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Generalized convergence theorems for monotone measures

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Abstract

In this paper, we propose three types of absolute continuity for monotone measures and present some of their basic properties. By means of these three types of absolute continuity, we establish generalized Egoroff's theorem, generalized Riesz's theorem and generalized Lebesgue's theorem in the framework involving the ordered pair of monotone measures. The Egoroff theorem, the Riesz theorem and the Lebesgue theorem in the traditional sense concerning a unique monotone measure are extended to the general case. These three generalized convergence theorems include as special cases several previous versions of Egoroff-like theorem, Riesz-like theorem and Lebesgue-like theorem for monotone measures, respectively.

Keywords: Non-additive measure; Absolute continuity; Egoroff's theorem; Riesz's theorem; Lebesgue's theorem

1. Introduction

In classical measure theory, there are several important convergence theorems for a sequence of measurable functions, such as the Egoroff theorem, the Riesz theorem, the Lebesgue theorem, etc. These theorems describe implications between three convergence concepts: almost everywhere convergence, almost uniform convergence, and convergence in measure for a sequence of real-valued measurable functions. We summarize these well-known results [1]: (1) Egoroff's theorem asserts that almost everywhere convergence implies almost uniform convergence on a measurable set of finite measure. (2) Riesz's theorem states that each sequence of measurable functions which converges in measure has a subsequence converging almost everywhere. (3) Lebesgue's theorem affirms that almost everywhere convergence implies convergence in measure on a measurable set of finite measure (sometimes the combination of (2) and (3) is also called F. Riesz–Lebesgue theorem, see [2]).

In general, many classical results are no longer valid when we move from σ-additive measures to general monotone measures. The above mentioned three theorems have been effectively extended in non-additive measure theory by

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using a variety of structural characteristics of set functions. These results were summarized in detail in [3]. For more details, we recommend [4–17].

We recall three typical results generalizing Egoroff's theorem, Riesz's theorem and Lebesgue's theorem from classical measure theory to monotone measure theory: (a) In [7] the concept of *condition* [**E**] of a monotone measure was introduced and it was proved that Egoroff's theorem holds in the case of monotone measures if and only if the monotone measures fulfil condition [**E**] (see also [8]). (b) In [6] the concept of *property* [**S**] of a monotone measure was introduced and it was shown that, in the context of monotone measures, the conclusion of the classical Riesz's theorem holds if and only if the monotone measures possess property [**S**]. (c) In [9] it was shown that the *strong order continuity* is not only a necessary, but also a sufficient condition for Lebesgue's theorem to hold in the case of monotone measures (see also [11]). From the above three statements we see that the condition [**E**], property [**S**] and strong order continuity of monotone measures are very important structural characteristics in non-additive measure theory.

As is well known, the monotone measures lose additivity, therefore the three convergence concepts we considered have so-called "pseudo-" variants, respectively: "pseudo-almost everywhere convergence", "pseudo-almost uniform convergence" and "convergence pseudo-in measure" (denoted by convergence *p.a.e.*, *p.a.u.* and *in* $p.\mu$, respectively). Thus, Egoroff's theorem, Riesz's theorem, Lebesgue's theorem, etc., each of them takes four different forms in the case of monotone measures (see [3,4,18]). The above three typical results (a), (b), and (c), which only concern convergence $a.e.$, $a.u.$, and *in measure*, are referred to as the standard-forms of convergence theorems. The other three pseudo-versions were established in the context of (pseudo-)convergence (see [3,4,18]).

When we consider the convergence and pseudo-convergence on the whole universe X and the monotone measure μ is finite (i.e., $\mu(X) < \infty$), then the convergence *p.a.e.*$[\mu]$ (resp. *p.a.u.*$[\mu]$ or *in* $p.\mu$) is equivalent to the convergence *a.e.*$[\overline{\mu}]$ (resp. *a.u.*$[\overline{\mu}]$ and *in* $\overline{\mu}$), where $\overline{\mu}$ is the conjugate of monotone measure μ. Thus, the pseudo-versions of convergence theorems involve two different monotone measures μ and $\overline{\mu}$ (although $\overline{\mu}$ is induced by μ). For instance, we recall a pseudo-form of Egoroff's theorem [3,10]: under certain assumptions for a monotone measure μ, the convergence $a.e.[\mu]$ implies convergence $a.u.[\overline{\mu}]$.

This motivates us to consider convergence theorems in a more general case involving a pair of monotone measures (λ, ν). For instance, we are trying to find a necessary and sufficient condition (related to the ordered pair (λ, ν) of monotone measures) that convergence $a.e.[\nu]$ implies convergence $a.u.[\lambda]$ such that the standard-form of Egoroff's theorem, as well as the pseudo-forms of Egoroff's theorem for monotone measures, are special cases of the situation we are considering.

In this paper we shall generalize Egoroff's theorem, Riesz's theorem and Lebesgue's theorem in the traditional sense (i.e., in classical measure theory and only concerning one measure) to the general case involving a pair of monotone measures. In the following section, we give some preliminaries and recall three important structural characteristics of monotone measures: condition [**E**], property [**S**] and strong order continuity. In Section 3, corresponding to the concepts of condition [**E**], property [**S**] and strong order continuity of monotone measures, we introduce respectively absolute continuity of types **E**, **R** and **L** for monotone measures. As we will see, the condition [**E**], property [**S**] and strong order continuity are special cases of these three types of absolute continuity, respectively. Further discussions to these three types of absolute continuity will be shown in Section 5. Our main results are presented in Section 4. By means of absolute continuity of types **E**, **R**, and **L**, we establish respectively generalized Egoroff's theorem, generalized Riesz's theorem and generalized Lebesgue's theorem, each of which is related to a pair of monotone measures. These three generalized convergence theorems include respectively as special cases the previous versions of the Egoroff theorem, the Riesz theorem and the Lebesgue theorem for monotone measures. Thus, unified approaches to Egoroff's theorems, Riesz's theorems and Lebesgue's theorems for monotone measures are presented, respectively.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X. A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a *monotone measure* ([4]) on measurable space $(X, \mathcal{A})$ if it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;

(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

Let $\mathcal{M}$ denote the set of all monotone measures defined on $(X, \mathcal{A})$. For $\lambda, \nu \in \mathcal{M}$, let (λ, ν) denote the ordered pair of λ and ν, i.e., $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$.

For a finite monotone measure $\mu \in \mathcal{M}$, i.e., $\mu(X) < \infty$, we define the conjugate $\overline{\mu}$ of μ by

$$\overline{\mu}(A) = \mu(X) - \mu(X - A), \quad A \in \mathcal{A}.$$

Obviously, $\overline{\mu} \in \mathcal{M}$ and $\overline{\overline{\mu}} = \mu$.

For more information concerning monotone measures and nonlinear integrals, we recommend [5,19–27].

We recall the *condition* [**E**], *property* [**S**] and *strong order continuity* of monotone measures, which play important roles in generalizing the well-known convergence theorems from classical measure theory to monotone measure theory [6,7,9] (see also [3]).

Definition 2.1. ([7]) Let $\mu \in \mathcal{M}$. μ is said to fulfil *condition* [**E**] (resp. *condition* [$\overline{\mathbf{E}}$]), if for every double sequence $(A_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ satisfying the condition: for any fixed $m = 1, 2, \ldots$, $A_n^{(m)} \searrow A^{(m)}$ $(n \to \infty)$ with $\mu\big(\bigcup_{m=1}^{+\infty} A^{(m)}\big) = 0$, there exist increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of natural numbers, such that

$$\lim_{k\to+\infty} \mu\Big(\bigcup_{i=k}^{+\infty} A_{n_i}^{(m_i)}\Big) = 0 \;\Big(\text{resp. } \lim_{k\to+\infty} \mu\Big(X - \bigcup_{i=k}^{+\infty} A_{n_i}^{(m_i)}\Big) = \mu(X)\Big). \tag{2.1}$$

Definition 2.2. ([6]) Let $\mu \in \mathcal{M}$. μ is said to have *property* [**S**] (resp. *property* [$\overline{\mathbf{S}}$]), if for any $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\lim_{n\to+\infty}\mu(A_n) = 0$, there exists a subsequence $(A_{n_i})_{i\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ such that

$$\mu\Big(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_{n_i}\Big) = 0 \;\Big(\text{resp. } \mu\Big(X - \bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_{n_i}\Big) = \mu(X)\Big). \tag{2.2}$$

Definition 2.3. ([9,28]) Let $\mu \in \mathcal{M}$. μ is said to be *strongly order continuous* (resp. *strongly order pseudo-continuous*), if for any $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $A \in \mathcal{A}$, $A_n \searrow A$ and $\mu(A) = 0$ implies

$$\lim_{n\to+\infty} \mu(A_n) = 0 \;\Big(\text{resp. } \lim_{n\to+\infty} \mu(X - A_n) = \mu(X)\Big). \tag{2.3}$$

Note 2.4. (1) The *strong order pseudo-continuity* of monotone measures is a new concept we have introduced.

(2) For pseudo-form of Condition [**E**], instead of the previous term "*condition* [**PSE**]" in [3], we use the term "*condition* [$\overline{\mathbf{E}}$]" in Definition 2.1 to correspond to the dual property concerning conjugate measures (in finite measures case), see Proposition 3.4 in the following sections and Theorem 4.1 (2), (3) and (4) in [10].

3. Three types of absolute continuity of monotone measures

In order to establish convergence theorems in general case relating to a pair of monotone measures, we need to generalize the concepts of the condition [**E**], property [**S**] and strong order continuity of monotone measures.

We introduce three types of absolute continuity for monotone measures: types **E**, **R** and **L**, which are closely related to *condition* [**E**], *property* [**S**] and *strong order continuity*, respectively. We will see in the next section that they play important roles in discussing the relations among several different convergence for a sequence of measurable functions.

Definition 3.1. Let $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$. λ is called to be absolutely continuous of Type **E** with respect to ν, denoted by $\lambda \ll_{\mathbf{E}} \nu$, if for every double sequence $(A_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ satisfying the condition: for any fixed $m = 1, 2, \ldots$, $A_n^{(m)} \searrow A^{(m)}$ $(n \to \infty)$ with

$$\nu\Big(\bigcup_{m=1}^{+\infty} A^{(m)}\Big) = 0, \tag{3.4}$$

there exist increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of natural numbers, such that

$$\lim_{k\to+\infty}\lambda\Big(\bigcup_{i=k}^{+\infty}A_{n_i}^{(m_i)}\Big)=0. \tag{3.5}$$

Definition 3.2. Let $(\lambda,\nu)\in\mathcal{M}\times\mathcal{M}$. λ is called to be absolutely continuous of Type **R** with respect to ν, denoted by $\lambda\ll_{\mathbf{R}}\nu$, if for any $(A_n)_{n\in\mathbb{N}}\subset\mathcal{A}$ with

$$\lim_{n\to+\infty}\nu(A_n)=0, \tag{3.6}$$

there exists a subsequence $(A_{n_i})_{i\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ such that

$$\lambda\Big(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}A_{n_i}\Big)=0. \tag{3.7}$$

Definition 3.3. Let $(\lambda,\nu)\in\mathcal{M}\times\mathcal{M}$. λ is called to be absolutely continuous of Type **L** with respect to ν, denoted by $\lambda\ll_{\mathbf{L}}\nu$, if for any $(A_n)_{n\in\mathbb{N}}\subset\mathcal{A}$, $A\in\mathcal{A}$,

$$A_n\searrow A \ \text{ and } \ \nu(A)=0 \tag{3.8}$$

implies

$$\lim_{n\to+\infty}\lambda(A_n)=0. \tag{3.9}$$

Obviously, Definitions 3.1, 3.2 and 3.3 are corresponding to Definitions 2.1, 2.2 and 2.3, respectively. When $\lambda=\nu$, the concepts of absolute continuity of types **E**, **R** and **L** go back to the concepts of condition [**E**], property [**S**] and strong order continuity, respectively, i.e., we have the following:

Proposition 3.4. *For any $\mu\in\mathcal{M}$, we have*
(1) $\mu\ll_{\mathbf{E}}\mu$ if and only if μ fulfils condition [**E**]*; when μ is finite, $\overline{\mu}\ll_{\mathbf{E}}\mu$ if and only if μ fulfils condition* [$\overline{\mathbf{E}}$]*;*
(2) $\mu\ll_{\mathbf{R}}\mu$ if and only if μ has property [**S**]*; when μ is finite, $\overline{\mu}\ll_{\mathbf{R}}\mu$ if and only if μ has property* [$\overline{\mathbf{S}}$]*;*
(3) $\mu\ll_{\mathbf{L}}\mu$ if and only if μ is strongly order continuous; when μ is finite, $\overline{\mu}\ll_{\mathbf{L}}\mu$ if and only if μ is strongly order pseudo-continuous.

We recall two basic types of absolute continuity of monotone measures, as follows:

Definition 3.5. ([29]) Let $(\lambda,\nu)\in\mathcal{M}\times\mathcal{M}$. We say that
(1) λ *is absolutely continuous of Type I with respect to* ν, denoted by $\lambda\ll_I\nu$, iff for any $A\in\mathcal{A}$, $\nu(A)=0$ implies $\lambda(A)=0$;
(2) λ *is absolutely continuous of Type VI with respect to* ν, denoted by $\lambda\ll_{VI}\nu$, iff for any $(A_n)_{n\in\mathbb{N}}\subset\mathcal{A}$, $\nu(A_n)\to 0$ $(n\to\infty)$ implies $\lambda(A_n)\to 0$ $(n\to\infty)$.

For any $(\lambda,\nu)\in\mathcal{M}\times\mathcal{M}$, $\lambda\ll_{VI}\nu$ implies $\lambda\ll_I\nu$. The converse may not be true for monotone measures. When λ and ν are finite σ-additive measures on $(X,\mathcal{A})$, then $\lambda\ll_{VI}\nu$ iff $\lambda\ll_I\nu$ ([2]).

In Section 5, we will continue to discuss more properties of these three types of absolute continuity.

4. Generalized convergence theorems

We reiterate that, as already mentioned in Section 1, Egoroff's theorem, Riesz's theorem and Lebesgue's theorem in classical measure theory, each of them is only related to one measure. In this section we generalize these three convergence theorems in the traditional sense to more general cases involving a pair of monotone measures on the same measurable space $(X,\mathcal{A})$.

Let $\mathcal{F}$ denote the class of all finite real-valued $\mathcal{A}$-measurable functions on $(X,\mathcal{A})$, and let f, $f_n\in\mathcal{F}$ $(n=1,2,\ldots)$ and $\mu\in\mathcal{M}$ be fixed. The following concepts can be found, for example, in [4,5].

We say that $(f_n)_{n\in\mathbb{N}}$ *converges almost everywhere to* f *on* X *with respect to* μ, and denote it by $f_n \xrightarrow{a.e.} f[\mu]$, if there is a subset $N \subset X$ such that $\mu(N) = 0$ and $f_n \to f$ on $X - N$; $(f_n)_{n\in\mathbb{N}}$ *converges pseudo-almost everywhere to* f *on* X *with respect to* μ, and denote it by $f_n \xrightarrow{p.a.e.} f[\mu]$, if there is a subset $N \subset X$ such that $\mu(X - N) = \mu(X)$ and $f_n \to f$ on $X - N$; $(f_n)_{n\in\mathbb{N}}$ *converges almost uniformly to* f *on* X *with respect to* μ, and denote it by $f_n \xrightarrow{a.u.} f[\mu]$, if for any $\epsilon > 0$ there is a set $A_\epsilon \in \mathcal{A}$ such that $\mu(X - A_\epsilon) < \epsilon$ and f_n converges to f uniformly on A_ϵ; $(f_n)_{n\in\mathbb{N}}$ *converges to* f *pseudo-almost uniformly on* X *with respect to* μ and, denote it by $f_n \xrightarrow{p.a.u.} f[\mu]$, if there exists $(A_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ with $\lim_{k\to+\infty} \mu(X - A_k) = \mu(X)$ such that f_n converges to f on $X - A_k$ uniformly for any fixed $k = 1, 2, \ldots$; $(f_n)_{n\in\mathbb{N}}$ *converge to* f *in measure* μ (resp. *pseudo-in measure* μ) on X, in symbols $f_n \xrightarrow{\mu} f$ (resp. $f_n \xrightarrow{p.\mu} f$), if for any $\sigma > 0$, $\lim_{n\to+\infty} \mu(\{x : |f_n(x) - f(x)| \geq \sigma\}) = 0$ (resp. $\lim_{n\to+\infty} \mu(\{x : |f_n - f| < \sigma\}) = \mu(X)$).

The convergence *a.e.* (resp. *a.u.* or *in* μ) on X and the convergence *p.a.e.* (resp. *p.a.u.* or *in* $p.\mu$) on X are dual to each other in the following sense, see [30].

Proposition 4.1. *Let* μ *be a finite monotone measure on* $(X, \mathcal{A})$. *Then*

(1) $f_n \xrightarrow{p.a.e.} f[\mu]$ *if and only if* $f_n \xrightarrow{a.e.} f[\overline{\mu}]$;
(2) $f_n \xrightarrow{p.a.u.} f[\mu]$ *if and only if* $f_n \xrightarrow{a.u.} f[\overline{\mu}]$;
(3) $f_n \xrightarrow{p.\mu} f$ *if and only if* $f_n \xrightarrow{\overline{\mu}} f$.

4.1. Generalized Egoroff's theorem for monotone measures

The *condition* [**E**] and *Egoroff condition* of a monotone measure were introduced, respectively, and it is shown that each of them is a necessary and sufficient condition that the Egoroff theorem remains valid for a monotone measure ([7,8]). Several pseudo-versions of Egoroff's theorem on monotone measure spaces were also established ([4,10,18]). A detailed overview of these results was shown in [3].

Now we present a general version of Egoroff's theorem which concerns a pair of monotone measures. Its proof is almost the same as the one of Theorem 1 in [7] (Egoroff's theorem for monotone measures). For readers convenience, we include its proof (see Appendix).

Theorem 4.2. *(Generalized Egoroff's theorem) Let* $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$. *Then the following are equivalent:*

(i) $\lambda \ll_{\mathbf{E}} \nu$;
(ii) for all $f \in \mathcal{F}$ *and all* $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, *we have*

$$f_n \xrightarrow{a.e.} f[\nu] \implies f_n \xrightarrow{a.u.} f[\lambda]. \tag{4.10}$$

When $(\lambda, \nu) = (\mu, \mu)$ (i.e., λ and ν are the same monotone measure $\mu \in \mathcal{M}$), then the generalized Egoroff theorem (Theorem 4.2) goes back to Egoroff's theorems for monotone measures in [7,8,17] (see also Theorem 4.3 in [3]):

Corollary 4.3. *(Egoroff's theorem) Let* μ *be a monotone measure on* $(X, \mathcal{A})$. *Then, for all* $f \in \mathcal{F}$ *and all* $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$,

$$f_n \xrightarrow{a.e.} f[\mu] \implies f_n \xrightarrow{a.u.} f[\mu] \tag{4.11}$$

if and only if μ *fulfils condition* [**E**] *(or Egoroff condition, or condition* [**M**]*).*

When $(\lambda, \nu) = (\overline{\mu}, \overline{\mu})$, $(\overline{\mu}, \mu)$ and $(\mu, \overline{\mu})$, respectively, where μ is a finite monotone measure, then the generalized Egoroff theorem includes as special cases three different pseudo-versions of Egoroff's theorem for monotone measures in the context of pseudo-convergence, see Theorem 4.1 (2), (3) and (4) in [10] (see also Theorem 4.8 in [3]).

Note that for any $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$, $\lambda \ll_{\mathbf{E}} \nu$ implies $\lambda \ll_{\mathbf{L}} \nu$. Therefore, the absolute continuity of Type **L** of λ with respect to ν is a necessary condition for the validity of (4.10) in Theorem 4.2 (Generalized Egoroff's theorem). As a special case we directly obtain the results presented in [7,8]: the strong order continuity is a necessary condition of Egoroff's theorem for monotone measures (see Proposition 1 in [7], or Proposition 3 in [8]).

Similarly, as special cases of Theorem 4.2 (when $(\lambda, \nu) = (\overline{\mu}, \overline{\mu})$, $(\overline{\mu}, \mu)$ and $(\mu, \overline{\mu})$, respectively) we respectively obtain necessary conditions of three pseudo-versions of the Egoroff theorem for monotone measures, see Corollary 4.2 (2), (3), and (4) in [10].

Example 4.4. Let $(X, \mathcal{A})$ and (λ, ν) be considered as in Example 5.5 (i) in the following Section 5. Note that both λ and ν do not fulfil condition [**E**], but $\lambda \ll_{\mathbf{E}} \nu$.

Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \{1, 2, \ldots, n\}, \\ 1 & \text{otherwise,} \end{cases}$$

$n = 1, 2, \ldots$, and $f(x) = 0$ $(x \in X)$. We can easily see that $f_n \xrightarrow{a.e.} f[\nu]$, and it implies $f_n \xrightarrow{a.e.} f[\lambda]$. Since λ does not fulfil condition [**E**], we can not directly use Egoroff's theorem (Corollary 4.3) to confirm the truth of the formula $f_n \xrightarrow{a.u.} f[\lambda]$. But, by using the generalized Egoroff theorem and noting that $\lambda \ll_{\mathbf{E}} \nu$, we obtain $f_n \xrightarrow{a.u.} f[\lambda]$.

Note 4.5. Similarly, corresponding to the *Egoroff condition* ([8]) and the *condition* [**M**] ([17]), respectively, we can introduce two types of absolute continuity in some sense such that each of them is equivalent with the validity of (4.10), respectively.

Note 4.6. When λ and ν are finite σ-additive measures on $(X, \mathcal{A})$, we obtain a generalized version of Egoroff's theorem on classical measure space, as follows:

Theorem 4.7. *(Generalized Egoroff's theorem for σ-additive measures) Let λ and ν be finite σ-additive measures on $(X, \mathcal{A})$. Then the following are equivalent:*

(i) $\lambda \ll_I \nu$.

(ii) $\lambda \ll_{VI} \nu$.

(ii) for all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, we have

$$f_n \xrightarrow{a.e.} f[\nu] \implies f_n \xrightarrow{a.u.} f[\lambda]. \tag{4.12}$$

4.2. Generalized Riesz's theorem for monotone measures

In [6] the concept of *property* [**S**] of a monotone measure was introduced and it was shown that, in the case of monotone measures, the conclusion of the classical Riesz theorem holds if and only if property [**S**] is satisfied. As a variant of property [**S**], the *property* [$\overline{\mathbf{S}}$] was introduced in [3] and three pseudo-versions of the Riesz theorem for monotone measures in the context of (pseudo-)convergence were shown. The other Riesz-like theorems for monotone measures in the sense of pseudo-convergence were also presented respectively under the conditions of *property* [**PS**] [3,6], *property* [**TS**] [30] and *property* [**TPS**] [30].

Now we establish a generalized Riesz theorem in the framework involving a pair of monotone measures.

Theorem 4.8. *(Generalized Riesz's theorem) Let $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$. Then the following are equivalent:*

(i) $\lambda \ll_{\mathbf{R}} \nu$;

(ii) for all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ with $f_n \xrightarrow{\nu} f$, there exists a subsequence $(f_{n_i})_{i\in N}$ of $(f_n)_{n\in N}$ such that $f_{n_i} \xrightarrow{a.e.} f[\lambda]$.

Proof. The proof is similar to that of Theorem 2.1 in [6] (see also Theorem 5 in [12]), therefore we omit its details. □

It is similar to the case of Egoroff's theorem, when $\lambda = \nu$, Riesz's theorem for monotone measures (see Theorem 2.1 in [6] or Theorem 5 in [12], see also Theorem 5.17 in [3]) is recovered by the generalized Riesz theorem (Theorem 4.8).

When we take $(\lambda, \nu) = (\overline{\mu}, \overline{\mu})$, $(\overline{\mu}, \mu)$ and $(\mu, \overline{\mu})$ in Theorem 4.8, respectively, as special results, the generalized Riesz theorem cover three pseudo-forms of Riesz's theorem on finite monotone measure spaces, see Theorem 5.20 in [3].

4.3. Generalized Lebesgue's theorem for monotone measures

The strong order continuity is a necessary and sufficient condition for which Lebesgue's theorem remains valid in the case of monotone measures [9]. Under the conditions of three kinds of continuity of monotone measures, such as continuity from below, order continuity and pseudo-order continuity, etc., three different versions of Lebesgue's theorem for monotone measures in the context of (pseudo-)convergence were presented in [11] (see also [4,18]).

In the following we present the generalized Lebesgue theorem for monotone measures.

Theorem 4.9. *(Generalized Lebesgue's theorem) Let $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$. Then the following are equivalent:*

(i) $\lambda \ll_{\mathbf{L}} \nu$;

(ii) for all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, we have

$$f_n \xrightarrow{a.e.} f[\nu] \implies f_n \xrightarrow{\lambda} f. \tag{4.13}$$

Proof. It is similar to the proof of Theorem 1 in [9], therefore we omit its details. □

When $\lambda = \nu$, as a direct result of the generalized Lebesgue theorem (Theorem 4.9) we get the following Lebesgue's theorem for monotone measures, which was shown in [9] (see Theorem 1 in [9] or Theorem 5.2 in [3]):

Corollary 4.10. *(Lebesgue's theorem) Let μ be a monotone measure on $(X, \mathcal{A})$. Then, for all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$*

$$f_n \xrightarrow{a.e.} f[\mu] \implies f_n \xrightarrow{\mu} f,$$

if and only if μ is strongly order continuous.

It is similar to the discussion of Egoroff-like theorems, for a finite monotone measure μ, take $(\lambda, \nu) = (\overline{\mu}, \overline{\mu})$, $(\overline{\mu}, \mu)$ and $(\mu, \overline{\mu})$ in Theorem 4.9 respectively, and combine Propositions 3.4 and 4.1, we obtain three different pseudo-versions of the Lebesgue theorem for monotone measures in the context of convergence and pseudo-convergence, as follows:

Corollary 4.11. *Let μ be a finite monotone measure on $(X, \mathcal{A})$. Then,*

(1) *For all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, $f_n \xrightarrow{p.a.e.} f[\mu]$ implies $f_n \xrightarrow{p.\mu} f$ if and only if $\overline{\mu}$ is strongly order continuous.*

(2) *For all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, $f_n \xrightarrow{a.e.} f[\mu]$ implies $f_n \xrightarrow{p.\mu} f$ if and only if μ is strongly order pseudo-continuous.*

(3) *For all $f \in \mathcal{F}$ and all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, $f_n \xrightarrow{p.a.e.} f[\mu]$ implies $f_n \xrightarrow{\mu} f$ if and only if $\overline{\mu}$ is strongly order pseudo-continuous.*

Remark 4.12. (1) In our discussions we only considered the (pseudo-) convergence (*a.e., p.a.e., a.u., p.a.u., in μ, in $p.\mu$*) on the whole space X. For the general case, i.e., for (pseudo-)convergence on any measurable subset of a measurable space $(X, \mathcal{A})$, we can consider the restriction of monotone measures to some subset, and introduce the corresponding concepts. As we have discussed, we can obtain the corresponding results and these results will cover the several different versions of convergence theorems presented in [4,11,18,30].

(2) The Egoroff theorem, the Riesz theorem and the Lebesgue theorem in classical measure theory were extended in some more general contexts: (i) lattice-valued monotone measures and lattice-valued measurable functions [31]; (ii) real-valued monotone measures and set-valued measurable functions [32]; (iii) Riesz space-valued monotone measures and real-valued functions [13,14]; (iv) fuzzy multimeasures and real-valued functions [15]; (v) ordered topological vector space-valued non additive measure [16]; (vi) set-valued fuzzy measures [33], etc. Under these contexts the extensions of our results are some subjects of further study.

5. Further discussion on absolute continuity

In this section we continue to discuss properties of absolute continuity of types **E**, **R** and **L**. We show the relationships between these three types of absolute continuity and their corresponding concepts: condition [**E**], property [**S**] and strong order continuity.

We can easily obtain the following propositions:

Proposition 5.1. *Let* $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$.

(1) *If* $\lambda \ll_{\mathbf{E}} \nu$*, then* $\lambda \ll_I \nu$.

(2) *If* $\lambda \ll_I \nu$ *and* λ *fulfils condition* [**E**]*, then* $\lambda \ll_{\mathbf{E}} \nu$.

(3) *If* $\lambda \ll_{VI} \nu$ *and* ν *fulfils condition* [**E**]*, then* $\lambda \ll_{\mathbf{E}} \nu$.

Proposition 5.2. *Let* $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$.

(1) *If* $\lambda \ll_{\mathbf{R}} \nu$*, then* $\lambda \ll_I \nu$.

(2) *If* $\lambda \ll_I \nu$ *and* ν *has property* [**S**]*, then* $\lambda \ll_{\mathbf{R}} \nu$.

(3) *If* $\lambda \ll_{VI} \nu$ *and* λ *has property* [**S**]*, then* $\lambda \ll_{\mathbf{R}} \nu$.

Proposition 5.3. *Let* $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$.

(1) *If* $\lambda \ll_{\mathbf{L}} \nu$*, then* $\lambda \ll_I \nu$.

(2) *If* $\lambda \ll_I \nu$ *and* λ *is strongly order continuous, then* $\lambda \ll_{\mathbf{L}} \nu$.

(3) *If* $\lambda \ll_{VI} \nu$ *and* ν *is strongly order continuous, then* $\lambda \ll_{\mathbf{L}} \nu$.

From Propositions 5.1(1), 5.2(1) and 5.3(1) we know that the absolute continuity of Type I (λ with respect to ν) is a necessary condition for each of the generalized Egoroff theorem, the generalized Riesz theorem and the generalized Lebesgue theorem. From the rest conditions (2) and (3) we obtain respectively sufficient conditions for these three generalized convergence theorems.

Remark 5.4. For any $(\lambda, \nu) \in \mathcal{M} \times \mathcal{M}$, the three types of absolute continuity relating to (λ, ν), types **E**, **R** and **L**, are corresponding to the concepts of *condition* [**E**], *property* [**S**] and *strong order continuity* of monotone measures, respectively. But the condition [**E**] of λ, the condition [**E**] of ν, and $\lambda \ll_{\mathbf{E}} \nu$ are logically independent from one to each other. The cases of *property* [**S**] and *strong order continuity* of monotone measures are similar.

Example 5.5. Let $X = X_1 \cup X_2$, $X_1 = \{1, 2, \ldots.\}$, $X_2 = \{1/2, 1/3, \ldots.\}$, and $\mathcal{A} = 2^X$.

(i) Define

$$\lambda(E) = \begin{cases} 0 & \text{if } E \subset \{1\} \cup X_2, \\ 1 & \text{if } 1 \in E \text{ and } E \cap (X_1 - \{1\}) \neq \emptyset, \\ \max\left\{\frac{1}{i} : i \in E \cap X_1\right\} & \text{if } 1 \notin E \text{ and } E \cap X_1 \neq \emptyset, \end{cases}$$

and

$$\nu(E) = \begin{cases} 0 & \text{if } E \subset X_2, \\ 1 & \text{otherwise.} \end{cases}$$

Then λ, ν are monotone measures on $(X, \mathcal{A})$ and both λ and ν do not fulfil condition [**E**]. It is not difficult to verify that $\lambda \ll_{\mathbf{E}} \nu$.

(ii) Define

$$\lambda(E) = \begin{cases} 0 & \text{if } E \subset X_2, \\ \max\left\{\frac{1}{i} : i \in E \cap X_1\right\} & \text{otherwise,} \end{cases}$$

and

$$\nu(E)=\begin{cases}0 & \text{if } E\subset X_1,\\ \max\left\{\frac{1}{i}:\frac{1}{i}\in E\cap X_2\right\} & \text{otherwise.}\end{cases}$$

Note that both λ and ν are possibility measures (i.e., $\lambda(A\cup B)=\max(\lambda(A),\lambda(B))$ holds for any $A, B\in\mathcal{A}$), and both λ and ν fulfil condition [**E**]. However, neither $\lambda \ll_{\mathbf{E}} \nu$ nor $\nu \ll_{\mathbf{E}} \lambda$ holds.

(iii) Define

$$\nu(E)=\begin{cases}0 & \text{if } E=\emptyset,\\ \max\left\{\frac{1}{i}:i\in E\right\} & \text{if } E\subset X_1 \text{ and } E\neq\emptyset,\\ \max\left\{\frac{1}{i}:\frac{1}{i}\in E\right\} & \text{if } E\subset X_2 \text{ and } E\neq\emptyset,\\ \max\left\{\max\limits_{i\in E\cap X_1}\left\{\frac{1}{i}\right\},\ \max\limits_{1/i\in E\cap X_2}\left\{\frac{1}{i}\right\}\right\} & \text{otherwise,}\end{cases}$$

and

$$\lambda(E)=\begin{cases}0 & \text{if } E\subset X_1 \text{ or } E\subset X_2,\\ \nu(E) & \text{otherwise.}\end{cases}$$

Then λ, ν are monotone measures on $(X,\mathcal{A})$ (ν is also a possibility measure) and for any $E\in\mathcal{A}$, $\lambda(E)\le\nu(E)$. It is not difficult to verify that λ does not fulfil condition [**E**], ν fulfils condition [**E**]. But we have $\lambda \ll_{\mathbf{E}} \nu$.

Example 5.6. Let $X=\{0,1,2,\ldots\}$ and $\mathcal{A}=2^X$. Define

$$\nu(E)=\sum_{i\in E}\frac{1}{2^i},\quad \forall E\in\mathcal{A},$$

and

$$\lambda(E)=\begin{cases}\nu(E) & \text{if } 0\notin E,\\ 0 & \text{if } E\subset\{0\},\\ 2 & \text{if } 0\in E \text{ and } E-\{0\}\neq\emptyset.\end{cases}$$

Then ν is an additive measure and λ is a monotone measure on $(X,\mathcal{A})$. Obviously, ν is strongly order continuous, λ is not strongly order continuous. But we have $\lambda \ll_{\mathbf{L}} \nu$.

Example 5.7. Let $X=[0,1]$, $\mathcal{A}$ be the class of all Lebesgue measurable sets on $[0,1]$ and m be Lebesgue's measure. Define a monotone measure λ on $\mathcal{A}$, as follows:

$$\lambda(E)=\begin{cases}0 & \text{if } m(E)=0,\\ 1 & \text{if } m(E)>0.\end{cases}$$

Then m is strongly order continuous, λ is not strongly order continuous, and $\lambda \ll_{\mathbf{L}} m$ is not true.

6. Conclusion

We have generalized the Egoroff theorem, the Riesz theorem and the Lebesgue theorem in the traditional sense concerning one monotone measure to general case involving a pair of monotone measures. Our main results are Theorem 4.2 (generalized Egoroff's theorem), Theorem 4.8 (generalized Riesz's theorem) and Theorem 4.9 (generalized Lebesgue's theorem). These three generalized convergence theorems cover respectively Egoroff's theorem (Corollary 4.3, see also Theorem 1 in [7]), Riesz's theorem (Theorem 1 in [6]) and Lebesgue's theorem (Corollary 4.10, see also Theorem 1 in [9]) for monotone measures and their pseudo-versions (Theorem 4.1 in [10], Theorem 5.20 in [3], Corollary 4.11, etc.). In this sense, we have presented unified approaches to Egoroff-like theorems, Riesz-like theorems and Lebesgue-like theorems for monotone measures, respectively.

Though the proofs of our results are almost the same as the related proofs of Egoroff's, Riesz's and Lebesgue's theorems for monotone measures (and thus we have given the proof only in the case of Theorem 4.2), we believe that our idea of considering ordered pair (λ, ν) of monotone measures instead of a unique measure μ is a valuable completion of the convergence part of the measure theory.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 4.2

$(i) \Rightarrow (ii)$: Suppose $\lambda \ll_{\mathbf{E}} \nu$ and $f_n \xrightarrow{a.e.} f[\nu]$. Let D be the set of these points x in X at which $\{f_n(x)\}$ dose not converge to $f(x)$. For any fixed $m = 1, 2, \ldots$, we denote

$$A_n^{(m)} = \bigcup_{i=n}^{+\infty} \left\{ x \in X : \, |f_i(x) - f(x)| \geq \frac{1}{m} \right\}$$

$n = 1, 2, \ldots$, then $D = \bigcup_{m=1}^{+\infty} \bigcap_{n=1}^{+\infty} A_n^{(m)}$. Write $A^{(m)} = \bigcap_{n=1}^{+\infty} A_n^{(m)}$, noting that $\nu(D) = 0$, then the double sequence $(A_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ satisfies the condition: for any fixed $m = 1, 2, \ldots$, as $n \to \infty$,

$$A_n^{(m)} \searrow A^{(m)} \text{ and } \nu\Big(\bigcup_{m=1}^{+\infty} A^{(m)}\Big) = 0. \tag{A.14}$$

Applying the condition $\lambda \ll_{\mathbf{E}} \nu$ to the double sequence $(A_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$, then there exist increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of natural numbers, such that

$$\lim_{k\to+\infty} \lambda\Big(\bigcup_{i=k}^{+\infty} A_{n_i}^{(m_i)}\Big) = 0.$$

For any $\epsilon > 0$, we take k_0 such that

$$\lambda\Big(\bigcup_{i=k_0}^{+\infty} A_{n_i}^{(m_i)}\Big) < \epsilon.$$

Let $A_\epsilon = X - \bigcup_{i=k_0}^{+\infty} A_{n_i}^{(m_i)}$, then $A_\epsilon \in \mathcal{A}$ and

$$\lambda(X - A_\epsilon) = \lambda\Big(\bigcup_{i=k_0}^{+\infty} A_{n_i}^{(m_i)}\Big) < \epsilon.$$

Now we just need to prove that $(f_n)_{n\in\mathbb{N}}$ converges to f on A_ϵ uniformly. Since

$$A_\epsilon = \bigcap_{i=k_0}^{+\infty} \bigcap_{j=n_i}^{+\infty} \left\{ x \in X : \, |f_j(x) - f(x)| < \frac{1}{m_i} \right\},$$

therefore, for any fixed $i \geq k_0$, $A_\epsilon \subset \bigcap_{j=n_i}^{+\infty} \left\{ x \in X : \, |f_j(x) - f(x)| < 1/m_i \right\}$. For any given $\sigma > 0$, we take i_0 $(\geq k_0)$ such that $1/m_{i_o} < \sigma$. Thus, as $j > n_{i_o}$, for any $x \in A_\epsilon$,

$$|f_j(x) - f(x)| < \frac{1}{m_{i_o}} < \sigma.$$

This shows that $(f_n)_{n\in\mathbb{N}}$ converges to f on A_ϵ uniformly.

$(ii) \Rightarrow (i)$: Suppose that for any $f \in \mathcal{F}$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$, $f_n \xrightarrow{a.e.} f[\nu]$ implies $f_n \xrightarrow{a.u.} f[\lambda]$.

Let $(A_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ be any given double sequence of sets and let it satisfy the condition: for any fixed $m = 1, 2, \cdots$,

$$A_n^{(m)} \searrow A^{(m)} \ (n \to \infty) \ \text{ and } \ \nu\Big(\bigcup_{m=1}^{+\infty} A^{(m)}\Big) = 0. \tag{A.15}$$

We put $\hat{A}_n^{(m)} = \bigcup_{i=1}^m A_n^{(i)} = A_n^{(1)} \cup A_n^{(2)} \cdots \cup A_n^{(m)} \quad (m, n \in \mathbb{N})$ and $\hat{A}^{(m)} = \bigcap_{n=1}^{+\infty} \hat{A}_n^{(m)} \quad (m = 1, 2, \cdots)$. Then we obtain a double sequence $(\hat{A}_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ satisfying the properties: for any fixed $n \in \mathbb{N}$, $\hat{A}_n^{(m)} \subset \hat{A}_n^{(m+1)}$, and for any fixed $m \in \mathbb{N}$, $\hat{A}_n^{(m)} \searrow \hat{A}^{(m)}$ as $n \to \infty$, and from $\bigcup_{m=1}^{+\infty} \hat{A}^{(m)} = \bigcup_{m=1}^{+\infty} A^{(m)}$, it follows that $\nu(\bigcup_{m=1}^{+\infty} \hat{A}^{(m)}) = 0$.

Now we construct a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$: for every $n \in \mathbb{N}$ we define

$$f_n(x) = \begin{cases} \frac{1}{m} & x \in \hat{A}_n^{(m+1)} - \hat{A}_n^{(m)} \quad m = 1, 2, \cdots \\ 2 & x \in \hat{A}_n^{(1)} \\ 0 & x \in X - \bigcup_{m=1}^{+\infty} \hat{A}_n^{(m)}. \end{cases}$$

Then for every $m = 1, 2, \cdots$,

$$\Big\{x : |f_n(x)| \le \frac{1}{m}\Big\} = X - \hat{A}_n^{(m)}.$$

Noting that for any fixed $m \in \mathbb{N}$, $\hat{A}_n^{(m)} \searrow \hat{A}^{(m)}$ as $n \to \infty$, we have

$$\Big\{x \in X : f_k(x) \to 0\Big\} = \bigcap_{m=1}^{+\infty} \bigcup_{n=1}^{+\infty} \bigcap_{k=n}^{+\infty} \Big\{x \in X : |f_k(x)| \le \frac{1}{m}\Big\} = X - \bigcup_{m=1}^{+\infty} \hat{A}^{(m)}.$$

Therefore, from $\nu(\bigcup_{m=1}^{+\infty} \hat{A}^{(m)}) = 0$, we have $f_n \xrightarrow{a.e.} 0\ [\nu]$ on X. It follows from the hypothesis that $f_n \xrightarrow{a.u.} 0\ [\lambda]$ on X. Thus, there exists a sequence $(H_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that for every j, $\lambda(X - H_j) < 1/j$ and as $n \to \infty$, f_n converges to 0 uniformly on H_j. Without loss of generality, we can assume $H_1 \subset H_2 \subset \cdots$ (otherwise, we can take $\bigcup_{i=1}^{j} H_i$ instead of H_j). Thus for every $j \in \mathbb{N}$, there exist $n_j \in \mathbb{N}$ such that for any $x \in H_j$, we have $|f_k(x)| \le 1/j$ whenever $k \ge n_j$. Therefore, for every $j \in \mathbb{N}$, we have

$$H_j \subset \bigcap_{k=n_j}^{+\infty} \Big\{x : |f_k(x)| \le \frac{1}{j}\Big\} = X - \hat{A}_{n_j}^{(j)},$$

and hence $\hat{A}_{n_j}^{(j)} \subset X - H_j \ \ j = 1, 2, \cdots$. Therefore, for any $k \ge 1$, we have

$$\bigcup_{j=k}^{+\infty} \hat{A}_{n_j}^{(j)} \subset \bigcup_{j=k}^{+\infty} (X - H_j).$$

Consequently, we have

$$\lambda\Big(\bigcup_{j=k}^{+\infty} \hat{A}_{n_j}^{(j)}\Big) \le \lambda\Big(\bigcup_{j=k}^{+\infty} (X - H_j)\Big) = \lambda\big(X - H_k\big) < \frac{1}{k}.$$

Thus we have chosen a subsequence $(\hat{A}_{n_j}^{(j)})_{j\in\mathbb{N}}$ of the double sequence $(\hat{A}_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ such that

$$\lim_{k\to+\infty} \lambda\Big(\bigcup_{j=k}^{+\infty} \hat{A}_{n_j}^{(j)}\Big) = 0.$$

Noting that for $m, n \in \mathbb{N}$, $A_n^m \subset \hat{A}_n^m$, then we have

$$\lim_{k\to+\infty} \lambda\Big(\bigcup_{j=k}^{+\infty} A_{n_j}^{(j)}\Big) = 0.$$

This shows that $\lambda \ll_{\mathbf{E}} \nu$. $\square$

References

[1] E. Hewitt, K. Stromberg, Real and Abstract Analysis, Springer, New York, 1965.
[2] J.J. Benedetto, W. Czaja, Integration and Modern Analysis, Birkhäuser, Basel, 2009.
[3] J. Li, R. Mesiar, E. Pap, E.P. Klement, Convergence theorems for monotone measures, Fuzzy Sets Syst. 281 (2015) 103–127.
[4] Z. Wang, G.J. Klir, Generalized Measure Theory, Springer, New York, 2009.
[5] E. Pap, Null-Additive Set Functions, Kluwer, Dordrecht, 1995.
[6] Q. Sun, Property (S) of fuzzy measure and Riesz's theorem, Fuzzy Sets Syst. 62 (1994) 117–119.
[7] J. Li, A further investigation for Egoroff's theorem with respect to monotone set functions, Kybernetika 39 (2003) 753–760.
[8] T. Murofushi, K. Uchino, S. Asahina, Conditions for Egoroff's theorem in non-additive measure theory, Fuzzy Sets Syst. 146 (2004) 135–146.
[9] J. Li, Order continuous of monotone set function and convergence of measurable functions sequence, Appl. Math. Comput. 135 (2003) 211–218.
[10] J. Li, M. Yasuda, On Egoroff's theorems on finite monotone non-additive measure spaces, Fuzzy Sets Syst. 153 (2005) 71–78.
[11] J. Song, J. Li, Lebesgue's theorems in non-additive measure theory, Fuzzy Sets Syst. 149 (2005) 543–548.
[12] J. Li, M. Yasuda, Q. Jiang, H. Suzuki, Z. Wang, G. Klir, Convergence of sequence of measurable functions on fuzzy measure space, Fuzzy Sets Syst. 87 (1997) 317–323.
[13] J. Kawabe, The Egoroff property and the Egoroff theorem in Riesz space-valued non-additive measure theory, Fuzzy Sets Syst. 158 (2007) 50–57.
[14] J. Kawabe, A study of Riesz space-valued non-additive measures, in: V.N. Huynh, et al. (Eds.), Integrated Uncertainty Management and Applications, in: AISC, vol. 68, 2010, pp. 91–102.
[15] A. Precupanu, A. Gavrilut, A set-valued Egoroff type theorem, Fuzzy Sets Syst. 175 (2011) 87–95.
[16] T. Watanabe, On sufficient conditions for the Egoroff theorem of an ordered topological vector space-valued non-additive measure, Fuzzy Sets Syst. 162 (2011) 79–83.
[17] M. Takahashi, T. Murofushi, S. Asahina, A new necessary and sufficient condition for the Egoroff theorem in non-additive measure theory, Fuzzy Sets Syst. 244 (2014) 34–40.
[18] Z. Wang, Asymptotic structural characteristics of fuzzy measure and their applications, Fuzzy Sets Syst. 16 (1985) 277–290.
[19] M. Sugeno, Theory of fuzzy integral and its applications, Ph.D. Thesis, Tokyo Institute of Technology, Tokyo, 1974.
[20] D. Denneberg, Non-additive Measure and Integral, Kluwer Academic Publisher, 1994.
[21] M. Grabisch, T. Murofushi, M. Sugeno, Fuzzy Measures and Integrals: Theory and Applications, Stud. Fuzziness Soft Comput., vol. 40, 2000.
[22] E.P. Klement, J. Li, R. Mesiar, E. Pap, Integrals based on monotone set functions, Fuzzy Sets Syst. 281 (2015) 88–102.
[23] E.P. Klement, W. Schwyhla, Correspondence between fuzzy measures and classical measures, Fuzzy Sets Syst. 7 (1982) 57–70.
[24] V. Torra, Y. Narukawa, M. Sugeno, Non-Additive Measure – Theory and Applications, Stud. Fuzziness Soft Comput., vol. 310, 2014.
[25] R. Mesiar, Fuzzy measures and integrals, Fuzzy Sets Syst. 156 (2005) 365–370.
[26] R. Mesiar, A. Mesiarová, Fuzzy integrals – what are they?, Int. J. Intell. Syst. 23 (2008) 199–212.
[27] E. Lehrer, R. Teper, The concave integral over large spaces, Fuzzy Sets Syst. 159 (2008) 2130–2144.
[28] Q. Jiang, Z. Wang, G. Klir, Null-additive fuzzy measures on S-compact spaces, Int. J. Gen. Syst. 25 (1996) 219–228.
[29] Z. Wang, G.J. Klir, Fuzzy measures defined by fuzzy integral and their absolute continuity, J. Math. Anal. Appl. 203 (1996) 150–165.
[30] M. Takahashi, T. Murofushi, Relationship between convergence concepts in fuzzy measure theory, in: Proc. 11th International Fuzzy Systems Association World Congress, Vol. I, 2005, pp. 467–473.
[31] X. Liu, G. Zhang, Lattice-valued fuzzy measure and lattice-valued fuzzy integral, Fuzzy Sets Syst. 62 (1994) 319–332.
[32] Y. Liu, Convergence of measurable set-valued function sequence on fuzzy measure space, Fuzzy Sets Syst. 112 (2000) 241–249.
[33] J. Wu, H. Liu, Autocontinuity of set-valued fuzzy measures and applications, Fuzzy Sets Syst. 175 (2011) 57–64.