6.1) and consider the set of functions $\mathscr{H}_{[-1,0]}$ given by

$$\mathscr{H}_{[-1,0]} = \big\{\mathscr{H}\colon [-1,0] \to \mathscr{F} \;\big|\; \mathscr{H}(-1) = \mathrm{id}_{[0,1]}, \mathscr{H}(0) = \mathbf{1}_{[0,1]}, \\ \mathscr{H} \text{ is continuous and strictly increasing}\big\}, \tag{6.3}$$

where the indicator function $\mathbf{1}_A\colon X \to \mathbb{R}$ is defined by $\mathbf{1}_A(x) = 1$ whenever $x \in A$, and $\mathbf{1}_A(x) = 0$ otherwise.

Note that each function $\mathscr{H} \in \mathscr{H}_{[-1,0]}$ given by $\mathscr{H}(\theta) = f_\theta$ represents a parameterized family of functions $(f_\theta\colon [0,1] \to [0,1])_{\theta\in[-1,0]}$ which is continuous and strictly increasing with respect to the parameter θ and which satisfies $f_{-1} = \mathrm{id}_{[0,1]}$ and $f_0 = \mathbf{1}_{[0,1]}$.

Example 6.3. Fix a copula $D \neq M$ and a function $\mathscr{H} \in \mathscr{H}_{[-1,0]}$ given by $\mathscr{H}(\theta) = f_\theta$, and assume that D is both ultramodular and Schur concave on Δ.

(i) Then for each copula $C \neq W$ we can define the family $\big(C_\theta^{[\mathscr{H},D]}\big)_{\theta\in[-1,0]}$ of copulas by

$$C_\theta^{[\mathscr{H},D]} = D(C, f_\theta(C^*)) \tag{6.4}$$

which satisfies $C_0^{[\mathscr{H},D]} = C$ and $C_{-1}^{[\mathscr{H},D]} = D(C, C^*) < C$.

(ii) We can extend the family of copulas $\big(C_\theta^{[\mathscr{H},D]}\big)_{\theta\in[-1,0]}$ given by (6.4) to the family of copulas $\big(C_\theta^{[\mathscr{H},[-\infty,0],D]}\big)_{\theta\in[-\infty,0]}$ as follows: by induction, define first for each $n \in \mathbb{N} \cup \{0\}$

$$C_{-n-1}^{[\mathscr{H},[-\infty,0],D]} = \begin{cases} C_{-1}^{[\mathscr{H},D]} = D(C, C^*) & \text{if } n = 0, \\ D\Big(C_{-n}^{[\mathscr{H},[-\infty,0],D]}, \big(C_{-n}^{[\mathscr{H},[-\infty,0],D]}\big)^*\Big) & \text{if } n \in \mathbb{N}, \end{cases}$$

and then

$$C_\theta^{[\mathscr{H},[-\infty,0],D]} = \begin{cases} \big(C_{-n}^{[\mathscr{H},[-\infty,0],D]}\big)_{\theta+n}^{[\mathscr{H},[-\infty,0],D]} = D\big(C_{-n}^{[\mathscr{H},[-\infty,0],D]}, f_{\theta+n}\big(\big(C_{-n}^{[\mathscr{H},[-\infty,0],D]}\big)^*\big)\big) \\ \qquad\qquad \text{if } \theta \in \,]-n-1, -n[\,, \\ W \qquad\qquad \text{if } \theta = -\infty. \end{cases}$$

(iii) If the copula C is invariant with respect to x-flipping, i.e., if $C^{x\text{flip}} = C$, we can extend the parameter set of the family $\big(C_\theta^{[\mathscr{H},[-\infty,0],D]}\big)_{\theta\in[-\infty,0]}$ even further, from $[-\infty, 0]$ to $\overline{\mathbb{R}}$:

$$C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]} = \begin{cases} C_\theta^{[\mathscr{H},[-\infty,0],D]} & \text{if } \theta \in [-\infty, 0]\,, \\ \big(C_{-\theta}^{[\mathscr{H},[-\infty,0],D]}\big)^{x\text{flip}} & \text{if } \theta \in \,]0, \infty]\,. \end{cases}$$

(iv) If the copula C is invariant with respect to y-flipping, i.e., if $C^{y\text{flip}} = C$, we can, in full analogy to (iii), extend the parameter set of the family $\big(C_\theta^{[\mathscr{H},[-\infty,0],D]}\big)_{\theta\in[-\infty,0]}$ from $[-\infty, 0]$ to $\overline{\mathbb{R}}$:

$$C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,y]} = \begin{cases} C_\theta^{[\mathscr{H},[-\infty,0],D]} & \text{if } \theta \in [-\infty, 0]\,, \\ \big(C_{-\theta}^{[\mathscr{H},[-\infty,0],D]}\big)^{y\text{flip}} & \text{if } \theta \in \,]0, \infty]\,. \end{cases}$$

Remark 6.4. Let $C, D\colon [0,1]^2 \to [0,1]$ be two copulas with $C \neq W$ and $D \neq M$, and assume that D is both ultramodular and Schur concave on Δ. Further, let $\mathscr{H} \in \mathscr{H}_{[-1,0]}$ be a function given by $\mathscr{H}(\theta) = f_\theta$.

(i) The function $\xi\colon [-1,0] \to \mathscr{C}$ given by $\xi(\theta) = C_\theta^{[\mathscr{H},D]}$ is continuous and strictly increasing, and we have $C_0^{[\mathscr{H},D]} = C$ and $C_{-1}^{[\mathscr{H},D]} = D(C, C^*) < C$.

(ii) Clearly, the formulas in Example 6.3(ii), on the one hand, and formula (6.4), on the other hand, yield the same results for $\theta \in [-1, 0]$ in this case, i.e.,

$$C_\theta^{[\mathscr{H},[-\infty,0],D]} = C_\theta^{[\mathscr{H},[-1,0],D]}.$$

Also, the function $\xi\colon [-\infty,0] \to \mathscr{C}$ given by $\xi(\theta) = C_\theta^{[\mathscr{H},[-\infty,0],D]}$ is continuous and strictly increasing.

(iii) Assume that the copula C is invariant with respect to x-flipping, i.e., $C^{x\text{flip}} = C$. Then the function $\xi\colon \overline{\mathbb{R}} \to \mathscr{C}$ given by $\xi(\theta) = C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ (see Example 6.3(iii)) is continuous and strictly increasing, and the family $\big(C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}\big)_{\theta\in\overline{\mathbb{R}}}$ of copulas is dependence-complete. However, it is comprehensive if and only if $C = \Pi$.
If the copula C is symmetric then $C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ is also symmetric for each $\theta \in \overline{\mathbb{R}}$. As a consequence of [75, Proposition 3.8], the copula $C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ is ultramodular on Δ and Schur concave on Δ for each $\theta \in [-\infty, 0]$. If C is symmetric then for each $\theta \in [-\infty, 0]$ the copula $C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ is even Schur concave on the whole unit square $[0,1]^2$. Finally, the copula $C_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ is NQD for each $\theta \le 0$ whenever C is NQD, and it is PQD for each $\theta \ge 0$ whenever C is PQD.
Analogous conclusions can be drawn when the copula C is invariant with respect to y-flipping, i.e., if $C^{y\text{flip}} = C$.

(iv) Moreover, if the copula D is absolutely continuous and if the functions f_θ are twice differentiable for each $\theta \in\,]-1,0[$, then the copula $\Pi_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ is absolutely continuous for each $\theta \in \mathbb{R}$. If, in particular, for each $\theta \in\,]-1,0[$ the function f_θ is a polynomial, then the copula $\Pi_\theta^{[\mathscr{H},\overline{\mathbb{R}},D,x]}$ is a polynomial copula [81] for each $\theta \in \mathbb{R}$.

Example 6.5. We keep the notations of Example 6.3 and restrict ourselves to the special case $C = D = \Pi$.

(i) Consider the function $\mathscr{H}^{(\text{lin})}\colon [-1,0] \to \mathscr{F}$ given by $\mathscr{H}^{(\text{lin})}(\theta) = f_\theta^{(\text{lin})}$, where the linear function $f_\theta^{(\text{lin})}\colon [0,1] \to [0,1]$ is given by $f_\theta^{(\text{lin})}(x) = 1 + \theta - \theta x$. Then for each $\theta \in [-1,0]$ the copula $\Pi_\theta^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ coincides with the EFGM copula $C_\theta^{\mathbf{EFGM}}$: for each $(x,y) \in [0,1]^2$ we get

$$\Pi_\theta^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}(x,y) = \Pi\big(\Pi(x,y), f_\theta^{(\text{lin})}(\Pi^*(x,y))\big) = xy(1+\theta(1-x-y+xy)) = C_\theta^{\mathbf{EFGM}}(x,y).$$

(ii) Recalling Example 5.9, we obtain in a similar way $\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} = \Pi_{\langle r\rangle}^{[\mathrm{Q}]}$ for each $r \in \overline{\mathbb{R}}$ and, as a consequence, for each $\theta \in [-1,1]$

$$\Pi_\theta^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} = \Pi_{\langle\theta\rangle}^{[\mathrm{Q}]} = C_\theta^{\mathbf{EFGM}}.$$

Remark 6.6. Let us summarize some of the properties of the set of copulas $\mathscr{C}_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} = \big\{\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} \mid r \in \overline{\mathbb{R}}\big\}$, the function $\xi^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}\colon \overline{\mathbb{R}} \to \mathscr{C}$ given by $\xi^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}(r) = \Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ and the family of copulas $\big(\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}\big)_{r\in\overline{\mathbb{R}}}$.

[EX] As a consequence of Example 6.5, the family $\big(\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}\big)_{r\in\overline{\mathbb{R}}}$ is an extension of the family of EFGM copulas. It is also comprehensive because of $\Pi_{-\infty}^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} = W$, $\Pi_0^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} = \Pi$ and $\Pi_\infty^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]} = M$.

[SP] The function $\xi^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ is continuous and strictly increasing because of Remark 6.4(iii).

[AP] Because of Remark 6.4(iii), the copula $\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ is symmetric for each $r \in \overline{\mathbb{R}}$ and, for each $r \in \mathbb{R}$, the copula $\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ is a polynomial and, therefore, absolutely continuous.

[CP] Because of Remark 6.4(iii), for each $r \in [-\infty,0]$ the copula $\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ is ultramodular on Δ and, due to its symmetry, Schur concave. As a consequence of Remark 6.4(iii), a copula $\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}$ is NQD if and only if $r \in [-\infty,0]$, and PQD if and only if $r \in [0,\infty]$.

[DP] The family $\big(\Pi_r^{[\mathscr{H}^{(\text{lin})},\overline{\mathbb{R}},\Pi]}\big)_{r\in\overline{\mathbb{R}}}$ is dependence-complete because of Remark 6.4(iii).

Example 6.7. We continue keeping the notations of Examples 6.3 and 6.5 and consider the function $\mathscr{H}^{(\mathrm{qu})}\colon [-1,0] \to \mathscr{F}$ given by $\mathscr{H}^{(\mathrm{qu})}(\theta) = f_\theta^{(\mathrm{qu})}$, where the functions $f_\theta^{(\mathrm{qu})}\colon [0,1] \to [0,1]$ are constructed by means of polynomial functions of degree 2 as follows:

$$f_\theta^{(\mathrm{qu})}(x) = \begin{cases} (1+\theta)(1+x^2) - (1+2\theta)x & \text{if } \theta \in \left[-1, -\frac{1}{2}\right], \\ 1+\theta-\theta x^2 & \text{if } \theta \in \left]-\frac{1}{2}, 0\right]. \end{cases}$$

Then we obtain $\Pi_n^{[\mathscr{H}^{(\mathrm{qu})},\overline{\mathbb{R}},\Pi]} = \Pi_n^{[\mathscr{H}^{(\mathrm{lin})},\overline{\mathbb{R}},\Pi]}$ for each integer $n \in \mathbb{Z}$. However, for $\theta = -0.5$ we get

$$\Pi_{-0.5}^{[\mathscr{H}^{(\mathrm{qu})},\overline{\mathbb{R}},\Pi]}(0.5, 0.5) = 0.195313 \neq 0.21875 = \Pi_{-0.5}^{[\mathscr{H}^{(\mathrm{lin})},\overline{\mathbb{R}},\Pi]}(0.5, 0.5),$$

i.e., $\Pi_{-0.5}^{\left[\mathscr{H}^{(\mathrm{qu})},\overline{\mathbb{R}},\Pi\right]} \neq \Pi_{-0.5}^{\left[\mathscr{H}^{(\mathrm{lin})},\overline{\mathbb{R}},\Pi\right]}$. In a similar way it can be shown that we have $\Pi_\theta^{\left[\mathscr{H}^{(\mathrm{qu})},\overline{\mathbb{R}},\Pi\right]} \neq \Pi_\theta^{\left[\mathscr{H}^{(\mathrm{lin})},\overline{\mathbb{R}},\Pi\right]}$ for each $\theta \in \mathbb{R} \setminus \mathbb{Z}$.

7. Polynomial copulas

As already mentioned several times, EFGM copulas are special polynomials (of degree 4 or less) restricted to the unit square $[0,1]^2$. Expanding this idea, a possible way is to replace the parameter θ in the definition of EFGM copulas given in (3.1) by a suitable polynomial $p\colon \mathbb{R}^2 \to \mathbb{R}$ in two variables, i.e., to consider functions $C^{[\mathrm{poly}]}\colon [0,1]^2 \to [0,1]$ of the form

$$C^{[\mathrm{poly}]}(x, y) = xy + p(x, y)x(1-x)y(1-y). \tag{7.1}$$

It is easy to see that not for each polynomial $p\colon \mathbb{R}^2 \to \mathbb{R}$ the function $C^{[\mathrm{poly}]}$ given by (7.1) is a copula: a trivial counterexample is a constant polynomial $p(x, y) = \theta$ with $\theta \in \mathbb{R} \setminus [-1, 1]$ in which case the function $C^{[\mathrm{poly}]}$ violates (2.2), i.e., C is not 2-increasing and, therefore, not a copula.

Observe that the functions $C^{[\mathrm{poly}]}$ given by (7.1) are related to the functions $C_h\colon [0,1]^2 \to \mathbb{R}$ given by

$$C_h(x, y) = xy + h(x, y),$$

where $h : [0,1]^2 \to \mathbb{R}$ is a suitable function (not necessarily a polynomial in two variables). Copulas of this type have been studied in [49,62,73,76]. The functions $C^{[\mathrm{poly}]}$ are also related to the functions $C_{(\phi,\theta)}\colon [0,1]^2 \to \mathbb{R}$ defined by

$$C_{(\phi,\theta)}(x, y) = xy + \theta\phi(x)\phi(y)$$

for some $\theta \in \mathbb{R}$ and some function $\phi\colon [0,1] \to [0,1]$, as considered in [1,2], and to the functions $H_\alpha^{(1)}, H_\alpha^{(2)}\colon [0,1]^2 \to \mathbb{R}$ given by

$$H_\alpha^{(1)}(x, y) = xy(1 + \alpha(1-x^p)(1-y^p)), \qquad H_\alpha^{(2)}(x, y) = xy(1 + \alpha(1-x)^p(1-y)^p),$$

where $\alpha \in \mathbb{R}$ and $p > 0$, which were investigated in [41,42] (see also [7,8], and for some early traces [45,46,78]). Under suitable assumptions, all these functions turn out to be (polynomial) copulas. Also copulas with quadratic sections [72] and cubic sections [68] are related to polynomial copulas.

Let us denote the set of polynomial copulas by $\mathscr{C}^{[\mathrm{poly}]}$. In [88] it was shown that each polynomial copula $C \in \mathscr{C}^{[\mathrm{poly}]}$ necessarily has the form $C = C^{[\mathrm{poly}]}$ given in (7.1) for some suitable polynomial $p\colon \mathbb{R}^2 \to \mathbb{R}$ in two variables.

Note that in the monograph [25] (using results from [74]) an alternative formula for polynomial copulas is given (which has been cited in [66, Subsection 4.4]). As shown in [81, Example 4.1], in general the conditions given in [66] are neither sufficient nor necessary for the functions considered there being copulas.

A polynomial $p\colon \mathbb{R}^2 \to \mathbb{R}$ in two variables given by

$$p(x, y) = \sum_{k=0}^{n} \sum_{i=0}^{k} a_{k,i}\, x^{k-i}\, y^i,$$

with coefficients $a_{0,0}, a_{1,0}, a_{0,1}, \ldots, a_{n,0}, a_{n,1}, \ldots, a_{n,n} \in \mathbb{R}$ and where $n \in \mathbb{N} \cup \{0\}$, is said to be of degree n. It is called an improper polynomial of degree n whenever $a_{n,0} = a_{n,1} = \cdots = a_{n,n} = 0$, and a proper polynomial of degree

n otherwise. For each improper polynomial $p\colon \mathbb{R}^2 \to \mathbb{R}$ of degree $n \in \mathbb{N}$, which is different from the zero polynomial, there is a number $m \in \{0, 1, \dots, n-1\}$ such that p is a proper polynomial of degree m.

Since each polynomial copula is of the form (7.1), it follows immediately that the product copula Π is the only polynomial copula of degree 2 (in which case p is the zero polynomial) and that there exists no proper polynomial of degree 3. The only polynomial copulas of degree 4 are the EFGM copulas (in which case $p(x, y) = \theta$ for some $\theta \in [-1, 1]$), and all EFGM copulas $C_\theta^{\mathbf{EFGM}}$ with $\theta \neq 0$ are proper polynomial copulas of degree 4.

If $n \in \mathbb{N} \setminus \{1, 3\}$ we shall write **PCn** for a parameter set of the family of polynomial copulas of degree n and

$$\mathscr{C}_{[n]}^{[\mathrm{poly}]} = \{C \in \mathscr{C} \mid C \text{ is a polynomial copula of degree } n\}.$$

It is obvious that $\mathbf{PC2} = \{0\}$ is a parameter set of the trivial family of polynomial copulas of degree 2 (consisting only of the product copula Π) and that $\mathbf{PC4} = [-1, 1]$ is a parameter set of the family of polynomial copulas of degree 4 (i.e., of the family of EFGM copulas). Generally, for $n \geq 4$ the parameter set **PCn** of the family of polynomial copulas of degree n is a subset of $\mathbb{R}^d$, where $d = \binom{n-2}{2}$.

In [81] the polynomial copulas of degree 5 have been studied in detail. To the best of our knowledge, no full characterization of the set of polynomial copulas of degree n, with $n \geq 6$, exists so far.

It is obvious that each polynomial function in two variables of degree 5 satisfying the boundary conditions (2.1) has the form

$$C_{(a,b,c)}(x, y) = xy + (ax + by + c)x(1-x)y(1-y) \tag{7.2}$$

for some triplet $(a, b, c) \in \mathbb{R}^3$. Obviously, each $C_{(a,b,c)}$ given by (7.2) is absolutely continuous, and its density $\varphi_{C_{(a,b,c)}}\colon [0, 1]^2 \to \mathbb{R}$ is given by

$$\varphi_{C_{(a,b,c)}}(x, y) = 1 + a(x - x^2)(1 - 2y) + b(1 - 2x)(y - y^2) + (ax + by + c)(1 - 2x)(1 - 2y). \tag{7.3}$$

Then, according to [81, Corollary 3.1], the function $C_{(a,b,c)}$ satisfies (2.2), i.e., it is 2-increasing and, therefore, a copula, if and only if $\varphi_{C_{(a,b,c)}}(x, y) \geq 0$ for all $(x, y) \in [0, 1]^2$. The set $\mathscr{C}_{[5]}^{[\mathrm{poly}]}$ of polynomial copulas of degree 5 is closed as a consequence of [28, Theorem 1.7.5], and we have $\mathscr{C}^{\mathbf{EFGM}} \subset \mathscr{C}_{[5]}^{[\mathrm{poly}]}$ because of $C_\theta^{\mathbf{EFGM}} = C_{(0,0,\theta)}$ and $C_{(0,0,\theta)} \in \mathscr{C}_{[5]}^{[\mathrm{poly}]}$ for each $\theta \in [-1, 1]$.

To determine the parameter set **PC5** of the family of polynomial copulas of degree 5 it is necessary to find all parameters $(a, b, c) \in \mathbb{R}^3$ such that the density $\varphi_{C_{(a,b,c)}}$ is non-negative, i.e., for all $(x, y) \in [0, 1]^2$

$$1 + a(x - x^2)(1 - 2y) + b(1 - 2x)(y - y^2) + (ax + by + c)(1 - 2x)(1 - 2y) \geq 0. \tag{7.4}$$

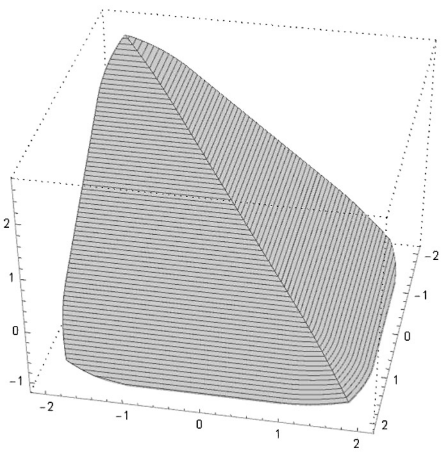
This can be seen as a quantifier elimination problem which, in [92], was shown to be decidable using computer algebra methods such as cylindrical algebraic decomposition. Collins [17] suggested a more efficient algorithm which is now implemented in Mathematica and elsewhere [13,80,89,90]. Additional details of Collins' algorithm are described in [9,16,47], and the way how to obtain the solution of our concrete inequality (7.4) using Mathematica (in particular the command `GenericCylindricalDecomposition`) is outlined in [81]. The results of these computations obtained in [81] are summarized in Appendix A, and a visualization of **PC5** using 3D plots is given in Fig. 3.

Remark 7.1. Let us summarize some of the properties of the parameter set **PC5**, the function $\xi_{[5]}^{[\mathrm{poly}]}\colon \mathbf{PC5} \to \mathscr{C}$ defined by $\xi_{[5]}^{[\mathrm{poly}]}(x, y, z) = C_{(x,y,z)}$, the set $\mathscr{C}_{[5]}^{[\mathrm{poly}]} = \{C_{(a,b,c)} \mid (a, b, c) \in \mathbf{PC5}\}$ of polynomials of degree 5, and the family $\big(C_{(a,b,c)}\big)_{(a,b,c)\in\mathbf{PC5}}$ of polynomials of degree 5:

[EX] By construction, the family $\big(C_{(a,b,c)}\big)_{(a,b,c)\in\mathbf{PC5}}$ is an extension of the family of EFGM copulas, and we have $\big(C_\theta^{\mathbf{EFGM}}\big)_{\theta\in[-1,1]} = \big(C_{(0,0,\theta)}\big)_{\theta\in[-1,1]}$.

[SP] The parameter set **PC5** is a closed convex subset of $\mathbb{R}^3$, and the set $\mathscr{C}_{[5]}^{[\mathrm{poly}]}$ is a closed convex subset of $\mathscr{C}$. Moreover, the function $\xi_{[5]}^{[\mathrm{poly}]}$ is continuous and it preserves convex combinations.

[AP] For each $(a, b, c) \in \mathbf{PC5}$ the copula $C_{(a,b,c)}$ is polynomial by construction and, therefore, absolutely continuous. A copula $C_{(a,b,c)} \in \mathscr{C}_{[5]}^{[\mathrm{poly}]}$ is symmetric if and only if $a = b$.



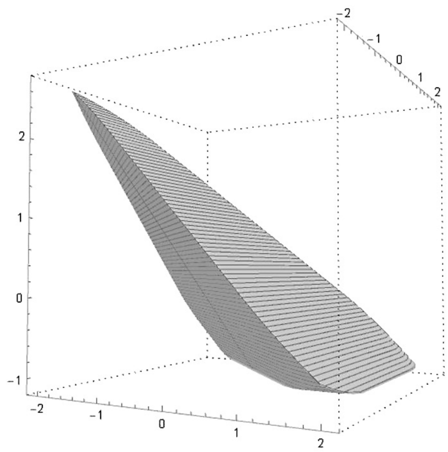


Fig. 3. 3D plots of the parameter set **PC5** from two different viewpoints (see [81] for more details).

[CP] For each subset A of $\mathbb{R}^3$, let $\mathrm{Conv}(A)$ denote the convex hull of A. Then we obtain for each $C_{(a,b,c)} \in \mathscr{C}_{[5]}^{[\mathrm{poly}]}$:

(i) $C_{(a,b,c)}$ is ultramodular if and only if (see [81, Proposition 5.3])

$$(a,b,c) \in \mathrm{Conv}\Big(\big\{(0,0,0), (-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}), (-\tfrac{1}{2}, 0, -\tfrac{1}{2}), (0, -\tfrac{1}{2}, -\tfrac{1}{2}), \\ (\tfrac{1}{2}, -\tfrac{1}{2}, -1), (-\tfrac{1}{2}, \tfrac{1}{2}, -1), (\tfrac{1}{2}, 0, -1), (0, \tfrac{1}{2}, -1), (\tfrac{1}{3}, \tfrac{1}{3}, -1)\big\}\Big).$$

(ii) $C_{(a,b,c)}$ is Schur concave if and only if $C_{(a,b,c)}$ is symmetric (see [81, Proposition 5.2]), i.e., $a = b$.

(iii) $C_{(a,b,c)}$ is NQD if and only if (see [81, Proposition 5.5])

$$(a,b,c) \in \mathrm{Conv}(\{(0,0,0), (0,-1,0), (-1,0,0), (1,-1,-1), (1,0,-1), (0,1,-1), (-1,1,-1)\}).$$

(iv) $C_{(a,b,c)}$ is PQD if and only if (see [81, Proposition 5.4])

$$(a,b,c) \in \mathrm{Conv}(\{(-1,-1,2), (-1,0,1), (0,-1,1), (0,0,0), (0,1,0), (1,1,0), (1,0,0)\}).$$

For the x-flipping, the y-flipping and the survival copula of a copula $C_{(a,b,c)} \in \mathscr{C}_{[5]}^{[\mathrm{poly}]}$ we get (compare [81, (5.17) and Corollaries 5.6, 5.7]):

$$(C_{(a,b,c)})^{x\mathrm{flip}} = C_{(a,-b,-a-c)}, \qquad (C_{(a,b,c)})^{y\mathrm{flip}} = C_{(-a,b,-b-c)}, \qquad (C_{(a,b,c)})^{\mathrm{surv}} = C_{(-a,-b,a+b+c)}.$$

[DP] For each polynomial copula of degree 5, i.e., for each triplet $(a,b,c) \in \mathbf{PC5}$, we get the following formulas for Spearman's rho, Kendall's tau, Blomqvist's beta and Gini's gamma (see [81, Corollary 6.2])

$$\varrho\big(C_{(a,b,c)}\big) = \frac{1}{6}(a+b+2c), \qquad \tau\big(C_{(a,b,c)}\big) = \frac{1}{9}(a+b+2c) - \frac{1}{450}ab,$$
$$\beta\big(C_{(a,b,c)}\big) = \frac{1}{8}(a+b+2c), \qquad \gamma\big(C_{(a,b,c)}\big) = \frac{2}{15}(a+b+2c).$$

As shown in [81, Proposition 6.3], we obtain the same ranges for these dependence parameters of polynomial copulas of degree 5 as in the case of EFGM copulas:

$$\big\{\varrho\big(C_{(a,b,c)}\big) \mid (a,b,c) \in \mathbf{PC5}\big\} = \left[-\frac{1}{3}, \frac{1}{3}\right], \qquad \big\{\tau\big(C_{(a,b,c)}\big) \mid (a,b,c) \in \mathbf{PC5}\big\} = \left[-\frac{2}{9}, \frac{2}{9}\right],$$
$$\big\{\beta\big(C_{(a,b,c)}\big) \mid (a,b,c) \in \mathbf{PC5}\big\} = \left[-\frac{1}{4}, \frac{1}{4}\right], \qquad \big\{\gamma\big(C_{(a,b,c)} \mid (a,b,c) \in \mathbf{PC5}\big\} = \left[-\frac{4}{15}, \frac{4}{15}\right].$$

According to [41], Spearman's rho cannot attain the values ± 1 for polynomial copulas with a fixed degree. Note that for polynomial copulas with arbitrary degree it was shown in [81, Proposition 6.4] that our four dependence parameters have a full range, i.e.,

$$\{\varrho(C) \mid C \in \mathscr{C}^{[\mathrm{poly}]}\} = \{\tau(C) \mid C \in \mathscr{C}^{[\mathrm{poly}]}\} = \{\beta(C) \mid C \in \mathscr{C}^{[\mathrm{poly}]}\} = \{\gamma(C) \mid C \in \mathscr{C}^{[\mathrm{poly}]}\} = \,]-1, 1[\,.$$

8. Concluding remarks

We have considered some well-known extensions of the family of EFGM copulas (as proposed in [43] and others which are based on perturbations), and also some more recent ones (based on quadratic constructions of copulas or on some forms of convexity, and polynomial copulas).

In all these cases we have analyzed how these extensions change the original properties [EFGM1]–[EFGM4] of the family of EFGM copulas, showing that this impact varies from one extension to the other.

As our survey is definitely not complete, it could be interesting to consider other extensions of EFGM copulas and additional properties (e.g., related to the dependence parameters).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Open access funding provided by Johannes Kepler University Linz. S. Saminger-Platz and E. P. Klement were supported by the “Technologie-Transfer-Förderung” of the Upper Austrian Government (Wi-2014-2007 10/13-Kx/Kai). A. Kolesárová, R. Mesiar and A. Šeliga acknowledge the support of the Slovak grants VEGA 1/0614/18, VEGA 1/0891/17 and APVV-18-0052.

We thank the three anonymous reviewers for taking their time for a careful check of the original submission and for providing a number of thoughtful comments and suggestions which helped the authors to improve an earlier version of the manuscript.

Appendix A. Non-negative density of polynomial copulas of degree five

The solution of the inequality (7.4) mentioned in Section 7 is given in the form of a full characterization of the level sets of the parameter set **PC5** (a more detailed description how to find this solution was presented in [81]).

Consider for a fixed number $c_0 \in \mathbb{R}$ the level set $\mathbf{PC5}^{[c_0]}$ of the parameter set **PC5** given by

$$\mathbf{PC5}^{[c_0]} = \{(a, b, c) \in \mathbf{PC5} \mid c = c_0\}.$$

Observe that $\mathbf{PC5}^{[c]} \neq \emptyset$ if and only if $c \in \left[-1, 1 + 2\sqrt{\frac{2}{3}}\right]$. Since **PC5** is a convex and closed subset of $\mathbb{R}^3$ it follows that each level set $\mathbf{PC5}^{[c]}$ is also a convex and closed subset of $\mathbb{R}^3$. Each level set $\mathbf{PC5}^{[c]}$ of the parameter set **PC5** will be fully characterized by the following data for the triplets $(a, b, c) \in \mathbf{PC5}^{[c]}$:

(i) the range of c in each of the five cases (I)–(V) to be distinguished,
(ii) the corresponding range of a (depending on c),
(iii) the corresponding lower and the upper bound for b (depending on c and a).

The solution of inequality (7.4) is summarized below where first a survey of the five cases for the values c is given which have to be distinguished, together the respective ranges of the value a (the latter depending on c). Finally, the complete formulas for the lower and upper bounds of b (provided that c and a are given) in each of these five cases are listed.

Solution. *For* $c \in \left[-1, 1+2\sqrt{\frac{2}{3}}\right]$ *we have to distinguish five cases* (I)–(V). *In each of these cases we first specify the corresponding interval for the values* a *(depending on* c*) and* b *(depending on* c *and* a*) such that* $(a, b, c) \in \mathbf{PC5}^{[c]}$.

Case (I)*: for each* $c \in \left[-1, -\frac{6}{7}\right[$ *we get*

$$\frac{-3-c-\sqrt{12-3(c-1)^2}}{2} \leq a \leq 1-c.$$

Case (II)*: for each* $c \in \left[-\frac{6}{7}, -\sqrt{\frac{2}{3}}\right[$ *we get*

$$\frac{-3-c-\sqrt{12-3(c-1)^2}}{2} \leq a \leq 1-c.$$

Case (III)*: for each* $c \in \left[-\sqrt{\frac{2}{3}}, 0\right[$ *we get*

$$\frac{-1-c-\sqrt{12-3(c+1)^2}}{2} \leq a \leq 1-c.$$

Case (IV)*: for each* $c \in [0, 2[$ *we get*

$$\frac{1-c-\sqrt{12-3(c-1)^2}}{2} \leq a \leq \frac{-3-c+\sqrt{12-3(c-1)^2}}{2}.$$

Case (V)*: for each* $c \in \left[2, 1+2\sqrt{\frac{2}{3}}\right]$ *we get*

$$\frac{1-c-\sqrt{12-3(c-1)^2}}{2} \leq a \leq \frac{-3-c+\sqrt{12-3(c-1)^2}}{2}.$$

For each case $(\mathrm{N}) \in \{(\mathrm{I}), (\mathrm{II}), (\mathrm{III}), (\mathrm{IV}), (\mathrm{V})\}$ *specified above we now give the complete formulas for the lower bounds* $\mathrm{LB}^{[c]}_{(\mathrm{N})}(a)$ *and the upper bounds* $\mathrm{UB}^{[c]}_{(\mathrm{N})}(a)$ *of the value* b *(depending on* c *and* a*), i.e.,*

$$\mathrm{LB}^{[c]}_{(\mathrm{N})}(a) \leq b \leq \mathrm{UB}^{[c]}_{(\mathrm{N})}(a),$$

such that $(a, b, c) \in \mathbf{PC5}^{[c]}$ *— compare the original publication in Tables* 1–2 *in* [81].

(I) *For* $c \in \left[-1, -\frac{6}{7}\right[$*, the lower and upper bounds of the parameter* b *(depending on* c *and* a*) are specified by the functions*

$$\mathrm{LB}^{[c]}_{(\mathrm{I})}, \mathrm{UB}^{[c]}_{(\mathrm{I})} \colon \left[\tfrac{-3-c-\sqrt{12-3(c-1)^2}}{2}, 1-c\right] \to \mathbb{R}$$

defined by

$$\mathrm{LB}^{[c]}_{(\mathrm{I})}(a) = \begin{cases} \frac{3-a-c-\sqrt{12-3(a+c+1)^2}}{2} & \text{if } \frac{-3-c-\sqrt{12-3(c-1)^2}}{2} \leq a < -2-c, \\ -1-a-c & \text{if } -2-c \leq a < 1, \\ \frac{-a-2c-\sqrt{12-3(a-2)^2}}{2} & \text{if } 1 \leq a < \frac{3+c+\sqrt{12-3(c-1)^2}}{2}, \\ \frac{-3-c-\sqrt{12-3(c-1)^2}}{2} & \text{otherwise}, \end{cases}$$

$$\mathrm{UB}^{[c]}_{(\mathrm{I})}(a) = \begin{cases} 1-c & \text{if } \frac{-3-c-\sqrt{12-3(c-1)^2}}{2} \leq a < 1, \\ \frac{-a-2c+\sqrt{12-3(a-2)^2}}{2} & \text{if } 1 \leq a < \frac{3-3c+\sqrt{36-3(c+3)^2}}{6}, \\ \frac{3-a-c+\sqrt{12-3(a+c+1)^2}}{2} & \text{otherwise}. \end{cases}$$

(II) *For* $c \in \left[-\frac{6}{7}, -\sqrt{\frac{2}{3}}\right[$, *the lower and upper bounds of the parameter* b *(depending on* c *and* a*) are specified by the functions*

$$\mathrm{LB}_{(\mathrm{II})}^{[c]}, \mathrm{UB}_{(\mathrm{II})}^{[c]} \colon \left[\tfrac{-3-c-\sqrt{12-3(c-1)^2}}{2}, 1-c\right] \to \mathbb{R}$$

defined by

$$\mathrm{LB}_{(\mathrm{II})}^{[c]}(a) = \begin{cases} \frac{3-a-c-\sqrt{12-3(a+c+1)^2}}{2} & \text{if } \frac{-3-c-\sqrt{12-3(c-1)^2}}{2} \le a < -2-c, \\ -1-a-c & \text{if } -2-c \le a < 1, \\ \frac{-a-2c-\sqrt{12-3(a-2)^2}}{2} & \text{if } 1 \le a < \frac{3+c+\sqrt{12-3(c-1)^2}}{2}, \\ \frac{-3-c-\sqrt{12-3(c-1)^2}}{2} & \text{otherwise}, \end{cases}$$

$$\mathrm{UB}_{(\mathrm{II})}^{[c]}(a) = \begin{cases} 1-c & \text{if } \frac{-3-c-\sqrt{12-3(c-1)^2}}{2} \le a < 1, \\ \frac{-a-2c+\sqrt{12-3(a-2)^2}}{2} & \text{if } 1 \le a < \frac{3-3c+\sqrt{36-3(c+3)^2}}{6}, \\ \frac{3-a-c+\sqrt{12-3(a+c+1)^2}}{2} & \text{otherwise}. \end{cases}$$

(III) *For* $c \in \left[-\sqrt{\frac{2}{3}}, 0\right[$, *the lower and upper bounds of the parameter* b *(depending on* c *and* a*) are specified by the functions*

$$\mathrm{LB}_{(\mathrm{III})}^{[c]}, \mathrm{UB}_{(\mathrm{III})}^{[c]} \colon \left[\tfrac{-1-c-\sqrt{12-3(c+1)^2}}{2}, 1-c\right] \to \mathbb{R}$$

defined by

$$\mathrm{LB}_{(\mathrm{III})}^{[c]}(a) = \begin{cases} \frac{3-a-c-\sqrt{12-3(a+c+1)^2}}{2} & \text{if } \frac{-1-c-\sqrt{12-3(c+1)^2}}{2} \le a < -2-c, \\ -1-a-c & \text{if } -2-c \le a < 1, \\ \frac{-a-2c-\sqrt{12-3(a-2)^2}}{2} & \text{otherwise}, \end{cases}$$

$$\mathrm{UB}_{(\mathrm{III})}^{[c]}(a) = \begin{cases} 1-c & \text{if } \frac{-1-c-\sqrt{12-3(c+1)^2}}{2} \le a < 1, \\ \frac{-a-2c+\sqrt{12-3(a-2)^2}}{2} & \text{if } 1 \le a < \frac{3-3c+\sqrt{36-3(c+3)^2}}{6}, \\ \frac{3-a-c+\sqrt{12-3(a+c+1)^2}}{2} & \text{otherwise}. \end{cases}$$

(IV) *For* $c \in [0, 2[$, *the lower and upper bounds of the parameter* b *(depending on* c *and* a*) are specified by the functions*

$$\mathrm{LB}_{(\mathrm{IV})}^{[c]}, \mathrm{UB}_{(\mathrm{IV})}^{[c]} \colon [-2, 1-c] \to \mathbb{R}$$

defined by

$$\mathrm{LB}_{(\mathrm{IV})}^{[c]}(a) = -1-a-c, \qquad \mathrm{UB}_{(\mathrm{IV})}^{[c]}(a) = 1-c.$$

(V) *For* $c \in \left[2, 1+2\sqrt{\frac{2}{3}}\right[$, *the lower and upper bounds of the parameter* b *(depending on* c *and* a*) are specified by the functions*

$$\mathrm{LB}_{(\mathrm{V})}^{[c]}, \mathrm{UB}_{(\mathrm{V})}^{[c]} \colon \left[\tfrac{1-c-\sqrt{12-3(c-1)^2}}{2}, \tfrac{-3-c+\sqrt{12-3(c-1)^2}}{2}\right] \to \mathbb{R}$$

defined by

$$\mathrm{LB}_{(\mathrm{V})}^{[c]}(a) = -1-a-c, \qquad \mathrm{UB}_{(\mathrm{V})}^{[c]}(a) = \frac{-3-c+\sqrt{12-3(c-1)^2}}{2}.$$

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