

On the optimal control of rate-independent soft crawlers

Giovanni Colombo^a, Paolo Gidoni^{b,*}

^a *Università di Padova, Dipartimento di Matematica “Tullio Levi-Civita”, via Trieste 63, 35121 Padova, Italy*

^b *Czech Academy of Sciences, Institute of Information Theory and Automation (UTIA), pod vodárenskou věží 4, 182 08, Prague 8, Czech Republic*



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ABSTRACT

Existence of optimal solutions and necessary optimality conditions for a controlled version of Moreau’s sweeping process are derived. The control is a measurable ingredient of the dynamics and the constraint set is a polyhedron. The novelty consists in considering time periodic trajectories, adding the requirement that the control has zero average, and considering an integral functional that lacks weak semicontinuity. A model coming from the locomotion of a soft-robotic crawler, that motivated our setting, is analysed in detail. In obtaining necessary conditions, a variant of the method of discrete approximations is used.

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R É S U M É

Nous démontrons l’existence de solutions optimales et des conditions d’optimalité nécessaires pour une version contrôlée du processus de rafle de Moreau. Le contrôle est un ingrédient mesurable de la dynamique et l’ensemble de contraintes est un polyèdre. Le nouveau aspect consiste à considérer des trajectoires périodiques dans le temps, en ajoutant la condition que le contrôle ait une moyenne nulle, et en considérant une fonctionnelle intégrale qui manque de faible semi-continuité. Un modèle issu de la locomotion d’un robot mou, qui a motivé ce travail, est analysé en détail. En obtenant les conditions nécessaires, nous utilisons une variante de la méthode des approximations discrètes.

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1. Introduction

Moreau’s sweeping process comprises a class of evolution inclusions that model the displacement of a point $x(t)$ dragged in a normal direction by a moving (convex or mildly non-convex) closed set: see, e.g., the survey paper [16] and references therein. If the point is also subject to an independent dynamics, then

* Corresponding author.

E-mail addresses: colombo@math.unipd.it (G. Colombo), gidoni@utia.cas.cz (P. Gidoni).

the evolution can be seen as a constrained motion, in which the reaction of the constraint is active. More precisely, the problem is stated as

$$\dot{x}(t) \in -\mathcal{N}_{C(t)}(x(t)) + g(t, x(t)) \quad \text{a.e. in } [0, T], \quad x(0) = x_0 \in C(0). \quad (1)$$

Here $\mathcal{N}_C(x)$ denotes the normal cone (of convex analysis if C is convex) to C at $x \in C$. The case where C is independent of time is particularly meaningful, because it is well known that the problem is equivalent to the so called projected differential equation

$$\dot{x}(t) = \pi_{T_C(x)}(g(t, x(t))), \quad x(0) = x_0 \in C, \quad (2)$$

where $\pi_{T_C(x)}(y)$ denotes the projection into the tangent cone to C at x of the vector y (see [5, Sec. 10.1]). The equivalence of (1) and (2), when $C(t) \equiv C$, both explains the role of the constraint in the dynamics and its intrinsic nonsmoothness (even discontinuity). Indeed, only one normal vector can be taken in (1), that is the smallest that cancels the (external) normal component of g , in order to keep the trajectory inside C . This latter fact follows from the emptiness of the normal, or tangent, cone to C at points outside C . Moreover, observe that the normal cone mapping $x \mapsto \mathcal{N}_C(x)$ is discontinuous – actually it has only closed graph – for two reasons: first because the boundary of C may not be C^1 -smooth (e.g. C is a polyhedron) and, second, because in the interior of C , if any, $\mathcal{N}_C(x) = \{0\}$, while at all boundary points $\mathcal{N}_C(x)$ contains at least a half line. A similar type of discontinuity appears in the right-hand side of (2). However, the monotone character of the normal cone mapping allows to prove forward-in-time existence (and uniqueness if the ODE $\dot{x} = g(t, x)$ allows so) of solutions to the Cauchy problem (1) under usual conditions imposed on g .

The simplest control problems involving Moreau's sweeping process occur when a control parameter $u(\cdot)$ appears within g : the dynamics then becomes

$$\dot{x}(t) \in -\mathcal{N}_{C(t)}(x(t)) + g(t, x(t), u(t)), \quad u(t) \in U \quad \text{a.e.}, \quad (3)$$

U being an assigned compact set. This paper is devoted to deriving necessary optimality conditions for a particular Bolza problem involving (3) together with further requirements on both x and u . This problem is motivated by maximizing the displacement in the locomotion of a bio-mimetic soft robotic crawler, whose mathematical model is presented in detail in Section 3. The robot can be described as a chain of N links, each formed by a spring coupled in series with an actuator, whose length is controlled. The movement is one dimensional and the evolution is supposed to be *quasi-static*, i.e., the mechanical system is modelled by a force balance law and therefore obeys a first order differential inclusion. After quite a few transformations, that are essentially known in the theory of *rate independent evolutions*, one arrives to the controlled dynamics (3), where the space dimension of the problem is the number of links. Since one wants to find an optimal *gait*, namely a periodic actuation to be repeated an arbitrarily large number of times, the fixed initial condition on the trajectory is substituted by a T -periodicity condition, T being fixed *a priori*. Moreover, in the final model the controls turn out to be the derivative of periodic Lipschitz functions, so that the zero mean condition

$$\int_0^T u(t) dt = 0 \quad (4)$$

must be imposed on feasible controls. Finally, C turns out to be a polyhedron. The functional to be maximized is an integral functional J involving two terms, the reaction of the constraint and the cost of the control (that of course appears with a minus sign):

$$J(x, u) := \int_0^T \left(f_1(g(t, x(t), u(t)) - \dot{x}(t)) - f_2(t, u(t)) \right) dt.$$

Here f_1 is Lipschitz and positively homogeneous with degree one and f_2 , for simplicity, is C^1 with respect to the second variable u . In our application, the first summand in the integrand of J is a function of the reaction of the constraint, measuring the displacement of the barycentre of the system of springs, while f_2 represents the cost of actuating the control. If, on one hand, it is natural to assume the convexity of f_2 , on the other hand the derivation of the explicit form of f_1 for our model, presented in Section 3.5, gives a functional that is not concave down with respect to \dot{x} and u . For instance, in the simple example presented in Section 4, the first summand of the integrand is $\frac{1}{2}|\dot{x} - u|$, so the integral functional is not weakly upper semicontinuous in $W^{1,2}([0, T]; \mathbb{R}^n)$, cf. also Remark 4. Therefore, the direct method cannot be used in order to ensure the existence of optimal state-trajectory pairs.

The first contribution of the present paper is proving an existence result (Theorem 1) for the maximization of J along trajectories of a controlled sweeping process of the type (3) by imposing a uniform bound on the total variation of admissible controls, that yields in turn the pointwise convergence of a maximizing sequence. This is a strong assumption, which however seems to be justified by the observation that optimal controls are expected to be bang-bang with finitely many switchings (see Section 4), or anyway with finite total variation (see Section 4.1). Moreover, this requirement does not completely trivialize the existence argument, because in order to allow passing to the limit on J along a maximizing sequence (x_ℓ, u_ℓ) one needs also the strong convergence of the sequence of derivatives \dot{x}_ℓ of the state variable. While for general differential inclusions this is not possible, the particular structure of the sweeping process allows to overcome this difficulty.

Our second contribution consists of necessary optimality conditions. The analysis of necessary conditions in this type of setting does not follow from the classical literature on state-constrained optimal-control problems (see, e.g., [46]), since the right-hand side of the dynamics is not Lipschitz (actually, is very far from being so), with respect to the state x . There are essentially two ways to attack the problem. The first one is based on a regularization of the dynamics and goes back essentially to [10], see also [3, 17, 49]. It provides an adjoint equation in the sense of measures together with a maximum principle of Pontryagin type, as it may be expected in such problems, but is – up to now – limited by requiring the set C to be smooth. The second one, that is due to Mordukhovich and collaborators (see, e.g., [13, 14] and references therein), is based on discrete approximations. This technique fits perfectly with our polyhedral setting, but provides only a weaker form of the maximum principle. In this paper we adapt to our problem the method of discrete approximations, by considering periodic trajectories and adding the control constraint (4). Moreover, taking inspiration from the fact that the normal vector in (3) cannot be chosen independently of the control, we simplify the discretization procedure by avoiding computing the normal cone to the normal cone $\mathcal{N}_C(x)$. Furthermore, our approximation technique allows general measurable controls, not being limited to controls with bounded variation as in [14]. Finally, we deal with the nonconcavity of the integral functional without relying at all on relaxation arguments. Actually, in this case relaxation results are difficult to obtain, since the integral functional involves also the derivative of the state, not only the control variable, and furthermore periodic solutions are considered. Nevertheless, the obtained necessary conditions are very similar to those derived in the framework of [14].

In a wider perspective, the subject of our investigation lies at the triple point among control problems with state constraints, rate-independent evolutions, and more pragmatic approaches to specific mechanical models. While we postpone to Section 3 the comparison with other approaches to similar locomotion models in the mechanical literature, we spend here a few words on the theoretical aspects.

Indeed, even if they do not follow from the classical theory, the results for the sweeping process discussed above, and including ours, may be considered as a recent branch of the theory of state-constrained control problems, see e.g. [46, 21], with which they share several analogies. The novel feature of problems like the

sweeping process is that the reaction of the constraints is part of the dynamics, so that sometimes they are referred to as problems with *active constraints*.

Instead of using the dynamics in normal form offered by the sweeping process, an alternative path is to exploit the variational structure of the problem. For our model of crawlers, this has been done in [23] to study wellposedness and in [26] for a vanishing inertia analysis derivation of the quasi-static limit. This approach has been extensively developed for models in continuum mechanics, for instance in elastoplasticity, thus considering an infinite dimensional evolution problem. We refer to the recent monography [37] for an extensive survey. Optimal control problems have been considered also within this framework. With respect to our first issue, namely the existence of an optimal solution, we mention [40], see also [31]. These works deal with quite general situations allowing a non-convex energy (whereas the sweeping process roughly corresponds to the case of quadratic energy), but with classical assumptions on the cost function. Our Theorem 1, instead, deals with the less standard situation, required by our model, where the objective functional is not upper semicontinuous in the relevant topology. Concerning the search for optimality conditions, we observe that also in the variational framework we may identify two main strategies, based respectively on regularization and discretization methods. Indeed, the vanishing viscosity approximation for rate-independent systems is strictly related to the penalization methods employed for sweeping processes [10,3,17]. Following the regularization approach, necessary optimality conditions for rate independent evolutions have been obtained in some recent papers (see [29,22,43,48] and references therein). We notice however that all these results are based in a strictly infinite-dimensional setting, that is not applicable to our problem; moreover, as we already discussed, existing regularization methods are not applicable to our framework. We notice that this is not surprising: already in [23], in the study of uniqueness conditions, it has been noticed that finite-dimensional models present some issues related to the regularity of the subdifferential of the dissipation potential, that disappear passing to infinite-dimensional models. Despite this key difference, we notice however that the sets of necessary conditions proved in [29,22,43] are comparable with ours, as an adjoint equation in weak form together with a kind of Pontryagin maximization property are obtained.

Regarding instead the discrete approximations of optimal control problems for rate independent evolutions, some results have been obtained in [1,32,41]. These works study the convergence of optimal solutions for the discretized problems, for which necessary conditions are derived, to optima of the continuous problem. In this way an optimization scheme useful for numerical implementation is obtained. Necessary optimality condition for the original problem has however not been considered in these papers.

Finally, let us observe further that the dynamics in (3) can be seen as a *hybrid* dynamics, in the sense that the system has finitely many states, or, in other words, is steered by finitely many different dynamics, according with the position of x : either $x(t)$ belongs to the interior, or to some edge or to some vertex of the polyhedron C . Results providing necessary optimality conditions for hybrid systems are well known (see, e.g., [20] and references therein). However, the system is required to visit each state only along finitely many intervals, while on the other hand it is easy to construct examples in which a solution of (3) lies on the boundary of C on a Cantor set with positive measure. Actually, several problems involving variational inequalities and/or complementarity conditions like the sweeping process are studied trying to exploit their own structure, rather than embedding them in a more general class, as it is witnessed by the comprehensive survey [12].

Finally, we describe the organization of the paper. The problem and the main results are stated in Section 2. The existence proof is presented in Section 5, while the proof of the theorem on necessary conditions appears in Sections 6, 7, 8, and 9. The intermediate Section 3 contains the general derivation of the model, while in Section 4 we discuss extensively the necessary condition obtained in Theorem 2 in the case of a one-link crawler, and make a few technical remarks and comments.

2. Statement of the problem and main results

2.1. Notation

Let A and S be sets, with $A \subset S$. We set, for $x \in S$,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The Lebesgue measure of $S \subset \mathbb{R}$ is denoted by $|S|$. Given an integrable function f on a set S with finite measure, we denote its average as

$$\overline{f} = \frac{1}{|S|} \int_S f(s) \, ds$$

Let X be a finite-dimensional normed space. The closed unit ball in X is denoted by B_X , and the interior of a set $S \subset X$ is denoted by $\text{int } S$. The convergence with respect to the Hausdorff distance between closed subsets of X will be considered (see [42, Section 4.C]). We denote with $\mathcal{C}([0, T], X)$ the space of continuous functions from $[0, T]$ to X , endowed with the $\|\cdot\|_\infty$ norm; with $\mathcal{C}^*([0, T], X)$ its dual space, and with $\mathcal{C}_+^*([0, T], \mathbb{R})$ the subset of positive measures on $[0, T]$.

Classical constructs of nonsmooth analysis will be used. In particular, for a set S , the cone of (limiting/Mordukhovich) normal vectors to S at $x \in S$ is denoted by $\mathcal{N}_S(x)$ (see [42, Definition 6.3]), while for $x \notin S$, we set $\mathcal{N}_S(x) = \emptyset$. For a Lipschitz function $f: X \rightarrow \mathbb{R}$, the (limiting/Mordukhovich) subdifferential of f at x is denoted by $\partial f(x)$ (see [42, Definition 8.6 (b)]); we also refer the interested reader to [39, Chapter 1], where the above concepts are used also in the context of *coderivatives* of set-valued maps.

By a process, or a state-control pair, for the controlled dynamics (3) we mean the couple (x, u) , where x is a solution of (3) corresponding to the (measurable) control u . The total variation of a function x with values in \mathbb{R}^d is denoted by $TV(u)$.

2.2. Statement of the problem

Let C be a given polyhedron in a Euclidean space $X = \mathbb{R}^n$, defined as

$$C = \bigcap_{j=1}^{\sigma} C_j, \tag{5}$$

where, for suitable unit vectors $x_*^j \in X$ and real numbers c_j ,

$$C_j := \{x \in X : \langle x_*^j, x \rangle \leq c_j\}.$$

Given $x \in C$, let us denote with $\mathcal{I}(x)$ the set of active constraints in x , namely $\mathcal{I}(x) = \{j = 1, \dots, \sigma : \langle x_*^j, x \rangle = c_j\}$. We assume that C has non-empty interior; in other words, the Positive Linear-Independence Constraint-Qualification (PLICQ) holds, i.e., if $\sum_{j \in \mathcal{I}(x)} \lambda_j x_*^j = 0$ and $\lambda_j \geq 0$, $j = 1, \dots, \sigma$, then $\lambda_j = 0$ for all j . In this case, the normal cone to C at $x \in C$ is

$$\mathcal{N}_C(x) = \left\{ v : v = \sum_{j \in \mathcal{I}(x)} \lambda_j x_*^j, \lambda_j \geq 0 \right\}.$$

The following assumptions will be considered.

(H_U) The control set $U \subset \mathbb{R}^d$ is a compact and convex set, and that $d \leq \dim X$. Moreover, since we will require a zero-average condition on $u(t)$, we assume $0 \in \text{int } U$.

We remark that in order to guarantee the existence of controls $u(t)$ with zero-average it is sufficient to assume $0 \in U$, but if the zero lies on the boundary then all the zero-average functions $u(t)$ have values in a lower dimensional convex set \tilde{U} with $0 \in \text{int } \tilde{U}$.

We consider the maps $g: [0, T] \times X \times U \rightarrow X$, $f_1: X \rightarrow \mathbb{R}$ and $f_2: [0, T] \times U \rightarrow \mathbb{R}$ with the following properties.

- (H_g) the map $t \mapsto g(t, x, u)$ is measurable for all $x \in X, u \in U$ and there exists $L \geq 0$ such that $|g(t, x, u)| \leq L$ for a.e. $t \in [0, T]$ and all $(x, u) \in X \times U$, $(x, u) \rightarrow g(t, x, u)$ is \mathcal{C}^1 -smooth and there exists $L' \geq 0$ such that $|D_x g(t, x, u)| \leq L'$ for a.e. $t \in [0, T]$ and all $(x, u) \in X \times U$;
- (H_{f_1}) the map $x \mapsto f_1(x)$ is Lipschitz continuous;
- (H_{f_2}) the map $t \mapsto f_2(t, u)$ is continuous for all $u \in U$ and the map $u \mapsto f_2(t, u)$ is continuously differentiable for a.e. $t \in [0, T]$ and all $u \in U$;

Problem (P) We set $T > 0$ and consider the problem

$$\begin{cases} \dot{x} \in -\mathcal{N}_C(x) + g(t, x, u) & \text{a.e. on } [0, T], \\ u(t) \in U & \text{a.e. and } \int_0^T u(t) dt = 0, \\ x(0) = x(T). \end{cases} \tag{6}$$

We wish to maximize the integral functional

$$J(x, u) := \int_0^T \left(f_1(g(t, x(t), u(t)) - \dot{x}(t)) - f_2(t, u(t)) \right) dt \tag{7}$$

among all processes (x, u) of (6).

2.3. Statement of the main results

The setting of the existence theorem is slightly more general than in the previous section. Therefore we list the assumptions directly in the statement of the result. We will make reference to problem (6), but the statement can be easily reformulated for a Cauchy problem, with or without the constraint on the mean of the control.

Theorem 1. Let $C \subset X = \mathbb{R}^n$ be compact and convex, let $U \subset \mathbb{R}^d$ be compact, and let $K > 0$. Let $g: [0, T] \times X \times U \rightarrow X$ be measurable with respect to t , continuous with respect to (x, u) and uniformly bounded. Let $f: [0, T] \times X^2 \times U \rightarrow X$ be measurable with respect to t and upper semicontinuous with respect to (x, \dot{x}, u) . Set

$$\mathcal{U}_K := \{u : [0, T] \rightarrow \mathbb{R}^d : u \text{ is measurable, } u(t) \in U \text{ a.e. and } TV(u) \leq K\}$$

and assume that the problem (6) admits solutions with $u \in \mathcal{U}_K$. Then the integral functional

$$\mathcal{J}(x, u) := \int_0^T f(t, x(t), \dot{x}(t), u(t)) dt$$

admits a maximizer among all processes (x, u) of problem (6) such that $u \in \mathcal{U}_K$.

Our necessary optimality conditions for problem (P) are applicable to a different class of processes, without constraints on the total variation of the admissible controls, namely to locally $W^{1,2}$ -optimal processes. We say that (\bar{x}, \bar{u}) is a locally $W^{1,2}$ -optimal process for (P) provided there exists $\bar{\varepsilon} > 0$ such that for all processes (x, u) of (6) with $\|x - \bar{x}\|_{W^{1,2}([0,T];X)} + \|u - \bar{u}\|_{L^2([0,T];X)} < \bar{\varepsilon}$ one has $J(x, u) \leq J(\bar{x}, \bar{u})$.

The result on necessary optimality conditions is the following

Theorem 2. *Let $C \subset X = \mathbb{R}^n$ be as in Subsection 2.2, let the assumptions (H_U) , (H_g) , (H_{f_1}) , (H_{f_2}) hold, and let (\bar{x}, \bar{u}) be a locally $W^{1,2}$ -optimal process for the problem (P). Then there exist*

- a number $\lambda \geq 0$,
- a function of bounded variation $p: [0, T] \rightarrow X$,
- positive and finite Radon measures $d\xi^j$ on $[0, T]$, $j = 1, \dots, \sigma$,
- a function $\psi \in L^1(0, T; X)$,
- a vector $\omega \in B_X$

that satisfy the following properties:

- (adjoint equation) $dp = -D_x g(t, \bar{x}(t), \bar{u}(t))^* dt + \sum_{j=1}^{\sigma} d\xi^j x_*^j$ (in $C^*([0, T]; X)$),
- (transversality) $p(T) = p(0)$,
- (weak maximality condition) $\psi(t) = -D_w g(t, \bar{x}(t), \bar{u}(t))^* p(t) - \omega - \lambda D_w f_2(t, \bar{u}(t)) \in \mathcal{N}_U(\bar{u}(t))$ a.e. on $[0, T]$,
- (support condition) for all $j = 1, \dots, \sigma$, $\text{supp}(d\xi^j) \subset \{t \in [0, T] : j \in \mathcal{I}(\bar{x}(t))\}$,
- (nontriviality condition) $(\lambda, p) \neq (0, 0)$.

The proof of Theorem 2 will be carried out in Sections 6–9.

Remark 3. One can consider a dynamics more general than (6), namely

$$\begin{cases} \dot{x}(t) \in -\mathcal{N}_C(x(t)) + g(t, x(t), u(t)) \text{ a.e. on } [0, T], \\ \dot{y}(t) = f_1(g(t, x(t), u(t)) - \dot{x}(t)) - f_2(t, y(t), u(t)), \\ u(t) \in U \text{ and } \int_0^T u(t) dt = 0, \\ x(0) = x(T), \quad y(0) = 0 \end{cases}$$

The object to be maximized in this case is

$$\varphi(y(T)),$$

for a suitable (e.g., u.s.c.) function φ . Also, the more general class of $W^{1,p}$ -optimal processes could be considered in Theorem 2. Furthermore, we believe that our proof of Theorem 2 can be adapted to the class of admissible controls that is considered in Theorem 1, since our approximation technique does preserve the total variation of the reference control (see Lemma 6 below). This amounts to adding only technical difficulties, that we wish to avoid here.

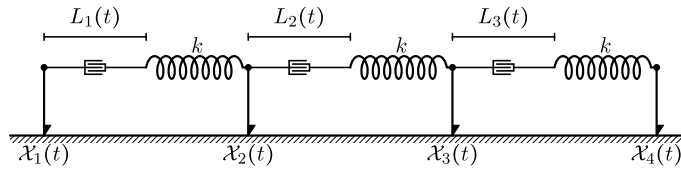


Fig. 1. A model of soft crawler.

3. A motivating locomotor model

3.1. Introduction

In the last years, an increasing attention has been directed to the analysis, control and optimization of the locomotion of simple devices, such as the chains of linked segments or blocks. The same trend can be observed both in swimming [6,38,50] or in locomotion on a solid surface, such as inching and crawling [2,9,23,47]. The employment of very simple mechanisms has two main motivations. The first one is that a simple mechanism allows an easier miniaturization of the device. The second advantage comes from the paradigm of *simplicity* in soft robotics, based on the idea that a simple mechanism with a low number of control parameters may still achieve a complex behaviour and adaptability to an unknown environment by exploiting the large deformation of a soft, elastic body [34]. This also motivates the strong role played by elasticity in our model, despite introducing several additional mathematical challenges.

In the specific case of crawling locomotion, several approaches have been applied to the search of optimal gaits. One strategy is to consider suitable approximations in the model, for instance neglecting elasticity or working on a small deformation regime, so that, with a certain degree of approximation, it is possible to have an explicit description of the dynamics in terms of the control function [2,19]. Another approach, introducing a feedback mechanism in order to apply adaptive control, is presented in [7,8]. A model-free control framework, based on the decomposition of possible gaits as paths between a finite number of basic states, has been proposed in [45].

In this paper we present a more mathematical approach, based on a maximum principle of Pontryagin type. On one hand this, compared to the more pragmatical approaches mentioned above, makes more difficult to obtain an explicit characterization of optimal gaits. On the other hand, we believe that the development of a more theoretical approach, in parallel to engineering studies appearing in literature, may contribute to a better understanding of the challenging issues raised by the optimal control of a soft bodied locomotor. In our opinion crawling locomotion is not only an interesting problem *per se*, but represent a less hostile framework in which we can learn to unravel difficult phenomena that appear in a more general setting.

Concerning the specific model considered in our paper, we will follow the approach developed in [23,24]. Our choice is motivated by the fact that such class of model includes the two main features observed in crawlers (a stick-slip interaction with the environment and an elastic body) without adding unnecessary elements. Moreover, even if here we consider only a smaller family of cases, the same formalism of sweeping process applies to a large class of behaviour, including continuous bodies and time-dependent friction [23], opening the way for future developments of our results.

3.2. A rate-independent model of soft crawler

Let us consider the mechanical system illustrated in Fig. 1, consisting of a chain of N blocks. Each couple of adjacent blocks is joined by a link composed by a spring in series with an actuator, namely an element of prescribed length $L_i(t)$, which is our control on the system. The body of the crawler can be therefore identified in the reference configuration by a set of N points $\{\xi_1, \dots, \xi_N\}$. We represent the state of the

crawler in the deformed configuration with a vector $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N)$ in \mathbb{R}^N , where \mathcal{X}_i stands for the displacement of the point ξ_i .

We consider the locomotion of our model in the regime of very slow (quasi-static) actuation, so that inertial forces can be neglected; a rigorous derivation of the quasi-static limit for our model has been obtained in [26]. Hence, the dynamics is described by a force balance between the friction forces acting on the body of the crawler and internal elastic forces associated to the deformations of the springs in the links, which can be written as

$$-D_{\mathcal{X}}\mathcal{E}(t, \mathcal{X}) \in \partial_{\dot{\mathcal{X}}}\mathcal{R}(\dot{\mathcal{X}}) \tag{8}$$

Here $\mathcal{E}(t, \mathcal{X})$ is the elastic energy of the crawler, and therefore can be expressed as the sum of the elastic energies $\mathcal{E}_i(t, \mathcal{X})$ of each link, namely

$$\mathcal{E}(t, \mathcal{X}) = \sum_{i=1}^{N-1} \mathcal{E}_i(t, \mathcal{X}) = \sum_{i=1}^{N-1} \frac{k}{2} (\mathcal{X}_{i+1} - \mathcal{X}_i - L_i(t))^2 \tag{9}$$

We assume that the actuation functions $L_i: [0, T] \rightarrow \mathbb{R}$ are Lipschitz continuous. The constant $k > 0$ is the elastic constant of the springs. Note that the same mathematical structure holds if we replace each actuator with an active control on the rest length of the corresponding spring, which is the case of robotic crawlers actuated e.g. by nematic elastomers [18,30].

Each of the points ξ_i is subject to an anisotropic dry friction, so that we can write the friction force F_i acting on ξ_i as

$$F_i = F_i(\dot{\mathcal{X}}_i) \in \begin{cases} \{\mu_i^-\} & \text{if } \dot{\mathcal{X}}_i < 0 \\ [-\mu_i^+, \mu_i^-] & \text{if } \dot{\mathcal{X}}_i = 0 \\ \{-\mu_i^+\} & \text{if } \dot{\mathcal{X}}_i > 0 \end{cases}$$

for some positive coefficients μ_i^\pm . Hence, friction forces can be expressed variationally in (8) as the subdifferential of the dissipation potential

$$\mathcal{R}(\dot{\mathcal{X}}) = \sum_{i=1}^N \mathcal{R}_i(\dot{\mathcal{X}}_i) \quad \text{with} \quad \mathcal{R}_i(\dot{\mathcal{X}}_i) = \begin{cases} -\mu_i^- \dot{\mathcal{X}}_i & \text{if } \dot{\mathcal{X}}_i \leq 0 \\ \mu_i^+ \dot{\mathcal{X}}_i & \text{if } \dot{\mathcal{X}}_i \geq 0 \end{cases} \tag{10}$$

We recall that $\partial_{\dot{\mathcal{X}}}\mathcal{R}(\dot{\mathcal{X}}) \subseteq \partial_{\dot{\mathcal{X}}}\mathcal{R}(0)$ since the function \mathcal{R} is positively homogeneous of degree one, and set

$$C_0 := \{\mathcal{X} \in X : -\mu_i^- \leq \langle e_i, \mathcal{X} \rangle \leq \mu_i^+ \quad \text{for } i = 1, \dots, N\} = \partial_{\dot{\mathcal{X}}}\mathcal{R}(0) \tag{11}$$

where e_1, \dots, e_n denotes the canonical base of X . Since the friction forces in (8) are bounded, we cannot allow too large initial elastic forces, hence we introduce the *admissibility condition* for the initial state:

$$-D_{\mathcal{X}}\mathcal{E}(t, \mathcal{X}_0) \in C_0 \tag{12}$$

In order to guarantee existence and uniqueness of solution for the Cauchy problem with an admissible initial state, we make the following assumption: for every subset of indices $J \subseteq \{1, \dots, N\}$ we have

$$\sum_{i \in J} \mu_i^+ - \sum_{i \in J^c} \mu_i^- \neq 0 \tag{13}$$

where J^c denotes the complement of J . We refer to [23, Section 2] for a complete proof and discussion. Since we will refer also later on to results obtained there, we observe for the reader’s convenience that the coordinate x and the sets C, C_{sh} in [23] correspond in this paper respectively to \mathcal{X}, C_0, C , where C will be introduced in the next subsection.

In order to study the locomotion of our model, it is useful to introduce the projections:

$$\begin{aligned} y &= \pi_Y(\mathcal{X}) := \frac{1}{N} \sum_{i=1}^N \mathcal{X}_i \in \mathbb{R} \\ z &= \pi_Z(\mathcal{X}) := (\mathcal{X}_2 - \mathcal{X}_1, \dots, \mathcal{X}_N - \mathcal{X}_{N-1}) =: (z_1, \dots, z_{N-1}) \in Z \cong \mathbb{R}^{N-1} \end{aligned} \tag{14}$$

In this way we can split the state of the crawler into two components: the term y describes the position of the crawler, whereas z describes its shape, namely the lengths of the $N - 1$ links in the deformed configuration.

Setting, without loss of generality $0 = y(0) = \pi_Y(\mathcal{X}(0))$, our problem consists of finding suitable choices of the actuations L_i that maximize $y(T) = \pi_Y(\mathcal{X}(T))$.

3.3. Formulation as a sweeping process

We now show how we can pass from the dynamics (8) for the model presented above to a sweeping process of the form (6) and discuss the other elements of problem (P).

We observe that, since the elastic energy \mathcal{E} is invariant for rigid translations, it depends only on the shape z , namely

$$\mathcal{E}(t, \mathcal{X}) = \left\langle \frac{k}{2} z - \ell_{\text{sh}}(t), z \right\rangle + \text{time-dependent term} \tag{15}$$

where we define $\ell_{\text{sh}}(t) = (kL_1(t), \dots, kL_{N-1}(t))$. The last term disappears in the dynamics (8), so can be neglected for our purposes.

We can reformulate the dynamics (8) as the variational inequality

$$\left\langle \frac{k}{2} \pi_Z(\dot{\mathcal{X}}(t)) - \dot{\ell}_{\text{sh}}(t), \pi_Z(V - \dot{\mathcal{X}}(t)) \right\rangle + \mathcal{R}(V) - \mathcal{R}(\dot{\mathcal{X}}(t)) \geq 0 \quad \text{for every } V \in X \tag{16}$$

cf. [36,37]. It is easily verified that a function $\mathcal{X}(t)$ satisfies (16) only if its projection $z(t) = \pi_Z(\mathcal{X}(t))$ satisfies:

$$\left\langle \frac{k}{2} z(t) - \dot{\ell}_{\text{sh}}(t), w - \dot{z}(t) \right\rangle + \mathcal{R}_{\text{sh}}(w) - \mathcal{R}_{\text{sh}}(\dot{z}(t)) \geq 0 \quad \text{for every } w \in Z \tag{17}$$

where the dissipation potential $\mathcal{R}_{\text{sh}}: Z \rightarrow \mathbb{R}$ is defined as

$$\mathcal{R}_{\text{sh}}(z) = \min\{\mathcal{R}(\mathcal{X}) : \mathcal{X} \in X, \pi_Z(\mathcal{X}) = z\} \tag{18}$$

The potential \mathcal{R}_{sh} is convex and positively homogeneous of degree one [23, Lemma 2.1].

We notice that, once (17) is solved, the solution to (16) can be recovered straightforwardly. Indeed, we observe that (13) allows to define a function $v_m: Z \rightarrow \mathbb{R}$ as the unique one satisfying

$$\mathcal{R}_{\text{sh}}(\pi_Z(\mathcal{X})) = \mathcal{R}(\mathcal{X}) \quad \text{if and only if} \quad \pi_Y(\mathcal{X}) = v_m(\pi_Z(\mathcal{X})) \tag{19}$$

cf. [23, Lemma 3.2]. We compute explicitly the function v_m in Subsection 3.5. This property allows us to recover the evolution of $y(t)$ from that of $z(t)$, as

$$\dot{y}(t) = \pi_Y(\dot{\mathcal{X}}(t)) = v_m(\pi_Z(\dot{\mathcal{X}}(t))) = v_m(\dot{z}(t)) \tag{20}$$

We can now reformulate the problem for the shape coordinates (17) in the differential inclusion formulation analogue to (8), namely

$$-kz + \ell_{\text{sh}}(t) \in \partial_z \mathcal{R}_{\text{sh}}(\dot{z}) \tag{21}$$

Let us denote by $\mathcal{R}_{\text{sh}}^*$ the Legendre transform of \mathcal{R}_{sh} . Setting $C := \partial_z \mathcal{R}_{\text{sh}}(0)$, by the Legendre-Fenchel equivalence we obtain

$$\dot{z} \in \partial_C \mathcal{R}_{\text{sh}}^*(-kz + \ell_{\text{sh}}(t)) = \mathcal{N}_C(-kz + \ell_{\text{sh}}(t)) \tag{22}$$

We observe that C is a polyhedron in Z of the form (5); indeed, by [23, Lemma 2.2] we deduce that

$$C = \{z \in Z : -\mu_i^- \leq \langle \pi_Z(e_i), z \rangle \leq \mu_i^+ \text{ for } i = 1, \dots, N\} \tag{23}$$

where e_1, \dots, e_N denotes the canonical base of \mathbb{R}^N .

Let us now consider the change of variables $x(t) = -kz(t) + \ell_{\text{sh}}(t)$ and set $u(t) := D_t \ell_{\text{sh}}(t)$. The locomotion of our system, by (20) and (22), is described by

$$\begin{cases} \dot{x}(t) \in -\mathcal{N}_C(x(t)) + u(t) \\ \dot{y}(t) = v_m(u(t) - \dot{x}(t)) \end{cases} \tag{24}$$

3.4. Formulation of the control problem (P)

Now that we have shown how the locomotion of our system can be described by the dynamics (24), we discuss the cost functional and the constraints required in our control problem (P).

A locomotion strategy, be it for crawling, swimming, walking or running, usually can be identified with a gait, namely a periodic pattern that is repeated a large number of times. Let us denote with T the period of a gait. In our model, this corresponds to assume that the function L_i are T -periodic, which in terms of the control u in (24) reads

$$\int_0^T u(t) dt = 0$$

It is also reasonable to assume that there are some constraints on the speed at which the shape change occurs, corresponding to a uniform Lipschitz constant for all the admissible actuations. This, for the dynamics (24), is exactly the constraint $u(t) \in U$, where the set U is of the form

$$U = \Pi_{\ell=1}^d [-a_\ell, a_\ell] \subset \mathbb{R}^d, \tag{25}$$

where $a_\ell > 0$ for all $\ell = 1, \dots, d$.

Since we are considering locomotion model in one-dimension, we want to maximize the advancement of the crawler produced by the chosen gait, plus possibly subtracting a cost for the actuation.

Due to the hysteretic behaviour of the sweeping process, a periodic input (in the above sense) does not necessarily produce a periodic change in the shape coordinates w , and the produced displacement $y(T) - y(0)$ of the crawler in a period depends on its initial shape $z(0)$. Hence the optimal gait may depend on the ability to exploit a specific initial state, and on the exact number of periods we are considering. This

does not suit our purposes, since we are interested in an arbitrarily long time behaviour. However it is known that sweeping processes with a periodic input converge asymptotically to a periodic output [33,27,28,15]. This has already been observed in the models with one link [24] and two links [25], also noticing that in the specific case of some “common sense” gaits the convergence to the asymptotic periodic orbit occurs within the first period.

Since we are interested in the long time behaviour, we can therefore optimize on the possible limit cycles (for the shape w) associated to a given gait, and evaluate the cost functional on a single period. This corresponds to optimize on the trajectories that satisfy the periodicity condition

$$x(0) = x(T)$$

Regarding the cost functional, denoting with $f_2(t, u)$ a possible cost of the actuation, we have

$$\begin{aligned} J(x, u) &= y(T) - y(0) - \int_0^T f_2(t, u(t)) dt \\ &= \int_0^T (v_m(u(t) - \dot{x}(t)) - f_2(t, u(t))) dt \end{aligned}$$

that corresponds to (7) for $g(t, x, u) = u$ and $f_2 = v_m$.

The reader may wonder why we are considering a periodic fixed time problem, instead of a free time or minimum time problem. The main motivation is that a gait works as an universal strategy, but might be slightly suboptimal for a specific prescribed problem. For instance, we expect that some case-by-case tuning on the first and last iterations of the gait, breaking periodicity, may provide a minor improvement to the solution. The natural way to avoid these complications is to optimize on the limit cycles of the system, as argued above. The reader may then wonder why to consider a fixed period. From the examples discussed in [18] and [24], one may observe that the action of the actuation of the crawler can be divided into two kinds of effects: a change in the tensions of the links ($\dot{x} \neq 0$) and movement of the contact points ($u - \dot{x} \neq 0$). A sufficiently complex change in the tension is necessary to reach suitable configurations that allow to move each contact point, and represents a sort of “fixed cost” necessary for locomotion. Every part of the period that is not used for the necessary tension change is best spent in pure locomotion ($u \in \mathcal{N}_C(w)$). A larger period hence increases the ratio of the time that is used for locomotion, leading to a better strategy. Hence, we do not expect a free period to affect in a significant way the qualitative structure of optimal solution. We also notice that a fixed period, combined with the bounds (25) on the rate of shape change, implicitly provide a bound on the maximum contraction and elongation of each link. This represents an additional physical constraint that should be incorporated in our model if such an assumption is removed. Since, as we will see, our problem is already quite complex as it is, we think that our approach is a preferable first step in the study of optimality for crawling locomotion.

3.5. Computation of the function v_m

We now compute explicitly the function v_m . Let us consider $\mathcal{X} \in C_0$ and denote with $J(\mathcal{X})$ the set of active constraints in \mathcal{X} , namely

$$J(\mathcal{X}) = \{j \in \{1, \dots, N\} : \mathcal{X}_j = -\mu_j^-\} \cup \{j \in \{N + 1, \dots, 2N\} : \mathcal{X}_{j-N} = \mu_{j-N}^+\}$$

(here the number of constraints is $\sigma = 2N$).

Let us consider a vector $v \in \mathbb{R}^N$. By the convexity of C_0 , there exists $\mathcal{X} \in C_0$ such that $v \in \mathcal{N}_{C_0}(\mathcal{X})$. Moreover we can write, for some non-negative $\lambda_i \geq 0$, $i = 1 \dots, 2N$,

$$v = \sum_{i=1}^N \lambda_i e_i + \sum_{i=N+1}^{2N} -\lambda_i e_{i-N} \tag{26}$$

We notice that, since the vectors e_i are linearly independent, the coefficients λ_i are uniquely determined. Moreover, they satisfy the active constraint condition

$$\lambda_i > 0 \Rightarrow i \in J(\mathcal{X}) \tag{27}$$

Let us set $\nu_i = \pi_Z(e_i)$ for $i = 1 \dots, N$ and $\nu_i = \pi_Z(-e_{i-N})$ for $i = N + 1 \dots, 2N$. We have

$$\pi_Z(v) = \sum_{i=1}^{2N} \lambda_i \nu_i \tag{28}$$

Let us recall that π_Z is a linear diffeomorphism between Z and $\pi_Y^{-1}(0)$, and that $C = \pi_Z(C_0 \cap \pi_Y^{-1}(0))$, see [23, Lemma 2.2]. We notice that, since $C_0 \cap \pi_Y^{-1}(0)$ is a section of a convex set, every vector $z \in Z$ can be written as $z = \pi_Z(v)$, with $v \in \mathcal{N}_{C_0}(\zeta)$ for some $\zeta \in C_0 \cap \pi_Y^{-1}(0)$. Moreover, condition (13) implies that such vector v is unique, see [23, Lemma 2.3]. These facts imply that every vector $z \in Z$ admits a decomposition (28), where the coefficients λ_i are uniquely determined.

We can finally use such decomposition of \dot{z} to give an explicit expression for v_m :

$$v_m \left(\sum_{i=1}^{2N} \lambda_i \nu_i \right) = \pi_Y \left(\sum_{i=1}^N \lambda_i e_i - \sum_{i=N+1}^{2N} \lambda_i e_{i-N} \right) = \sum_{i=1}^N \frac{\lambda_i}{N} - \sum_{i=N+1}^{2N} \frac{\lambda_i}{N} \tag{29}$$

where we used the fact that $\pi_Y(e_i) = \frac{1}{N}$ and (27). We observe that (29) in particular implies that v_m is Lipschitz continuous.

Remark 4 (*Properties of v_m*). We observe that, by construction, the function v_m is positively homogeneous of degree one. According to the choice of the parameters μ_i^\pm , it can be convex, concave, or more often neither.

Let us consider for example the case $N = 3$, with homogeneous friction for the three contact points, namely $\mu_i^+ = \mu^+$ and $\mu_i^- = \mu^-$ for $i = 1, 2, 3$. Excluding the critical values according to (13), we have three situations. If $\mu^- > 2\mu^+$ then v_m is convex and positive outside the origin, meaning that the crawler can move only forward. Symmetrically, if $\mu^+ > 2\mu^-$ then v_m is concave and negative outside the origin, meaning that the crawler can move only backward. In the intermediate case $\frac{1}{2}\mu^- < \mu^+ < 2\mu^-$, the function v_m is neither convex nor concave, and assumes both positive and negative values, with the crawler able to move in both directions. In particular, the origin is a “monkey-saddle point” (namely, a saddle point with three ridges and three ravines). Remarkably, the mathematically desirable case of concave v_m is also the less meaningful physically: indeed, the crawler can only move backward whereas we want to optimize its movement forwards, so that, for any reasonable actuation cost f_2 , the optimal strategy is trivially to stay idle and not move.

4. Application to a one-link crawler and remarks

We analyse now the information provided by Theorem 2 for the model introduced in Section 3. For simplicity, we consider only the one-link crawler. In this case, taking into account (29), the optimal control problem reads as follows.

Given an interval $C := [a, b]$, $T > 0$ and a smooth convex function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$\text{maximize } J(x, u) := \int_0^T \left(\frac{1}{2} |u(t) - \dot{x}(t)| - f(u(t)) \right) dt$$

subject to

$$\dot{x} \in -\mathcal{N}_C(x) + u \quad \text{a.e. on } [0, T], \quad x(T) = x(0) \in C, \quad (30)$$

where $u(t) \in [-1, 1]$ a.e. and $\int_0^T u(t) dt = 0$.

Let (\bar{x}, \bar{u}) be an optimal trajectory-control pair. Applying Theorem 2, we obtain the following necessary conditions:

there exist $\lambda \geq 0$, a BV function $p: [0, T] \rightarrow \mathbb{R}$, two finite positive Radon measures $d\xi^1$ and $d\xi^2$, $\omega \in \mathbb{R}$, and $\psi \in L^1([0, T]; \mathbb{R})$ such that:

- 1) $\text{supp}(d\xi^1) \subseteq \{t \in [0, T] : \bar{x}(t) = a\}$;
- 2) $\text{supp}(d\xi^2) \subseteq \{t \in [0, T] : \bar{x}(t) = b\}$;
- 3) $dp = -d\xi^1 + d\xi^2$ and $p(T) = p(0)$;
- 4) $\psi(t) = -p(t) - \omega - \lambda D_u f(\bar{u}(t)) \in \mathcal{N}_{[-1, 1]}(\bar{u}(t))$ a.e.;
- 5) $(\lambda, p) \neq (0, 0)$.

Observe first that there is a degenerate case, namely $p(t) \equiv -\omega \neq 0$ and $\lambda = 0$, that is satisfied by all trajectories of (30), with $d\xi^1 = d\xi^2 = 0$ and $\psi \equiv 0$, for any cost f .

Now we analyse a few nondegenerate cases, in the (desirable) event they occur. To simplify the analysis, we take either $f(u) \equiv 0$ or $f(u) = \frac{1}{2}u^2$.

Let us first focus on the trajectories $x(\cdot)$ in the interior of C , namely the ones such that $a < x(t) < b$ for all t . If $f \equiv 0$, we observe that they are all local extrema, because in this case $\dot{x}(t) = u(t)$ a.e. and the functional J vanishes in a neighbourhood of (x, u) .

In the case $f(u) = \frac{1}{2}u^2$, instead, then necessary conditions provide more information. Indeed, assume again that $a < x(t) < b$ for all t . Then, by the support conditions 1) and 2) and the adjoint equation 3), p is constant, so that $p + \omega$ is constant as well. Assume now the nondegeneracy condition $\lambda > 0$ is valid. Then the extremality condition 4) reads as

$$0 \in p + \omega + \lambda u(t) + \mathcal{N}_{[-1, 1]}(u(t)) \quad \text{for a.e. } t.$$

The right-hand side of the above expression is a strictly monotone function of u , thus there exists one and only one u^* such that $p + \omega + \lambda u^* + \mathcal{N}_{[-1, 1]}(u^*) = 0$, i.e., $u(t) \equiv u^*$. Since all feasible controls must have zero mean, $u \equiv 0$. Therefore, in this first nondegenerate case, the only extremal solutions that lie in the interior of $[a, b]$ for all t are constant. Observe that the above analysis remains valid if $f(\cdot)$ is strictly convex with minimum at 0. Observe also that, still assuming $\lambda > 0$, the above argument implies that \bar{u} is constant (not necessarily zero) along any interval I in which \bar{x} lies in the interior of $[a, b]$.

Let us now consider the trajectories that touch the boundary of $[a, b]$. In particular, let us notice that, in order to achieve “true” locomotion both the boundary points must be touched by the trajectory, since one contact point can be moved only if $x(t) = a$ and the other only if $x(t) = b$: a necessary and sufficient condition for this to happen is $T > 2(b - a)$. We assume again $f \equiv 0$. In this case, the adjoint vector p may not be constant, being however constant in every interval where \bar{x} lies in the interior of $[a, b]$. To proceed, we assume the further nondegeneracy condition

$$p(t) + \omega \neq 0 \text{ for all } t \in [0, T]. \tag{31}$$

Under this condition, 4) implies that $\bar{u}(t) = -\text{sign}(p(t) + \omega)$ a.e. Let $\mathcal{I} := \{t : a < \bar{x}(t) < b\}$. Since p is constant in every connected component of \mathcal{I} , $\bar{u} \in \{\pm 1\}$ is constant as well in any such component. Therefore, \mathcal{I} is a finite union of open intervals, each of them having length $b - a$, except possibly the first and the last one, whose lengths however sum up to $b - a$ as well due to the periodicity condition on \bar{x} . Summarizing, in this case optimal controls are bang-bang with finitely many switchings.

4.1. Remarks on an assumption of Theorem 1

The above example also illustrates why it is reasonable to expect optimal controls $\bar{u}(t)$ to have bounded variation. Indeed, we show that, given a state-control pair (x, u) with unbounded variation, we can always modify it to obtain a control pair (\tilde{x}, \tilde{u}) with $\|x - \tilde{x}\|_{L^\infty([0, T])}$ arbitrarily small, and such that $J(\tilde{x}, \tilde{u}) = J(x, u)$ if $f_2 = 0$, while $J(\tilde{x}, \tilde{u}) > J(x, u)$ if $f_2 = \frac{1}{2}u^2$.

Indeed, let us consider a control $u(t)$ with unbounded variation, and let t^* be a time such that in every neighbourhood of t^* the function $u(t)$ has unbounded variation. We distinguish two cases.

Firstly, consider the case $u(t^*) \in \text{int } U$ and take a sufficiently small interval $[t_1, t_2]$ such that $t_1 < t^* < t_2$ and $u(t) \in \text{int } U$ for every $t \in [t_1, t_2]$. Then we define a new state-control pair as

$$(\hat{x}, \hat{u}) := \begin{cases} \left(\frac{(t_2-t)x(t_1)+(t-t_1)x(t_2)}{t_2-t_1}, \int_{t_1}^{t_2} u(s) ds \right) & \text{for } t \in [t_1, t_2] \\ (x(t), u(t)) & \text{elsewhere} \end{cases}$$

We observe that (\hat{x}, \hat{u}) satisfies (30), with $J(\hat{x}, \hat{u}) = J(x, u)$ if $f_2 = 0$, and $J(\hat{x}, \hat{u}) > J(x, u)$ if $f_2 = \frac{1}{2}u^2$. Moreover (\hat{x}, \hat{u}) has bounded variation in $[t_1, t_2]$.

Secondly, we consider the case when $u(t^*)$ lies in the boundary of U ; for simplicity we discuss the case $u(t^*) = b$. We take a sufficiently small interval $[t_1, t_4]$ such that $t_1 < t^* < t_4$ and $u(t) \in (a, b]$ for every $t \in [t_1, t_4]$. Moreover we set $t_2 := \min\{t \in [t_1, t_4] : x(t) = b\}$ and $t_3 := \max\{t \in [t_1, t_4] : x(t) = b\}$. Then we define a new state-control pair as

$$(\hat{x}, \hat{u}) := \begin{cases} \left(\frac{(t_2-t)x(t_1)+(t-t_1)b}{t_2-t_1}, \int_{t_1}^{t_2} u(s) ds \right) & \text{for } t \in [t_1, t_2] \\ \left(b, \int_{t_2}^{t_3} u(s) \mathbb{1}_{u^{-1}(b)}(s) ds \right) & \text{for } t \in [t_2, t_3] \\ \left(\frac{(t_4-t)b+(t-t_3)x(t_4)}{t_4-t_3}, \int_{t_3}^{t_4} u(s) ds \right) & \text{for } t \in [t_3, t_4] \\ (x(t), u(t)) & \text{elsewhere} \end{cases}$$

Also in this case it is easy to see that (\hat{x}, \hat{u}) satisfies (30), with $J(\hat{x}, \hat{u}) = J(x, u)$ if $f_2 = 0$, and $J(\hat{x}, \hat{u}) > J(x, u)$ if $f_2 = \frac{1}{2}u^2$. Moreover \hat{u} has bounded variation in $[t_1, t_4]$.

We notice that, by the compactness of $[0, T]$, with a finite number of such modifications, we can obtain the control pair (\tilde{x}, \tilde{u}) as desired. Furthermore, by considering sufficiently small intervals, using the well-known fact that the solutions of (6) are Lipschitz continuous (see, e.g., [44]), we can obtain an arbitrarily small $\|x - \tilde{x}\|_{L^\infty([0, T])}$.

4.2. Remarks on the necessary conditions of Theorem 2

1. Observe that, differently from classical state constrained Bolza problems, the Hamiltonian contains only one summand of the integral cost: the part involving $f_1(g(t, x(t), u(t)) - \dot{x}(t))$ is missing, due to a cancellation that occurs in the proof of Theorem 10.

2. It is well known (see, e.g., [46, Sec. 10.6] and [4]) that necessary optimality conditions for state constrained control problems may be satisfied by *all* state-control pairs. The zero mean condition on the control u provides the further multiplier $\omega \in \mathbb{R}^d$, and this is why the stronger nontriviality condition (31) plays a role. However, up to now there are no sufficient conditions for (31) to hold. Similarly, conditions ensuring $\lambda > 0$ need to be studied, since classical results of this type do not apply to our setting.

5. Proof of Theorem 1

The proof of the existence result is based on a strong convergence argument that is essentially contained in [11] (see also Sec. 1.3 in [33]). We present here a version of this result that is fit for our setting.

Lemma 5. *Let $C \subset X$ be closed and convex and let g be as in the statement of Theorem 1. Let $u_\ell, u \in L^2([0, T]; \mathbb{R}^d)$, $\ell \in \mathbb{N}$ be such that*

$$u_\ell \rightarrow u \text{ in } L^2.$$

Let $x_\ell^0 \in C$ be such that

$$x_\ell^0 \rightarrow x^0$$

and let $x_\ell: [0, T] \rightarrow X$ be a solution of the Cauchy problem

$$\begin{cases} \dot{x}_\ell & \in -\mathcal{N}_C(x_\ell) + g(t, x_\ell, u_\ell) \\ x_\ell(0) & = x_\ell^0. \end{cases}$$

Then there exist a subsequence $\{x_{\ell_k}\}$ such that

$$x_{\ell_k} \rightarrow x \text{ strongly in } W^{1,2}([0, T]; X),$$

and the limit function $x(t)$ is a solution of the Cauchy problem

$$\begin{cases} \dot{x} & \in -\mathcal{N}_C(x) + g(t, x, u) \\ x(0) & = x^0. \end{cases} \tag{32}$$

Indeed, for every weak $W^{1,2}$ -limit point x of the sequence (x_ℓ) there exists a subsequence $\{x_{\ell_k}\}$ strongly converging to x , and x is a solution of (32).

Proof. It is well known (see, e.g., [35]) that $\{x_\ell\}$ is uniformly bounded in $W^{1,2}([0, T]; X)$, so that, up to a subsequence, x_ℓ converges weakly in $W^{1,2}([0, T]; X)$ to some $x: [0, T] \rightarrow X$. Again by well known arguments (see, e.g., [35]), x is a solution of (32). Moreover,

$$g(t, x_\ell, u_\ell) \rightarrow g(t, x, u) \text{ in } L^2. \tag{33}$$

Set $\xi_\ell := g(t, x_\ell, u_\ell) - \dot{x}_\ell$ and observe that $\xi_\ell(t) \in \mathcal{N}_C(x_\ell(t))$ a.e. Therefore, for a.e. t , all ℓ and all $h > 0$ small enough one has both

$$\left\langle \xi_\ell(t), \frac{x_\ell(t+h) - x_\ell(t)}{h} \right\rangle \leq 0$$

and

$$\left\langle \xi_\ell(t), \frac{x_\ell(t-h) - x_\ell(t)}{h} \right\rangle \leq 0.$$

By passing to the limit as $h \rightarrow 0^+$, one obtains that

$$\langle \xi_\ell(t), \dot{x}(t) \rangle = 0 \quad \text{a.e.} \tag{34}$$

The sequence $\{\xi_\ell\}$ converges weakly in L^2 to $\xi := g(t, x, u) - \dot{x}$. Since $\xi(t) \in \mathcal{N}_C(x(t))$ for a.e. t , the same argument as above yields

$$\langle \xi(t), \dot{x}_\ell(t) \rangle = 0 \quad \text{a.e.} \tag{35}$$

By (33), $\dot{x}_\ell + \xi_\ell \rightarrow \dot{x} + \xi$ strongly in $L^2([0, T]; X)$. Since $\dot{x}_\ell - \xi_\ell$ converges to $\dot{x} - \xi$ weakly in L^2 , the strong convergence is equivalent to

$$\|\dot{x}_\ell - \xi_\ell\|_{L^2} \rightarrow \|\dot{x} - \xi\|_{L^2}. \tag{36}$$

To show (36), observe that, by (34) and (35),

$$\|\dot{x}_\ell - \xi_\ell\|_{L^2}^2 = \|\dot{x}_\ell\|_{L^2}^2 + \|\xi_\ell\|_{L^2}^2 = \|\dot{x}_\ell + \xi_\ell\|_{L^2}^2 \rightarrow \|\dot{x} + \xi\|_{L^2}^2 = \|\dot{x} - \xi\|_{L^2}^2,$$

and the proof is concluded. \square

Theorem 1 now follows easily.

Proof of Theorem 1. Let $\{(x_\ell, u_\ell)\}$ be a maximizing sequence for \mathcal{J} among solutions of (6) with $u_\ell \in \mathcal{U}_K$ for all ℓ . By Helly’s theorem, there exists \bar{u} such that, up to a subsequence, $u_\ell \rightarrow \bar{u}$ pointwise, so that $\bar{u}(t) \in U$ for all t , and actually $u_\ell \rightarrow \bar{u}$ in L^2 . Moreover, $\int_0^T \bar{u}(t) dt = 0$. Since C is compact, up to taking another subsequence we may assume that $x_\ell(0) \rightarrow x^0$ for some $x^0 \in C$. Then, by Lemma 5, $x_\ell \rightarrow \bar{x}$ strongly in $W^{1,2}$, where \bar{x} is the solution of (6) corresponding to \bar{u} and with initial condition $\bar{x}(0) = x^0$ (\bar{x} is easily seen to be T -periodic). Then

$$\mathcal{J}(\bar{x}, \bar{u}) \geq \lim_{\ell \rightarrow \infty} \mathcal{J}(x_\ell, u_\ell),$$

so that (\bar{x}, \bar{u}) is obviously an optimal state-control pair. \square

6. Discrete approximations of trajectories

In order to apply the discretization approach, we need first to establish a result on discrete approximations of solutions of the general sweeping process

$$\dot{x} \in -\mathcal{N}_C(x) + g(t, x, u) \quad \text{a.e.}, \quad u(\cdot) \text{ measurable with } u(t) \in U, \tag{37}$$

where the polyhedron $C \subset X$, the function g and the control set U satisfy the assumption of Section 2.2. A similar result was obtained in [14]; here the assumptions on the reference process (\bar{x}, \bar{u}) are weakened, as $\dot{\bar{x}}$ and \bar{u} are no longer supposed to have bounded variation.

Let us first state a lemma on piece-wise constant approximations of functions with prescribed average.

Lemma 6. *Let $T > 0$ and $\bar{u} \in L^1([0, T]; \mathbb{R}^n)$. For each $m \in \mathbb{N}$, set $t_m^i = \frac{iT}{2^m}$, $i = 0, 1, \dots, 2^m$, and $I_m^i = [t_m^i, t_m^{i+1})$. Define*

$$u_m(t) := \sum_{i=0}^{2^m-1} \int_{I_m^i} \bar{u}(s) \, ds \, \mathbb{1}_{I_m^i}(t). \tag{38}$$

Then $u_m \rightarrow \bar{u}$ a.e. on $[0, T]$ and $\int_0^T u_m = \int_0^T \bar{u}$.

Proof. Set $h_m = \frac{T}{2^m}$. Let $t \in [0, T)$ and let $I_m(t) := [\tau_m(t), \tau_m(t) + h_m)$ be the unique interval I_m^i such that $t \in I_m^i$. By Lebesgue differentiation theorem, $\lim_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} \bar{u}(s) \, ds = \lim_{h \rightarrow 0^+} \frac{1}{h} \int_{t-h}^t \bar{u}(s) \, ds = \bar{u}(t)$ almost everywhere. Thus, for almost every t ,

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{I_m(t)} \bar{u} &= \lim_{m \rightarrow \infty} \frac{1}{h_m} \left(\int_{\tau_m(t)}^t \bar{u} + \int_t^{\tau_m(t)+h_m} \bar{u} \right) \\ &= \lim_{m \rightarrow \infty} \frac{1}{h_m} \left[(t - \tau_m(t)) \left(\bar{u}(t) + \frac{1}{t - \tau_m(t)} \int_{\tau_m(t)}^t \bar{u}(s) \, ds - \bar{u}(t) \right) \right. \\ &\quad \left. + (\tau_m(t) + h_m - t) \left(\bar{u}(t) + \frac{1}{\tau_m(t) + h_m - t} \int_t^{\tau_m(t)+h_m} \bar{u}(s) \, ds - \bar{u}(t) \right) \right] \\ &= \bar{u}(t) + \lim_{m \rightarrow \infty} \left(\frac{t - \tau_m(t)}{h_m} o(1) + \frac{\tau_m(t) + h_m - t}{h_m} o(1) \right) \\ &= \bar{u}(t). \end{aligned}$$

Finally, we observe that the average of \bar{u} is preserved by the discretization. \square

Recalling that, as it is well known, all solutions of (37) are Lipschitz with the common Lipschitz constant L , we now state the main result of the section.

Theorem 7. Under the assumptions of Section 2.2, let (\bar{x}, \bar{u}) be a process for (6). Let $m \in \mathbb{N}$ and let I_m^i , $i = 0, \dots, 2^m - 1$ be as in Lemma 6. Set also $h_m = \frac{T}{2^m}$. Then there exist sequences

$$\{c_m^j\}_{j=1, \dots, \sigma, i=0, \dots, 2^m-1} \subset \mathbb{R}, \quad x_m : [0, T] \rightarrow X, \quad r_m : [0, T] \rightarrow [0, +\infty) \quad (m \in \mathbb{N}),$$

with the following properties:

- a) $\max_{i=0, \dots, 2^m-1} |c_m^i - c^i| \leq \frac{LT}{2^m}$;
- b) $x_m(\cdot)$ is continuous and is affine on each I_m^i , $i = 0, \dots, 2^m - 1$, $x_m(0) = \bar{x}(0)$, $x_m(T) = \bar{x}(T)$, and, for $i = 0, \dots, 2^m - 1$,

$$\begin{aligned} \frac{x_m(t_m^{i+1}) - x_m(t_m^i)}{h_m} &\in - \int_{I_m^i} (g(s, \bar{x}(s), \bar{u}(s)) - \dot{\bar{x}}(s)) \, ds + g(t_m^i, x_m(t_m^i), u_m(t_m^i)) + \int_{I_m^i} r_m(s) \, ds \, B_X \\ &\subset -\mathcal{N}_{C_m^i}(x_m(t_m^i)) + g(t_m^i, x_m(t_m^i), u_m(t_m^i)) + \int_{I_m^i} r_m(s) \, ds \, B_X, \end{aligned}$$

where u_m is defined as in (38) and the polyhedra C_m^i will be defined in (41) below;

c) the sequence of polyhedral valued maps

$$C_m(t) := \sum_{i=0}^{2^m-1} C_m^i \mathbb{1}_{I_m^i}(t)$$

converges to C with respect to the Hausdorff metric, uniformly w.r.t. t , and C_m^i satisfies the Positive Linear-Independence Constraint-Qualification for all $i = 0, \dots, \sigma$, provided m is large enough;

d) $r_m \rightarrow 0$ a.e. on $[0, T]$ and $\|r_m\|_\infty$ is bounded uniformly w.r.t. m ;

e) $x_m \rightarrow \bar{x}$ strongly in $W^{1,2}([0, T]; X)$.

f) $J(x_m, u_m) \rightarrow J(\bar{x}, \bar{u})$;

g) $\int_0^T u_m(t) dt = 0$.

Proof. Fix $m \in \mathbb{N}$ and define, for $i = 0, \dots, 2^m - 1$ and $t \in I_m^i$,

$$\begin{aligned} \omega_m^i &:= \frac{\bar{x}(t_m^{i+1}) - \bar{x}(t_m^i)}{h_m} \\ x_m(t) &:= \bar{x}(t_m^i) + (t - t_m^i)\omega_m^i \\ &= \bar{x}(t_m^i) + (t - t_m^i) \int_{I_m^i} \dot{\bar{x}}(s) ds. \end{aligned}$$

Observe that, for each $i = 0, \dots, 2^m$, $x_m(t_m^i) = \bar{x}(t_m^i) \in C$, and, by Lemma 6, $x_m \rightarrow \bar{x}$ strongly in $W^{1,2}([0, T]; X)$. Define

$$\omega_m(t) := \sum_{i=0}^{2^m-1} \omega_m^i \mathbb{1}_{I_m^i}(t), \quad t \in [0, T].$$

Fix $i = 0, \dots, 2^m - 1$ and set, for $j = 1, \dots, \sigma$,

$$c_m^{ji} := \begin{cases} c^j & \text{if } \langle \bar{x}(t), x_*^j \rangle < c^j \text{ for all } t \in I_m^i \\ \langle \bar{x}(t_m^i), x_*^j \rangle & \text{otherwise,} \end{cases}$$

and

$$c_m^j(t) = \sum_{i=0}^{2^m-1} c_m^{ji} \mathbb{1}_{I_m^i}(t).$$

We claim that $c_m^j(t) \rightarrow c^j$ uniformly on $[0, T]$ as $m \rightarrow \infty$. Indeed, set for $t \in [0, T]$

$$\tau_m(t) := \max\{t_m^i : i = 0, \dots, 2^m - 1, t_m^i \leq t\} \text{ and } I_m(t) = [\tau_m(t), \tau_m(t) + h_m),$$

and fix $j = 1, \dots, \sigma$ and $\bar{t} \in [0, T]$. If $\langle \bar{x}(\bar{t}), x_*^j \rangle < c^j$, then eventually $\langle \bar{x}(t), x_*^j \rangle < c^j$ for all $t \in [\tau_m(\bar{t}), \tau_m(\bar{t}) + h_m)$, so that, eventually, $c_m^j(\bar{t}) = c^j$. Let now $\langle \bar{x}(\bar{t}), x_*^j \rangle = c^j$. Then $c_m^j(\bar{t})$, that is equal to $c_m^j(\tau_m(\bar{t}))$, satisfies the conditions

$$\langle \bar{x}(\tau_m(\bar{t})), x_*^j \rangle = c_m^j(\tau_m(\bar{t})) \leq \langle \bar{x}(\bar{t}), x_*^j \rangle = c^j,$$

where both the equality and the inequality follow from our definition of $c_m^j(t)$. Then

$$|c_j - c_m^j(\tau_m(\bar{t}))| = |\langle \bar{x}(\bar{t}) - \bar{x}(\tau_m), x_*^j \rangle| \leq Lh_m.$$

This in turn implies a).

Set now

$$g_m(t) := g(\tau_m(t), \bar{x}(\tau_m(t)), u_m(t)) \quad \text{and} \quad \zeta_m(t) = g_m(t) - \omega_m(t), \quad t \in [0, T]. \tag{39}$$

We recall that, for a.e. $t \in [0, T]$

$$g(t, \bar{x}(t), \bar{u}(t)) - \dot{\bar{x}}(t) = \sum_{j=1}^{\sigma} \lambda_j(t) x_*^j$$

for suitable measurable $\lambda_j(\cdot)$. Moreover we can take $\lambda_j(\cdot) \geq 0$, and such that $\lambda_j(t) = 0$ for each t for which $\langle \bar{x}(t), x_*^j \rangle < c^j$, namely $j \notin \mathcal{I}(\bar{x}(t))$. Observe that

$$\begin{aligned} \zeta_m(t) &= \int_{[\tau_m(t), \tau_m(t)+h_m)} (g(s, \bar{x}(s), \bar{u}(s)) - \dot{\bar{x}}(s)) \, ds - \int_{[\tau_m(t), \tau_m(t)+h_m)} (g(s, \bar{x}(s), \bar{u}(s)) - g_m(s)) \, ds \\ &= \sum_{j=1}^{\sigma} x_*^j \int_{[\tau_m(t), \tau_m(t)+h_m)} \lambda_j(s) \, ds - \int_{[\tau_m(t), \tau_m(t)+h_m)} (g(s, \bar{x}(s), \bar{u}(s)) - g_m(s)) \, ds. \end{aligned} \tag{40}$$

Define now, for $j = 1, \dots, \sigma, i = 0, \dots, 2^m - 1$,

$$C_m^{ji} := \{x \in X : \langle x, x_*^j \rangle \leq c_m^{ji}\}, \quad C_m^i = \bigcap_{j=1}^{\sigma} C_m^{ji}, \quad C_m^j(t) := \sum_{i=0}^{2^m-1} C_m^{ji} \mathbb{1}_{I_m^i}(t) \tag{41}$$

and observe that, by our construction,

$$\sum_{j=1}^{\sigma} x_*^j \int_{I_m(t)} \lambda_j(s) \, ds \in \mathcal{N}_{C_m^j(\tau_m(t))}(x_m(\tau_m(t))) = \mathcal{N}_{C_m^j(\tau_m(t))}(x_m(t))$$

for a.e. t . Moreover, for all $t \in [0, T]$,

$$\begin{aligned} \int_{I_m(t)} (g(s, \bar{x}(s), \bar{u}(s)) - g_m(s)) \, ds &= \int_{I_m(t)} (g(s, \bar{x}(s), \bar{u}(s)) - g(\tau_m(s), \bar{x}(\tau_m(t)), \bar{u}(s))) \, ds \\ &\quad + \int_{I_m(t)} (g(\tau_m(s), \bar{x}(\tau_m(t)), \bar{u}(s)) - g_m(s)) \, ds, \end{aligned}$$

where we have used the fact that, for $s \in I_m(t)$, $\tau_m(s) = \tau_m(t)$. Now, by the uniform continuity of g w.r.t. x and u in the domain of interest the proof of b) is concluded.

The remaining claims are immediate consequences of the construction and of Lemma 6, taking into account that $\|\dot{\zeta}_m\|_{\infty}$ is bounded uniformly w.r.t. m . In particular, the Hausdorff convergence in item c) follows from the fact that active constraints are eventually constant, while statement f) follows from the pointwise a.e. convergence of the sequences $\dot{\zeta}_m$ and u_m . Moreover, u_m has zero mean by construction. \square

7. A discrete optimization problem

Let (\bar{x}, \bar{u}) be a locally $W^{1,2}$ -optimal process for problem (P), i.e., there exists $\bar{\varepsilon} > 0$ such that for all processes (x, u) of (6) with $\|x - \bar{x}\|_{W^{1,2}([0,T];X)} + \|u - \bar{u}\|_{L^2([0,T];X)} < \bar{\varepsilon}$ one has $J(x, u) \leq J(\bar{x}, \bar{u})$. Let $\{(x_m, u_m)\}$ be the sequence of approximations of (\bar{x}, \bar{u}) constructed according to Lemma 6 and Theorem 7 and set, for $m \in \mathbb{N}$,

$$\alpha_m = \sum_{i=0}^{2^m-1} \int_{I_m^i} \left(\left| \frac{x_m^{i+1} - x_m^i}{h_m} - \dot{\bar{x}}(t) \right|^2 + |u_m^i - \bar{u}(t)|^2 \right) dt.$$

By Theorem 7, $\alpha_m \rightarrow 0$ as $m \rightarrow \infty$. Set

$$\kappa_m = \frac{1}{\sqrt{\alpha_m}}. \tag{42}$$

Consider the following family of finite dimensional optimization problems (P_m) :

Problem (P_m) Let $\bar{\varepsilon}$ be given by the definition of locally $W^{1,2}$ optimal process; let $m \in \mathbb{N}$ be given; let I_m^i , $i = 0, \dots, 2^m - 1$, and u_m be as in Lemma 6; let $r_m(\cdot)$ and c_m^j , $j = 1, \dots, \sigma$, $i = 0, \dots, 2^m - 1$, be as in Theorem 7, and set $c_m^{j2^m} := c_m^j$, $j = 1, \dots, \sigma$. Writing $(z_m; w_m; \rho_m) = (z_m^0, \dots, z_m^{2^m}; w_m^0, \dots, w_m^{2^m-1}; \rho_m^0, \dots, \rho_m^{2^m-1})$, we want to

$$\begin{aligned} \text{maximize } J_m(z_m, w_m) := & h_m \sum_{i=0}^{2^m-1} \left(f_1 \left(g(t_m^i, z_m^i, w_m^i) - \frac{z_m^{i+1} - z_m^i}{h_m} \right) - f_2(t_m^i, w_m^i) \right) \\ & - \frac{\kappa_m}{2} \left(|z_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} \left(\left| \frac{z_m^{i+1} - z_m^i}{h_m} - \dot{\bar{x}}(t) \right|^2 + |w_m^i - \bar{u}(t)|^2 \right) dt \right) \end{aligned}$$

over discrete processes (z_m, w_m, ρ_m) such that

$$\langle x_m^j, z_m^i \rangle \leq c_m^j, \quad i = 0, \dots, 2^m - 1, \quad j = 1, \dots, \sigma; \tag{m_1}$$

$$\begin{aligned} \frac{z_m^{i+1} - z_m^i}{h_m} = & - \int_{I_m^i} (g(s, \bar{x}(s), \bar{u}(s)) - \dot{\bar{x}}(s)) ds + g(t_m^i, z_m^i, w_m^i) \\ & + \rho_m^i \int_{I_m^i} r_m(s) ds, \quad i = 0, \dots, 2^m - 1; \end{aligned} \tag{m_2}$$

$$z_m^{2^m} = z_m^0; \tag{m_3}$$

$$|z_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} \left(\left| \frac{z_m^{i+1} - z_m^i}{h_m} - \dot{\bar{x}}(t) \right|^2 + |w_m^i - \bar{u}(t)|^2 \right) dt \leq \frac{\bar{\varepsilon}}{2}; \tag{m_4}$$

$$\sum_{i=0}^{2^m-1} w_m^i = 0; \tag{m_5}$$

$$w_m^i \in U, \quad i = 0, \dots, 2^m - 1; \tag{m_6}$$

$$\rho_m^i \in B_X, \quad i = 0, \dots, 2^m - 1. \tag{m_7}$$

By standard finite dimensional programming arguments, problem (P_m) admits optimal processes.

The following result is pivotal in the method of discrete approximations. It is similar to, e.g., Theorem 4.3 in [14], with a difference: we do not assume relaxation stability. Indeed our method allows to treat a non-concave integral functional, as the functional that appears in Sections 3.4 and 4, without passing through the relaxed problem.

Theorem 8. *Let the assumptions of Theorem 2 hold. Let (\bar{x}, \bar{u}) be a $W^{1,2}$ -optimal process for Problem (P) and let (\bar{x}_m, \bar{u}_m) be optimal processes for Problems (P_m) . With an abuse of notation, consider \bar{x}_m , resp. \bar{u}_m , as piecewise affinely, resp. piecewise constantly, extended to the whole of $[0, T]$. Then*

$$\begin{aligned} \bar{x}_m &\rightarrow \bar{x} \text{ strongly in } W^{1,2}([0, T]; X) \\ \bar{u}_m &\rightarrow \bar{u} \text{ strongly in } L^2([0, T]; \mathbb{R}^d). \end{aligned}$$

More precisely,

$$\lim_{m \rightarrow \infty} \kappa_m \left(|\bar{x}_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} (|\dot{\bar{x}}_m(t) - \dot{\bar{x}}(t)|^2 + |\bar{u}_m(t) - \bar{u}(t)|^2) dt \right) = 0. \tag{43}$$

Consequently, $J_m(\bar{x}_m, \bar{u}_m) \rightarrow J(\bar{x}, \bar{u})$.

Proof. Suppose by contradiction that, possibly along a subsequence,

$$\lim_{m \rightarrow \infty} \kappa_m \left(|\bar{x}_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} (|\dot{\bar{x}}_m(t) - \dot{\bar{x}}(t)|^2 + |\bar{u}_m(t) - \bar{u}(t)|^2) dt \right) = \gamma \in (0, +\infty]. \tag{44}$$

Thanks to (m_4) , the sequence $\{(\bar{x}_m, \bar{u}_m)\}$ is bounded in $W^{1,2}([0, T]; X) \times L^2([0, T]; \mathbb{R}^d)$, so that, up to taking a subsequence, it converges to some (\tilde{x}, \tilde{u}) weakly in the same space.

By our definition of κ_m ,

$$\lim_{m \rightarrow \infty} \kappa_m \left(|x_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} (|\dot{x}_m(t) - \dot{\bar{x}}(t)|^2 + |u_m(t) - \bar{u}(t)|^2) dt \right) = 0, \tag{45}$$

where we recall that (x_m, u_m) is the sequence of approximations of (\bar{x}, \bar{u}) constructed in Lemma 6 and in Theorem 7, hence, in particular, $x_m^0 = \bar{x}(0)$. Since (\bar{x}_m, \bar{u}_m) is an optimal process for (P_m) ,

$$J_m(\bar{x}_m, \bar{u}_m) \geq J_m(x_m, u_m) \quad \forall m \in \mathbb{N}. \tag{46}$$

By (45) and by strong convergence (see Theorem 7),

$$J_m(x_m, u_m) \rightarrow J(\bar{x}, \bar{u}) \text{ as } m \rightarrow \infty. \tag{47}$$

Moreover, by (m_2) , (m_4) and our assumptions, the sequence

$$h_m \sum_{i=0}^{2^m-1} \left(f_1 \left(g(t_m^i, \bar{x}_m^i, \bar{u}_m^i) - \frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} \right) - f_2(t_m^i, \bar{u}_m^i) \right)$$

is uniformly bounded, so that the sequence

$$J_m(\bar{x}_m, \bar{u}_m) + \frac{\kappa_m}{2} \left(|\bar{x}_m(0) - \bar{x}(0)|^2 + \int_0^T (|\dot{\bar{x}}_m(t) - \dot{\bar{x}}(t)|^2 + |\bar{u}_m(t) - \bar{u}(t)|^2) dt \right)$$

is uniformly bounded from above. Moreover, thanks to (46) and (47), the same sequence is also uniformly bounded from below. This implies in turn that the sequence

$$\kappa_m \left(|\bar{x}_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} (|\dot{\bar{x}}_m(t) - \dot{\bar{x}}(t)|^2 + |\bar{u}_m(t) - \bar{u}(t)|^2) dt \right)$$

is uniformly bounded, so that, in particular, $\gamma < +\infty$. As a consequence, $(\tilde{x}, \tilde{u}) = (\bar{x}, \bar{u})$, and the convergence $(\bar{x}_m, \bar{u}_m) \rightarrow (\bar{x}, \bar{u})$ is indeed strong in $W^{1,2}([0, T]; X) \times L^2([0, T]; \mathbb{R}^d)$. Thanks to this fact,

$$\lim_{m \rightarrow \infty} J_m(\bar{x}_m, \bar{u}_m) = J(\bar{x}, \bar{u}).$$

Therefore, (44) implies that

$$J(\bar{x}, \bar{u}) - \frac{\gamma}{2} \geq J(\bar{x}, \bar{u}),$$

and this contradiction completes the proof. \square

8. Necessary conditions for the discrete approximate problem

Throughout this section, the assumptions of Theorem 2 are supposed to hold.

Fix $m \geq 1$. In order to proceed with deriving necessary optimality conditions for problems (P_m) , we will introduce some further notations. We set

$$\mathcal{X} := (x_m^0, \dots, x_m^{2^m-1}; w_m^0, \dots, w_m^{2^m-1}; \rho_m^0, \dots, \rho_m^{2^m-1}; \Delta_m^0, \dots, \Delta_m^{2^m-1}) \in X^{2^m} \times \mathbb{R}^{2^m d} \times X^{2^m} \times X^{2^m}.$$

With the understanding that the periodicity condition (m_3) has been directly incorporated in the structure of the problem by identifying $x_m^0 \equiv x_m^{2^m}$, and that the incremental ratios are described by $\Delta_m^i = \frac{x_m^{i+1} - x_m^i}{h_m}$, we will write the functional J_m as depending on \mathcal{X} .

We define furthermore the following maps, for $i = 0, \dots, 2^m - 1$: for $x \in X$, $w \in U$, and $\rho \in B_X$, we set

$$\Gamma_m^i(x, w, \rho) := - \int_{I_m^i} (g(s, \bar{x}(s), \bar{u}(s)) - \dot{\bar{x}}(s)) ds + g(t_m^i, x, w) + \rho \int_{I_m^i} r_m(s) ds \quad (\in X).$$

The computation of necessary conditions for (P_m) will be carried out in two steps. In the first step we will show how to set (P_m) in the framework of classical results on finite dimensional optimization, while in the second one the calculations for this particular case will be carried out.

Theorem 9. *Let $\bar{\mathcal{X}} = (\bar{x}_m, \bar{w}_m, \bar{\rho}_m, \bar{\Delta}_m)$ be an optimal process for (P_m) . Then there exist $\lambda_m > 0$, $\omega_m \in \mathbb{R}^d$, $\psi_m^i \in \mathbb{R}^d$, $\beta_m^i, \eta_m^i \in X$, $\xi_m^i = (\xi_m^{i1}, \dots, \xi_m^{i\sigma}) \in \mathbb{R}_+^\sigma$, for $i = 0, \dots, 2^m - 1$, $p_m^i \in X$ for $i = 0, \dots, 2^m$, and $\mathcal{X}_m^* \in X^{2^m} \times \mathbb{R}^{2^m d} \times X^{2^m} \times X^{2^m}$, such that*

$$\xi_m^{ij} (\langle x_*^j, \bar{x}_m^i \rangle - c_m^{ij}) = 0, \quad i = 0, \dots, 2^m - 1, \quad j = 1, \dots, \sigma, \tag{48}$$

and, for $i = 0, \dots, 2^m - 1$,

$$\begin{aligned} & \left(\kappa_m^i (\bar{x}(0) - \bar{x}_m^i) - \lambda_m D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i) \eta_m^i - \sum_{j=1}^{\sigma} \frac{\xi_m^{ij}}{h_m} x_m^j + \frac{p_m^{i+1} - p_m^i}{h_m}, \right. \\ & \quad - \lambda_m (D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i) \eta_m^i + \kappa_m (\bar{w}_m^i - u_m^i) + D_w f_2(t_m^i, w_m^i)) - \frac{\omega_m}{h_m} - \frac{\psi_m^i}{h_m}, \\ & \quad \left. - \frac{\beta_m^i}{h_m}, p_m^i + \lambda_m \left(\eta_m^i + \kappa_m \left(\frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} - \int_{I_m^i} \dot{x}(t) dt \right) \right) \right) \in \mathcal{N}_{\text{graph}(\Gamma_m^i)}(\bar{\mathcal{X}}), \end{aligned} \tag{49}$$

where

$$\kappa_m^i = \kappa_m \text{ for } i = 0 \text{ and } \kappa_m^i = 0 \text{ for } i \neq 0, \text{ where } \kappa_m \text{ is defined in (42),} \tag{50}$$

$$p_m^{2^m} = p_m^0, \tag{51}$$

and

$$\psi_m^i \in \mathcal{N}_U(\bar{w}_m^i), \quad \beta_m^i \in \mathcal{N}_{B_X}(\bar{\rho}_m^i), \quad \eta_m^i \in \partial f_1 \left(g(t_m^i, \bar{x}_m^i, \bar{w}_m^i) - \frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} \right). \tag{52}$$

Proof. We begin by arranging in two different categories the constraints in problem (P_m) . The variable \mathcal{X} fulfils the following requirements:

$$\Phi(\mathcal{X}) := |x_m^0 - \bar{x}(0)|^2 + \sum_{i=0}^{2^m-1} \int_{I_m^i} |(\Delta_m^i, w_m^i) - (\dot{x}(t), \bar{u}(t))|^2 dt - \frac{\varepsilon}{2} \leq 0 \tag{53}$$

$$\delta_m^i(\mathcal{X}) := x_m^{i+1} - x_m^i - h_m \Delta_m^i = 0, \quad i = 0, \dots, 2^m - 2 \tag{54}$$

$$\delta_m^{2^m-1}(\mathcal{X}) := x_m^0 - x_m^{2^m-1} - h_m \Delta_m^{2^m-1} = 0 \tag{55}$$

$$h_m^{ij}(\mathcal{X}) := \langle x_m^j, x_m^i \rangle - c_m^{ij} \leq 0, \quad i = 0, \dots, 2^m - 1, \quad j = 1, \dots, \sigma \tag{56}$$

together with

$$\mathcal{X} \in \Xi_m^i := \{ \mathcal{X} : \Delta_m^i = \Gamma_m^i(x_m^i, w_m^i, \rho_m^i) \}, \quad i = 0, \dots, 2^m - 1 \tag{57}$$

$$\mathcal{X} \in \mathcal{B}_m^i := \{ \mathcal{X} : |\rho_m^i| \leq 1 \}, \quad i = 0, \dots, 2^m - 1 \tag{58}$$

$$\mathcal{X} \in \Omega_m^i := \{ \mathcal{X} : w_m^i \in U \}, \quad i = 0, \dots, 2^m - 1 \tag{59}$$

$$\mathcal{X} \in \Omega_m := \{ \mathcal{X} : \sum_{i=0}^{2^m-1} w_m^i = 0 \}. \tag{60}$$

Recalling Theorem 8, the constraint (53) is eventually inactive and therefore will be neglected in the computations of necessary conditions. Applying classical results in mathematical programming we obtain a set of necessary conditions for (P_m) that read as follows.

There exist $\lambda_m > 0$, $\omega_m \in \mathbb{R}^d$, $\psi_m^i \in \mathbb{R}^d$, $\xi_m^i = (\xi_m^{i1}, \dots, \xi_m^{i\sigma}) \in \mathbb{R}_+^\sigma$, $p_m^i \in X$, and $\mathcal{X}_i^* \in X^{2^m} \times \mathbb{R}^{2^m d} \times X^{2^m} \times X^{2^m}$, $i = 0, \dots, 2^m$ such that

$$\mathcal{X}_i^* \in \mathcal{N}_{\Xi_m^i}(\bar{\mathcal{X}}) + \hat{\mathcal{N}}_{\mathcal{B}_m^i}(\bar{\mathcal{X}}) + \hat{\mathcal{N}}_{\Omega_m^i}(\bar{\mathcal{X}}), \quad i = 0, \dots, 2^m - 1 \tag{61}$$

$$\mathcal{X}_{2^m}^* \in \hat{\mathcal{N}}_{\Omega_m}(\bar{\mathcal{X}}), \tag{62}$$

where

$$\begin{aligned} \hat{\mathcal{N}}_{\mathcal{B}_m^i}(\mathcal{X}) &= (0, \dots, 0, \beta_m^i, 0, \dots, 0), \quad \text{with } \beta_m^i \in \mathcal{N}_{\mathcal{B}_m^i}(\bar{\rho}_m^i) \\ \hat{\mathcal{N}}_{\Omega_m^i}(\mathcal{X}) &= (0, \dots, 0, \psi_m^i, 0, \dots, 0), \quad \text{with } \psi_m^i \in \mathcal{N}_U(\bar{w}_m^i) \\ \hat{\mathcal{N}}_{\Omega_m}(\mathcal{X}) &= (0, \dots, 0, \underline{\omega}_m, 0, \dots, 0), \quad \text{with } \underline{\omega}_m = (\omega_m, \dots, \omega_m) \in \mathbb{R}^d, \end{aligned}$$

together with

$$\begin{aligned} -\sum_{i=0}^{2^m} \mathcal{X}_i^* \in \lambda_m \partial J_m(\bar{\mathcal{X}}) + \sum_{i=0}^{2^m-1} \sum_{j=1}^{\sigma} \xi_m^{ij} \nabla h_m^{ij}(\bar{\mathcal{X}}) + \sum_{i=0}^{2^m-1} \nabla \delta_m^i(\bar{\mathcal{X}})^* p_m^i, \\ \text{where } \xi_m^{ij} h_m^{ij}(\bar{\mathcal{X}}) = 0, \quad i = 0, \dots, 2^m - 1, j = 1, \dots, \sigma. \end{aligned} \tag{63}$$

We now write componentwise the above expression, making first explicit the (sub)gradients. Invoking the nonsmooth chain rule (see, e.g., Theorem 10.6 and Example 10.8 in [42]) we obtain

$$\begin{aligned} \partial J_m(\bar{\mathcal{X}}) \subset & \left(h_m D_x g(t_m^0, \bar{x}_m^0, \bar{w}_m^0)^* \partial f_1(g(t_m^0, \bar{x}_m^0, \bar{w}_m^0) - \bar{\Delta}_m^0) + \kappa_m(\bar{x}(0) - \bar{x}_m^0), \right. \\ & [h_m D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \partial f_1(g(t_m^i, \bar{x}_m^i, \bar{w}_m^i) - \bar{\Delta}_m^i)]_{i=1, \dots, 2^m-1}; \\ & \left. [h_m (D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \partial f_1(g(t_m^i, \bar{x}_m^i, \bar{w}_m^i) - \bar{\Delta}_m^i) \right. \\ & \quad \left. - D_w f_2(t_m^i, \bar{w}_m^i) + \kappa_m \int_{I_m^i} (\bar{u}(t) - \bar{w}_m^i) dt) \right]_{i=0, \dots, 2^m-1}; \\ & 0_{X^{2^m}}; \left[-h_m \partial f_1(g(t_m^i, \bar{x}_m^i, \bar{w}_m^i) - \bar{\Delta}_m^i) + \kappa_m \int_{I_m^i} (\dot{\bar{x}}(t) - \bar{\Delta}_m^i) dt \right]_{i=0, \dots, 2^m-1} \Big). \end{aligned}$$

Moreover

$$\begin{aligned} (\nabla h_m^{ij}(\bar{\mathcal{X}}))_{X^i} &= x_*^j, \quad i = 0, \dots, 2^m - 1, j = 1, \dots, \sigma, \\ \left(\sum_{\ell=0}^{2^m-1} \nabla \delta_m^\ell(\bar{\mathcal{X}})^* p_m^\ell \right)_{X^i} &= \begin{cases} p_m^{i-1} - p_m^i & \text{for } 1 \leq i \leq 2^m - 1 \\ p_m^{2^m-1} - p_m^0 & \text{for } i = 0 \end{cases} \\ \left(\sum_{\ell=0}^{2^m-1} \nabla \delta_m^\ell(\bar{\mathcal{X}})^* p_m^\ell \right)_{\Delta^i} &= -h_m p_m^i, \quad i = 0, \dots, 2^m - 1. \end{aligned}$$

Thus we obtain from (63), for a suitable $\eta_m^\ell \in \partial f_1(g(t_m^\ell, \bar{x}_m^\ell, \bar{w}_m^\ell) - \bar{\Delta}_m^\ell)$,

$$\begin{aligned} -\sum_{i=0}^{2^m} x_i^{*\ell} = -x_\ell^{*\ell} &= \lambda_m h_m D_x g(t_m^\ell, \bar{x}_m^\ell, \bar{w}_m^\ell)^* \eta_m^\ell + \sum_{j=1}^{\sigma} \xi_m^{\ell j} x_*^j - p_m^\ell + p_m^{\ell-1}, \quad \ell = 1, \dots, 2^m - 1, \\ -\sum_{i=0}^{2^m} x_i^{*0} = -x_0^{*0} &= \lambda_m h_m D_x g(t_m^0, \bar{x}_m^0, \bar{w}_m^0)^* \eta_m^0 + \sum_{j=1}^{\sigma} \xi_m^{0j} x_*^j - p_m^0 + p_m^{2^m-1}. \end{aligned}$$

For $\ell = 0, \dots, 2^m - 1$ we have moreover

$$\begin{aligned}
 -\sum_{i=0}^{2^m} w_i^{*\ell} &= -w_\ell^{*\ell} = \lambda_m h_m (D_w g(t_m^\ell, \bar{x}_m^\ell, \bar{w}_m^\ell)^* \eta_m^\ell + u_m^\ell - \bar{w}_m^\ell + D_w f_2(t_m^\ell, \bar{w}_m^\ell)), \\
 -\sum_{i=0}^{2^m} \rho_i^{*\ell} &= -\rho_\ell^{*\ell} = 0, \\
 -\Delta_\ell^{*\ell} &= -h_m p_m^\ell - \lambda_m \left(h_m (\eta_m^\ell + \bar{\Delta}_m^\ell) - \int_{I_m^i} \dot{x}(t) dt \right).
 \end{aligned}$$

Observe now that (61) and (62) can be rewritten as

$$(x_{i+1}^{*i+1}, w_i^{*i} - \psi_m^i - \omega_m, \rho_i^{*i} - \beta_m^i, \Delta_i^{*i}) \in \mathcal{N}_{\text{graph}(\Gamma^i)}(\bar{\mathcal{X}}^i), \quad i = 0, \dots, 2^m - 1 \tag{64}$$

for suitable vectors $\psi_m^i \in \mathcal{N}_U(\bar{w}^i)$ and $\beta_m^i \in \mathcal{N}_{B_X}(\bar{\rho}^i)$. Dividing by h_m the left-hand side of (64) and taking into account the above list of necessary conditions, one arrives to (48) and (49). The proof is concluded. \square

In the next result we obtain more explicit necessary conditions by computing the normal cone in the right-hand side of (49).

Theorem 10. *Let $\bar{\mathcal{X}} = (\bar{x}_m, \bar{w}_m, \bar{\rho}_m, \bar{\Delta}_m)$ be an optimal process for (P_m) . Then there exist $\lambda_m \in \mathbb{R}$, $\omega_m \in \mathbb{R}^d$, $\psi_m^i \in \mathbb{R}^d$, $\beta_m^i, \eta_m^i \in X$, $\xi_m^i = (\xi_m^{i1}, \dots, \xi_m^{i\sigma}) \in \mathbb{R}_+^\sigma$ for $i = 0, \dots, 2^m - 1$, $p_m^i \in X$ for $i = 0, \dots, 2^m$, and $\mathcal{X}_m^* \in X^{2^m} \times \mathbb{R}^{2^m d} \times X^{2^m} \times X^{2^m}$, such that (48) and (52) hold, together with*

$$\lambda_m > 0 \tag{65}$$

and, for, $i = 0, \dots, 2^m - 1$,

$$\frac{p_m^{i+1} - p_m^i}{h_m} = -D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* (p_m^i - \lambda_m \vartheta_m^{x,i}) + \sum_{j \in I_m^i} \frac{\xi_m^{ij}}{h_m} x_*^j + \lambda_m \kappa_m^i (\bar{x}_m^0 - \bar{x}(0)), \tag{66}$$

where $I_m^i = \{j = 1, \dots, \sigma : \langle x_*^j, x_m^i \rangle = c_m^{ij}\}$, κ_m^i is as in (50), and we have set

$$\vartheta_m^{x,i} := \int_{I_m^i} \dot{x}(t) dt - \frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m}, \tag{67}$$

$$\begin{aligned}
 \frac{\psi_m^i}{h_m} &= \lambda_m \left(\kappa_m^i (u_m^i - \bar{w}_m^i) - D_w f_2(t_m^i, \bar{w}_m^i) - D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \vartheta_m^{x,i} \right) \\
 &\quad - \frac{\omega_m}{h_m} + D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* p_m^i \in \mathcal{N}_U(\bar{w}_m^i)
 \end{aligned} \tag{68}$$

$$\frac{\beta_m^i}{h_m} = \vartheta_m^{r,i} (p_m^i + \lambda_m (\eta_m^i - \vartheta_m^{x,i})) \in \mathcal{N}_{B_X}(\bar{\rho}_m^i) \tag{69}$$

where we have set

$$\vartheta_m^{r,i} := \int_{I_m^i} r_m(t) dt. \tag{70}$$

Moreover, in (66) we have

$$p_m^{2^m} = p_m^0. \tag{71}$$

Proof. The computation of the normal cone to the graph of Γ_m^i , recalling (49), yields for $i = 0, \dots, 2^m - 1$

$$\begin{aligned} & \begin{pmatrix} \kappa_m^i(\bar{x}(0) - \bar{x}^0) - \lambda_m D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \eta_m^i - \sum_{j \in I_m^i} \frac{\xi_m^{ij}}{h_m} x_*^j + \frac{p_m^{i+1} - p_m^i}{h_m} \\ -\lambda_m (D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \eta_m^i + \kappa_m(\bar{w}_m^i - u_m^i) + D_w f_2(t_m^i, \bar{w}_m^i)) - \frac{\omega_m}{h_m} - \frac{\psi_m^i}{h_m} \\ -\frac{\beta_m^i}{h_m} \end{pmatrix} \\ &= - \begin{pmatrix} D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \\ D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \\ \vartheta_m^{r,i} \mathbb{I} \end{pmatrix} (p_m^i + \lambda_m(\eta_m^i - \vartheta_m^{x,i})) \end{aligned}$$

where \mathbb{I} denotes the identity matrix in X . By computing the above product and recalling the terminal condition from Theorem 9, the assertions follow. \square

9. Proof of Theorem 2: passing to the limit

We conclude the proof of Theorem 2 by performing a limiting procedure along the necessary conditions for problems (P_m) that were proved in Theorem 10.

Proof of Theorem 2. Referring to the statement of Theorem 10, we set

- $p_m(t) = p_m^i + (t - t_m^i)(p_m^{i+1} - p_m^i)$, for $t \in [t_m^i, t_m^{i+1})$, $i = 0, \dots, 2^m - 1$
- $p_m(T) = p_m(0)$
- $\xi_m^j(t) = \sum_{i=0}^{2^m-1} \frac{\xi_m^{ij}}{h_m} \mathbb{1}_{[t_m^i, t_m^{i+1})}(t)$, $t \in [0, T)$, $j = 1, \dots, \sigma$
- $\psi_m(t) = \sum_{i=0}^{2^m-1} \frac{\psi_m^i}{h_m} \mathbb{1}_{[t_m^i, t_m^{i+1})}(t)$, $t \in [0, T)$
- $\eta_m(t) = \sum_{i=0}^{2^m-1} \eta_m^i \mathbb{1}_{[t_m^i, t_m^{i+1})}(t)$, $t \in [0, T)$
- $\beta_m(t) = \sum_{i=1}^{2^m-1} \frac{\beta_m^i}{h_m} \mathbb{1}_{[t_m^i, t_m^{i+1})}(t)$, $t \in [0, T)$
- $\vartheta_m(t) = \sum_{i=0}^{2^m-1} \frac{\vartheta_m^i}{h_m} \mathbb{1}_{[t_m^i, t_m^{i+1})}(t)$, $t \in [0, T)$, where $\vartheta_m^i = (\vartheta_m^{x,i}, \vartheta_m^{r,i})$
- $\omega_m^* = \frac{\omega_m}{h_m}$.

Observe first that, by (67), (70), and (43), $\vartheta_m \rightarrow 0$ in $L^1([0, T]; X^2)$ and $\kappa_m(\bar{x}_m^0 - \bar{x}(0)) \rightarrow 0$. In particular, there exists $\Lambda \in \mathbb{R}$ such that, for every m ,

$$\kappa_m(\bar{x}_m^0 - \bar{x}(0)) + \sum_{i=0}^{2^m-1} \|\vartheta_m^i\| \leq \Lambda. \tag{72}$$

Since all conditions appearing in the statement of Theorem 10 are positively homogeneous of degree one, thanks to (65) we assume without loss of generality that

$$\lambda_m + |p_m(T)| + \sum_{i=0}^{2^m-1} \left| \sum_{j=1}^{\sigma} \xi_m^{ij} x_*^j \right| + \frac{|\omega_m|}{h_m} + \sum_{i=1}^{2^m-1} |\psi_m^i| = 1, \tag{73}$$

i.e.,

$$\lambda_m + |p_m(T)| + \left\| \sum_{j=1}^{\sigma} \xi_m^j x_*^j \right\|_{L^1([0, T]; X)} + |\omega_m^*| + \|\psi_m\|_{L^1([0, T]; X)} = 1.$$

By compactness, there exists a subsequence, that we do not relabel, and there exist $\lambda \geq 0, \omega \in X, d\xi^j \in C_+^*([0, T]; \mathbb{R}), i = 1, \dots, \sigma$, such that

$$\begin{aligned} \lambda_m &\rightarrow \lambda \\ \omega_m^* &\rightarrow \omega \\ \sum_{j=1}^{\sigma} x_*^j d\xi_m^j &\rightarrow \sum_{j=1}^{\sigma} x_*^j d\xi^j \quad \text{weakly-star in } C^*([0, T]; X) \end{aligned}$$

Observe that, thanks to the Positive Linear-Independence Constraint-Qualification and to the complementarity conditions (48) we have also

$$d\xi_m^j \rightarrow d\xi^j \quad \text{weakly-star in } C^*([0, T]; \mathbb{R}).$$

The main point of the proof is showing that the sequence $\{p_m : m \in \mathbb{N}\}$ is uniformly bounded in $W^{1,1}([0, T]; X)$, so that a subsequence of $\{p_m\}$ will converge weakly to a BV function p . This fact, in turn, will imply that the further sequences $\{\psi_m\}$ and $\{\beta_m\}$ will converge (strongly) in the appropriate spaces, thanks to (68) and (69). The convergence argument will be divided into three steps.

Step 1. The sequence $\{p_m : m \in \mathbb{N}\}$ is bounded in $L^\infty([0, T]; X)$.

Proof of Step 1. We start by rewriting (66) as

$$\begin{aligned} p_m^{i+1} &= (\mathbb{I} - h_m D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^*) p_m^i + \sum_{j=1}^{\sigma} \xi_m^{ij} x_*^j + \lambda_m h_m D_x g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \vartheta_m^{x,i} \\ &\quad + \lambda_m h_m \kappa_m^i (\bar{x}_m^0 - \bar{x}(0)), \quad i = 0, \dots, 2^m - 1, \end{aligned} \tag{74}$$

where we recall that $p_m^{2^m} = p_m^0$. Set $\gamma_m^i = |p_m^i|, i = 0, \dots, 2^m - 1, m \in \mathbb{N}$. By (71) and (73),

$$\gamma_m^0 \leq 1 \quad \forall m \in \mathbb{N}.$$

By (74) we obtain

$$\gamma_m^1 \leq (1 + h_m L') + \lambda_m h_m L' |\vartheta_m^{x,1}| + \left| \sum_{j=1}^{\sigma} \xi_m^{1j} x_*^j \right| + \lambda_m h_m \Lambda =: d_m^1,$$

and, for $i = 2, \dots, 2^m - 1$,

$$\gamma_m^i \leq (1 + h_m L') \gamma_m^{i-1} + \lambda_m h_m L' |\vartheta_m^{x,i}| + \left| \sum_{j=1}^{\sigma} \xi_m^{ij} x_*^j \right| =: (1 + h_m L') \gamma_m^{i-1} + d_m^i.$$

By induction, we obtain from the above conditions that, for each $k = 1, \dots, 2^m - 1$

$$\gamma_m^k \leq \sum_{i=1}^k d_m^i (1 + h_m L')^{k-i} = \sum_{\ell=0}^{k-1} d_m^{k-\ell} (1 + h_m L')^\ell \leq e^{TL'} \sum_{i=1}^k d_m^i.$$

Therefore, for each $k = 1, \dots, 2^m - 1$, recalling (72),

$$\gamma_m^k \leq e^{TL'} \left[\lambda_m h_m (L' + 1) \Lambda + \sum_{i=1}^{2^m} \left| \sum_{j=1}^{\sigma} \xi_m^{ij} x_*^j \right| \right].$$

Therefore, the sequence $\{\gamma_m^k : k = 0, \dots, 2^m\}$ is bounded uniformly w.r.t. m , and the proof of Step 1 is concluded.

Step 2. The sequence $\{\dot{p}_m : m \in \mathbb{N}\}$ is bounded in $L^1([0, T]; X)$ uniformly w.r.t. m .

Proof of Step 2. By (66) and Step 1,

$$\|\dot{p}_m\|_{L^1} \leq L' \lambda_m \Lambda + L' c + \left\| \sum_{j=1}^{\sigma} \xi_m^j x_*^j \right\|_{L^1([0, T]; X)},$$

for a suitable constant c . Recalling (73), the claim follows.

Up to taking another subsequence, by standard compactness results we can now assume that

$$\begin{aligned} p_m(t) &\rightarrow p(t) \quad \text{for all } t \in [0, T] \\ \dot{p}_m dt &\xrightarrow{*} dp \quad \text{in } \mathcal{C}^*([0, T]; X), \end{aligned}$$

for a suitable BV function $p: [0, T] \rightarrow X$.

Step 3. The sequence $\{\psi_m\}$ converges strongly in $L^1([0, T]; X)$ to a function ψ that satisfies the weak maximality condition. Furthermore, $\beta_m \rightarrow 0$ strongly in $L^1([0, T]; X)$.

Proof of Step 3. We obtain from (68) that, for all m and all $i = 0, \dots, 2^m - 1$,

$$\begin{aligned} \frac{\psi_m^i}{h_m} &= -\omega_m^* + D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* p_m^i \\ &+ \lambda_m (\kappa_m(u_m^i - \bar{w}_m^i) - D_w g(t_m^i, \bar{x}_m^i, \bar{w}_m^i)^* \vartheta_m^{x,i} - D_w f_2(t_m^i, \bar{w}_m^i)) \in \mathcal{N}_U(\bar{w}_m^i). \end{aligned} \tag{75}$$

We recall that, by Theorem (43), $\kappa_m(\bar{w}_m - \bar{u}) \rightarrow 0$ in $L^1([0, T]; X)$. Therefore, recalling also Lemma 6 and the above discussion of convergence of ϑ_m , ω_m^* , p_m , the right-hand side of (75) converges strongly in $L^1([0, T]; X)$ to

$$-\omega + D_w g(t, \bar{x}(t), \bar{w}(t))^* p(t) - \lambda D_w f_2(t, \bar{u}(t)) =: \psi(t).$$

By the graph closedness of the normal cone \mathcal{N}_U we obtain also that $\psi(t) \in \mathcal{N}_U(\bar{u}(t))$ a.e., hence concluding the proof of Step 3.

The above arguments also allow to pass to the limit along (66) and (71) in the suitable topologies and obtain the adjoint equation and the transversality condition.

We are therefore left with proving the nontriviality and the support conditions. To prove the first one, suppose by contradiction that both λ and p vanish. Then, by the adjoint equation, $\sum_{j=1}^{\sigma} x_*^j d\xi^j = 0$. Therefore, by the weak maximality condition, $\psi(t) \equiv -\omega$ is constant. Assume by contradiction that $\omega \neq 0$, so that necessarily $\bar{u}(t) \in \partial U$ for a.e. t . Then, since $0 \in \text{int } U$,

$$\langle \omega, \bar{u}(t) \rangle < \langle \omega, 0 \rangle = 0 \quad \text{for a.e. } t \in [0, T].$$

By integrating the above inequality we contradict the assumption that $\int_0^T \bar{u}(t) dt = 0$. The above argument, therefore, shows that for all m large enough (73) must be violated, hence concluding the proof of the nontriviality condition.

To prove the support condition, fix $j = 1, \dots, \sigma$ and set

$$E_j := \{t : \langle x_*^j, \bar{x}(t) \rangle < c^j\}.$$

Assume that $E_j \neq \emptyset$ and let $K \subset E_j$ be compact. For all m large enough, $\langle x_*^j, \bar{x}_m(t) \rangle < c_m^j(t)$ for all $t \in K$. By (66), $\xi_m^j(t) = 0$ on K , so that the support condition is proved.

The proof of Theorem 2 is concluded. \square

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