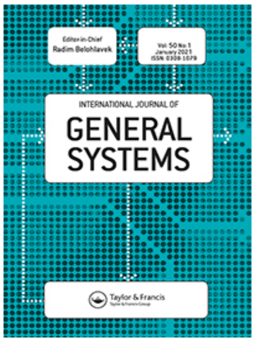

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Generalizing expected values to the case of L^*-fuzzy events

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ABSTRACT
Starting with [Goguen, J.A. 1967. Journal of Mathematical Analysis and Applications], several generalizations of the original definition of a fuzzy set have been proposed. In one popular case, one considers as truth values the points in the lower left triangle of the unit square, where their first coordinate is interpreted as "degree of membership", and their second coordinate as "degree of non-membership". Generalizing ideas in [Zadeh, L.A. 1968. Journal of Mathematical Analysis and Applications], [Grzegorzewski, P. and E. Mrówka. 2002. In: Soft Methods in Probability, Statistics and Data Analysis, Heidelberg: Physica], [Grzegorzewski, P. 2013. Information Sciences] and [Klement, E.P. and R. Mesiar. 2015. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems], the concept of expected values (based on capacities) of fuzzy events in this general sense is introduced and investigated. Expected values satisfying additional properties such as positive-linearity, comonotone additivity and comonotone maxitivity are studied, as well as an extension to real-valued expected values.

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Expected value; L^*-fuzzy event; capacity; Choquet integral; Sugeno integral; linearity; comonotone additivity; comonotone maxitivity

1. Introduction

When trying to determine the "size" of an L^*-fuzzy event, we follow the ideas of Zadeh (1968), where a first approach to probability measures of fuzzy events was introduced and studied, and of Klement and Mesiar (2015), where expected values of fuzzy events were investigated in general.

In Zadeh (1968), a fixed probability measure was considered and the Lebesgue (–Stieltjes) integral (with respect to this probability measure) was applied to the membership function of some fuzzy event in order to define the probability measure of this fuzzy event.

In a natural generalization of classical statistics where the expected value of the characteristic (i.e. membership) function of a set coincides with the probability measure of this

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set, the probability measure of a fuzzy event defined in Zadeh (1968) can be interpreted as the expected value of this event (with respect to the underlying probability measure).

This procedure can be further generalized by considering more general types of measures (such as capacities (Choquet 1954) or fuzzy measures (Sugeno 1974)) and more general types of integrals (such as the Choquet (Choquet 1954), Shilkret (Shilkret 1971) or Sugeno integral (Sugeno 1974), or even a more abstract framework for integrals such as the universal integrals studied in Klement, Mesiar, and Pap (2010)) to define expected values of fuzzy events (Klement and Mesiar 2015).

Various more general types of measures (which are monotone non-decreasing, but in general neither (finitely or σ-)additive nor continuous) have been studied in the literature (Medolaghi 1907; Carathéodory 1918; Vitali 1925; Choquet 1954; Dempster 1967; Shilkret 1971; Sugeno 1974; Shafer 1976; Ralescu and Adams 1980; Bhaskara Rao and Rao 1983; Stolz 1883; Klement and Weber 1991; Denneberg 1994; Pap 1995; Grabisch 1997; Höhle and Rodabaugh 1999; Klement and Weber 1999; Mesiar 1999; Grabisch, Murofushi, and Sugeno 2000; Pap 2002; Mesiar 2005; Pap 2008; Lehrer 2009; Wang and Klir 2009; Li and Mesiar 2011; Mesiar and Stupňanová 2013; Li, Mesiar, and Pap 2014; Torra, Narukawa, and Sugeno 2014).

In the most general case, these measures are (up to some boundary conditions) only monotone non-decreasing with respect to set inclusion.

In this paper, we shall consistently use the term capacity for a monotone non-decreasing set function (defined on a σ-algebra) which also satisfies the standard boundary conditions. In this way, we shall avoid any confusion with similar notions like monotone set function, monotone measure, non-additive measure or fuzzy measure (even if these notions were used in the original sources).

For capacities with additional properties, we shall either use their standard names (such as probability measure) or will explicitly mention the additional properties: the Lebesgue measure on the unit interval and, more generally, each probability measure are examples of σ-additive capacities.

The main aim of this paper is a deep study of expected values of L^*-fuzzy sets (for details concerning the lattice L^* and the corresponding L^*-fuzzy sets see Section 2). Observe that in a series of publications (Atanassov 1984, 1986, 1999) the so-called "intuitionistic fuzzy sets" were introduced and studied. This type of fuzzy sets is based on the lattice L^* and therefore, following the ideas in Goguen (1967), we consistently use the term "L^*-fuzzy sets" in order to avoid any confusion with the intuitionistic fuzzy logic developed in Takeuti and Titani (1984) and Takeuti and Titani (1987).

The paper is organized as follows. In the following section, the necessary preliminaries concerning fuzzy sets and L^*-fuzzy sets are given. In Section 3, we present capacities and several related integrals (which, in general, are non-linear). The main part of the paper is contained in Section 4 where we introduce and discuss expected values of fuzzy events and L^*-fuzzy events. The following sections are devoted to the investigation of expected values of L^*-fuzzy events satisfying some additional properties: in Section 5 the linearity of L^*-fuzzy events and weaker forms thereof, in particular, idempotency, are considered, and in Section 6, comonotone additivity and maxitivity of the expected values is assumed to hold. Finally, in Section 7 we extend our concept to real-valued expected values of L^*-fuzzy events.

2. Preliminaries

The set of all crisp (or Cantorian) subsets of a universe of discourse X will be denoted by $\mathscr{P}(X)$ (power set of X). Since we can identify each crisp subset A of X with its characteristic function $\mathbf{1}_A \colon X \to \{0,1\}$ the complete bounded lattice $(\mathscr{P}(X), \subseteq)$ is isomorphic to the lattice $(\{0,1\}^X, \leq)$, where $\{0,1\}^X$ is the set of all functions from X to $\{0,1\}$ and $\leq$ the component-wise standard order. For the rest of this paper, we will write $\leq$ for the standard linear order on subsets of the real line $\mathbb{R}$ (such as $\{0,1\}$ and the unit interval $[0,1]$). From now on, we will denote the unit interval $[0,1]$ simply by $\mathbb{I}$.

In Zadeh (1965), the unit interval $\mathbb{I}$ was suggested as set of truth values. The set of all fuzzy subsets of X will be denoted by $\mathscr{F}(X)$. A fuzzy subset $A \in \mathscr{F}(X)$ is described by its membership function $\mu_A \colon X \to \mathbb{I}$, where $\mu_A(x) \in \mathbb{I}$ defines the degree of membership of the object x in the fuzzy subset A, and the complete bounded lattice $(\mathscr{F}(X), \subseteq)$ is isomorphic to $(\mathbb{I}^X, \leq)$, where $\mathbb{I}^X$ is the set of all functions from X to $\mathbb{I}$.

The lattice $(\mathscr{P}(X), \subseteq)$ of crisp subsets of X can be embedded into the lattice $(\mathscr{F}(X), \subseteq)$ of fuzzy sets of X: the function $\mathrm{emb}_{\mathscr{P}(X)} \colon \mathscr{P}(X) \to \mathscr{F}(X)$, where the membership function of the fuzzy set $\mathrm{emb}_{\mathscr{P}(X)}(A)$ is given by $\mu_{\mathrm{emb}_{\mathscr{P}(X)}(A)} = \mathbf{1}_A$, provides a natural embedding.

In a further generalization (Goguen 1967), it was suggested to use an abstract set L as set of truth values and to describe an L-fuzzy subset A of X by means of its membership function $\mu_A \colon X \to L$. Several important examples for L were discussed in Goguen (1967), such as complete lattices or complete lattice-ordered semigroups. For a recent overview of generalizations of fuzzy sets, see Bustince et al. (2016).

In most cases, a bounded lattice $(L, \leq_L)$ is considered, i.e. a non-empty set L equipped with a partial order $\leq_L$ such that there exist a bottom (or smallest) element $\mathbf{0}_L$ and a top (or greatest) element $\mathbf{1}_L$ in L, and such that each finite subset of L has a meet (or greatest lower bound) and a join (or least upper bound) in L. If, additionally, each arbitrary subset of L has a meet and a join then the lattice is called complete.

For notions and results in the theory of general lattices, we refer to the book (Birkhoff 1973). We only recall that in a lattice there is a natural relationship between the partial order, on the one hand, and the lattice theoretical operations meet and join, on the other hand: the assertion $a \leq_L b$ is equivalent to each of $a \wedge_L b = a$ and $a \vee_L b = b$.

In the rest of the paper, we shall mainly work with the following set L^* of truth values and the partial order $\leq_{L^*}$ on L^* (compare Atanassov 1986; Deschrijver and Kerre 2003; Wang and He 2000):

$$L^* = \{(a,b) \in \mathbb{I}^2 \mid a + b \leq 1\}, \tag{1}$$

$$(a,b) \leq_{L^*} (c,d) \iff a \leq c \text{ and } b \geq d. \tag{2}$$

Obviously, $(L^*, \leq_{L^*})$ is a complete bounded lattice: the meet $\wedge_{L^*}$ and the join $\vee_{L^*}$ in $(L^*, \leq_{L^*})$ are given by

$$(a,b) \wedge_{L^*} (c,d) = (\min(a,c), \max(b,d)), \tag{3}$$

$$(a,b) \vee_{L^*} (c,d) = (\max(a,c), \min(b,d)), \tag{4}$$

and $\mathbf{0}_{L^*} = (0, 1)$ and $\mathbf{1}_{L^*} = (1, 0)$ are the bottom and top elements of $(L^*, \leq_{L^*})$, respectively. Obviously, $(\mathbb{I}, \leq)$ can be embedded into $(L^*, \leq_{L^*})$: the function $\text{emb}_{\mathbb{I}} \colon \mathbb{I} \to L^*$ given by $\text{emb}_{\mathbb{I}}(x) = (x, 1 - x)$ is a possible embedding.

Note that the order $\leq_{L^*}$ is not linear. However, it is possible to construct refinements of $\leq_{L^*}$ which are linear (De Miguel et al. 2016).

An L^*-fuzzy subset A of a universe of discourse X is characterized by its L^*-membership function $\mu_A^{L^*} \colon X \to L^*$. The L^*-membership function $\mu_A^{L^*} \colon X \to L^*$ has two component functions, say $\mu_A, \nu_A \colon X \to \mathbb{I}$, i.e. for each $x \in X$ we have $\mu_A^{L^*}(x) = (\mu_A(x), \nu_A(x))$ and $\mu_A(x) + \nu_A(x) \leq 1$. Both μ_A and ν_A can be seen as membership functions of suitable fuzzy subsets $A^+, A^- \in \mathscr{F}(X)$, respectively, and then for each $x \in X$ we get $\mu_{A^+}(x) = \mu_A(x)$, $\mu_{A^-}(x) = \nu_A(x)$, and $\mu_{A^+}(x) + \mu_{A^-}(x) \leq 1$, i.e. $A^+ \subseteq \complement(A^-)$, where $\complement(A^-)$ is the complement of the fuzzy set A^- in the sense of Zadeh (1965), i.e. $\mu_{\complement(A^-)} = 1 - \mu_{(A^-)}$. The value $\mu_{A^+}(x)$ is usually called the degree of membership of the object x in the L^*-fuzzy set A, while $\mu_{A^-}(x)$ is said to be the degree of non-membership of the object x in the L^*-fuzzy set A.

We already mentioned in Section 1 that L^*-fuzzy sets were called "intuitionistic fuzzy sets" in Atanassov (1984, 1986, 1999), and we also explained there why we avoid this name.

Denoting the set of all L^*-fuzzy subsets of X by $\mathscr{F}_{L^*}(X)$, each $A \in \mathscr{F}_{L^*}(X)$ is characterized by its L^*-membership function $\mu_A^{L^*} \colon X \to L^*$, i.e. we may write $\mu_A^{L^*} = (\mu_A, \nu_A) = (\mu_{A^+}, \mu_{A^-})$. As a consequence, we obtain

$$\mathscr{F}_{L^*}(X) = \{(A^+, A^-) \in (\mathscr{F}(X))^2 \mid A^+ \subseteq \complement(A^-)\}.$$

This means that we can identify an L^*-fuzzy subset $A \in \mathscr{F}_{L^*}(X)$ with a pair of fuzzy sets (A^+, A^-) satisfying $A^+ \subseteq \complement(A^-)$. Subsequently, for two L^*-fuzzy subsets $A = (A^+, A^-)$ and $B = (B^+, B^-)$ of X we have

$$A \subseteq_{L^*} B \quad \text{if and only if } \left(A^+ \subseteq B^+ \text{and } B^- \subseteq A^-\right),$$

and the complement of an L^*-fuzzy subset $A = (A^+, A^-)$ is the L^*-fuzzy set $\complement A = (A^-, A^+)$. Note that we are using the same symbol $\complement$ for the complement of different types of sets (crisp sets, fuzzy sets, L^*-fuzzy sets) since its meaning will always be clear from the context.

It is immediately seen that $(\mathscr{F}_{L^*}(X), \subseteq_{L^*})$ is a complete bounded lattice with bottom element $\emptyset = (\emptyset, X)$ and top element $X = (X, \emptyset)$, and the lattice $(\mathscr{F}_{L^*}(X), \subseteq_{L^*})$ of L^*-fuzzy sets is isomorphic to $({L^*}^X, \leq_{L^*})$. Clearly, $(\mathscr{F}(X), \subseteq)$ can be embedded into $(\mathscr{F}_{L^*}(X), \subseteq_{L^*})$: an embedding $\text{emb}_{\mathscr{F}(X)} \colon \mathscr{F}(X) \to \mathscr{F}_{L^*}(X)$ is given by $\text{emb}_{\mathscr{F}(X)}(A) = (A, \complement A)$.

The bounded lattice $(L^*, \leq_{L^*})$ turns out to be isomorphic to a number of other lattices considered in the literature.

Remark 2.1: The bounded lattice $(L^*, \leq_{L^*})$ is isomorphic to each of the following two lattices:

(i) to the upper left triangle Δ in $\mathbb{I}^2$ (with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$), equipped with the pointwise partial order $\leq_\Delta$, i.e.

$$\Delta = \{(a, b) \in \mathbb{I}^2 \mid 0 \leq a \leq b \leq 1\},$$

$$(a, b) \leq_\Delta (c, d) \iff a \leq c \text{ and } b \leq d,$$

with bottom and top elements $\mathbf{0}_\Delta = (0,0)$ and $\mathbf{1}_\Delta = (1,1)$, respectively; here $\varphi\colon L^* \to \Delta$ given by $\varphi(x,y) = (x, 1-y)$ is an isomorphism between the lattices $(L^*, \leq_{L^*})$ and $(\Delta, \leq_\Delta)$;

(ii) to the bounded lattice $(L^\diamond, \leq_{L^\diamond})$ of all closed subintervals of $\mathbb{I}$ given by

$$L^\diamond = \{[a,b] \mid 0 \leq a \leq b \leq 1\},$$
$$[a,b] \leq_{L^\diamond} [c,d] \iff a \leq c \text{ and } b \leq d,$$

with bottom and top elements $\mathbf{0}_{L^\diamond} = [0,0]$ and $\mathbf{1}_{L^\diamond} = [1,1]$, respectively; here $\varphi\colon L^* \to L^\diamond$ given by $\varphi(x,y) = \left[x, 1-y\right]$ is an isomorphism between the lattices $(L^*, \leq_{L^*})$ and $(L^\diamond, \leq_{L^\diamond})$.

Observe that the lattices $(L^*, \leq_{L^*})$ and $(L^\diamond, \leq_{L^\diamond})$ are isomorphic, i.e. have the same mathematical structure, but their semantics may be quite different. The values of the membership function of an L^*-fuzzy set are typically given by a pair of numbers $(\mu(x), \nu(x)) \in \mathbb{I}^2$ representing the degree of membership and the degree of non-membership of the object x in the L^*-fuzzy set A. In the case of interval-valued fuzzy sets (based on $(L^\diamond, \leq_{L^\diamond})$), the values (a,b) of its membership function are usually identified with $[a,b]$, i.e. with a subinterval of $\mathbb{I}$ (in this sense interval-valued fuzzy sets are special examples of fuzzy set of type 2 – for a discussion of these generalizations of fuzzy sets, see, e.g. Zadeh 1975 and Walker and Walker 2009).

We only mention that there are infinitely many other bounded lattices which are isomorphic to the lattice $(L^*, \leq_{L^*})$. For example, the lattices $(P^*, \leq_{L^*})$, $(F^*, \leq_{L^*})$ and $(O^{q*}, \leq_{L^*})$ where

$$P^* = \{(a,b) \in \mathbb{I}^2 \mid a^2 + b^2 \leq 1\}, \quad \varphi_{P^*}(x,y) = (x^2, y^2)$$
$$F^* = \{(a,b) \in \mathbb{I}^2 \mid a^3 + b^3 \leq 1\}, \quad \varphi_{F^*}(x,y) = (x^3, y^3)$$
$$O^{q*} = \{(a,b) \in \mathbb{I}^2 \mid a^q + b^q \leq 1\} \quad \text{for some q} \geq 1, \quad \varphi_{O^{q*}}(x,y) = (x^q, y^q)$$

are all isomorphic to $(L^*, \leq_{L^*})$ (and the functions $\varphi_{P^*}\colon P^* \to L^*$, $\varphi_{F^*}\colon F^* \to L^*$, and $\varphi_{O^{q*}}\colon O^{q*} \to L^*$ as given above provide trivial isomorphisms).

Although the corresponding concepts of fuzzy sets based on P^*, F^* and O^{q*} are not only isomorphic mathematical objects but usually also share the same semantics, they sometimes are dealt with under various (fantasy) names such as “Pythagorean fuzzy sets” in Yager and Abbasov (2013), Yager (2014) and Dick, Yager, and Yazdanbakhsh (2016), “Fermatean fuzzy sets” in Senapati and Yager (2020a), Senapati and Yager (2019b), and “qth rung orthopair fuzzy subsets” in Yager (2017) (compare also Atanassov and Vassilev 2018).

In fact, for each involutive negation $n\colon \mathbb{I} \to \mathbb{I}$ (i.e. an order reversion bijection satisfying $n \circ n = \mathrm{id}_\mathbb{I}$) one can introduce in a straightforward way a lattice $((L^*)_n, \leq_{L^*})$ (which obviously is isomorphic to the bounded lattice $(L^*, \leq_{L^*})$) by putting

$$(L^*)_n = \{(a,b) \in \mathbb{I}^2 \mid a \leq n(b)\}.$$

Then, considering for each $q \in [1, \infty[$ the involutive negation $n_q\colon \mathbb{I} \to \mathbb{I}$ given by

$$n_q(x) = \left(1 - x^q\right)^{\frac{1}{q}},$$

we immediately obtain

$$L^* = (L^*)_{n_1}, \quad P^* = (L^*)_{n_2}, \quad F^* = (L^*)_{n_3}, \quad O^{q*} = (L^*)_{n_q}.$$

3. Capacities and some integrals based on them

For expected values of crisp, fuzzy and L^*-fuzzy events our basic setting is a measurable space (Bauer 1981; Halmos 1950), i.e. a pair $(X, \mathscr{A})$ consisting of a non-empty set X (the universe of discourse) and a σ-algebra $\mathscr{A}$ of subsets of X. Recall that a σ-algebra (Bauer 1981) is a collection of subsets of X which contains the empty set $\emptyset$ and the universe X and which is closed under complementation and countable unions.

Definition 3.1: If $(X, \mathscr{A})$ is a measurable space, then a *capacity* on $(X, \mathscr{A})$ is a function $m\colon \mathscr{A} \to \mathbb{I}$ which satisfies

(i) $m(\emptyset) = 0$ and $m(X) = 1$, (*boundary conditions*)
(ii) $m(A) \leq m(B)$ whenever $A \subseteq B$. (*monotone non-decreasing*)

Note again that a capacity as given in Definition 3.1, is neither required to be (finitely or σ-)additive nor to be continuous in any sense (recall that a set function $m\colon \mathscr{A} \to \mathbb{I}$ is said to be (finitely) additive whenever $m(A \cup B) = m(A) + m(B)$ for all disjoint subsets A, B of X, and σ-additive whenever

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

for each sequence $(A_1, A_2, \dots)$ of pairwise disjoint subsets of X).

We shall write $\mathscr{D}$ for the class of all measurable spaces. Given a specific measurable space $(X, \mathscr{A})$, the set of all capacities $m\colon \mathscr{A} \to \mathbb{I}$ will be denoted by $\mathscr{M}(X, \mathscr{A})$, and the set of all measurable functions $f\colon X \to \mathbb{I}$ by $\mathscr{F}(X, \mathscr{A})$. Recall that a function $f\colon X \to \mathbb{I}$ is *measurable* (with respect to the σ-algebra $\mathscr{A}$ on X and the σ-algebra $\mathscr{B}(\mathbb{I})$ of Borel subsets of $\mathbb{I}$) if, for each $\alpha \in \mathbb{I}$, we have $\{x \in X \mid f(x) \geq \alpha\} \in \mathscr{A}$.

Example 3.2: Let $(X, \mathscr{A})$ be a measurable space, $h\colon X \to \mathbb{I}$ a measurable function and $m\colon \mathscr{A} \to \mathbb{I}$ a capacity on $(X, \mathscr{A})$.

(i) The Choquet integral $\mathbf{Ch}_m$ (Choquet 1954), the Sugeno integral $\mathbf{Su}_m$ There are many concepts for integrals with respect to capacities: Sugeno (1974), and the Shilkret integral $\mathbf{Sh}_m$ (Shilkret 1971) with respect to m (see also Benvenuti, Mesiar, and Vivona 2002; Pap 2002) are given by, respectively,

$$\mathbf{Ch}_m(h) = \int_0^1 m(\{h \geq t\})\, \mathrm{d}t, \tag{5}$$

$$\mathbf{Su}_m(h) = \bigvee_{t \in \mathbb{I}} (t \wedge m(\{h \geq t\})), \tag{6}$$

$$\mathbf{Sh}_m(h) = \bigvee_{t \in \mathbb{I}} (t \cdot m(\{h \geq t\})) \tag{7}$$

where the integral on the right-hand side of (5) is a standard Riemann integral.

(ii) In Klement, Mesiar, and Pap (2010), the concept of so-called universal integrals was proposed, which can be defined for arbitrary measurable spaces $(X, \mathscr{A})$, for arbitrary measurable functions $h\colon X \to \mathbb{I}$ and for arbitrary capacities m on $(X, \mathscr{A})$. These universal integrals provide a common framework for the Choquet, Sugeno and Shilkret integrals given in (5)–(7), but also for other distinguished integrals which can be found in the literature: the Sugeno–Weber integral (Weber 1986), the Choquet-like integrals (Mesiar 1995), some copula-based integrals such as the Imaoka integral and the inverse Sugeno integral (Imaoka 1997, 2000) (see also Klement et al. 2010; Klement and Mesiar 2012b, 2012a), and many others.

4. Expected values

Following some ideas in Zadeh (1968) and Grzegorzewski and Mrówka (2002); Grzegorzewski (2013), on the one hand, and Klement and Mesiar (2015), on the other hand, we review first the concept of expected values of fuzzy events and extend it to the case of L^*-expected values, i.e. to expected values of L^*-fuzzy events.

Given a measurable space $(X, \mathscr{A})$, then a fuzzy subset A of X with a measurable membership function $\mu_A\colon X \to \mathbb{I}$ (which is equivalent to the condition that for each $\alpha \in\,]0,1]$ the α-level set $A^{[\alpha]} = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is contained in $\mathscr{A}$), is called a fuzzy event. The set of all fuzzy events will be denoted by $\mathscr{A}_{\mathbb{I}}$. Clearly, $(\mathscr{A}_{\mathbb{I}}, \subseteq)$ is a bounded sublattice of $(\mathscr{F}(X), \subseteq)$ with bottom element $\emptyset$ and top element X.

The classical expected value of a random variable $Y\colon X \to \mathbb{R}$ is given by

$$E_P(Y) = \int_X Y \,\mathrm{d}P,$$

where $P\colon \mathscr{A} \to \mathbb{I}$ is a probability measure on the measurable space $(X, \mathscr{A})$ and the integral is a Lebesgue(–Stieltjes) integral. This concept was used in Zadeh (1968) to define the probability of a fuzzy event A, simply replacing the random variable Y by the membership function of A (see (8) below). Since then, several approaches to the expected value of fuzzy events were proposed and studied. In Klement and Mesiar (2015), a general concept for expected values of fuzzy events was introduced. We recall here those notions and results which will be needed in the following sections.

Definition 4.1: Let $(X, \mathscr{A})$ be a measurable space. An expected value of fuzzy events is an order homomorphism $\mathbf{E}$ from $(\mathscr{A}_{\mathbb{I}}, \subseteq)$ to $(\mathbb{I}, \leq)$, i.e. a function $\mathbf{E}\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ satisfying

(i) $\mathbf{E}(\emptyset) = 0$ and $\mathbf{E}(X) = 1$,
(ii) $\mathbf{E}(A) \leq \mathbf{E}(B)$ whenever $A \subseteq B$.

The following basic example of an expected value of fuzzy events was proposed by Zadeh in Zadeh (1968) under the name “probability of fuzzy events”.

Example 4.2: Starting with a probability measure P on $(X, \mathscr{A})$, the expected value of fuzzy sets $\mathbf{E}_P \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ is given by

$$\mathbf{E}_P(A) = \int_X \mu_A \, \mathrm{d}P, \tag{8}$$

where the integral on the right hand side is the Lebesgue integral.

It is evident that for each expected value $\mathbf{E}$ of fuzzy events the function $m_{\mathbf{E}} \colon \mathscr{A} \to \mathbb{I}$ given by $m_{\mathbf{E}}(A) = \mathbf{E}(A)$ is a capacity on $(X, \mathscr{A})$.

Given a capacity $m \in \mathscr{M}(X, \mathscr{A})$, the monotonicity in Definition 4.1 implies that the smallest expected value $(\mathbf{E}_m)^{\perp}$ of fuzzy events with respect to m is given by $(\mathbf{E}_m)^{\perp}(A) = m(A^{[1]})$, where $A^{[1]} = \{\mu_A = 1\}$ is the kernel of A, while the greatest expected value $(\mathbf{E}_m)^{\top}$ of fuzzy events with respect to m is given by $(\mathbf{E}_m)^{\top}(A) = m(\{\mu_A > 0\})$, where $\{\mu_A > 0\}$ is the support of A.

Not only the Lebesgue integral as in (8), also other integrals such as the Choquet integral, the Sugeno integral and the Shilkret integral (see Example 3.2(i)) can be used to construct expected values of fuzzy events:

Example 4.3: Let $(X, \mathscr{A})$ be a measurable space and $m \colon \mathscr{A} \to \mathbb{I}$ a capacity.

(i) For a fuzzy event $A \in \mathscr{A}_{\mathbb{I}}$, the expected values of A based on the Choquet, Sugeno and Shilkret integral (Choquet, Sugeno and Shilkret expectation) (compare (5)–(7)) are given by, respectively,

$$(\mathrm{Ch})\mathbf{E}_m(A) = \mathbf{Ch}_m(\mu_A) \tag{9}$$

$$(\mathrm{Su})\mathbf{E}_m(A) = \mathbf{Su}_m(\mu_A) \tag{10}$$

$$(\mathrm{Sh})\mathbf{E}_m(A) = \mathbf{Sh}_m(\mu_A) \tag{11}$$

(ii) More generally, if $\mathbf{I}$ is a universal integral on $\mathbb{I}$ based on a semicopula (Durante and Sempi 2005) and the capacity m (as introduced and studied in (Klement, Mesiar, and Pap 2010)), then the function $(\mathrm{I})\mathbf{E}_m \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ given by (see Klement and Mesiar 2015, (5.1))

$$(\mathrm{I})\mathbf{E}_m(A) = \mathbf{I}(m, \mu_A) \tag{12}$$

is an expected value on fuzzy events.

If $(X, \mathscr{A})$ is a measurable space, then an L^*-fuzzy set $A = (A^+, A^-) \in \mathscr{F}_{L^*}(X)$ will be called an L^*-fuzzy event whenever both A^+ and A^- are fuzzy events, i.e. A^+ and A^- belong to $\mathscr{A}_{\mathbb{I}}$. The set of all L^*-fuzzy events will be denoted $\mathscr{A}_{L^*}$. Clearly, $(\mathscr{A}_{L^*}, \subseteq_{L^*})$ is a bounded sublattice of $(\mathscr{F}_{L^*}(X), \subseteq_{L^*})$. Its bottom and top elements are the L^*-fuzzy sets $\emptyset = (\emptyset, X)$ (i.e.$\mu_{\emptyset}^{L^*}(x) = \mathbf{0}_{L^*}$ for each $x \in X$) and $X = (X, \emptyset)$ (i.e. $\mu_X^{L^*}(x) = \mathbf{1}_{L^*}$ for each $x \in X$), respectively.

The concept of a capacity and of an expected value of fuzzy events is extended to the case of L^*-fuzzy events in a straightforward way:

Definition 4.4: Let $(X, \mathscr{A})$ be a measurable space. An L^*-capacity m on $\mathscr{A}$ is an order homomorphism from $(\mathscr{A}, \subseteq)$ to $(L^*, \leq_{L^*})$, i.e. a function $m\colon \mathscr{A} \to L^*$ satisfying

(i) $m(\emptyset) = \mathbf{0}_{L^*}$ and $m(X) = \mathbf{1}_{L^*}$,
(ii) $m(A) \leq_{L^*} m(B)$ whenever $A \subseteq B$.

Corollary 4.5: *If $(X, \mathscr{A})$ is a measurable space and $m\colon \mathscr{A} \to L^*$ an L^*-capacity, then there exist two capacities $m_1, m_2\colon \mathscr{A} \to \mathbb{I}$ satisfying $m_1 \leq m_2$ such that for all crisp sets $A \in \mathscr{A}$ we have*

$$m(A) = (m_1(A), 1 - m_2(A)). \tag{13}$$

Definition 4.6: Let $(X, \mathscr{A})$ be a measurable space. An L^*-expected value (or expected value of L^*-fuzzy events) is an order homomorphism $\mathbf{E}$ from $(\mathscr{A}_{L^*}, \subseteq_{L^*})$ to $(L^*, \leq_{L^*})$, i.e. a function $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ satisfying

(i) $\mathbf{E}(\emptyset) = \mathbf{0}_{L^*}$ and $\mathbf{E}(X) = \mathbf{1}_{L^*}$,
(ii) $\mathbf{E}(A) \leq_{L^*} \mathbf{E}(B)$ whenever $A \subseteq_{L^*} B$.

Remark 4.7: Denote the set of all L^*-expected values by Λ and consider the pointwise order on Λ inherited from $\leq_{L^*}$ (which we also shall denote by $\leq_{L^*}$). Then we have:

(i) $(\Lambda, \leq_{L^*})$ is a complete distributive and bounded lattice with bottom and top elements $\underline{\mathbf{E}}\colon \mathscr{A}_{L^*} \to L^*$ and $\overline{\mathbf{E}}\colon \mathscr{A}_{L^*} \to L^*$ given by, respectively,

$$\underline{\mathbf{E}}(A) = \begin{cases} (0,1) & \text{if } A = \emptyset, \\ (1,0) & \text{otherwise,} \end{cases} \qquad \overline{\mathbf{E}}(A) = \begin{cases} (1,0) & \text{if } A = X, \\ (0,1) & \text{otherwise.} \end{cases}$$

(ii) Λ is a convex set, i.e. for any L^*-expected values $\mathbf{F}, \mathbf{G} \in \Lambda$ and all $\lambda \in \mathbb{I}$, also $\lambda \cdot \mathbf{F} + (1 - \lambda) \cdot \mathbf{G}$ is an element of Λ. Note that for $(a, b), (c, d) \in L^*$ their convex sum is defined as

$$\lambda \cdot (a, b) + (1 - \lambda) \cdot (c, d) = (\lambda \cdot a + (1 - \lambda) \cdot c, \lambda \cdot b + (1 - \lambda) \cdot d).$$

Starting with a probability measure, an L^*-expected value can be constructed considering in both coordinates the expected value of fuzzy events given by (8) going back to Zadeh (1968). This approach was proposed in Grzegorzewski and Mrówka (2002); Grzegorzewski (2013), and its axiomatic characterization was discussed in Ciungu and Riečan (2010), compare also Riečan (2006).

Example 4.8: Let $(X, \mathscr{A})$ be a measurable space and $P\colon \mathscr{A} \to \mathbb{I}$ be a probability measure. Then the function $\mathbf{E}_P^{L^*}\colon \mathscr{A}_{L^*} \to L^*$ given by

$$\mathbf{E}_P^{L^*}((A^+, A^-)) = \left(\int_X \mu_{A^+}\, \mathrm{d}P, \int_X \mu_{A^-}\, \mathrm{d}P\right) \tag{14}$$

is an L^*-expected value.

Remark 4.9: Let $(X, \mathscr{A})$ be a measurable space and $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ an L^*-expected value. In a natural way, $\mathbf{E}$ gives rise to two expected values of fuzzy events and to some (L^*-)capacities.

(i) The L^*-expected value $\mathbf{E}$ induces an L^*-capacity $m_{\mathbf{E}}\colon \mathscr{A} \to L^*$ on $\mathscr{A}$ given by

$$m_{\mathbf{E}}(A) = \mathbf{E}(A).$$

(ii) The L^*-expected value $\mathbf{E}$ induces two expected values $\mathbf{E}_1, \mathbf{E}_2\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ of fuzzy events as follows:

$$\mathbf{E}_1(A) = \pi_1(\mathbf{E}((A, \complement A))), \quad \mathbf{E}_2(A) = 1 - \pi_2(\mathbf{E}((A, \complement A))), \tag{15}$$

where we use the embedding $\mathrm{emb}_{\mathscr{F}(X)}$ from $(\mathscr{F}(X), \subseteq)$ into $(\mathscr{F}_{L^*}(X), \subseteq_{L^*})$ given by

$$\mathrm{emb}_{\mathscr{F}(X)}(A) = (A, \complement A),$$

and where π_1 and π_2 denote the first and second projection from L^* to $\mathbb{I}$, respectively. Observe that (15) implies that for all fuzzy events $A \in \mathscr{A}_{\mathbb{I}}$

$$\mathbf{E}((A, \complement A)) = (\mathbf{E}_1(A), 1 - \mathbf{E}_2(A)), \tag{16}$$

i.e. we necessarily have $\mathbf{E}_1 \leq \mathbf{E}_2$.

(iii) Using the two expected values $\mathbf{E}_1$ and $\mathbf{E}_2$ of fuzzy events given in (15), the L^*-expected value $\mathbf{E}$ induces two capacities $m_{\mathbf{E}_1}, m_{\mathbf{E}_2}\colon \mathscr{A} \to \mathbb{I}$ on $\mathscr{A}$ given by

$$m_{\mathbf{E}_1}(A) = \mathbf{E}_1(A) \quad \text{and} \quad m_{\mathbf{E}_2}(A) = \mathbf{E}_2(A). \tag{17}$$

Definition 4.10: Let $(X, \mathscr{A})$ be a measurable space and $m\colon \mathscr{A} \to L^*$ be an L^*-capacity. We say that an L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ is based on the L^*-capacity m if for all crisp sets $A \in \mathscr{A}$ we have

$$m(A) = \mathbf{E}((A, \complement A)). \tag{18}$$

Observe that, keeping the notations of Definition 4.10, the validity of (18) for all $A \in \mathscr{A}$ is equivalent to each of the following assertions:

(i) the restrictions $m_1{\restriction}_{\mathscr{A}}$ and $m_2{\restriction}_{\mathscr{A}}$ coincide with the restrictions $\mathbf{E}_1{\restriction}_{\mathscr{A}}$ and $\mathbf{E}_2{\restriction}_{\mathscr{A}}$, respectively, i.e. for all crisp events $A \in \mathscr{A}$ we have $m_1(A) = \mathbf{E}_1(A)$ and $m_2(A) = \mathbf{E}_2(A)$,

(ii) $m_1 = m_{\mathbf{E}_1}$ and $m_2 = m_{\mathbf{E}_2}$,

where $m_1, m_2\colon \mathscr{A} \to \mathbb{I}$ are the two capacities satisfying (13) which exist according to Corollary 4.5, and $\mathbf{E}_1, \mathbf{E}_2\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ are the two expected values of fuzzy events given by (15), and the two capacities $m_{\mathbf{E}_1}, m_{\mathbf{E}_2}\colon \mathscr{A} \to \mathbb{I}$ on $\mathscr{A}$ are given by (17).

Definition 4.11: Let $(X, \mathscr{A})$ be a measurable space and $m\colon \mathscr{A} \to \mathbb{I}$ be a capacity. We say that an L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ is based on the capacity m if for the two capacities

$m_1, m_2 \colon \mathscr{A} \to \mathbb{I}$ satisfying (13) which exist according to Corollary 4.5 we have $m_1 = m_2 = m$.

Proposition 4.12: *Let* $\mathbf{E}_1, \mathbf{E}_2 \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ *be two expected values of fuzzy events satisfying* $\mathbf{E}_1 \leq \mathbf{E}_2$*. Then the smallest and greatest* L^**-expected values* $\mathbf{E}^{\perp}, \mathbf{E}^{\top} \colon \mathscr{A}_{L^*} \to L^*$ *such that* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *can be obtained from both* $\mathbf{E}^{\perp}$ *and* $\mathbf{E}^{\top}$ *via (15) are defined, for each* L^**-fuzzy event* $A = (A^+, A^-)$*, by*

$$\mathbf{E}^{\perp}(A) = (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(A^+)), \tag{19}$$

$$\mathbf{E}^{\top}(A) = (\mathbf{E}_1(\complement(A^-)), 1 - \mathbf{E}_2(\complement(A^-))). \tag{20}$$

Proof: Fix two L^*-expected values $\mathbf{E}_1, \mathbf{E}_2 \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ satisfying $\mathbf{E}_1 \leq \mathbf{E}_2$, and define the two functions $\mathbf{E}^{\perp}, \mathbf{E}_{\top} \colon \mathscr{A}_{L^*} \to L^*$ as in (19) and (20), respectively.

Then the boundary conditions in Definition 4.6(i) follow from the boundary conditions in Definition 4.1(i):

$$\begin{aligned}
\mathbf{E}^{\perp}(\emptyset) = \mathbf{E}^{\top}(\emptyset) = (\mathbf{E}_1(\emptyset), 1 - \mathbf{E}_2(\emptyset)) = (0,1) = \mathbf{0}_{L^*},\\
\mathbf{E}^{\perp}(X) = \mathbf{E}^{\top}(X) = (\mathbf{E}_1(X), 1 - \mathbf{E}_2(X)) = (1,0) = \mathbf{1}_{L^*}.
\end{aligned}$$

For L^*-fuzzy events $A = (A^*, A^-)$ and $B = (B^*, B^-)$ with $A \subseteq_{L^*} B$, i.e. with $A^+ \subseteq B^+$ and $B^- \subseteq A^-$, the monotonicity of both $\mathbf{E}^{\perp}$ and $\mathbf{E}^{\top}$ according to Definition 4.6(ii) follows from the monotonicity of $\mathbf{E}_1$ and $\mathbf{E}_2$ because of

$$\begin{aligned}
\mathbf{E}^{\perp}(A) = (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(A^+)) \leq_{L^*} (\mathbf{E}_1(B^+), 1 - \mathbf{E}_2(B^+)) = \mathbf{E}^{\perp}(B),\\
\mathbf{E}^{\top}(A) = (\mathbf{E}_1(\complement(A^-)), 1 - \mathbf{E}_2(\complement(A^-))) \leq_{L^*} (\mathbf{E}_1(\complement(B^-)), 1 - \mathbf{E}_2(\complement(B^-))) = \mathbf{E}^{\perp}(B),
\end{aligned}$$

implying that $\mathbf{E}^{\perp}$ and $\mathbf{E}^{\top}$ are expected values of L^*-fuzzy events. Obviously, both triplets $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}^{\perp})$ and $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}^{\top})$ are solutions of the functional equations in (15).

Finally, for each L^*-expected value $\mathbf{E} \colon \mathscr{A}_{L^*} \to L^*$ such that $\mathbf{E}_1$ and $\mathbf{E}_2$ can be obtained from $\mathbf{E}$ via (15), formula (16) and the monotonicity of $\mathbf{E}$ together with $A^+ \subseteq \complement(A^-)$ imply for each L^*-fuzzy event (A^+, A^-)

$$\begin{aligned}
\mathbf{E}^{\perp}((A^+, A^-)) &= (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(A^+)) = \mathbf{E}((A^+, \complement(A^+))),\\
\mathbf{E}((\complement(A^-), A^-)) &= (\mathbf{E}_1(\complement(A^-)), 1 - \mathbf{E}_2(\complement(A^-))) = \mathbf{E}^{\top}((A^+, A^-)),\\
\mathbf{E}((A^+, \complement(A^+))) &\leq_{L^*} \mathbf{E}((A^+, A^-)) \leq_{L^*} \mathbf{E}((\complement(A^-), A^-)),
\end{aligned}$$

thus completing the proof. ■

Remark 4.13: Keeping the notations of Proposition 4.12, from the constructions (19) and (20) it follows that the two extremal L^*-expected values $\mathbf{E}^{\perp}$ and $\mathbf{E}^{\top}$ and, subsequently, all L^*-expected values satisfying that $\mathbf{E}_1$ and $\mathbf{E}_2$ can be obtained from $\mathbf{E}$ via (15) coincide for all L^*-fuzzy events (A^+, A^-) satisfying $A^- = \complement(A^+)$, i.e.for all fuzzy events $A \in \mathscr{A}_{\mathbb{I}}$ we have

$$\mathbf{E}^{\perp}((A, \complement A)) = \mathbf{E}((A, \complement A)) = \mathbf{E}^{\top}((A, \complement A)).$$

Recall that the membership function $\mu_A^{L^*}: X \to L^*$ of an L^*-fuzzy set $A = (A^+, A^-)$ can be seen as a pair of functions $\mu_A, \nu_A : X \to \mathbb{I}$, i.e. $\mu_A^{L^*} = (\mu_A, \nu_A)$, where $\mu_A(x)$ describes the degree of membership of x in A and $\nu_A(x)$ describes the degree of non-membership of x in A.

Example 4.14: Let $(X, \mathscr{A})$ be a measurable space.

(i) Consider the L^*-expected value $\mathbf{E}: \mathscr{A}_{L^*} \to L^*$ given by

$$\mathbf{E}(A) = \left(\bigvee_{x \in X} \frac{1 + 2\mu_A(x) - \nu_A(x)}{3}, \bigwedge_{x \in X} \frac{1 - \mu_A(x) + 2\nu_A(x)}{3} \right).$$

Then for each fuzzy event $A \in \mathscr{A}_{\mathbb{I}}$ we get

$$\mathbf{E}_1(A) = \bigvee_{x \in X} \mu_A(x) = \mathbf{E}_2(A),$$

and for each L^*-fuzzy event $A \in \mathscr{A}_{L^*}$

$$\mathbf{E}^{\perp}(A) = \left(\bigvee_{x \in X} \mu_A(x), 1 - \bigvee_{x \in X} \mu_A(x) \right),$$

$$\mathbf{E}^{\top}(A) = \left(1 - \bigwedge_{x \in X} \nu_A(x), \bigwedge_{x \in X} \nu_A(x) \right).$$

(ii) For each L^*-expected value $\mathbf{E}: \mathscr{A}_{L^*} \to L^*$ consider the L^*-expected values $\mathbf{E}^{\diamond}, \mathbf{E}_{\diamond}: \mathscr{A}_{L^*} \to L^*$ given by

$$\mathbf{E}^{\diamond}((A^+, A^-)) = \mathbf{E}((A^+, (A^-)^{(2)})),$$

$$\mathbf{E}_{\diamond}((A^+, A^-)) = \mathbf{E}(((A^+)^{(2)}, A^+)),$$

where, for a fuzzy subset A of X, the membership function of the fuzzy set $A^{(2)}$ is given by $(\mu_A)^2$. Then we have

$$(\mathbf{E}^{\diamond})_1 = \mathbf{E}_1 \quad \text{and} \quad (\mathbf{E}^{\diamond})_2 \geq \mathbf{E}_2,$$

$$(\mathbf{E}_{\diamond})_1 \leq \mathbf{E}_1 \quad \text{and} \quad (\mathbf{E}_{\diamond})_2 = \mathbf{E}_2.$$

Note, however, that the expected values of fuzzy events $(\mathbf{E}^{\diamond})_2$ and $\mathbf{E}_2$, on the one hand, and $(\mathbf{E}_{\diamond})_1$ and $\mathbf{E}_1$, on the other hand, coincide for crisp events, i.e. for each $A \in \mathscr{A}$ we have $(\mathbf{E}_{\diamond})_1(A) = \mathbf{E}_1(A)$ and $(\mathbf{E}^{\diamond})_2(A) = \mathbf{E}_2(A)$.

(iii) Applying the construction given in (ii) to the L^*-expected value $\mathbf{E}$ considered in (i), then for each fuzzy event $A \in \mathscr{A}_{\mathbb{I}}$ we get

$$(\mathbf{E}_{\diamond})_1(A) = \bigvee_{x \in X} (\mu_A(x))^2 \leq \bigvee_{x \in X} \mu_A(x) = \mathbf{E}_1(A),$$

$$(\mathbf{E}^{\diamond})_2(A) = \bigvee_{x \in X} \left(2\mu_A(x) - (\mu_A(x))^2\right) \geq \bigvee_{x \in X} \mu_A(x) = \mathbf{E}_2(A),$$

showing that the inequalities above may be strict (in fact, these inequalities are strict whenever $0 < \bigvee_{x \in X} \mu_A(x) < 1$).

Starting from a capacity $m \in \mathscr{M}(X,\mathscr{A})$, then Proposition 4.12 and the monotonicity in Definition 4.6(ii) allows us to obtain the smallest L^*-expected value $\mathbf{E}_m^{\perp}$ and the largest L^*-expected value $\mathbf{E}_m^{\top}$ based on m as follows:

Corollary 4.15: *Given a measurable space $(X,\mathscr{A})$ and a capacity $m\colon \mathscr{A} \to \mathbb{I}$. Then the smallest L^*-expected value $\mathbf{E}_m^{\perp}\colon \mathscr{A}_{\mathbb{I}} \to L^*$ based on m and the greatest L^*-expected value $\mathbf{E}_m^{\top}\colon \mathscr{A}_{\mathbb{I}} \to L^*$ based on m are given by*

$$\mathbf{E}_m^{\perp}((A^+,A^-)) = \big(m\big((A^+)^{[1]}\big), 1 - m\big((A^+)^{[1]}\big)\big),$$
$$\mathbf{E}_m^{\top}((A^+,A^-)) = (1 - m(\{\mu_{A^-} < 1\}), m(\{\mu_{A^-} < 1\})),$$

where $(A^+)^{[1]} \in \mathscr{P}(X)$ is the kernel of the fuzzy subset A^+ of X and $\mu_{A^-}\colon X \to \mathbb{I}$ is the membership function of the fuzzy subset A^- of X.

The following extension of an expected value of fuzzy events to an L^*-expected value is a consequence of Ciungu and Riečan (2010).

Corollary 4.16: *Let $(X,\mathscr{A})$ be a measurable space and $\mathbf{E}\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ an expected value of fuzzy events. Then the function $\mathbf{E}^{L^*}\colon \mathscr{A}_{L^*} \to L^*$ given by*

$$\mathbf{E}^{L^*}((A^+,A^-)) = (\mathbf{E}(A^+), 1 - \mathbf{E}(\complement(A^-))) \tag{21}$$

is an L^*-expected value.

Definition 4.17: Let $(X,\mathscr{A})$ be a measurable space and $m\colon \mathscr{A} \to \mathbb{I}$ be a capacity.

(i) Consider the expected values of fuzzy events

$$(\mathrm{Ch})\mathbf{E}_m, (\mathrm{Su})\mathbf{E}_m, (\mathrm{Sh})\mathbf{E}_m\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$$

based on the Choquet, the Sugeno and the Shilkret integral with respect to m (as given in Example 4.3(i)), respectively. Then the functions

$$(\mathrm{Ch})\mathbf{E}_m^{L^*}, (\mathrm{Su})\mathbf{E}_m^{L^*}, (\mathrm{Sh})\mathbf{E}_m^{L^*}\colon \mathscr{A}_{L^*} \to L^*$$

constructed by means of (21) are called the L^*-expected values based on the Choquet, the Sugeno and the Shilkret integral with respect to m, respectively.

(ii) If $(\mathrm{I})\mathbf{E}_m\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ is the expected value of fuzzy events based on a universal integral $\mathbf{I}$ on $\mathbb{I}$ with respect to a semicopula (Durante and Sempi 2005) and to the capacity m (as given by (12) in Example 4.3(ii)), then for each L^*-capacity $m^{L^*}\colon \mathscr{A} \to L^*$ related to the capacities $m_1, m_2\colon \mathscr{A} \to \mathbb{I}$ with $m_1 \leq m_2$ satisfying (13), the function $(\mathrm{I})\mathbf{E}_{m^{L^*}}\colon \mathscr{A}_{L^*} \to L^*$ given by

$$(\mathrm{I})\mathbf{E}_{m^{L^*}}((A^+,A^-)) = \big((\mathrm{I})\mathbf{E}_{m_1}(A^+), 1 - (\mathrm{I})\mathbf{E}_{m_2}(\complement(A^-))\big) \tag{22}$$

is an L^*-expected value, and we call it the L^*-expected value based on the universal integral $\mathbf{I}$ with respect to m.

Example 4.18: If $(X, \mathscr{A})$ is a measurable space and $m\colon \mathscr{A} \to L^*$ an L^*-capacity then Corollary 4.5 tells us that there exist two capacities $m_1, m_2\colon \mathscr{A} \to \mathbb{I}$ satisfying $m_1 \leq m_2$ such that (13) holds, i.e. for all crisp sets $A \in \mathscr{A}$ we have $m(A) = (m_1(A), 1 - m_2(A))$. Then for each L^*-fuzzy event $A \in \mathscr{A}_{L^*}$ characterized by the pair (μ_A, ν_A) of functions $\mu_A, \nu_A\colon X \to \mathbb{I}$ representing membership and non-membership in A, respectively, the L^*-expected value $(\mathrm{Su})\mathbf{E}_m^{L^*}$ of A based on the Sugeno integral is given by

$$\begin{aligned}
(\mathrm{Su})\mathbf{E}_m^{L^*}(A) &= \bigvee_{B\in\mathscr{A}} L\Big(m(B) \wedge_{L^*} \bigwedge_{x\in B} L(\mu_A(x), \nu_A(x))\Big) \\
&= \left(\bigvee_{B\in\mathscr{A}} \Big(m_1(B) \wedge \bigwedge_{x\in B} \mu_A(x)\Big), \bigwedge_{B\in\mathscr{A}} \Big((1-m_2)(B) \vee \bigvee_{x\in B} \nu_A(x)\Big)\right) \\
&= \left(\mathbf{Su}_{m_1}(\mu_A), 1 - \bigvee_{B\in\mathscr{A}} \Big(m_2(A) \wedge \bigwedge_{x\in B} (1-\nu_A(x))\Big)\right) \\
&= \big(\mathbf{Su}_{m_1}(\mu_A), 1 - \mathbf{Su}_{m_2}(1-\nu_A)\big) \\
&= \big(\mathbf{Su}_{m_1}(\mu_A), \mathbf{Su}_{m_2^d}(\nu_A)\big),
\end{aligned}$$

where the dual capacity $m_2^d\colon \mathscr{A} \to \mathbb{I}$ of m_2 is given by $m_2^d(A) = 1 - m_2(\complement A)$. A similar equality holds also in the case of the Choquet integral, i.e.

$$(\mathbf{Ch})\mathbf{E}_m^{L^*}(A) = \big(\mathbf{Ch}_{m_1}(\mu_A), \mathbf{Ch}_{m_2^d}(\nu_A)\big),$$

but not for the Shilkret integral.

In multicriteria decision making, expected values of fuzzy events, similarly as expected values of random variables, can be considered as utility functions, inducing a pre-order on these sets. Although then any two alternatives (expressed in the form of two fuzzy sets or of two random variables) are comparable, they may be non-distinguishable. In the case of L^*-fuzzy sets, for any expected value we can introduce a weak partial order, and thus a general preference structure (including possible incomparabilities) on any set of alternatives expressed in the form of L^*-fuzzy subsets of a fixed set X (which, e.g. can be a set of criteria or a set of experts). Clearly, different expected values yield different preference structures, in general. In the following example, some of them are shown explicitly.

Example 4.19: Fix $X = \{1, 2, 3\}$, and let $m^*, m_*, m_u\colon \mathscr{P}(X) \to \mathbb{I}$ be the greatest capacity, the smallest capacity and the uniform capacity on X, respectively. Observe that m^* and m_* are $\{0, 1\}$-valued capacities, in which case the Sugeno and the Choquet integrals coincide:

$$\begin{aligned}
\mathbf{Su}_{m^*}(f) &= \mathbf{Ch}_{m^*}(f) = \max\{f(1), f(2), f(3)\} = \max f, \\
\mathbf{Su}_{m_*}(f) &= \mathbf{Ch}_{m_*}(f) = \min\{f(1), f(2), f(3)\} = \min f.
\end{aligned}$$

For the uniform capacity m_u (which is a probability measure), we get

$$\mathbf{Su}_{m_u}(f) = \begin{cases} \min f & \text{if } \min f \geq \frac{2}{3}, \\ \min(\operatorname{med} f, \frac{2}{3}) & \text{if } \min f < \frac{2}{3} \text{ and } \operatorname{med} f \geq \frac{1}{3}, \\ \min(\max f, \frac{1}{3}) & \text{otherwise,} \end{cases}$$

$$\mathbf{Ch}_{m_u}(f) = \frac{1}{3}(f(1) + f(2) + f(3)).$$

Moreover, we have $(m^*)^d = m_*$ and $(m_u)^d = m_u$.

Consider three L^*-fuzzy subsets A_1, A_2 and A_3 of X (which may represent three alternatives a_1, a_2 and a_3 in the decision problem under consideration) given by

$$\big(\mu_{A_i}^{L^*}(1), \mu_{A_i}^{L^*}(2), \mu_{A_i}^{L^*}(3)\big) = \begin{cases} \big((0.2, 0.7), (0.5, 0.3), (0.4, 0.4)\big) & \text{if } i = 1, \\ \big((0.1, 0.9), (0.3, 0.6), (0.6, 0.4)\big) & \text{if } i = 2, \\ \big((0.3, 0.4), (0.3, 0.4), (0.3, 0.4)\big) & \text{if } i = 3. \end{cases}$$

Now we consider five simple cases of L^*-capacities $m\colon \mathscr{P}(X) \to L^*$ and L^*-expected values $\mathbf{E}\colon \mathscr{P}(X)_{L^*} \to L^*$, and we obtain the following different scenarios (for a visualization see Figure 1):

(i) If m is generated by m_* and m_* and if $\mathbf{E} = \mathbf{Su}_m^{L^*} = \mathbf{Ch}_m^{L^*}$, then the L^*-expected values of A_1, A_2 and A_3 are given by $\mathbf{E}(A_1) = (0.2, 0.9)$, $\mathbf{E}(A_2) = (0.1, 0.9)$ and $\mathbf{E}(A_3) = (0.3, 0.4)$. Hence $a_2 <_{L^*} a_1 <_{L^*} a_3$, i.e. a_3 is the best alternative, while a_2 is considered to be the worst alternative in this case.

(ii) If m is generated by m^* and m^* and if $\mathbf{E} = \mathbf{Su}_m^{L^*} = \mathbf{Ch}_m^{L^*}$, then we get the values $\mathbf{E}(A_1) = (0.5, 0.3)$, $\mathbf{E}(A_2) = (0.6, 0.4)$ and $\mathbf{E}(A_3) = (0.3, 0.4)$. Hence $a_3 <_{L^*} a_1$ and $a_3 <_{L^*} a_2$, but a_1 and a_2 are incomparable, i.e. a_3 is the worst alternative, and we cannot distinguish between two best alternatives a_1 and a_2.

(iii) If m is generated by m_* and m^* and if $\mathbf{E} = \mathbf{Su}_m^{L^*} = \mathbf{Ch}_m^{L^*}$, then the result is $\mathbf{E}(A_1) = (0.2, 0.3)$, $\mathbf{E}(A_2) = (0.1, 0.4)$ and $\mathbf{E}(A_3) = (0.3, 0.4)$. Hence $a_2 <_{L^*} a_1$, $a_2 <_{L^*} a_3$, and a_1 and a_3 are incomparable, i.e. a_2 is the worst alternative, and a_1 and a_3 are two indistinguishable best alternatives.

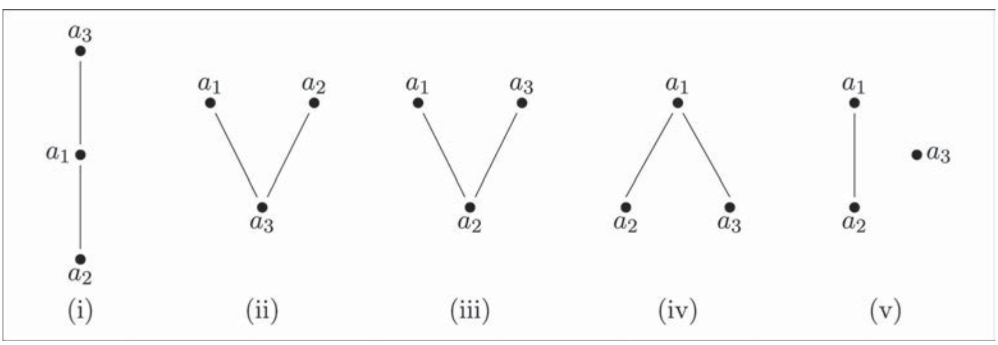

Figure 1. The five cases considered in Example 4.19.

(iv) If m is generated by m_u and m_u and if $\mathbf{E} = \mathbf{Su}_m^{L^*}$ then $\mathbf{E}(A_1) = (0.4, 0.4)$, $\mathbf{E}(A_2) = (\frac{1}{3}, 0.6)$ and $\mathbf{E}(A_3) = (0.3, 0.4)$. Hence $a_2 <_{L^*} a_1$, $a_3 <_{L^*} a_1$, and a_2 and a_3 are incomparable, i.e. a_1 is the best alternative, and a_2 and a_3 are two indistinguishable worst alternatives.

(v) If m is generated by m_u and m_u and if $\mathbf{E} = \mathbf{Ch}_m^{L^*}$ then $\mathbf{E}(A_1) = (\frac{11}{30}, \frac{14}{30})$, $\mathbf{E}(A_2) = (\frac{10}{30}, \frac{19}{30})$ and $\mathbf{E}(A_3) = (\frac{9}{30}, \frac{12}{30})$. Hence $a_2 <_{L^*} a_1$, and a_3 is incomparable with both a_1 and a_2.

As we can see, we have a large variety of possibilities how to build preference structures by means of L^*-expected values, which enrich the background of the theory of decision making.

5. Linearity and weaker forms thereof

The Lebesgue integral is a prototypical linear functional, so the classical expected value with respect to a probability measure is always linear. Since the truth values in L^* and, therefore, the values of membership functions of L^*-fuzzy sets cannot be negative, it suffices to consider positive-linearity for L^*-expected values.

Definition 5.1: Let $(X, \mathscr{A})$ be a measurable space. An L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ is called positive-linear if, for all L^*-fuzzy events $A, B \in \mathscr{A}_{L^*}$ and for all $\alpha, \beta \in [0, \infty[$ with $\alpha \cdot A + \beta \cdot B \in \mathscr{A}_{L^*}$, we have

$$\mathbf{E}(\alpha \cdot A + \beta \cdot B) = \alpha \cdot \mathbf{E}(A) + \beta \cdot \mathbf{E}(B),$$

where the membership function of the linear combination $\alpha \cdot A + \beta \cdot B$ is given by

$$\begin{aligned}\mu^{L^*}_{\alpha \cdot A+\beta \cdot B} &= (\mu_{(\alpha \cdot A+\beta \cdot B)^+}, \mu_{(\alpha \cdot A+\beta \cdot B)^-}) \\ &= (\alpha \cdot \mu_{A^+} + \beta \cdot \mu_{B^+}, \alpha \cdot \mu_{A^-} + \beta \cdot \mu_{B^-}).\end{aligned}$$

Clearly, (L^*-)expected values as introduced in Definition 4.6 are not necessarily positive-linear. Moreover, it turns out that all continuous positive-linear L^*-expected values have the form (14):

Proposition 5.2: *Let* $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ *be an* L^**-expected value. The following are equivalent:*

(i) $\mathbf{E}$ *is continuous and positive-linear;*

(ii) *there exists a probability measure* $P\colon \mathscr{A} \to \mathbb{I}$ *such that* $\mathbf{E} = \mathbf{E}_P^{L^*}$*, i.e. for all* L^**-fuzzy events* $(A^+, A^-) \in \mathscr{A}_{L^*}$ *we have*

$$\mathbf{E}((A^+, A^-)) = \left(\int_X \mu_{A^+}\, \mathrm{d}P, \int_X \mu_{A^-}\, \mathrm{d}P\right).$$

Proof: The only non-trivial part of the proof is to show that (i) implies (ii).

If an L^*-expected value $\mathbf{E}$ satisfies (i), then for all L^*-fuzzy events (A^+, A^-) we have

$$\mathbf{E}((A^+, A^-)) = \mathbf{E}((A^+, \emptyset) + (\emptyset, A^-)) = \mathbf{E}((A^+, \emptyset)) + \mathbf{E}((\emptyset, A^-)).$$

Moreover, the functions $H_1, H_2 \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ defined by $H_1(A) = \mathbf{E}((A, \emptyset))$ and $H_2(A) = \mathbf{E}((\emptyset, A))$ satisfy $H_1(X) = H_2(X) = 1$, and they are continuous and additive. Therefore, there exist probability measures P_1 and P_2 on $(X, \mathscr{A})$ such that

$$H_1(A) = \int_X \mu_A \, \mathrm{d}P_1 \quad \text{and} \quad H_2(A) = \int_X \mu_A \, \mathrm{d}P_2,$$

implying

$$\mathbf{E}((A^+, A^-)) = \left(\int_X \mu_{A^+} \, \mathrm{d}P_1, \int_X \mu_{A^-} \, \mathrm{d}P_2 \right). \tag{23}$$

Now choose an arbitrary fuzzy event $B \in \mathscr{A}_{\mathbb{I}}$ and consider the L^*-fuzzy event $(B, \complement B) \in \mathscr{A}_{L^*}$. Then from (23) it follows that

$$1 \geq \int_X \mu_B \, \mathrm{d}P_1 + \int_X (1 - \mu_B) \, \mathrm{d}P_2 = \int_X \mu_B \, \mathrm{d}P_1 + 1 - \int_X \mu_B \, \mathrm{d}P_2,$$

i.e. for all fuzzy events $B \in \mathscr{A}_{\mathbb{I}}$ we get

$$\int_X \mu_B \, \mathrm{d}P_1 \leq \int_X \mu_B \, \mathrm{d}P_2$$

or, equivalently, for the probability measures P_1 and P_2 we have $P_1 \leq P_2$ which is possible only if $P_1 = P_2 = P$, i.e. assertion (ii) follows. ■

The following result can easily be derived from (15)–(16).

Corollary 5.3: *Let* $\mathbf{E} \colon \mathscr{A}_{L^*} \to L^*$ *be an* L^**-expected value. The following are equivalent:*

(i) *there exist two expected values of fuzzy events* $\mathbf{E}_1, \mathbf{E}_2 \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ *with* $\mathbf{E}_1 \leq \mathbf{E}_2$ *such that for all* L^**-fuzzy events* $(A^+, A^-) \in \mathscr{A}_{L^*}$

$$\mathbf{E}((A^+, A^-)) = (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(\complement A^-));$$

(ii) *for all* L^**-fuzzy events* $(A^+, A^-) \in \mathscr{A}_{L^*}$

$$\mathbf{E}((A^+, A^-)) = \mathbf{E}((A^*, \emptyset)) + \mathbf{E}((\emptyset, A^-)).$$

As already mentioned, the classical expected value of a random variable (based on the Lebesgue integral with respect to some probability measure) is always linear and, subsequently, idempotent, i.e. the expected value of a constant random variable equals exactly this constant.

For (L^*-)expected values as introduced in Definition 4.6 this is not true, in general (for a concrete counterexample see Example 5.7 below).

Denoting for $\alpha \in \mathbb{I}$ the membership function $\mu_{\alpha \cdot X} \colon X \to \mathbb{I}$ of the constant fuzzy event $\alpha \cdot X$ by $\mu_{\alpha \cdot X} = \alpha \cdot \mathbf{1}_X$, an expected value $\mathbf{E} \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ of fuzzy events is usually called idempotent if for all $\alpha \in \mathbb{I}$

$$\mathbf{E}(\alpha \cdot X) = \alpha. \tag{24}$$

Definition 5.4: Denote, in a measurable space $(X, \mathscr{A})$, for $(\alpha, \beta) \in L^*$ the membership function $\mu^{L^*}_{(\alpha,\beta)\cdot X} \colon X \to L^*$ of the constant L^*-fuzzy event $(\alpha, \beta) \cdot X$ by $\mu^{L^*}_{(\alpha,\beta)\cdot X} = (\alpha \cdot \mathbf{1}_X, \beta \cdot \mathbf{1}_X)$. Then an L^*-expected value $\mathbf{E} \colon \mathscr{A}_{L^*} \to L^*$ is called idempotent if for all $(\alpha, \beta) \in L^*$

$$\mathbf{E}((\alpha, \beta) \cdot X) = (\alpha, \beta). \tag{25}$$

Example 5.5: Many of the L^*-expected values considered so far turn out to be idempotent:

(i) If P is a probability measure on the measurable space $(X, \mathscr{A})$, then the L^*-expected value $\mathbf{E}^{L^*}_P \colon \mathscr{A}_{L^*} \to \mathbb{I}$ given by (14) obviously is idempotent.

(ii) More generally, consider a universal integral $\mathbf{I}$ on $\mathbb{I}$ (as introduced and studied in Klement, Mesiar, and Pap (2010)) based on a semicopula (Durante and Sempi 2005) and a capacity $m \colon \mathscr{A} \to \mathbb{I}$. Then the function $(\mathrm{I})\mathbf{E}_m \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ given by (12) is an idempotent expected value on fuzzy events (see Klement and Mesiar 2015, (5.1)) and, therefore, for each L^*-capacity $m^{L^*} \colon \mathscr{A} \to L^*$ related to the capacities $m_1, m_2 \colon \mathscr{A} \to \mathbb{I}$ with $m_1 \leq m_2$ satisfying (13), the function $(\mathrm{I})\mathbf{E}_{m^{L^*}} \colon \mathscr{A}_{L^*} \to L^*$ given by (22) is an idempotent L^*-expected value.

(iii) Since the universal integrals cover, among others, the Choquet (1954), the Sugeno (1974) and the Shilkret (1971) integral, also the L^*-expected values $(\mathrm{Ch})\mathbf{E}^{L^*}_m \colon \mathscr{A}_{L^*} \to L^*$, $(\mathrm{Su})\mathbf{E}^{L^*}_m \colon \mathscr{A}_{L^*} \to L^*$ and $(\mathrm{Sh})\mathbf{E}^{L^*}_m \colon \mathscr{A}_{L^*} \to L^*$ based on the Choquet, the Sugeno and the Shilkret integral, respectively, and constructed by means of (21) are idempotent as a consequence of (ii).

An immediate consequence of Definition 5.4 is that the idempotency of an L^*-expected value $\mathbf{E}$ is inherited by the expected values of fuzzy events in (15):

Corollary 5.6: *If an L^*-expected value $\mathbf{E} \colon \mathscr{A}_{L^*} \to L^*$ is idempotent, then the two expected values $\mathbf{E}_1, \mathbf{E}_2 \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ of fuzzy events given by* (15)*, respectively, are also idempotent.*

The converse implication in Corollary 5.6 does not hold, in general:

Example 5.7: Choose $X = \{1, 2\}$ and $\mathscr{A} = \mathscr{P}(X))$, and define for L^*-fuzzy events $A \in \mathscr{A}_{L^*}$, characterized by the pair of functions

$$(\mu_A, \nu_A) = (\alpha_1 \cdot \mathbf{1}_{\{1\}} + \alpha_2 \cdot \mathbf{1}_{\{2\}}, \beta_1 \cdot \mathbf{1}_{\{1\}} + \beta_2 \cdot \mathbf{1}_{\{2\}}),$$

the L^*-expected value $\mathbf{E} \colon \mathscr{A}_{L^*} \to L^*$ by

$$\mathbf{E}(A) = \left(\frac{\alpha_1 + \alpha_2}{2}, \frac{4(\beta_1 \wedge \beta_2)(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)}{(1 + \alpha_1 + \beta_1)(1 + \alpha_2 + \beta_2)} \right).$$

For a constant L^*-fuzzy event A, i.e. if $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ we obtain

$$\mathbf{E}(A) = \left(\alpha, \frac{4\beta(\alpha+\beta)^2}{(1+\alpha+\beta)^2}\right),$$

and it is easy to see that $\mathbf{E}(A) = (\alpha, \beta)$ only if $\beta = 0$ or $\alpha + \beta = 1$, i.e. $\mathbf{E}$ is not idempotent.
On the other hand, using (15) we get $\mathbf{E}_1(\delta \cdot X) = \mathbf{E}_2(\delta \cdot X) = \delta$ for each constant fuzzy event $\delta \cdot X$, so both $\mathbf{E}_1$ and $\mathbf{E}_2$ are idempotent.

If the L^*-expected value $\mathbf{E}$ has a special form then a simple computation shows that the converse implication in Corollary 5.6 holds, too:

Corollary 5.8: *If $\mathbf{E}_1, \mathbf{E}_2\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ are idempotent expected values of fuzzy events with $\mathbf{E}_1 \leq \mathbf{E}_2$, then the L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ given by*

$$\mathbf{E}((A^+, A^-)) = (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(\complement(A^-)))$$

is also idempotent.

6. Comonotone additivity and comonotone maxitivity

A characteristic property of the Choquet integral (5) (which is not positive-linear, in general) is its comonotone additivity (Klement, Mesiar, and Pap 2010).
Recall that two functions $f, g\colon X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$ we have

$$(f(x) - f(y)) \cdot (g(x) - g(y)) \geq 0.$$

For two fuzzy subsets $A, B \in \mathscr{F}(X)$ comonotonicity means that their membership functions $\mu_A, \mu_B\colon X \to \mathbb{I}$ are comonotone, and for two L^*-fuzzy subsets $A = (A^+, A^-)$ and $B = (B^+, B^-)$ in $\mathscr{F}_{L^*}(X)$ the comonotonicity of A and B means that A^+ and B^+ as well as A^- and B^- are comonotone pairs of fuzzy sets.
Comonotone additive expected values of fuzzy events have been considered in, e.g. Klement and Mesiar (2015); Klement et al. (2015). We extend this to the case of L^*-expected values.

Definition 6.1: Let $(X, \mathscr{A})$ be a measurable space. An L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ is called comonotone additive if for all comonotone L^*-fuzzy events $A, B \in \mathscr{A}_{L^*}$ with $A + B \in \mathscr{A}_{L^*}$ we have $\mathbf{E}(A + B) = \mathbf{E}(A) + \mathbf{E}(B)$.

It is not difficult to see that, given a capacity m on the measurable space $(X, \mathscr{A})$, the L^*-expected value $(\mathrm{Ch})\mathbf{E}_m^{L^*}\colon \mathscr{A}_{L^*} \to L^*$ based on the Choquet integral with respect to m (constructed by means of (21)) is comonotone additive. As a trivial consequence, for each probability measure P on $(X, \mathscr{A})$, also the L^*-expected value $\mathbf{E}_P^{L^*}\colon \mathscr{A}_{L^*} \to \mathbb{I}$ given by (14) is comonotone additive.
The following result is a consequence of the axiomatic characterization of the Choquet integral in Schmeidler (1986, 1989) and of Corollary 5.3.

Proposition 6.2: *Let* $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ *be an* L^**-expected value. The following are equivalent:*

(i) $\mathbf{E}$ *is comonotone additive and there exist two expected values of fuzzy events* $\mathbf{E}_1, \mathbf{E}_2\colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ *with* $\mathbf{E}_1 \leq \mathbf{E}_2$ *such that for all* L^**-fuzzy events* $(A^+, A^-) \in \mathscr{A}_{L^*}$

$$\mathbf{E}((A^+, A^-)) = (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(\complement(A^-)));$$

(ii) *there exist two capacities* $m_1, m_2\colon \mathscr{A} \to \mathbb{I}$ *with* $m_1 \leq m_2$ *such that for all* L^**-fuzzy events* $(A^+, A^-) \in \mathscr{A}_{L^*}$

$$\begin{aligned}\mathbf{E}((A^+, A^-)) &= (\mathbf{Ch}_{m_1}(\mu_{A^+}), 1 - \mathbf{Ch}_{m_2}(1 - \mu_{A^-}))\\ &= (\mathbf{Ch}_{m_1}(\mu_{A^+}), \mathbf{Ch}_{m_2^d}(\mu_{A^-})),\end{aligned}$$

where m_2^d *is the dual capacity of* m_2*, i.e.* $m_2^d(A) = 1 - m_2(\complement A)$ *for each* $A \in \mathscr{A}$*.*

Observe that for a comonotone additive L^*-expected value $\mathbf{E}$ we always have $\mathbf{E}_1 = \mathbf{Ch}_{m_1}$ and $\mathbf{E}_2 = \mathbf{Ch}_{m_2}$ for some capacities m_1 and m_2 satisfying $m_1 \leq m_2$. However, as shown by the following example, this does not necessarily imply $\mathbf{E}((A^+, A^-)) = (\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(\complement(A^-)))$.

Example 6.3: We keep the notations of Example 5.7 for the measurable space $(X, \mathscr{A})$ and for the L^*-fuzzy events $A \in \mathscr{A}_{L^*}$. Then the L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ given by

$$\mathbf{E}((A^+, A^-)) = (\alpha_1 \wedge \alpha_2, (1 - \alpha_1) \wedge (1 - \alpha_2))$$

is comonotone additive, but

$$(\mathbf{E}_1(A^+), 1 - \mathbf{E}_2(\complement(A^-))) = (\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2) \neq \mathbf{E}((A^+, A^-)).$$

Due to the axiomatic characterization of the Sugeno integral on bounded distributive lattices in Couceiro and Marichal (2010a), Couceiro and Marichal (2010b) and Halaš, Mesiar, and Pócs (2017), we obtain the following characterization of a class of L^*-expected values on a finite universe X which are comonotone maxitive and comonotone minitive (see (26) and (27) below).

In analogy to comonotone additivity (which preserves additivity for comonotone functions), we can also consider operations preserving meets and joins of comonotone functions.

An L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ is called comonotone maxitive if

$$\mathbf{E}(A \vee_{L^*} B) = \mathbf{E}(A) \vee_{L^*} \mathbf{E}(B), \tag{26}$$

and comonotone minitive if

$$\mathbf{E}(A \wedge_{L^*} B) = \mathbf{E}(A) \wedge_{L^*} \mathbf{E}(B) \tag{27}$$

for all comonotone L^*-fuzzy events $A, B \in \mathscr{A}_{L^*}$.

Proposition 6.4: *Let X be a finite set, $\mathscr{A} = \mathscr{P}(X)$ and $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ an idempotent function which is both comonotone maxitive and comonotone minitive. Then $\mathbf{E}$ is the L^*-Sugeno integral with respect to the L^*-capacity $m\colon \mathscr{A} \to L^*$ given by*

$$m(A) = \mathbf{E}((A, \complement A)),$$

i.e. for each $A \in \mathscr{A}_{L^*}$ we have

$$\mathbf{E}(A) = \bigvee_{(a,b)\in L^*}^{L^*} ((a, b) \wedge_{L^*} m(\{x \in X \mid \mu_A^{L^*}(x) \geq_{L^*} (a, b)\})). \tag{28}$$

As already observed, each L^*-capacity $m\colon \mathscr{A} \to L^*$ is related to two capacities $m_1, m_2\colon \mathscr{A} \to \mathbb{I}$ satisfying $m_1 \leq m_2$ and (13). Then formula (28) can be rewritten as

$$\mathbf{E}(A) = ((\mathrm{Su})\mathbf{E}_{m_1}(A^+), (\mathrm{Su})\mathbf{E}_{m_2^d}(A^-)), \tag{29}$$

where $m_2^d\colon \mathscr{A} \to \mathbb{I}$ is the dual capacity of m_2 given by $m_2^d(A) = 1 - m_2(\complement A)$. Evidently, for each L^*-capacity $m\colon \mathscr{A} \to L^*$, the function $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ defined by (29) is an L^*-expected value which, in the case of $m_1 = m_2$, coincides with the L^*-expected value based on the Sugeno integral.

7. Real-valued expectations

So far we have considered expected values of L^*-fuzzy events keeping in mind the point of view that the expected value of an L^*-fuzzy event should be an element of L^*.

Taking into account the hierarchical structure of the three lattices $(\mathbf{2}, \leq)$, $(\mathbb{I}, \leq)$ and $(L^*, \leq_{L^*})$, on the one hand, and of crisp, fuzzy and L^*-fuzzy subsets of the universe X, on the other hand, crisp sets can be seen as special fuzzy sets, and fuzzy sets are special L^*-fuzzy sets.

It therefore makes sense to consider also L^*-expectations assuming values in $\mathbb{I}$ only rather than in L^*.

Definition 7.1: Let $(X, \mathscr{A})$ be a measurable space. A real-valued expected value of L^*-fuzzy events ($\mathbb{R}L^*$-expected value for short) is an order homomorphism $\mathbb{E}$ from $(\mathscr{A}_{L^*}, \subseteq_{L^*})$ to $(\mathbb{I}, \leq)$, i.e. a function $\mathbb{E}\colon \mathscr{A}_{L^*} \to \mathbb{I}$ satisfying

(i) $\mathbb{E}(\emptyset) = 0$ and $\mathbb{E}(X) = 1$,
(ii) $\mathbb{E}(A) \leq \mathbb{E}(B)$ whenever $A \subseteq_{L^*} B$.

Again it is easy to see that the set of all $\mathbb{R}L^*$-expected values on $(X, \mathscr{A})$ is a convex set and a bounded lattice with bottom and top elements $\mathbb{E}^\perp$ and $\mathbb{E}^\top$ given by, respectively,

$$\mathbb{E}^\perp(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbb{E}^\top(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, for each $\lambda \in]0,1[$, $\mathbb{E}_\lambda = \lambda \cdot \mathbb{E}^\top + (1-\lambda)\cdot \mathbb{E}^\perp$ is an $\mathbb{R}L^*$-expected value, and it is given by

$$\mathbb{E}_\lambda(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A = X, \\ \lambda & \text{otherwise.} \end{cases}$$

Example 7.2: There are several natural ways to construct an $\mathbb{R}L^*$-expected value from an L^*-expected value.

(i) Starting from an L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ and an order homomorphism $\varphi\colon L^* \to \mathbb{I}$ from $(L^*, \leq_{L^*})$ to $(\mathbb{I}, \leq)$, it is immediately clear that the function $\mathbb{E}_{\varphi,\mathbf{E}}\colon \mathscr{A}_{L^*} \to \mathbb{I}$ defined by $\mathbb{E}_{\varphi,\mathbf{E}} = \varphi \circ \mathbf{E}$ is an $\mathbb{R}L^*$-expected value.

(ii) Fix $\lambda \in \mathbb{I}$ and, using the so-called K_λ-operator Atanassov (1986) (compare also Bustince et al. 2013), define the homomorphism $\varphi_\lambda\colon L^* \to \mathbb{I}$ from $(L^*, \leq_{L^*})$ to $(\mathbb{I}, \leq)$ by

$$\varphi_\lambda((a,b)) = (1-\lambda)\cdot a + \lambda \cdot (1-b).$$

Note that $\varphi_0((a,b)) = a$, i.e. φ_0 is just the first projection, and $\varphi_1((a,b)) = 1-b$, i.e. φ_1 is the negation of the second projection.

If $P\colon \mathscr{A} \to \mathbb{I}$ is a probability measure on $(X, \mathscr{A})$ and if $\mathbf{E}_P^{L^*}\colon \mathscr{A}_{L^*} \to L^*$ is the L^*-expected value given by (14), then for the corresponding $\mathbb{R}L^*$-expected value $\mathbb{E}_{\varphi_\lambda,\mathbf{E}_P}\colon \mathscr{A}_{L^*} \to \mathbb{I}$ we have for each L^*-fuzzy event $A = (A^+, A^-) \in \mathscr{A}_{L^*}$

$$\mathbb{E}_{\varphi_\lambda,\mathbf{E}_P^{L^*}}(A) = \int_X ((1-\lambda)\cdot A^+ + \lambda\cdot(\complement(A^-)))\,\mathrm{d}P = \int_X \varphi_\lambda \circ \mu_A^{L^*}\,\mathrm{d}P.$$

Recall that, given an L^*-fuzzy event $A \in \mathscr{A}_{L^*}$ with membership function $\mu_A^{L^*}\colon X \to L^*$ and an order homomorphism $\varphi\colon L^* \to \mathbb{I}$ from $(L^*, \leq_{L^*})$ to $(\mathbb{I}, \leq)$, then the function $\varphi \circ \mu_A^{L^*}\colon X \to \mathbb{I}$ is the membership function of a fuzzy event which we shall denote $\varphi(A)$, i.e.$\varphi \circ \mu_A^{L^*}(x) = \varphi(A)(x)$.

Therefore, for each expected value $\mathbf{E}\colon \mathscr{A}_\mathbb{I} \to \mathbb{I}$ we can consider the $\mathbb{R}L^*$-expected value $\mathbb{E}^{\varphi,\mathbf{E}}\colon \mathscr{A}_{L^*} \to \mathbb{I}$ given by

$$\mathbb{E}^{\varphi,\mathbf{E}}(A) = \mathbf{E}(\varphi(A)).$$

Example 7.3: Keeping the notations from Example 7.2(ii) and considering the expected value of fuzzy events $\mathbf{E}_P\colon \mathscr{A}_\mathbb{I}$ given by (8), it is immediately clear that

$$\mathbb{E}_{\varphi,\mathbf{E}_P^{L^*}} = \mathbb{E}^{\varphi,\mathbf{E}_P}.$$

Example 7.4: Let X be a finite set and $\mathscr{A} = \mathscr{P}(X)$. Consider the L^*-expected value $\mathbf{E}\colon \mathscr{A}_{L^*} \to L^*$ given by

$$\mathbf{E}((A^+, A^-)) = \Big(\min_{x\in X} \mu_{A^+}(x), \max_{x\in X} \mu_{A^-}(x)\Big).$$

Then the $\mathbb{R}L^*$-expected value $\mathbb{E}_{\varphi_{0.5},\mathbf{E}}\colon \mathscr{A}_{L^*} \to \mathbb{I}$ is given by

$$\mathbb{E}_{\varphi_{0.5},\mathbf{E}}((A^+, A^-)) = 0.5\cdot\Big(1 + \min_{x\in X} \mu_{A^+}(x) - \max_{x\in X} \mu_{A^-}(x)\Big).$$

From $\varphi_{0.5}((A^+, A^-)) = 0.5 \cdot (X + A^+ - A^-)$ it follows that, whenever $\mathrm{card}(X) \geq 2$, there is no expected value of fuzzy events $\mathbf{E}_0 \colon \mathscr{A}_{\mathbb{I}} \to \mathbb{I}$ such that $\mathbb{E}_{\varphi_{0.5},\mathbf{E}} = \mathbb{E}^{\varphi_{0.5},\mathbf{E}_0}$.

8. Concluding remarks

We have introduced a general view on expected values of L^*-fuzzy events, including the presentation of several examples and a deep study of these expected values with some particular properties. Besides L^*-valued expectations of L^*-fuzzy events, we have considered also real-valued expectations of these events. Our approach can be helpful in several applications of L^*-fuzzy events, in particular in decision making when some preference structures should be constructed.

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References

Atanassov, K. 1984. "Intuitionistic Fuzzy Sets." In *VII ITKR's Session, Sofia, June 1983*, edited by V. Sgurev. Sofia (deposed in Central Science and Technical Library, Bulgarian Academy of Sciences, 1697/84, in Bulgarian).

Atanassov, K. T. 1986. "Intuitionistic Fuzzy Sets." *Fuzzy Sets and Systems* 20: 87–96.

Atanassov, K. T. 1999. *Intuitionistic Fuzzy Sets*. Heidelberg: Physica-Verlag.

Atanassov, K. T., and P. Vassilev. 2018. "On the Intuitionistic Fuzzy Sets of *n*th type." In *Advances in Data Analysis with Computational Intelligence Methods: Dedicated to Professor Jacek Żurada*, edited by A. E. Gawęda, J. Kacprzyk, L. Rutkowski, and G. G. Yen, 265–274. Cham: Springer.

Bauer, H. 1981. *Probability Theory and Elements of Measure Theory*. London: Academic Press.

Benvenuti, P., R. Mesiar, and D. Vivona. 2002. "Monotone Set Functions-Based Integrals." In *Handbook of Measure Theory*, edited by E. Pap, Chap. 33, 1329–1379. Amsterdam: Elsevier Science.

Bhaskara Rao, K. P. S., and M. Bhaskara Rao. 1983. *Theory of Charges. A Study of Finitely Additive Measures*. New York: Academic Press.

Birkhoff, G. 1973. *Lattice Theory*. Providence: American Mathematical Society.

Bustince, H., E. Barrenechea, M. Pagola, J. Fernandez, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera, and B. De Baets. 2016. "A Historical Account of Types of Fuzzy Sets and Their Relationships." *IEEE Transactions on Fuzzy Systems* 24: 179–194.

Bustince, H., M. Galar, B. Bedregal, A. Kolesárová, and R. Mesiar. 2013. "A New Approach to Interval-valued Choquet Integrals and the Problem of Ordering in Interval-Valued Fuzzy Set Applications." *IEEE Transactions on Fuzzy Systems* 21: 1150–1162.

Carathéodory, C. 1918. *Vorlesungen über reelle Funktionen*. Berlin: Teubner.

Choquet, G. 1954. "Theory of Capacities." *Annales de l'Institut Fourier (Grenoble)* 5: 131–295.

Ciungu, L. C., and B. Riečan. 2010. "Representation Theorem for Probabilities on IFS-events." *Information Sciences* 180: 793–798.

Couceiro, M., and J.-L. Marichal. 2010a. "Characterizations of Discrete Sugeno Integrals as Polynomial Functions Over Distributive Lattices." *Fuzzy Sets and Systems* 161: 694–707.

Couceiro, M., and J.-L. Marichal. 2010b. "Representations and Characterizations of Polynomial Functions on Chains." *Journal of Multiple-Valued Logic and Soft Computing* 16: 65–86.

De Miguel, L., H. Bustince, J. Fernandez, E. Induráin, A. Kolesárová, and R. Mesiar. 2016. "Construction of Admissible Linear Orders for Interval-valued Atanassov Intuitionistic Fuzzy Sets with An Application to Decision Making." *Information Fusion* 27: 189–197.

Dempster, A. P. 1967. "Upper and Lower Probabilities Induced by a Multivalued Mapping." *Annals of Mathematical Statistics* 38: 325–339.

Denneberg, D. 1994. *Non-additive Measure and Integral*. Dordrecht: Kluwer Academic Publishers.

Deschrijver, G., and E. E. Kerre. 2003. "On the Relationship Between Some Extensions of Fuzzy Set Theory." *Fuzzy Sets and Systems* 133: 227–235.

Dick, S., R. R. Yager, and O. Yazdanbakhsh. 2016. "On Pythagorean and Complex Fuzzy Set Operations." *IEEE Transactions on Fuzzy Systems* 24: 1009–1021.

Durante, F., and C. Sempi. 2005. "Semicopulæ." *Kybernetika* 41: 315–328.

Goguen, J. A. 1967. "*L*-fuzzy Sets." *Journal of Mathematical Analysis and Applications* 18: 145–174.

Grabisch, M. 1997. "*k*-order Additive Discrete Fuzzy Measures and Their Representation." *Fuzzy Sets and Systems* 92: 167–189.

Grabisch, M., T. Murofushi, and M. Sugeno, eds. 2000. *Fuzzy Measures and Integrals. Theory and Applications*. Heidelberg: Physica-Verlag.

Grzegorzewski, P. 2013. "On Some Basic Concepts in Probability of IF-events." *Information Sciences* 232: 411–418.

Grzegorzewski, P., and E. Mrówka. 2002. "Probability of Intuitionistic Fuzzy Events." In *Soft Methods in Probability, Statistics and Data Analysis*, edited by P. Grzegorzewski, O. Hryniewicz, and M. Á. Gil, 105–115. Heidelberg: Physica-Verlag.
Halaš, R., R. Mesiar, and J. Pócs. 2017. "Generalized Comonotonicity and New Axiomatizations of Sugeno Integrals on Bounded Distributive Lattices." *International Journal of Approximate Reasoning* 81: 183–192.
Halmos, P. R. 1950. *Measure Theory*. New York: Van Nostrand Reinhold.
Höhle, U., and S. E. Rodabaugh, eds. 1999. *Mathematics of Fuzzy Sets. Logic, Topology, and Measure Theory*. The Handbook of Fuzzy Sets Series. Boston: Kluwer Academic Publishers.
Imaoka, H. 1997. "On a Subjective Evaluation Model by a Generalized Fuzzy Integral." *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 5: 517–529.
Imaoka, H. 2000. "Comparison Between Three Fuzzy Integrals." In *Fuzzy Measures and Integrals. Theory and Applications*, edited by M. Grabisch, T. Murofushi, and M. Sugeno, 273–286. Heidelberg: Physica-Verlag.
Klement, E. P., A. Kolesárová, R. Mesiar, and A. Stupňanová. 2010. "Lipschitz Continuity of Discrete Universal Integrals Based on Copulas." *International Journal of Uncertainty, Fuzziness Knowledge-Based Systems* 18: 39–52.
Klement, E. P., J. Li, R. Mesiar, and E. Pap. 2015. "Integrals Based on Monotone Set Functions." *Fuzzy Sets and Systems* 281: 88–102.
Klement, E. P., and R. Mesiar. 2012a. "Discrete Integrals and Axiomatically Defined Functionals." *Axioms* 1: 9–20.
Klement, E. P., and R. Mesiar. 2012b. "On the Axiomatization of Some Classes of Discrete Universal Integrals." *Knowledge-Based Systems* 28: 13–18.
Klement, E. P., and R. Mesiar. 2015. "On the Expected Value of Fuzzy Events." *International Journal of Uncertainty, Fuzziness Knowledge-Based Systems* 23 (Suppl. 1): 57–74.
Klement, E. P., R. Mesiar, and E. Pap. 2010. "A Universal Integral as Common Frame for Choquet and Sugeno Integral." *IEEE Transactions on Fuzzy Systems* 18: 178–187.
Klement, E. P., and S. Weber. 1991. "Generalized Measures." *Fuzzy Sets and Systems* 40: 375–394.
Klement, E. P., and S. Weber. 1999. "Fundamentals of a Generalized Measure Theory." In *Mathematics of Fuzzy Sets. Logic, Topology, and Measure Theory*. The Handbook of Fuzzy Sets Series., edited by U. Höhle, and S. E. Rodabaugh, Chap. 11, 633–651. Boston: Kluwer Academic Publishers.
Lehrer, E. 2009. "A New Integral for Capacities." *Economic Theory* 39: 157–176.
Li, J., and R. Mesiar. 2011. "Lusin's Theorem on Monotone Measure Spaces." *Fuzzy Sets and Systems* 175: 75–86.
Li, J., R. Mesiar, and E. Pap. 2014. "Atoms of Weakly Null-additive Monotone Measures and Integrals." *Information Sciences* 257: 183–192.
Medolaghi, P. 1907. "La Logica Matematica Ed Il Calcolo Delle Probabilità." *Bollettino dell'Associazione Italiana per l'Incremento delle Scienze Attuariali* 18: 20–39.
Mesiar, R. 1995. "Choquet-like Integrals." *Journal of Mathematical Analysis and Applications* 194: 477–488.
Mesiar, R. 1999. "*k*-order Additive Measures." *International Journal of Uncertainty, Fuzziness Knowledge-Based Systems* 7: 561–568.
Mesiar, R. 2005. "Fuzzy Measures and Integrals." *Fuzzy Sets and Systems* 156: 365–370.
Mesiar, R., and A. Stupňanová. 2013. "Decomposition Integrals." *International Journal of Approximate Reasoning* 54: 1252–1259.
Pap, E. 1995. *Null-Additive Set Functions*. Dordrecht: Kluwer Academic Publishers.
Pap, E, ed. 2002. *Handbook of Measure Theory*. Amsterdam: Elsevier Science.
Pap, E. 2008. "Generalized Real Analysis and Its Applications." *International Journal of Approximate Reasoning* 47: 368–386.
Ralescu, D., and G. Adams. 1980. "The Fuzzy Integral." *Journal of Mathematical Analysis and Applications* 75: 562–570.
Riečan, B. 2006. "On a Problem of Radko Mesiar: General Form of IF-probabilities." *Fuzzy Sets and Systems* 157: 1485–1490.

Schmeidler, D. 1986. "Integral Representation Without Additivity." *Proceedings of the American Mathematical Society* 97: 255–261.
Schmeidler, D. 1989. "Subjective Probability and Expected Utility Without Additivity." *Econometrica* 57: 571–587.
Senapati, T., and R. R. Yager. 2019b. "Fermatean Fuzzy Weighted Averaging/geometric Operators and Its Application in Multi-criteria Decision-making Methods." *Engineering Applications of Artificial Intelligence* 85: 112–121.
Senapati, T., and R. R. Yager. 2020a. "Fermatean Fuzzy Sets." *Journal of Ambient Intelligence and Humanized Computing* 11: 663–674.
Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton: Princeton University Press.
Shilkret, N. 1971. "Maxitive Measure and Integration." *Indagationes Mathematicae* 33: 109–116.
Stolz, O. 1883. "Über einen zu einer unendlichen Punktmenge gehörigen Grenzwert." *Mathematische Annalen* 23: 152–156.
Sugeno, M. 1974. "Theory of Fuzzy Integrals and its Applications." PhD diss., Tokyo Institute of Technology.
Takeuti, G., and S. Titani. 1984. "Intuitionistic Fuzzy Logic and Intuitionistic Fuzzy Set Theory." *Journal Symbolic Logic* 49: 851–866.
Takeuti, G., and S. Titani. 1987. "Globalization of Intuitionistic Set Theory." *Annals of Pure and Applied Logic* 33: 195–211.
Torra, V., Y. Narukawa, and M. Sugeno., eds. 2014. *Non-Additive Measures. Theory and Applications*. Cham: Springer.
Vitali, G. 1925. "Sulla Definizione Di Integrale Delle Funzioni Di Una Variabile." *Annali di Matematica Pura ed Applicata* 2: 111–121.
Walker, C. L., and E. A. Walker. 2009. "Sets with Type-2 Operations." *International Journal of Approximate Reasoning* 50: 63–71.
Wang, G.-J., and Y.-Y. He. 2000. "Intuitionistic Fuzzy Sets and L-fuzzy Sets." *Fuzzy Sets and Systems* 110: 271–274.
Wang, Z., and G. J. Klir. 2009. *Generalized Measure Theory*. New York: Springer.
Weber, S. 1986. "Two Integrals and Some Modified Versions: Critical Remarks." *Fuzzy Sets and Systems* 20: 97–105.
Yager, R. R. 2014. "Pythagorean Membership Grades in Multi-criteria Decision Making." *IEEE Transactions on Fuzzy Systems* 22: 958–965.
Yager, R. R. 2017. "Generalized Orthopair Fuzzy Sets." *IEEE Transactions on Fuzzy Systems* 25: 1222–1230.
Yager, R. R., and A. M. Abbasov. 2013. "Pythagorean Membership Grades, Complex Numbers, and Decision Making." *International Journal of Intelligent Systems* 28: 436–452.
Zadeh, L. A. 1965. "Fuzzy Sets." *Information and Control* 8: 338–353.
Zadeh, L. A. 1968. "Probability Measures of Fuzzy Events." *Journal of Mathematical Analysis and Applications* 23: 421–427.
Zadeh, L. A. 1975. "The Concept of a Linguistic Variable and Its Applications to Approximate Reasoning. Part I." *Information and Control* 8: 199–251.