



Elastoplasticity of gradient-polyconvex materials

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Abstract. We propose a model for rate-independent evolution in elastoplastic materials under external loading, which allows large strains. In the setting of strain-gradient plasticity with multiplicative decomposition of the deformation gradient, we prove the existence of the so-called energetic solution. The stored energy density function is assumed to depend on gradients of minors of the deformation gradient which makes our results applicable to shape-memory materials, for instance.

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1. Introduction and notation

Elastoplasticity at large strains is an area of ongoing research, bringing together contributions from modelling, analysis and numerical simulations. For the mathematical analysis of elastoplastic models, it is often convenient to use powerful tools from the calculus of variations, which are now able to treat quasistatic evolutionary problems as well (see e.g. [49], [41] as pioneering works). In principle, the existence of solutions could be ensured by assuming some generalized convexity of the strain energy, but general material behavior may contradict this assumption. For example, this is manifested in shape-memory alloys (SMA) [4], magnetostrictive [14] and ferroelectric materials [54], even if convexity is understood in a generalized sense, such as *polyconvexity* (first defined in [1]).

As a remedy, one can then recourse to higher-gradient regularizations, where the stored energy density W also depends e.g. on the second gradient of the deformation. From a mathematical point of view, this adds compactness to the model, which is instrumental in proving the existence of solutions by the direct method. [2] Materials with such constitutive equations are referred to as non-simple and were introduced by Toupin [57, 58]. Since then, the concept has been elaborated by many authors so that its thermodynamical side is also better understood [7, 13, 18, 19, 25, 45, 50].

Gradient-polyconvex (GPC) materials form a special class of non-simple solids and appeared in [5] for the first time. Their stored energy density is not a function of the full second gradient $\nabla^2 y$ of the deformation $y: \Omega \rightarrow \mathbb{R}^3$, which maps the reference configuration Ω to \mathbb{R}^3 , but it only depends, in a convex way, on the weak gradients of $\text{cof} \nabla y$ (and that of $\det \nabla y$, if desirable). (See Section 3 for the definition of a cofactor matrix.) To interpret the condition physically, note that since $\det \nabla y$ measures the local change in volume between the reference and current configuration of the material and $\text{cof} \nabla y$ describes the transformation of area [27, p. 78], the stored energy W depending on their gradients offers a control of how abruptly these changes vary in space. Gradient polyconvexity has since been applied to the evolution of SMA [35] and a numerical implementation of GPC material models is also available [32]. GPC allows us to consider stored energy densities without assuming any notion of convexity in the deformation gradient variable. This makes it suitable for modelling of shape-memory alloys, for instance, because the resulting functional is lower semicontinuous in the underlying weak topology. An alternative

approach to energy functionals that are not lower semicontinuous is relaxation [10], however, available results prevent us from considering energies tending to infinity for extreme compression. In this article, we study an elastoplastic model using gradient polyconvexity.

The idea of strain-gradient plasticity is similar to that of nonsimple materials, as it also uses higher-order terms to prevent physical quantities from unrealistic fine-scale oscillations. Incorporating gradients of plastic variables in the constitutive equations is common in the engineering literature [44, p. 250] and we refer an interested reader to [3, 15, 26, 28, 36, 48, 59] or [16] and the references therein. Gradient terms account for non-local interactions of dislocations and we include them as well, as they offer a suitable regularization to our model.

For our problem, we formulate the so-called *energetic solution* (let us name [46] as an early reference; other related sources are cited in [44]). One advantage of the energetic formulation is that it avoids derivatives of constitutive equations and time derivatives of the solution itself. [41] The variational nature of this solution concept also combines well with homogenization and relaxation. Two conditions lie at the core of the energetic formulation: a *stability inequality*, which couples minimization of the elastic energy with a principle of maximum dissipation, and an *energy balance*. The two requirements together imply that a usual *plastic flow rule* is satisfied for sufficiently smooth solutions. Our model describes large-strain multiplicative elastoplasticity similarly to [38].

The plan of the paper is as follows: in Sect. 2 we review some basic facts from the modelling of *materials with internal variables*, loosely following [21] and [30]. For the sake of completeness, we also motivate the definition of an energetic solution, although this has been done more thoroughly in previous works of Mielke *et al.* The main part of this paper is Section 3, where we study the rate-independent behavior of elastoplastic GPC materials under external loads and prove the existence result. The section is concluded with an example from crystal plasticity, which illustrates the usability of our findings.

Our approach draws inspiration from [39], where the existence of energetic solutions in large-strain elastoplasticity is proved in the presence of plastic strain gradients and a polyconvex W . However, as our stored energy is gradient polyconvex, our findings apply to the setting of multiwell energies, as encountered e.g. in shape-memory alloys, cf. [6], [43] or [37] for instance. See also the end of Subsection 3.1. We remark that energetic formulations in elastoplasticity have also been propounded: in the linear framework [22], without strain gradients [8], using a finite quasiconvex energy [33], involving a plastic Cauchy-Green tensor [23, 24], for numerical computations [45], and elsewhere. The paper [12] discusses different assumptions in quasistatic large-strain elastoplastic evolutions. Lastly, in [34], rate-independent dislocation-free plasticity is treated.

In what follows, $L^\beta(\Omega; \mathbb{R}^n)$, $1 \leq \beta < +\infty$ denotes the usual Lebesgue space of mappings $\Omega \rightarrow \mathbb{R}^n$ whose modulus is integrable with the power β and $L^\infty(\Omega; \mathbb{R}^n)$ is the space of measurable and essentially bounded mappings $\Omega \rightarrow \mathbb{R}^n$. Further, $W^{1,\beta}(\Omega; \mathbb{R}^n)$ standardly represents the space of mappings which live in $L^\beta(\Omega; \mathbb{R}^n)$ and their gradients belong to $L^\beta(\Omega; \mathbb{R}^{n \times n})$. Finally, $W_0^{1,\beta}(\Omega; \mathbb{R}^n)$ is a subspace of $W^{1,\beta}(\Omega; \mathbb{R}^n)$ of maps with a zero trace on $\partial\Omega$. The weak convergence in $L^\beta(\Omega; \mathbb{R}^n)$ is defined as follows: $y_k \rightarrow y$ weakly in $L^\beta(\Omega; \mathbb{R}^n)$ (weakly star for $\beta = +\infty$) if $\int_\Omega y_k(x) \cdot \varphi(x) \, dx \rightarrow \int_\Omega y(x) \cdot \varphi(x) \, dx$ for all $\varphi \in L^{\beta'}(\Omega; \mathbb{R}^n)$ where $\beta' = \beta/(\beta - 1)$ if $1 < \beta < +\infty$, $\beta' = 1$ if $\beta = +\infty$ and $\beta' = +\infty$ for $\beta = 1$. Weak convergence of mappings and their gradients in L^β then defines the weak convergence in $W^{1,\beta}(\Omega; \mathbb{R}^n)$. We also write $w\text{-}\lim_{k \rightarrow +\infty} y_k = y$ or $y_k \rightharpoonup y$ to denote weak convergence. Finally, $C(\Omega)$ or $C(\mathbb{R}^{n \times n})$ stand for function spaces of functions continuous on Ω or $\mathbb{R}^{n \times n}$, respectively.

If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex but possibly nonsmooth we define its subdifferential at a point $x_0 \in \mathbb{R}^n$ as the set of all $v \in \mathbb{R}^n$ such that $f(x) \geq f(x_0) + v \cdot (x - x_0)$ for all $x \in \mathbb{R}^n$. The subdifferential of f will be denoted $\partial^{\text{sub}} f$ and its elements will be called subgradients of f at x_0 .

2. Motivation: modelling inelastic processes with internal variables

Consider $\Omega \subset \mathbb{R}^n$, a bounded Lipschitz domain representing the so-called reference configuration of a solid body, and a mapping $y: \Omega \rightarrow \mathbb{R}^n$, the deformation which the body is subjected to. We explain the idea of elastoplasticity and the concept of energetic solutions following freely the exposition in [21] and restricting ourselves to simple materials.

According to Han and Reddy [30, p. 34], plastic deformation ‘is most conveniently described in the framework of materials with internal variables’. Those material models are not only governed by external (controllable) variables, such as temperature or strain, but also incorporate a vector $z \in Z \subset \mathbb{R}^m$ of internal variables, which describe e.g. an ongoing chemical reaction, elastoplastic behavior or material damage. (Details can be found in [29] or in [30, p. 39].) The *hyperelastic* stored energy density \mathcal{W} then has the form $\mathcal{W} = \mathcal{W}(F, z)$, if we consider a *simple material*, i.e. a material with constitutive equations involving only the first gradient $F = \nabla y$ of the deformation y .

The thermodynamic conception is usually that differentiating \mathcal{W} with respect to F , we get the mechanical stress, whereas the derivative $-\partial_z \mathcal{W}$ gives another stress-like variable – the so-called thermodynamic force

$$Q := -\frac{\partial}{\partial z} \mathcal{W}(F, z) \tag{2.1}$$

associated with the internal variable z . We can imagine that Q tries to restructure the material irreversibly, which would lead to changing the value of z . The development of convex analysis [47] allowed quite a general formulation of an evolution rule for the internal variable z . Assuming the existence of a nonnegative convex potential of dissipative forces $\delta = \delta(\dot{z})$, where \dot{z} denotes the time derivative of z , we write the *flow law* as

$$Q(t) \in \partial^{\text{sub}} \delta(\dot{z}(t)) \tag{2.2}$$

everywhere in Ω . A common simplifying assumption is *rate-independent* behavior. It is suitable for some particular materials or for the modelling of processes with low rates of external loading. In simple terms, rate-independence means that rescaling the loading in time only results in a corresponding time-rescaling of the deformation and no additional viscous, inertial or thermal effects arise. Rate-independence translates into positive one-homogeneity of δ , i.e. $\delta(\alpha \dot{z}) = \alpha \delta(\dot{z})$ for all $\alpha > 0$.

Remark 2.1. Since the subdifferential is *monotone* (see [51]), by (2.2) we have

$$(Q(t) - \theta) \cdot (\dot{z}(t) - \xi) \geq 0. \tag{2.3}$$

for all $\theta \in \partial^{\text{sub}} \delta(\xi)$. Choosing $\xi = 0$ and observing that the one-homogeneity of δ implies $\delta(\dot{z}) = \omega \cdot \dot{z}$ for all $\omega \in \partial^{\text{sub}} \delta(\dot{z})$ we get

$$\delta(\dot{z}(t)) = Q(t) \cdot \dot{z}(t) \geq \theta \cdot \dot{z}(t) \tag{2.4}$$

for all $\theta \in \partial^{\text{sub}} \delta(0)$. Hence we derived the so-called *maximum dissipation principle* (see e.g. [31] or [52, 53]) which states that the plastic dissipation that takes place in reality is not less than any possible dissipation due to thermodynamic forces ‘available’ in the so-called *elastic domain* $\partial^{\text{sub}} \delta(0)$.

Hereafter, ν is the outer unit normal to $\partial\Omega$, and $\partial\Omega \supset \Gamma_0, \Gamma_1$ which are disjoint. Let $f(t): \Omega \rightarrow \mathbb{R}^n$ be the (volume) density of external *body* forces and $g(t): \Gamma_1 \subset \partial\Omega \rightarrow \mathbb{R}^n$ be the (surface) density of surface forces. The conservation of momentum yields the *equilibrium equations*

$$-\text{div} \left(\frac{\partial}{\partial F} \mathcal{W}(\nabla y(t), z(t)) \right) = f(t) \text{ in } \Omega, \tag{2.5}$$

$$y(t, x) = y_0(x) \text{ on } \Gamma_0, \tag{2.6}$$

$$\frac{\partial}{\partial F} \mathcal{W}(\nabla y(t), z(t)) \nu(x) = g(t, x) \text{ on } \Gamma_1. \tag{2.7}$$

If $z_0 \in Z$ is an initial condition for the internal variable, the system of equations (2.5)-(2.7) together with (2.8):

$$-\frac{\partial}{\partial z} \mathcal{W}(\nabla y(t), z(t)) \in \partial \delta(\dot{z}(t)), \quad z(0) = z_0, \tag{2.8}$$

governs the mechanical behavior characterized by the unknowns $y(t), z(t)$.

Unfortunately, the system (2.5)-(2.7) is ill-posed in many situations. See the works of Suquet [55] and Temam [56] for early analysis of this problem.

To get existence results, it seems necessary to include the gradient of the internal variable, i.e. use the strain energy density

$$\tilde{\mathcal{W}}(\nabla y, z, \nabla z) := \mathcal{W}(\nabla y, z) + \epsilon |\nabla z|^\alpha$$

for $\alpha \geq 1$ and $\epsilon > 0$. Let us briefly recall how an energy balance, which appears in the *energetic* formulation of this regularized problem, is obtained, under sufficient smoothness and integrability assumptions on all the present mappings. For details, see [21, 41].

The functional

$$\mathcal{I}(t, y(t), z(t)) := \int_{\Omega} \mathcal{W}(\nabla y(t), z(t)) \, dx + \epsilon \int_{\Omega} |\nabla z(t)|^\alpha \, dx - L(t, y(t)), \tag{2.9}$$

expresses the potential energy in our system, where the work done by external forces is

$$L(t, y(t)) := \int_{\Omega} f(t) \cdot y(t) \, dx + \int_{\Gamma_1} g(t) \cdot y(t) \, dS. \tag{2.10}$$

We also introduce the total dissipation along z ,

$$\text{Diss}(z; [0, t]) := \int_0^t \int_{\Omega} \delta(\dot{z}(s)) \, dx ds.$$

Calculating the thermodynamic force $\partial_z \tilde{\mathcal{W}}$, using the relation $\delta(\dot{z}) = \omega \cdot \dot{z}$, $\omega \in \partial^{\text{sub}} \delta(\dot{z})$, mentioned above (2.4), we can deduce from (2.5)-(2.7) the following energy balance:

$$\mathcal{I}(t, y(t), z(t)) + \text{Diss}(z; [0, t]) = \mathcal{I}(0, y(0), z(0)) + \int_0^t \dot{L}(s, y(s)) \, ds.$$

Due to low regularity in time, a more general expression for $\text{Diss}(z; [0, t])$ must be used in practice, though – see (2.14) below.

To this end, we define a dissipation distance between two values $z_0, z_1 \in Z$ of the internal variable:

$$D(z_0, z_1) := \inf_z \left\{ \int_0^1 \delta(\dot{z}(s)) \, ds; \quad z(0) = z_0, \quad z(1) = z_1 \right\}, \tag{2.11}$$

where $z \in C^1([0, 1]; Z)$, and set

$$\mathcal{D}(z_1, z_2) = \int_{\Omega} D(z_1(x), z_2(x)) \, dx \tag{2.12}$$

for $z_1, z_2 \in Z := \{z: \Omega \rightarrow \mathbb{R}^m; \quad z(x) \in Z \text{ a.e. in } \Omega\}$, as in [41].

In order to find a quasistatic evolution of the system Mielke, Theil, and Levitas [46] came up with the following definition of the energetic solution which conveniently overcomes nonsmoothness mentioned above. Moreover, it fully exploits the possible variational structure of the problem and allows for very general energy and dissipation functionals. This concept has versatile applications to many problems in

continuum mechanics of solids. Additionally, working with \mathcal{I} and \mathcal{D} directly allows us to include also higher derivatives of y into the model or to require integrability of some functions of ∇y as also done in this contribution.

2.1. Energetic solution

Let \mathbb{Y} be the set of admissible deformations (usually a subset of a Sobolev space) and suppose that the evolution of $y(t) \in \mathbb{Y}$ and $z(t) \in \mathbb{Z}$ is studied during a time interval $[0, T]$. The following two properties characterize the energetic solution due to Mielke *et al.* [46].

(i) Stability inequality:

$\forall t \in [0, T], \tilde{z} \in \mathbb{Z}, \tilde{y} \in \mathbb{Y}$:

$$\mathcal{I}(t, y(t), z(t)) \leq \mathcal{I}(t, \tilde{y}, \tilde{z}) + \mathcal{D}(z(t), \tilde{z}) \quad (2.13)$$

(ii) Energy balance: $\forall 0 \leq t \leq T$

$$\begin{aligned} \mathcal{I}(t, y(t), z(t)) + \text{Var}(\mathcal{D}, z; [0, t]) &= \mathcal{I}(0, y(0), z(0)) + \int_0^t \dot{L}(\xi, y(\xi)) \, d\xi, \\ \text{where } \text{Var}(\mathcal{D}, z; [s, t]) &:= \sup \left\{ \sum_{i=1}^N \mathcal{D}(z(t_i), z(t_{i-1})); \{t_i\} \text{ partition of } [s, t] \right\}. \end{aligned} \quad (2.14)$$

Definition 2.2. The mapping $t \mapsto (y(t), z(t)) \in \mathbb{Y} \times \mathbb{Z}$ is an energetic solution to the problem (\mathcal{I}, δ, L) if the stability inequality and energy balance are satisfied for all $t \in [0, T]$.

Remark 2.3. The mechanical idea behind stability inequality (i) is the following: imagine first that $\tilde{z} := z(t)$, then $\mathcal{D}(z(t), \tilde{z})$ vanishes, since no change in the internal variables implies no dissipation. Consequently, (i) simplifies to $\mathcal{I}(t, y(t), z(t)) \leq \mathcal{I}(t, \tilde{y}, \tilde{z})$ for all $\tilde{y} \in \mathbb{Y}$ and $y(t)$ is a global minimizer of $\mathcal{I}(t, \cdot, z(t))$ over \mathbb{Y} . So we see that in this case, (i) has the meaning of an elastic equilibrium. If $\tilde{z} \neq z(t)$, then the amount of dissipated energy between the states \tilde{z} and $z(t)$ must, by (i), at least compensate for, if not outweigh the associated loss in the total energy, which is a version of the principle of maximum dissipation. [41]

We will write $\mathbb{Q} := \mathbb{Y} \times \mathbb{Z}$ and set $q := (y, z)$. Next let us define the set of *stable states* at time t as

$$\mathcal{S}(t) := \{q \in \mathbb{Q}; \forall \tilde{q} \in \mathbb{Q}: \mathcal{I}(t, q) \leq \mathcal{I}(t, \tilde{q}) + \mathcal{D}(q, \tilde{q})\} \quad (2.15)$$

and

$$\mathcal{S}_{[0, T]} := \bigcup_{t \in [0, T]} \{t\} \times \mathcal{S}(t). \quad (2.16)$$

Moreover, a sequence $\{(t_k, q_k)\}_{k \in \mathbb{N}}$ is called *stable* if $q_k \in \mathcal{S}(t_k)$.

Remark 2.4. Energetic solutions are applicable to convex as well as nonconvex problems. In nonconvex problems, however, they do not necessarily provide proper predictions of the mechanical behaviour of the system, as jumps of the solution in time appear ‘too early’ when compared to physical experiments in several applications, for example in models predicting damage and fracture. In fact, energetic solutions jump immediately when there is enough energy available. In this sense, they represent the worst-scenario approach. Nevertheless, they are very flexible and applicable to nonlinear problems in continuum mechanics. We refer to [44] for further discussions on this topic.

3. Applications to elastoplasticity

This section shows how the energetic approach can be applied to an elastoplastic problem of gradient-polyconvex materials.

3.1. Gradient polyconvexity

Gradient polyconvexity was first defined in [5] in analogy with classical polyconvexity [1]. The difference is that here we assume that the stored energy density is convex in gradients of minors of the deformation gradient but not in minors alone. More precisely, the following definition is taken from [5]. We recall that for an invertible $F \in \mathbb{R}^{n \times n}$ we define the cofactor of F , $\text{cof} F = (\det F)F^{-\top} \in \mathbb{R}^{n \times n}$. As is shown in [5] there are maps $y \in W^{1,1}(\Omega; \mathbb{R}^3)$ such that $\det \nabla y$ and $\text{cof} \nabla y$ are Lipschitz continuous but $y \notin W^{2,1}(\Omega; \mathbb{R}^3)$. The same conclusion can be reached for every $n \geq 3$. On the other hand, if $n = 2$ then $\nabla \text{cof} \nabla y$ has the same entries (up to the minus sign) as $\nabla^2 y$.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. Let $\mathcal{W}_1: \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. The functional

$$J(y) = \int_{\Omega} \mathcal{W}_1(\nabla y(x), \nabla[\text{cof} \nabla y(x)]) \, dx, \tag{3.1}$$

defined for any measurable function $y: \Omega \rightarrow \mathbb{R}^n$ for which the weak derivatives $\nabla y, \nabla[\text{cof} \nabla y]$ exist and are integrable is called *gradient polyconvex* if the function $\mathcal{W}_1(F, \cdot)$ is convex for every $F \in \mathbb{R}^{n \times n}$.

We assume that for some $c > 0$, and numbers $\alpha > n - 1, s > 0$ it holds that for every $F \in \mathbb{R}^{n \times n}$ and every $H \in \mathbb{R}^{n \times n \times n}$ we have that

$$\mathcal{W}_1(F, H) \geq \begin{cases} c(|F|^\alpha + (\det F)^{-s} + |H|^{\alpha/(n-1)}) & \text{if } \det F > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.2}$$

where $|\cdot|$ denotes the Euclidean norm. These growth assumptions can be surely weakened but we stick to them in order to simplify our presentation.

The idea behind condition (3.2) is that the energy blows up if the deformation does not preserve orientation or if the measures of strain on the right-hand side grow to extreme values. Coercivity conditions involving $|F|$ and $\det F$ are commonly used in nonlinear elasticity (see e.g. [9]) and reflect variations in volume or changes of the distances of points caused by the deformation. The term H which is a placeholder for $\nabla \text{cof} \nabla y$ penalizes spatial changes of $\text{cof} \nabla y$ and, consequently, it aims at suppressing abrupt areal changes in the deformed configuration.

An important feature of gradient polyconvexity is that *no* convexity assumptions on \mathcal{W}_1 are needed in the F -variable, so that very general material laws can be considered including multiwell energy functions [6] or the St. Venant–Kirchhoff energy density [9]. Indeed, assume that we are given matrices $F_i \in \mathbb{R}^{n \times n}$ with positive determinants that define mutually different preferred right Cauchy–Green strains $C_i = F_i^\top F_i$ of martensitic variants for $1 \leq i \leq M$ and define

$$\mathcal{W}_1(F, H) = \begin{cases} \min_{i=1, \dots, M} C_i : (C - C_i) + \varepsilon |H|^{\alpha/(n-1)} + \frac{1}{\det F} & \text{if } \det F > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where C_i is the tensor of elastic constants of the i -th variant and $C = F^\top F$. Clearly, \mathcal{W}_1 is admissible in (3.1) and models a multiwell energy density of a shape-memory alloy.

We call materials whose stored energy J obeys (3.1) also gradient-polyconvex.

3.2. Assumptions on problem data

We work under the assumption of multiplicative decomposition of the deformation gradient $F = F_e F_p$, where F_e is the elastic part and F_p the plastic one (see Section 3.3 for more details). As in [26] we will consider so-called *separable materials*, i.e. materials where the elastoplastic energy density has the form

$$\mathcal{W}(F_e, H, F_p, \nabla F_p, p, \nabla p) := \mathcal{W}_1(F_e, H) + \mathcal{W}_2(F_p, \nabla F_p, p, \nabla p). \quad (3.3)$$

We will assume that \mathcal{W}_1 is continuous and satisfies (3.2) while for \mathcal{W}_2 we require:

- (i) The plastic part \mathcal{W}_2 is continuous in its all arguments.
- (ii) Suppose that there are two constants $C, c > 0$ so that the following assumption holds for constants $c_1 > 0$, $\beta > n$, and $\omega > n$:

$$\begin{aligned} C(1 + |F_p|^\beta + |G|^\beta + |p|^\omega + |\pi|^\omega) &\geq \mathcal{W}_2(F_p, G, p, \pi) \\ &\geq c(|F_p|^\beta + |G|^\beta + |p|^\omega + |\pi|^\omega) - c_1. \end{aligned} \quad (3.4)$$

- (iii) There is $c_2 > 0$, $v^* \in \mathbb{R}^m$ and a modulus of continuity $\hat{\omega}$ such that for all $\hat{\alpha} > 0$, $F_p \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times n \times n}$, $p \in \mathbb{R}^m$ and $\pi \in \mathbb{R}^{m \times n}$:

$$|\mathcal{W}_2(F_p, G, p + \hat{\alpha}v^*, \pi) - \mathcal{W}_2(F_p, G, p, \pi)| \leq \hat{\omega}(\hat{\alpha})(\mathcal{W}_2(F_p, G, p, \pi) + c_2). \quad (3.5)$$

Furthermore, let us suppose that for every $F_e, F_p \in \mathbb{R}^{n \times n}$ and $p \in \mathbb{R}^m$, the functions $\mathcal{W}_1(F_e, \cdot)$ and $\mathcal{W}_2(F_p, \cdot, p, \cdot)$ are convex.

The dissipation distance $\mathcal{D}: \mathbb{Z} \times \mathbb{Z} \rightarrow [0, +\infty]$ takes the form (2.12) for a function $D: \Omega \times (\text{SL}(n) \times \mathbb{R}^m)^2$ and we only change the definition of \mathbb{Z} to (3.11) below. We make the following assumptions on \mathcal{D} :

- (i) Lower semicontinuity:

$$\mathcal{D}(z, \tilde{z}) \leq \liminf_{k \rightarrow \infty} \mathcal{D}(z_k, \tilde{z}_k), \quad (3.6)$$

whenever $z_k \rightharpoonup z$ and $\tilde{z}_k \rightharpoonup \tilde{z}$.

- (ii) Positivity:

$$\text{If } \{z_k\} \subset Z \text{ is bounded and } \min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \rightarrow 0 \text{ then } z_k \rightarrow z. \quad (3.7)$$

- (iii) For all $z_1, z_2 \in \mathbb{Z}$: $\mathcal{D}(z_1, z_2) = 0$ if and only if $z_1 = z_2$.

- (iv) Triangle inequality: $\mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3)$ for all $z_1, z_2, z_3 \in \mathbb{Z}$.

We refer the reader to [39] to see that (ii) follows from (i) and (iii). After stating Proposition 3.6, we specify further assumptions on D ((3.16) or (3.A)–(3.C)). Besides, it is naturally required that D be such that (i)–(iv) holds.

In order to prove the existence of a solution to (3.18) we must impose some data qualifications. In what follows, we assume that

$$f \in C^1 \left([0, T]; L^{\tilde{d}}(\Omega; \mathbb{R}^n) \right), \quad (3.8)$$

$$g \in C^1 \left([0, T]; L^{\hat{d}}(\Gamma_1; \mathbb{R}^n) \right), \quad (3.9)$$

where $\tilde{d} \geq [nd/(n-d)]' = nd/(nd-n+d)$ if $1 \leq d < n$ or $\tilde{d} > 1$ otherwise. Similarly, we suppose that $\hat{d} \geq [(nd-d)/(n-d)]' = (nd-d)/(nd-n)$ if $d < n$ or $\hat{d} > 1$ otherwise.

3.3. Formulation of the problem

From now on, $y: \Omega \rightarrow \mathbb{R}^n$ will represent the deformation of a material body, whose reference configuration is a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Since y models both *elastic* and *plastic* behaviour, we split the deformation gradient $F = \nabla y$ as $F = F_e F_p$, where F_e stands for an elastic part and $F_p \in \text{SL}(n) := \{A \in \mathbb{R}^{n \times n}; \det A = 1\}$ is a plastic part, which irreversibly transforms the material. To capture e.g. *back*

stresses, we use the vector $p \in \mathbb{R}^m$ of hardening internal variables. Written together, $z(x) = (F_p(x), p(x))$ is a plastic variable, lying in $SL(n) \times \mathbb{R}^m$ for almost all $x \in \Omega$.

The energy functional \mathcal{I} is given by

$$\mathcal{I}(t, y(t), z(t)) := \int_{\Omega} \mathcal{W}(\nabla y F_p^{-1}, \nabla[(\text{cof} \nabla y) F_p^{\top}], F_p, \nabla F_p, p, \nabla p) \, dx - L(t, y(t)), \tag{3.10}$$

with L defined in (2.10).

Our stored energy density \mathcal{W} does not explicitly depend on the spatial variable x , but treating the inhomogeneous case would not need many modifications.

Let us remark that $(\text{cof} \nabla y) F_p^{\top}$ is the cofactor of the elastic part F_e , since by the product rule for cofactor matrices [9, p. 4] and by $F_p \in SL(n)$, we have

$$\text{cof} F_e = \text{cof}(F F_p^{-1}) = (\text{cof} F) \text{cof}(F_p^{-1}) = (\text{cof} F) \det(F_p^{-1}) (F_p^{-1})^{-\top} = (\text{cof} F) F_p^{\top}.$$

The admissible deformations y lie in

$$\mathbb{Y} := \{y \in W^{1,d}(\Omega; \mathbb{R}^n); y = y_0 \text{ on } \Gamma_0\},$$

where $\Gamma_0 \subset \partial\Omega$ with a positive surface measure and $y_0 \in W^{1-1/d,d}(\Gamma_0; \mathbb{R}^n)$ is given. Assuming that $\Gamma_1 \subset \partial\Omega$ as in Sect. 2, we suppose $\Gamma_0 \cap \Gamma_1 = \emptyset$. For the internal states z let us define the set

$$\mathbb{Z} := \{(F_p, p) \in W^{1,\beta}(\Omega; \mathbb{R}^{n \times n}) \times W^{1,\omega}(\Omega; \mathbb{R}^m) : F_p(x) \in SL(n) \text{ for a.e. } x \in \Omega\}. \tag{3.11}$$

For ease of notation, we write $q = (y, z) \in \mathbb{Q} = \mathbb{Y} \times \mathbb{Z}$ and understand $\mathcal{I}(t, \cdot)$, $L(t, \cdot)$ and \mathcal{D} as functions of q , that is:

$$\mathcal{I}(t, q(t)) = \int_{\Omega} \mathcal{W}(\nabla y F_p^{-1}, \nabla[(\text{cof} \nabla y) F_p^{\top}], F_p, \nabla F_p, p, \nabla p) \, dx - L(t, q(t)),$$

$$L(t, q(t)) := L(t, y(t)),$$

$$\mathcal{D}(q_1, q_2) := \mathcal{D}(z_1, z_2)$$

if $q_1 = (y_1, z_1)$ and $q_2 = (y_2, z_2)$.

In order to prove the existence of an energetic solution to our problem we will need the following results of technical nature.

3.4. Auxiliary results

We start this section by the following reverse Young inequality.

Lemma 3.2. *Suppose that $a > 0$, $b > 0$, $\delta > 0$, $r > 1$. Then*

$$\frac{a}{b} \geq r \delta^{\frac{r}{r-1}} a^{\frac{1}{r}} - (r-1) \delta^{\frac{r}{(r-1)^2}} b^{\frac{1}{r-1}}.$$

Proof. Young's inequality states that given a pair of positive numbers α, β and $1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\alpha \beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}. \tag{3.12}$$

Set $p = r$, $\alpha = a^{\frac{1}{r}}$, $\beta = \delta^{\frac{r}{r-1}} b$ in (3.12). Then $q = \frac{r}{r-1}$ and Young's inequality yields

$$\frac{r-1}{r} \delta^{\frac{r}{(r-1)^2}} b^{\frac{r}{r-1}} + \frac{1}{r} a \geq \delta^{\frac{r}{r-1}} a^{\frac{1}{r}} b,$$

which after multiplying by $\frac{r}{b}$ implies the desired result. □

It will also be useful to give a name to a kind of convergence which makes $\mathcal{I}(t, \cdot)$ lower semicontinuous.

Definition 3.3. We say that the sequence $\{q_k\}_{k \in \mathbb{N}} \subset \mathbb{Q}$, $q_k = (y_k, z_k) = (y_k, F_{p_k}, p_k)$ *gpc-converges* to $q_* = (y_*, z_*) = (y_*, F_{p_*}, p_*)$ if $z_k \rightarrow z_*$ in \mathbb{Z} , $\nabla \text{cof}(\nabla y_k F_{p_k}^{-1}) \rightarrow \nabla \text{cof}(\nabla y_* F_{p_*}^{-1})$ in $L^{\alpha/(n-1)}(\Omega; \mathbb{R}^{n \times n \times n})$ and $\nabla y_k \rightarrow \nabla y_*$ in measure. We write $q_k \xrightarrow{\text{gpc}} q_*$ for short.

Lemma 3.4. Let $t_k \rightarrow t_*$ with $t_k, t_* \in [0, T]$, $k \in \mathbb{N}$, and $q_k \xrightarrow{\text{gpc}} q_*$, $\{q_k\}_{k \in \mathbb{N}} \subset \mathbb{Q}$. Then $\mathcal{I}(t_*, q_*) \leq \liminf_{k \rightarrow \infty} \mathcal{I}(t_k, q_k)$.

Proof. This is an immediate consequence of [17, Corollary 7.9]. Note that we can construct a subsequence such that $\nabla y_{k_j} F_{p_{k_j}}^{-1} \rightarrow \nabla y_* F_{p_*}^{-1}$ almost everywhere. \square

Even though we do not have the weak lower semicontinuity of \mathcal{I} in general, we can get it for a subsequence provided the respective values of \mathcal{I} are bounded.

Lemma 3.5. Provided that $\alpha^{-1} + \beta^{-1} \leq d^{-1} < (n-1)^{-1}$ and $d > \frac{\beta(n-1)}{\beta-1}$, let $t_k \in [0, T]$, $k \in \mathbb{N}$, and $q_k \rightarrow q_*$ in \mathbb{Q} . Suppose there is $C_I > 0$ such that for all $k \in \mathbb{N}$ the bound $\mathcal{I}(t_k, q_k) \leq C_I$ holds true. Then there exists a subsequence $\{q_{k_j}\}_{j \in \mathbb{N}}$ of $\{q_k\}_{k \in \mathbb{N}}$ that *gpc-converges* to the same limit q_* .

Proof. The proof goes the same way as in Proposition 3.8. To keep the flow of ideas uninterrupted there, we postpone the presentation to that section. \square

Proposition 3.6. Let \mathcal{I} be lower semicontinuous with respect to *gpc-convergence* and let (3.8) and (3.9) hold. Let it for all $(t_*, q_*) \in [0, T] \times \mathbb{Q}$ and all stable sequences $\{(t_k, q_k)\}_{k \in \mathbb{N}}$ such that $w\text{-}\lim_{k \rightarrow \infty} (t_k, q_k) = (t_*, q_*)$ be true that for all $\tilde{q} \in \mathbb{Q}$ there is $\{\tilde{q}_k\} \subset \mathbb{Q}$ such that

$$\limsup_{k \rightarrow \infty} (\mathcal{I}(t_k, \tilde{q}_k) + \mathcal{D}(q_k, \tilde{q}_k)) \leq \mathcal{I}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q}). \quad (3.13)$$

Then for any stable sequence $\{(t_k, q_k)\}_{k \in \mathbb{N}}$ such that $w\text{-}\lim_{k \rightarrow \infty} (t_k, q_k) = (t_*, q_*)$ and $\mathcal{I}(t_k, q_k) \leq C$ for some $C > 0$, we have $\lim_{k \rightarrow \infty} \mathcal{I}(t_k, q_k) = \mathcal{I}(t_*, q_*)$ and $q_* \in \mathcal{S}(t_*)$.

Proof. We follow the proof of Prop. 4.3 in [39]. Take $\tilde{q} := q_*$ in (3.13), which yields a sequence $\{\tilde{q}_k\}_{k \in \mathbb{N}}$. Then we get, by the stability of q_k ,

$$\limsup_{k \rightarrow \infty} \mathcal{I}(t_k, q_k) \leq \limsup_{k \rightarrow \infty} (\mathcal{I}(t_k, \tilde{q}_k) + \mathcal{D}(q_k, \tilde{q}_k)) \leq \mathcal{I}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q}) = \mathcal{I}(t_*, q_*). \quad (3.14)$$

The assumptions (3.8) and (3.9) on f and g further give

$$\lim_{k \rightarrow \infty} |\mathcal{I}(t_k, q_k) - \mathcal{I}(t_*, q_k)| = \lim_{k \rightarrow \infty} |L(t_k, q_k) - L(t_*, q_k)| = 0. \quad (3.15)$$

Since \mathcal{I} is lower semicontinuous with respect to *gpc-convergence* (and by Lemma 3.5 we can pass to a *gpc-convergent* subsequence, without relabeling it), we deduce by equation (3.15) that

$$\liminf_{k \rightarrow \infty} \mathcal{I}(t_k, q_k) = \lim_{k \rightarrow \infty} (\mathcal{I}(t_k, q_k) - \mathcal{I}(t_*, q_k)) + \liminf_{k \rightarrow \infty} \mathcal{I}(t_*, q_k) \geq \mathcal{I}(t_*, q_*).$$

This combined with (3.14) establishes the weak continuity along a stable sequence: $\mathcal{I}(t_k, q_k) \rightarrow \mathcal{I}(t_*, q_*)$. In the end, pick a $\tilde{q} \in \mathbb{Q}$ and apply (3.13) to it:

$$\mathcal{I}(t_*, q_*) = \lim_{k \rightarrow \infty} \mathcal{I}(t_k, q_k) \leq \liminf_{k \rightarrow \infty} (\mathcal{I}(t_k, \tilde{q}_k) + \mathcal{D}(q_k, \tilde{q}_k)) \leq \mathcal{I}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q});$$

hence, the stability of q_* is proved. \square

A natural question is how to ensure the validity of (3.13). If $\mathcal{D}: \mathbb{Q} \times \mathbb{Q} \rightarrow [0, +\infty)$, i.e. no irreversibility constraint is imposed on plastic processes, then it is sufficient if D from (2.12) satisfies

$$\begin{aligned} & D \text{ is continuous and} \\ & D(z_1, z_2) \leq c + C(|F_{p_1}|^{\beta^* - \epsilon} + |F_{p_2}|^{\beta^* - \epsilon} + |p_1|^{\omega^* - \epsilon} + |p_2|^{\omega^* - \epsilon}), \end{aligned} \quad (3.16)$$

where $\epsilon > 0$ is small enough and $\beta^* := n\beta/(n-\beta)$ if $n > \beta$ and $\beta^* > 1$ if $\beta \geq n$. Similarly, $\omega^* := n\omega/(n-\omega)$ if $n > \omega$ and $\omega^* > 1$ if $\omega \geq n$. Then the compact embedding provides the continuity of \mathcal{D} . This shows that (3.13) is valid with a constant sequence $\tilde{q}_k = \tilde{q}$.

If $\mathcal{D}: \mathbb{Q} \times \mathbb{Q} \rightarrow [0, +\infty]$, the assumptions are more elaborate. Following [39] we impose the following sufficient conditions on D from (2.11):

(3.A) $D(\cdot, \cdot): \mathbb{D} \rightarrow [0, +\infty)$ is continuous, where $\mathbb{D} := \{(z_1, z_2); D(z_1, z_2) < +\infty\}$,

(3.B) For every $R > 0$ there is $K > 0$ such that: $D(z_1, z_2) < K$ if $z_1, z_2 \in \mathbb{D}$ and $|z_1|, |z_2| < R$, and

(3.C) The direction $v^* \in \mathbb{R}^m$ from (3.5) has the property that for all $\alpha, R > 0$ there is $\rho > 0$ such that for every z, z_0, z_1 :

$$|z - z_0| < \rho, (z_0, z_1) \in \mathbb{D} \text{ and } |z_0|, |z_1| < R \text{ implies } (z, z_1 + (0, \alpha v^*)) \in \mathbb{D}.$$

Proposition 3.7. *Let $\beta, \omega > n$. Let D satisfy (3.A)–(3.C). Then (3.13) holds.*

Proof. The reasoning follows the lines of [39]. If $\mathcal{D}(q_*, \tilde{q}) = +\infty$ in (3.13), the proof is finished. So, we assume that

$$\mathcal{D}(q_*, \tilde{q}) \in \mathbb{R}.$$

If $q_k \rightarrow q_*$, we observe that

$$\rho_k := \|F_{pk} - F_{p^*}\|_{C(\bar{\Omega}; \mathbb{R}^{n \times n})} + \|p_k - p^*\|_{C(\bar{\Omega}; \mathbb{R}^m)} \rightarrow 0. \tag{3.17}$$

by the compact embedding. Then $|z_k| + |z_*| + |\tilde{z}| < R$ for some $R > 0$ if k is large enough. Define $\tilde{z}_k := (\tilde{F}_p, \tilde{p} + \alpha_k v^*)$ where $\alpha_k \rightarrow 0$ and relates to ρ_k as in (3.C) (we may need to redefine the ρ_k from (3.17) by passing to a subsequence, which is without loss of generality). Thus, $(z_k, \tilde{z}_k) \in \mathbb{D}(x)$ a.e. in Ω and we have $|z_k|, |\tilde{z}_k| < R$. The continuity of D gives the convergence of $D(x, z_k, \tilde{z}_k) \rightarrow D(x, z_*, \tilde{z})$ pointwise so that $\mathcal{D}(q_k, \tilde{q}_k) \rightarrow \mathcal{D}(q_*, \tilde{q})$ by condition (3.B) and the dominated convergence theorem. Furthermore, properties of \mathcal{W}_2 and L (assumptions (3.8), (3.9)) imply that $\mathcal{I}(t_k, \tilde{q}_k) \rightarrow \mathcal{I}(t_*, \tilde{q}_*)$. Summing up, we deduce that (3.13) is satisfied with equality. \square

3.5. Incremental problems

Next, we define the following sequence of incremental problems. We consider a *stable* initial condition $q_\tau^0 := q^0 \in \mathbb{Q}$.

Let us take $\tau > 0$, a time step, chosen in the way that $N = T/\tau \in \mathbb{N}$. For $1 \leq k \leq N$, $t_k := k\tau$, find $q_\tau^k \in \mathbb{Q}$ such that q_τ^k solves

$$\left. \begin{array}{l} \text{minimize } q \mapsto \mathcal{I}(t_k, q) + \mathcal{D}(q_\tau^{k-1}, q) \\ \text{subject to } q \in \mathbb{Q}. \end{array} \right\} \tag{3.18}$$

Proposition 3.8. *Let $\alpha^{-1} + \beta^{-1} \leq d^{-1} < (n - 1)^{-1}$, $d > \frac{\beta(n-1)}{\beta-1}$. Let the assumptions on \mathcal{W} and \mathcal{D} be satisfied. Let further (3.8) and (3.9) be satisfied. Then the problem (3.18) has a solution for all $k = 1, \dots, T/\tau$. In addition, for the solution $q_\tau^k = (y_\tau^k, z_\tau^k)$ we get $\det \nabla y_\tau^k > 0$ a.e. in Ω .*

Proof. Given $q_\tau^{k-1} \in \mathbb{Q}$ from the previous time step, suppose that $\{q_j\} \subset \mathbb{Q}$ is a minimizing sequence for $q \mapsto \mathcal{I}(t_k, q) + \mathcal{D}(q_\tau^{k-1}, q)$. The assumption (3.4) implies that $\{z_j\}$ is uniformly bounded in $W^{1,\beta}(\Omega; \mathbb{R}^{n \times n}) \times W^{1,\omega}(\Omega; \mathbb{R}^m)$. Hence, as $\beta, \omega > 1$ we can extract a weakly converging subsequence (not relabeled) $z_j \rightharpoonup z$ in $W^{1,\beta}(\Omega; \mathbb{R}^{n \times n}) \times W^{1,\omega}(\Omega; \mathbb{R}^m)$. The strong convergence of $z_j \rightarrow z := (F_p, p)$ in $L^\beta(\Omega; \mathbb{R}^{n \times n}) \times L^\omega(\Omega; \mathbb{R}^m)$ ensures that $F_p(x) \in \text{SL}(n)$ almost everywhere. Write $z_j = (F_p^j, p^j)$,

$q_j = (y^j, z_j)$. Exploiting the submultiplicativity of the Euclidean norm, we estimate using Lemma 3.2

$$\begin{aligned} \int_{\Omega} |\nabla y^j(x)(F_p^j)^{-1}|^\alpha dx &\geq \frac{\|\nabla y^j\|_{L^d(\Omega; \mathbb{R}^{n \times n})}^\alpha}{\|F_p^j\|_{L^\beta(\Omega; \mathbb{R}^{n \times n})}^\alpha} \\ &\geq \frac{\alpha}{d} \delta^{\alpha/(\alpha-d)} \|\nabla y^j\|_{L^d(\Omega; \mathbb{R}^{n \times n})}^d - \frac{\alpha-d}{d} \delta^{\alpha^2/(\alpha-d)^2} \|F_p^j\|_{L^\beta(\Omega; \mathbb{R}^{n \times n})}^\beta. \end{aligned} \tag{3.19}$$

The L^d -term on the right hand side is bounded due to (3.4) and the boundedness of $\{y^j\}$ in $W^{1,d}(\Omega; \mathbb{R}^n)$ follows by the Poincaré inequality if $\delta > 0$ is taken small. Hence $y^j \rightharpoonup y$ in $W^{1,d}(\Omega; \mathbb{R}^n)$ (up to a subsequence). This then also implies that $\text{cof} \nabla y^j \rightharpoonup \text{cof} \nabla y$ in $L^{d/(n-1)}(\Omega; \mathbb{R}^{n \times n})$; cf. [9]. Due to reflexivity of $W^{1,\alpha/(n-1)}(\Omega; \mathbb{R}^{n \times n})$ we get for a non-relabelled subsequence that $(\text{cof} \nabla y^j) F_p^{j\top} \rightharpoonup \Xi$ in $W^{1,\alpha/(n-1)}(\Omega; \mathbb{R}^{n \times n})$ for some Ξ . However, the weak convergence of $\{\text{cof} \nabla y^j\}_j$ and the strong convergence of $\{F_p^j\}$ allow us to identify $\Xi = (\text{cof} \nabla y) F_p^\top$. Moreover, the growth condition (3.2) implies that $\det \nabla y > 0$ a.e. in Ω (see [5]). Cramer’s rule together with $\det F_p^j = 1$ and $\det^{n-1}(\nabla y^j (F_p^j)^{-1}) = \det(\text{cof}[\nabla y^j (F_p^j)^{-1}]) = \det(\text{cof} \nabla y^j)$ give

$$\frac{\text{cof}[\nabla y^j (F_p^j)^{-1}]}{\det \nabla y^j} = (\nabla y^j (F_p^j)^{-1})^{-\top}$$

and the transpose of the left-hand side converges pointwise (for a subsequence again) to $(\nabla y (F_p)^{-1})^{-1}$ a.e. in Ω . This, together with the fact that $\det \nabla y > 0$ a.e., implies that $\nabla y^j (F_p^j)^{-1} \rightarrow \nabla y (F_p)^{-1}$ a.e. in Ω . Then by [17, Corollary 7.9] we see that \mathcal{I} is weakly lower semicontinuous.

The assumptions on \mathcal{D} ensure that it is sequentially weakly lower semicontinuous on \mathbb{Q} , too. By the direct method of the calculus of variations, we conclude that a minimizer q_τ^k exists. \square

Remark 3.9. Note that in fact, we proved that a minimizing sequence has a gpc-convergent subsequence.

3.6. Interpolation in time

We denote by q_τ a piecewise constant interpolation of $q_\tau^k =: (y_\tau^k, z_\tau^k)$, i.e. $q_\tau(t) = q_\tau^k$ if $t \in [k\tau, (k+1)\tau)$ and $k = 0, \dots, T/\tau - 1$. Finally, $q_\tau(T) := q_\tau^N$. Analogously, $L_\tau(t, q_\tau(k\tau)) := L(k\tau, q_\tau(k\tau))$ is a piecewise constant interpolation of L and $\mathcal{I}_\tau(t, q_\tau(k\tau)) := \mathcal{I}(k\tau, q_\tau(k\tau))$ is a piecewise constant interpolation of \mathcal{I} . Propositions stated below are classical in the context of energetic solutions. We refer to [21, 44] for proofs.

Proposition 3.10. *Under the assumptions of Proposition 3.8, problem (3.18) has a solution $q_\tau(t)$ which is stable, i.e., for all $t \in [0, T]$ and for every $q \in \mathbb{Q}$,*

$$\mathcal{I}_\tau(t, q_\tau(t)) \leq \mathcal{I}_\tau(t, q) + \mathcal{D}(q_\tau(t), q). \tag{3.20}$$

Moreover, for all $t_I \leq t_{II}$ from the set $\{k\tau\}_{k=0}^N$, the following discrete energy inequalities hold if one extends the definition of $q_\tau(t)$ by setting $q_\tau(t) := q^0$ if $t < 0$:

$$\begin{aligned} - \int_{t_I}^{t_{II}} \dot{L}(t, q_\tau(t-\tau)) dt &\leq \mathcal{I}(t_{II}, q_\tau(t_{II})) + \text{Var}(\mathcal{D}, q_\tau; [t_I, t_{II}]) - \mathcal{I}(t_I, q_\tau(t_I)) \\ &\leq - \int_{t_I}^{t_{II}} \dot{L}(t, q_\tau(t)) dt. \end{aligned} \tag{3.21}$$

We would like to pass with the step size τ to zero and for this we need certain *a priori* bounds.

Proposition 3.11. *Let (3.8) and (3.9) be satisfied. Then there is $\kappa > 0$ such that for any $\tau > 0$:*

$$\|y_\tau\|_{L^\infty(0,T;W^{1,d}(\Omega;\mathbb{R}^n))} < \kappa, \tag{3.22}$$

$$\text{Var}(\mathcal{D}, q_\tau; [0, T]) < \kappa, \tag{3.23}$$

$$\|z_\tau\|_{L^\infty(0,T;W^{1,\alpha}(\Omega;\mathbb{R}^{n \times n}) \times W^{1,\beta}(\Omega;\mathbb{R}^m))} < \kappa. \tag{3.24}$$

3.7. Limit passage

The following lemma is proved in [38].

Lemma 3.12. *Let $\mathcal{D}: \mathbb{Z} \times \mathbb{Z} \rightarrow [0, +\infty]$ satisfy (3.6) and (3.7). Let \mathcal{K} be a weakly sequentially compact subset of \mathbb{Z} . Then for every sequence $\{z_k\}_{k \in \mathbb{N}}$, $z_k: [0, T] \rightarrow \mathcal{K}$ for which $\sup_{k \in \mathbb{N}} \text{Var}(\mathcal{D}, z_k; [0, T]) < C$, $C > 0$, there exists a subsequence (not relabelled), a function $z: [0, T] \rightarrow \mathcal{K}$, and a function $\Delta: [0, T] \rightarrow [0, C]$ such that:*

- (i) $\text{Var}(\mathcal{D}, z_k; [0, t]) \rightarrow \Delta(t)$ for all $t \in [0, T]$,
- (ii) $z_k \rightarrow z$ in \mathbb{Z} for all $t \in [0, T]$, and
- (iii) $\text{Var}(\mathcal{D}, z; [t_0, t_1]) \leq \Delta(t_1) - \Delta(t_0)$ for all $0 \leq t_0 < t_1 \leq T$.

Let us denote $\mathbb{X} := L^\beta(\Omega; \mathbb{R}^{n \times n}) \times L^\omega(\Omega; \mathbb{R}^m)$. Finally, we proved the existence of an energetic solution.

Theorem 3.13. *Let $\alpha^{-1} + \beta^{-1} \leq d^{-1} < (n - 1)^{-1}$, $d > \frac{\beta(n-1)}{\beta-1}$. Let $q^0 \in \mathbb{Q}$ be a stable initial condition. Let the assumptions on \mathcal{W} , \mathcal{D} , f and g from Section 3.2 hold. Let further (3.16) or (3.A), (3.B), (3.C) hold. Then there is a process $q: [0, T] \rightarrow \mathbb{Q}$ with $q(t) = (y(t), z(t))$ such that q is an energetic solution according to Definition 2.2. The following convergence statements also hold:*

- (i) for a t -dependent (not relabelled) subsequence $w\text{-}\lim_{\tau \rightarrow 0} y_\tau(t) = y(t)$ in $W^{1,d}(\Omega; \mathbb{R}^n)$ for all $t \in [0, T]$,
- (ii) for a (not relabelled) subsequence $\lim_{\tau \rightarrow 0} z_\tau(t) = z(t)$ in \mathbb{X} for all $t \in [0, T]$,
- (iii) for a (not relabelled) subsequence $\lim_{\tau \rightarrow 0} \mathcal{I}_\tau(t, q_\tau(t)) = \mathcal{I}(t, q(t))$ for all $t \in [0, T]$, and
- (iv) for a (not relabelled) subsequence $\lim_{\tau \rightarrow 0} \text{Var}(\mathcal{D}, q_\tau; [0, t]) = \text{Var}(\mathcal{D}, q; [0, t])$ for all $t \in [0, T]$.

Proof. We have adapted the proof from [21, 42] and divided it into three steps.

Step 1: Assertion (i) follows from the *a priori* estimate in Proposition 3.11 and (ii) results from Lemma 3.12. Hence we have a limit $q(t) = (y(t), z(t))$. We easily get that $q(t) \in \mathbb{Q}$ for all $t \in [0, T]$.

Put $S(t, \tau) := \max_{k \in \mathbb{N} \cup \{0\}} \{k\tau; k\tau \leq t\}$. Then $\lim_{\tau \rightarrow 0} S(t, \tau) = t$ and by definition $q_\tau(t) := q_\tau(S(t, \tau)) \in \mathcal{S}(S(t, \tau))$. As we have seen, \mathcal{D} can be assumed such that (3.13) holds. Hence by Proposition 3.6, the limit is stable, i.e. $q(t) \in \mathcal{S}(t)$ (thanks to our *a priori* estimates, we can pass to a gpc-convergent subsequence to get lower semicontinuity).

Step 2: Notice that $\theta_\tau(t) := \frac{\partial \mathcal{I}}{\partial t}(t, q_\tau(t))$ is bounded in $L^\infty(0, T)$ by (3.8), (3.9) and (3.22), so we deduce a weak*-convergent subsequence, which we do not relabel, with the limit θ . By Fatou's lemma $\theta_i \leq \theta$.

Our interpolation satisfies $q_\tau(t) = q_\tau(k\tau)$ for $0 \leq t - k\tau < \tau$. Successively using (3.21), (3.8), (3.9) and (3.22), we get for some $C, C_1 > 0$

$$\begin{aligned} \mathcal{I}(t, q_\tau(t)) + \text{Var}(\mathcal{D}, q_\tau; [0, t]) &\leq \mathcal{I}(k\tau, q_\tau(k\tau)) + \text{Var}(\mathcal{D}, q_\tau; [0, k\tau]) + C\tau \\ &\leq \mathcal{I}(0, q_\tau(0)) - \int_0^{k\tau} \dot{\mathcal{I}}(s, q_\tau(s)) \, ds + C\tau \\ &\leq \mathcal{I}(0, q_\tau(0)) - \int_0^t \dot{\mathcal{I}}(s, q_\tau(s)) \, ds + C_1\tau. \end{aligned}$$

On account of Lemma 3.12 (i) and Proposition 3.6 we find that for $\tau \rightarrow 0$

$$\mathcal{I}(t, q(t)) + \Delta(t) \leq \mathcal{I}(0, q(0)) - \int_0^t \theta(s) \, ds.$$

Following [42] we set $\theta_i(t) := \liminf_{\tau \rightarrow 0} \theta_\tau(t)$. Since $\Delta(t) \geq \text{Var}(\mathcal{D}, q; [0, t])$ and by Fatou's lemma $\int_0^t \theta(s) \, ds \geq \int_0^t \theta_i(s) \, ds$, for a.a. $t \in [0, T]$ we get

$$\mathcal{I}(t, q(t)) + \text{Var}(\mathcal{D}, q; [0, t]) \leq \mathcal{I}(0, q(0)) - \int_0^t \theta_i(s) \, ds.$$

The convergence $\theta_\tau(s) = \dot{L}(s, q_\tau(s)) \rightarrow \dot{L}(s, q(s))$ and extracting an s -dependent subsequence of $\{\theta_\tau(s)\}$ converging to $\theta_i(s)$ show that $\theta_i(s) = \dot{L}(s, q(s))$. We finally get the upper energy estimate

$$\mathcal{I}(t, q(t)) + \text{Var}(\mathcal{D}, q; [0, t]) \leq \mathcal{I}(0, q(0)) - \int_0^t \dot{L}(s, q(s)) \, ds. \tag{3.25}$$

Step 3: The fact that $q(t)$ is stable for all $t \in [0, T]$ will be useful in proving the lower estimate. Consider a partition of a time interval $[t_I, t_{II}] \subset [0, T]$ such that $t_I = \vartheta_0^M < \vartheta_1^M < \vartheta_2^M < \dots < \vartheta_{K_M}^M = t_{II}$ and $\max_i(\vartheta_i^M - \vartheta_{i-1}^M) =: \vartheta^M \rightarrow 0$ as $M \rightarrow \infty$. Let us test the stability of $q(\vartheta_{k-1}^M)$ with $q(\vartheta_k^M)$, $k = 1, \dots, K_M$. We get

$$\begin{aligned} \sum_{k=1}^{K_M} [L((\vartheta_{k-1}^M, q(\vartheta_{k-1}^M)) - L(\vartheta_k^M, q(\vartheta_k^M))] &\leq \mathcal{I}(t_{II}, q(t_{II})) - \mathcal{I}(t_I, q(t_I)) \\ &+ \sum_{k=1}^{K_M} \mathcal{D}(q(\vartheta_{k-1}^M), q(\vartheta_k^M)). \end{aligned} \tag{3.26}$$

Thus,

$$\begin{aligned} \sum_{k=1}^{K_M} - \int_{\vartheta_{k-1}^M}^{\vartheta_k^M} \dot{L}(s, q(\vartheta_k^M)) \, ds &\leq \mathcal{I}(t_{II}, q(t_{II})) - \mathcal{I}(t_I, q(t_I)) \\ &+ \text{Var}(\mathcal{D}, q; [t_I, t_{II}]). \end{aligned} \tag{3.27}$$

We finally rearrange the terms as

$$\begin{aligned} \sum_{k=1}^{K_M} \int_{\vartheta_{k-1}^M}^{\vartheta_k^M} \dot{L}(s, q(\vartheta_k^M)) \, ds &= \sum_{k=1}^{K_M} \dot{L}(\vartheta_k^M, q(\vartheta_k^M))(\vartheta_k^M - \vartheta_{k-1}^M) \\ &+ \sum_{k=1}^{K_M} \int_{\vartheta_{k-1}^M}^{\vartheta_k^M} (\dot{L}(s, q(\vartheta_k^M)) - \dot{L}(\vartheta_k^M, q(\vartheta_k^M))) \, ds. \end{aligned} \tag{3.28}$$

The last sum on the right-hand side of (3.28) goes to zero with $\vartheta^M \rightarrow 0$ because the time derivative of the external loading is uniformly continuous in time by (3.8) and (3.9). The first sum on the right-hand side converges to $\int_{t_I}^{t_{II}} \dot{L}(s, q(s)) \, ds$ by [11, Lemma 4.12].

Altogether, the upper and lower estimates give us the energy balance

$$\mathcal{I}(t, q(t)) + \text{Var}(\mathcal{D}, q; [0, t]) = \mathcal{I}(0, q(0)) - \int_0^t \dot{L}(s, q(s)) \, ds. \quad (3.29)$$

Moreover,

$$\begin{aligned} \mathcal{I}(0, q(0)) - \int_0^t \theta_i(s) \, ds &\leq \mathcal{I}(t, q(t)) + \text{Var}(\mathcal{D}, q; [0, t]) \leq \mathcal{I}(t, q(t)) + \Delta(t) \\ &\leq \mathcal{I}(0, q(0)) - \int_0^t \theta(s) \, ds \leq \mathcal{I}(0, q(0)) - \int_0^t \theta_i(s) \, ds. \end{aligned} \quad (3.30)$$

Thus in fact, all inequalities in (3.30) are equalities and we get (iv). Proposition 3.6 also implies (iii). \square

Finally, we include a simple example from single-crystal plasticity, covered by our approach, in the spirit of [26] and [40].

Example 3.1. (single slip without hardening). Consider two orthonormal vectors $a, b \in \mathbb{R}^3$, describing the motion of edge dislocations in such a way that a is the glide direction and b is the slip-plane normal. Further we focus on a particular case of the so-called *separable* material where

$$\mathcal{W}(F_e, H, z) = \mathcal{W}_1(F_e) + c|H|^2 + c \det(F_e)^{-s} + \varepsilon |F_p|^6 + \varepsilon |\nabla F_p|^6, \quad c, s, \varepsilon > 0,$$

with $F_p(x, t) = \mathbb{I} + \gamma(x, t)a \otimes b$ where γ is the plastic slip and we can choose \mathcal{W}_1 to be e.g. the stored energy density of the Saint Venant–Kirchhoff material, i.e. $\mathcal{W}_1(F_e) = \frac{1}{8}(F_e F_e - \mathbb{I}) : \mathbb{C} : (F_e F_e - \mathbb{I})$ so that $\mathcal{W}_1(F_e) \geq c(|F_e|^4 - 1)$. The vectors a, b are not fixed in the reference configuration in general (more precisely, they lie in an intermediate lattice space, see [27]). The slip-plane normal \tilde{b} in the reference configuration is given by $\tilde{b} = (F_p)^\top b$. However, in the special case of a single slip we obtain $\tilde{b} = b$ and the slip-plane normal remains unchanged during plastic transformations.

As F_p is completely described by γ , we identify $z := \gamma$ and use the dissipation potential

$$\delta(\dot{\gamma}) = \kappa |\dot{\gamma}|,$$

where $\kappa > 0$ represents the resistance to the slip.

This corresponds to the dissipation distance

$$\mathcal{D}(\gamma_1, \gamma_2) = \int_\Omega \kappa |\gamma_1(x) - \gamma_2(x)| \, dx.$$

An extension to multi-slip, described by several glide directions $\{a_i\}$ and slip-plane normals $\{b_i\}$, $1 \leq i \leq N$, would be feasible, cf. [40].

As a last remark, note that our stored energy \mathcal{W} could also depend on F_e^{-1} , cf. [5].

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References

- [1] Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **63**, 337–403 (1977)
- [2] Ball, J.M., Currie, J.C., Olver, P.L.: Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Funct. Anal.* **41**, 135–174 (1981)
- [3] Bažant, Z.P., Jirásek, M.: Nonlocal integral formulation of plasticity and damage: a survey of progress. *J. Engrg. Mech.* **128**, 1119–1149 (2002)
- [4] Benešová, B., Kružík, M.: Weak lower semicontinuity of integral functionals and applications. *SIAM Rev.* **59**, 703–766 (2017)
- [5] Benešová, B., Kružík, M., Schlömerkemper, A.: A note on locking materials and gradient polyconvexity. *Math. Models Meth. Appl. Sci.* **28**, 2367–2401 (2018)
- [6] Bhattacharya, K.: *Microstructure of Martensite. Why it forms and how it gives rise to the shape-memory effect.* Oxford University Press, Oxford (2003)
- [7] Capriz, G.: Continua with latent microstructure. *Arch. Ration. Mech. Anal.* **90**, 43–56 (1985)
- [8] Carstensen, C., Hackl, K., Mielke, A.: Nonconvex potentials and microstructures in finite-strain plasticity. *Proc. Roy. Soc. Lond. A* **458**, 299–317 (2002)
- [9] Ciarlet, P.G.: *Mathematical Elasticity Vol.I: Three-Dimensional Elasticity.* North-Holland, Amsterdam (1988)
- [10] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Springer, Berlin (1989)
- [11] Dal Maso, G., Francfort, G., Toader, R.: Quasistatic Crack Growth in Nonlinear Elasticity. *Arch. Ration. Mech. Anal.* **176**, 165–225 (2005)
- [12] Davoli, E., Francfort, G.: A Critical revisiting of finite elasto-plasticity. *SIAM J. Math. Anal.* **47**, 526–565 (2015)
- [13] Dell’Isola, F., Sciarra, G., Vidoli, S.: Generalized Hooke’s law for isotropic second gradient materials. *Proc. R. Soc. Lond. A* **465**, 2177–2196 (2009)
- [14] DeSimone, A.: Coarse-grained models of materials with non-convex free-energy: two case studies. *Comput. Methods Appl. Mech. Engrg.* **193**, 5129–5141 (2004)
- [15] Dillon, O.W., Kratochvíl, J.: A strain gradient theory of plasticity. *Int. J. Solid Struct.* **6**, 1513–1533 (1970)
- [16] Ebobisse, F., Neff, P., Aifantis, E.: Existence result for a dislocation based model of single crystal gradient plasticity with isotropic or linear kinematic hardening. *Quar. J. Mech. Appl. Math.* **71**, 99–124 (2018)
- [17] Fonseca, I., Leoni, G.: *Modern Methods in the Calculus of Variations: L^p Spaces.* Springer, New York (2007)
- [18] Forest, S.: Micromorphic approach for gradient elasticity, viscoplasticity, and damage. *J. Eng. Mech.* **135**, 117 (2009)
- [19] Forest, S., Sab, K.: Finite-deformation second-order micromorphic theory and its relations to strain and stress gradient models. *Math. Mech. Solids* **25**(7), 1429–1449 (2017)
- [20] Francfort, G., Giacomini, A., Marigo, J.J.: A case study for uniqueness of elasto-plastic evolutions: The bi-axial test. *J. Math. Pures et Appl.* **105**, 198–227 (2016)
- [21] Francfort, G., Mielke, A.: Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. Reine Angew. Math.* **595**, 55–91 (2006)
- [22] Giacomini, A., Lusardi, L.: Quasistatic evolution for a model in strain gradient plasticity. *SIAM J. Math. Anal.* **40**, 1201–1245 (2008)
- [23] Grandi, D., Stefanelli, U.: Finite plasticity in $P^\top P$. Part I: constitutive model. *Continuum Mech. Thermodyn.* **29**, 97–116 (2017)
- [24] Grandi, D., Stefanelli, U.: Finite plasticity in $P^\top P$. Part II: Quasi-Static Evolution and Linearization. *SIAM J. Math. Anal.* **49**(2), 1356–1384 (2017)
- [25] Green, A.E., Rivlin, R.S.: Multipolar continuum mechanics. *Arch. Ration. Mech. Anal.* **17**, 113–147 (1964)
- [26] Gurtin, M.E.: On the plasticity of single crystals: free energy, microforces, plastic-strain gradients. *J. Mech. Phys. Solids.* **48**, 989–1036 (2000)
- [27] Gurtin, M.E., Fried, E., Anand, L.: *The Mechanics and Thermodynamics of Continua.* Cambridge Univ. Press, New York (2009)
- [28] Gurtin, M.E., Anand, L.: A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. I. Small deformations. *J. Mech. Phys. Solids* **53**, 1624–1649 (2005)
- [29] Halphen, B., Nguyen, Q.S.: Sur les matériaux standards généralisés. *J. Méc.* **14**, 39–63 (1975)
- [30] Han, W., Reddy, B.D.: *Plasticity (Mathematical Theory and Numerical Analysis).* Interdisciplinary Applied Mathematics, vol. 9. Springer-Verlag, New York (1999)
- [31] Hill, R.: A variational principle of maximum plastic work in classical plasticity. *Q. J. Mech. Appl. Math.* **1**, 18–28 (1948)
- [32] Horák, M., Kružík, M.: Gradient polyconvex material models and their numerical treatment. *Int. J. Solids Struct.* **195**, 57–65 (2020)
- [33] Kratochvíl, J., Kružík, M.: Energetic approach to large strain gradient crystal plasticity. *Acta Polytechnica* **52**, 9–14 (2012)

- [34] Kružík, M., Melching, D., Stefanelli, U.: Quasistatic evolution for dislocation-free finite plasticity. *ESAIM Control. Optim. Calc. Var.* 26, article No. 123, 23 pages, (2020) <https://doi.org/10.1051/cocv/2020031>
- [35] Kružík, M., Pelech, P., Schlömerkemper, A.: Gradient polyconvexity in evolutionary models of shape-memory alloys. *J. Optim. Th. Appl.* **184**, 5–20 (2020)
- [36] Kružík, M., Roubíček, T.: *Mathematical Methods in Continuum Mechanics of Solids*. Springer, Switzerland (2019)
- [37] Kružík, M., Zimmer, J.: A model of shape memory alloys taking into account plasticity. *IMA J. Appl. Math.* **76**, 193–216 (2011)
- [38] Mainik, A., Mielke, A.: Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations.* **22**, 73–99 (2005)
- [39] Mainik, A., Mielke, A.: Global existence for rate-independent gradient plasticity at finite strain. *J. Nonlin. Sci.* **19**(3), 221–248 (2009)
- [40] Mielke, A.: Finite Elastoplasticity Lie Groups and Geodesics on $SL(d)$. In: Newton, P., Holmes, P., Weinstein, A. (eds.) *Geometry, Mechanics, and Dynamics*, pp. 61–90. Springer, New York (2002)
- [41] Mielke, A.: Energetic formulation of multiplicative elasto-plasticity using dissipation distances. *Cont. Mech. Thermodyn.* **15**, 351–382 (2003)
- [42] Mielke, A.: Evolution of rate-independent systems. In: Dafermos, C.M., Feireisl, E. (eds.) *Handbook of Differential Equations: Evolutionary equations*, vol. 2, pp. 461–559. Elsevier/North-Holland, Amsterdam (2005)
- [43] Mielke, A., Roubíček, T.: A rate-independent model for inelastic behavior of shape-memory alloys. *Multiscale Model. Simul.* **1**, 571–597 (2003)
- [44] Mielke, A., Roubíček, T.: *Rate-Independent Systems: Theory and Application*. Springer, New York (2015)
- [45] Mielke, A., Roubíček, T.: Rate-independent elastoplasticity at finite strains and its numerical approximation. *Math. Models Methods in Appl. Sci.* **26**, 2203–2236 (2016)
- [46] Mielke, A., Theil, F., Levitas, V.I.: A variational formulation of rate-independent phase transformations using an extremum principle. *Arch. Ration. Mech. Anal.* **162**, 137–177 (2002)
- [47] Moreau, J.J.: Application of convex analysis to the treatment of elastoplastic systems. In: Germain, P., Nayroles, B. (eds.) *Applications of Methods of Functional Analysis to Problems in Mechanics*. Springer-Verlag, Berlin (1976)
- [48] Mühlhaus, H.-B., Aifantis, E.C.: A variational principle for gradient plasticity. *Int. J. Solid. Structures.* **28**, 845–857 (1991)
- [49] Ortiz, M., Repetto, E.A.: Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids.* **47**, 397–462 (1999)
- [50] Podio-Guidugli, P.: Contact interactions, stress, and material symmetry, for nonsimple elastic materials. *Theor. Appl. Mech.* **28–29**, 26–276 (2002)
- [51] Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. *Pacific J. Math.* **33**, 209–216 (1970)
- [52] Simo, J.: A framework for finite strain elastoplasticity based on maximum plastic dissipation and multiplicative decomposition. Part I. Continuum formulation. *Comp. Meth. Appl. Mech. Engrg.* **66**, 199–219 (1988)
- [53] Simo, J.: A framework for finite strain elastoplasticity based on maximum plastic dissipation and multiplicative decomposition. Part II. Comput. Aspects. *Comp. Meth. Appl. Mech. Engrg.* **68**, 1–31 (1988)
- [54] Shu, Y.C., Bhattacharya, K.: Domain patterns and macroscopic behaviour of ferroelectric materials. *Philosophical Magazine B.* **81**, 2021–2054 (2001)
- [55] Suquet, P.-M.: Sur les équations de la plasticité : existence et régularité des solutions. *J. Méc.* **20**, 3–39 (1981)
- [56] Temam, R.: Problèmes mathématiques en plasticité. In: *Méthodes Mathématiques de l’Informatique*, vol. 12. Gauthier-Villars, Montrouge (1983)
- [57] Toupin, R.A.: Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **11**, 385–414 (1962)
- [58] Toupin, R.A.: Theory of elasticity with couple stress. *Arch. Ration. Mech. Anal.* **17**, 85–112 (1964)
- [59] Tsagrakis, I., Aifantis, E.C.: Recent developments in gradient plasticity. *J. Engrg. Mater. Tech.* **124**, 352–357 (2002)

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