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EQUILIBRIUM OF IMMERSED HYPERELASTIC SOLIDS

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Abstract. We discuss different equilibrium problems for hyperelastic solids immersed in a fluid at rest. In particular, solids are subjected to gravity and hydrostatic pressure on their immersed boundaries. By means of a variational approach, we discuss free-floating bodies, anchored solids, and floating vessels. Conditions for the existence of local and global energy minimizers are presented.

1. **Introduction.** The equilibrium of partially immersed bodies is a classical problem in mechanics and has attracted very early attention. Indeed, the basic observation in this context has to be traced back more than two millennia to the work of Archimedes [1]. In the first of his two books *On Floating Bodies*, he formulated his celebrated buoyancy principle which is regarded as the germinal moment of hydrostatics. In his second book, he discusses the floating of a rigid convex paraboloid with horizontal basis, probably inspired by the study of floating vessels.

Strangely enough, the mathematical literature on floating bodies is rather scant. After Archimedes, an early discussion dates to Laplace [16], who considered the case of a drop of mercury floating on water. The rigid body case has been investigated in [11], both at the equilibrium level and for harmonic motions. In more recent years, a question by S. Ulam [17, Problem 19] triggered investigations on the stability of convex bodies of given density [9, 15, 24]. Moreover, attention has been given to the capillary case, where the fluid surface is not assumed to be flat and contact

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conditions arise [5, 18, 19, 20, 23]. Criteria for the stable floating of a convex rigid body in two and three dimensions have been recently analyzed in [6, 7].

Driven by its obvious practical relevance, the case of floating *deformable* bodies has attracted huge attention from the engineering community. Correspondingly, the literature on *hydroelasticity* [3] is rather extended. This theme fits into the general frame of *fluid-structure interactions* and the reader is referred to the recent [12, 22] for a collection of topics and references.

To the best of our knowledge, no analysis is available for the case of a hyperelastic body deforming under the combined effect of gravity and fluid pressure. We would like to fill this gap by recording in this note some remarks on the existence of local and global equilibria. After having collected some basic material in Sections 2-4, we explore a suite of different settings, ranging from incompressible to compressible free-floating solids (Sections 5-6), to solids at anchor (Section 7), to bounded fluid reservoirs (Section 8), to the case of ship-like bodies (Section 9).

In all of our discussion we follow the variational approach, by systematically restricting our attention to energetic arguments and by refraining from considering directly the corresponding differential problems. On the one hand, this reflects our personal take, which favors the relevance of variational theories. This, in particular, leads to recovering a variational version of Archimedes' celebrated principle. On the other hand, this choice allows for an effective and compact tractation of many different settings, which happen to be clearly distinguished and readily amenable by this approach.

A caveat on presentation style: in the following, we articulate a rigorous discussion, avoiding however the classical statement-proof structure. We hope that the reader will enjoy our informal tone, which, in our view, better reflects the exploratory nature of our considerations.

2. **Basic setting and energy.** The actual shape of the solid is described by its continuous deformation $y : \Omega \to \mathbb{R}^3$ from its reference configuration $\Omega \subset \mathbb{R}^3$, the latter being a bounded, connected, and smooth set. We will use the symbol X for points in the reference configuration Ω and x for points in actual space, which we endow with the orthonormal system (e_1, e_2, e_3). In coordinates, deformation y is then written as (y_1, y_2, y_3). For a measurable set $\omega \subset \overline{\Omega}$, we denote by $\omega^y = y(\omega)$ its image under the continuous map y. The actual configuration $y(\Omega)$ of the solid is hence Ω^y. Both solid and fluid are subjected to a constant gravity in direction $-e_3$ and we classically indicate with g the corresponding force density, which we assume to be constant. In the basic setting, the fluid is assumed to be incompressible and to fill the region $\{x_3 \le 0\} \setminus \Omega^y$, i.e. the part of $\{x_3 \le 0\}$ outside of the body Ω^y, see Figure 1. Correspondingly, we say that the solid *floats* if $\sup y_3 > 0$, and that it *barely floats* if $\sup y_3 = 0$. Moreover, the solid is said to be *immersed* if $\inf y_3 < 0$ and *completely immersed* if $\sup y_3 \le 0$.

The total energy of the solid is given by the sum of its *elastic* potential, the *hydrostatic* potential, and the *gravitational* potential, namely

$$E(y) = \int_\Omega W(\nabla y(X))\,\mathrm{d}X + \int_{\Omega^y} g\rho_f x_3^-\,\mathrm{d}x + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X. \tag{1}$$

We have used the symbol $x^- = \max\{0, -x\}$ for the negative part. Moreover, we have indicated with $\rho_f > 0$ the (constant) density of the fluid, and with $\rho_s > 0$ the (constant) referential density of the solid. In particular, energy E features the occurrence of both Lagrangian and Eulerian terms.

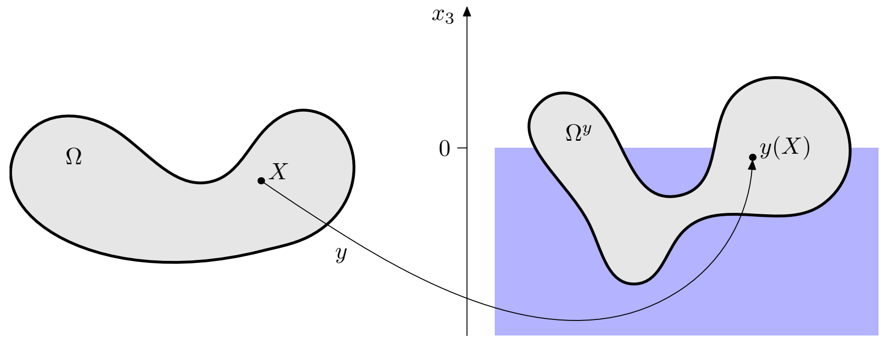

FIGURE 1. The basic setting.

Before moving on, let us justify the form of the energy (1) by computing the Euler-Lagrange equations for E. Let the invertible deformation y be given and let $v\colon \Omega \to \mathbb{R}^3$ represent a smooth variation. Assuming sufficient smoothness and making use of [14, Prop. 1.2.8, p. 23] for taking the variation of the hydrostatic term we deduce that

$$\begin{aligned}\langle \delta E(y), v\rangle = &\int_\Omega \mathrm{D}W(\nabla y(X)) : \nabla v(X)\,\mathrm{d}X\\ &+ \int_{\partial\Omega} g\rho_f y_3^-(X)\,\mathrm{cof}\,\nabla y(X) N(X)\cdot v(X)\,\mathrm{d}S(X)\\ &+ \int_\Omega g\rho_s e_3\cdot v(X)\,\mathrm{d}X\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the duality product in deformation space, the symbol $:$ is the standard contraction product among tensors, and $N(X)$ denotes the outward pointing normal to $\partial\Omega$. For y critical, namely $\delta E(y) = 0$, we formally deduce the equilibrium system for the first Piola-Kirchhoff stress $P(X) = \mathrm{D}W(\nabla y(X))$ as

$$-\mathrm{div}_X\, P(X) + g\rho_s e_3 = 0 \quad \text{in } \Omega, \tag{2}$$

$$P(X)N(X) = -g\rho_f y_3^-(X)\,\mathrm{cof}\,\nabla y(X)N(X) \quad \text{on } \partial\Omega. \tag{3}$$

Relations (2)-(3) express the equilibrium of forces in the bulk and at the boundary, respectively. The boundary relation (3) is better understood in actual variables. The equilibrium system (2)-(3) can be equivalently restated in terms of the Cauchy stress $T(y(X)) = P(X)(\mathrm{cof}\,\nabla y(X))^{-1}$ as

$$-\mathrm{div}_x\, T(x) + g\rho_s e_3(\det \nabla y^{-1})(x) = 0 \quad \text{in } \Omega^y, \tag{4}$$

$$T(x)n(x) = -g\rho_f x_3^- n(x) \quad \text{on } \partial\Omega^y, \tag{5}$$

where $x = y(X)$ and $n(x) = \mathrm{cof}\,\nabla y(X)N(X)/|\mathrm{cof}\,\nabla y(X)N(X)|$ is the outward pointing normal to $\partial\Omega^y$ at x and y^{-1} is the inverse of y. Here, we used properties of Piola's transform, see [14, Thm. 1.1.9, p. 9]. In particular, the hydrostatic pressure increases linearly with depth and no tension is exerted at its boundary for $y_3 \geq 0$.

The case of a nonhomogeneous solid could be handled by simply letting the density ρ_s be a given integrable function in X. Moreover, the density of the fluid can also be taken to be space-dependent. Note that this does not directly apply to water, whose density varies very little with depth, but may be relevant in the case of gases. Eventually, one can consider the case of a porous solid, which might have two densities, say ρ_s^w and ρ_s^d, depending on the fact that it is *wet* or *dry*. By assuming that the wet portion of the body corresponds with the immersed one, we can model this case by simply replacing $\rho_s y_3(X)$ by $-\rho_s^w y_3^-(X) + \rho_s^d y_3^+(X)$ in the gravitational term of E. Here and in the following we let $x^+ = \max\{x, 0\}$. Note that here we are excluding capillarity effects in the solid, which would wet also portions of the solid with $y_3 > 0$.

3. **Requirements on the energy.** Let us specify some requirements on the elastic energy density $W\colon \mathrm{GL}_+(3) := \{F \in \mathbb{R}^{3\times 3}\colon \det F > 0\} \to [0, \infty]$ which will be assumed throughout. We ask W to vanish at the identity tensor $I \in \mathbb{R}^{3\times 3}$, and to be coercive, unbounded for $\det F \to 0+$, polyconvex, and frame indifferent. Namely, we ask for c_0, $c_1 > 0$, $p > 3$, $s > 0$, and a convex function $\mathbb{W} : \mathrm{GL}_+(3) \times \mathrm{GL}_+(3) \times (0, \infty) \to [0, \infty]$ so that, for all $F \in \mathrm{GL}_+(3)$ we have

$$W(F) \geq c_0|F|^p + c_1(\det F)^{-s} - 1/c_0, \tag{6}$$

$$W(F) = \mathbb{W}(F, \operatorname{cof} F, \det F), \tag{7}$$

$$W(QF) = W(F) \quad \forall Q \in \mathrm{SO}(3) \tag{8}$$

where $\mathrm{SO}(3) := \{A \in \mathbb{R}^{3\times 3}\colon AA^\top = I,\ \det A = 1\}$ is the set of orientation-preserving rotations.

Given the coercivity (6), finite-energy deformations y necessarily belong to the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^3)$ and are tacitly identified with their unique continuous representative. In particular, the set Ω^y in (1) is well-defined as the image of the continuous function y.

We ask admissible deformations $y \in W^{1,p}(\Omega; \mathbb{R}^3)$ to additionally fulfill the classical Ciarlet-Nečas condition [4]

$$\int_\Omega \det \nabla y(X)\, \mathrm{d}X \leq |\Omega^y|, \tag{9}$$

where, here and below $|\cdot|$ stands also for the Lebesgue measure in $\mathbb{R}^3$. This condition implies that the map y is almost everywhere injective, namely, there exists $\omega \subset \Omega$ such that $|\omega| = 0$ and $y(x_1) \neq y(x_2)$ for every $x_1, x_2 \in \Omega \setminus \omega$ satisfying $x_1 \neq x_2$. Note that (9) is closed under weak $W^{1,p}(\Omega; \mathbb{R}^3)$ convergence.

As the embedding of $W^{1,p}(\Omega; \mathbb{R}^3)$ in $C(\overline{\Omega}; \mathbb{R}^3)$ is compact, one readily checks that energy E from (1) is lower semicontinuous with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$. Indeed, if y_k converges to y weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$ and uniformly, one has that $\operatorname{cof} \nabla y_k \rightharpoonup \operatorname{cof} \nabla y$ weakly in $L^{p/2}(\Omega; \mathbb{R}^{3\times 3})$, $\det \nabla y_k \rightharpoonup \det \nabla y$ weakly in $L^{p/3}(\Omega)$, and $((y_k)_3)^- \to y_3^-$ uniformly, so that

$$\liminf_{k\to\infty} E(y_k) = \liminf_{k\to\infty} \left(\int_\Omega \mathbb{W}(\nabla y_k(X), \operatorname{cof} \nabla y_k(X), \det \nabla y_k(X))\, \mathrm{d}X \right.$$
$$\left. + \int_\Omega g\rho_f y_3^-(X) \det \nabla y(X)\, \mathrm{d}X + \int_\Omega g\rho_s y_3(X)\, \mathrm{d}X \right) \geq E(y).$$

Having settled lower semicontinuity, the discussion on existence of energy minimizers will focus on identifying conditions ensuring the coercivity of the energy E with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$. Indeed, once such coercivity is established, existence of global energy minimizers would follow by the direct method.

As almost everywhere injective can still be noninjective, one could consider strengthening the coercivity of the energy as

$$W(F) \geq c_0|F|^p + c_0\frac{|F|^{3q}}{(\det F)^q} + c_1(\det F)^{-s} - \frac{1}{c_0}$$

for some $q > 2$. Under this stronger coercivity condition, finite-energy deformations are of *finite-distortion* [10, Def. 1.11, p. 14]. In particular, owing to [10, Thm. 3.4, p. 43] they are *open*, namely, map open sets to open sets. The last step is then to check that almost everywhere injective open deformations y are actually homeomorphisms from Ω to Ω^y [8, Thm. 3.5]. This in particular entails that y is injective.

4. **Coercivity and the archimedes principle.** As already mentioned, coercivity of the energy in the weak $W^{1,p}(\Omega;\mathbb{R}^3)$ topology entails existence of minimizers. Note that the elastic energy controls the L^p norm of ∇y via (6). Thus, in order to deduce the coercivity of E with respect to the weak $W^{1,p}(\Omega;\mathbb{R}^3)$ topology, we just need to ascertain that admissible deformations are uniformly bounded on some energy sublevel. The elastic part of the energy is invariant under rigid motions and, since $p > 3$, an L^p bound on ∇y entails a bound on the *diameter* $\mathrm{diam}(\Omega^y)$ of Ω^y, namely

$$\mathrm{diam}(\Omega^y) \leq c\|\nabla y\|_{L^p(\Omega)}. \tag{10}$$

Here and in the following, we will use the symbols c, c', c'' to indicate positive constants, possibly depending on data but independent of the deformation, and changing from line to line.

Owing to the diameter bound (10), the boundedness of y will follow as soon as one checks that the position of the barycenter $\overline{y}$ of the deformed body

$$\overline{y} = (\overline{y}_1, \overline{y}_2, \overline{y}_3) := \frac{1}{|\Omega|}\int_\Omega y(X)\,\mathrm{d}X \in \mathbb{R}^3$$

is bounded by the energy. Since the hydrostatic and the gravitational terms depend just on the y_3 component, it is not restrictive to assume that

$$\overline{y}_1 = \overline{y}_2 = 0.$$

Hence, one needs to check the boundedness of $\overline{y}_3$ only.

The gravitational potential strictly decreases by translating the solid in direction $-e_3$. As the hydrostatic term vanishes out of the fluid, i.e., for $y_3 > 0$, one readily proves that solids minimizing the energy are necessarily immersed, namely $\inf y_3 < 0$. This in turn implies $\sup y_3 \leq \inf y_3 + \mathrm{diam}(\Omega^y) < \mathrm{diam}(\Omega^y)$. By definition of the barycenter this yields

$$\overline{y}_3 < \mathrm{diam}(\Omega^y). \tag{11}$$

The issue is then to control $\overline{y}_3$ from below. Assume then that the solid is completely immersed and rewrite E as

$$\begin{aligned} E(y) &= \int_\Omega W(\nabla y(X))\,\mathrm{d}X + g\int_\Omega (-\rho_f J^y(X) + \rho_s)\,y_3(X)\,\mathrm{d}X \\ &= \int_\Omega W(\nabla y(X))\,\mathrm{d}X + g\int_\Omega (-\rho_f J^y(X) + \rho_s)\,(y_3(X) - \overline{y}_3)\,\mathrm{d}X \\ &\qquad + g|\Omega|\left(-\rho_f \overline{J}^y + \rho_s\right)\overline{y}_3, \end{aligned} \tag{12}$$

where we have used the short-hand notations

$$J^y(X) = \det \nabla y(X) \quad \text{and} \quad \overline{J}^y = \frac{1}{|\Omega|}\int_\Omega J^y(X)\,\mathrm{d}X.$$

The first two terms in the right-hand side of (12) are invariant under translations in direction e_3. The last term is decreasing as $\overline{y}_3$ increases iff

$$\rho_f \overline{J}^y > \rho_s. \tag{13}$$

Relation (13) is hence a necessary and sufficient buoyancy condition.

Let now y be a critical point for E and consider variations of the form $y + \alpha e_3$ for $\alpha \in \mathbb{R}$. From $f'(0) = 0$ for $f(\alpha) = E(y + \alpha e_3)$ we deduce from (1) that

$$g\rho_f\,|\Omega^y \cap \{x_3 \le 0\}| = g\rho_s\,|\Omega|. \tag{14}$$

The left and the right term in this equation are respectively the weight of the displaced fluid and the weight of the solid, so that relation (14) is nothing but the classical *Archimedes principle*.

5. **Incompressible solids.** We consider an *incompressible* free-floating solid by requiring $J^y = 1$ almost everywhere (for instance by letting $W(F) = \infty$ if $\det F \neq 1$). Note that the incompressibility constraint $J^y = 1$ a.e. is stable under weak $W^{1,p}(\Omega;\mathbb{R}^3)$ convergence. In this case, condition (13) reduces to a relation between ρ_s and ρ_f.

In case $\rho_f > \rho_s$ we have that solids minimizing the energy necessarily float. In particular, the energy controls the full $W^{1,p}$ norm of the deformation and the direct method ensures the existence of a ground state y. One can check directly that Ω^y is floating, for one has

$$|\Omega^y \cap \{x_3 \le 0\}| \overset{(14)}{=} (\rho_s/\rho_f)|\Omega| < |\Omega|.$$

If $\rho_f < \rho_s$, the energy E is not bounded from below since it decreases linearly for translations of the solid in direction $-e_3$ once the solid is completely immersed, see (12). In this case, no global minimizer of E exists.

In the critical case $\rho_f = \rho_s$, one can rewrite the energy as

$$E(y) = \int_\Omega W(\nabla y(X))\,dX + \int_\Omega g\rho_s y_3^+(X)\,\mathrm{d}X \ge 0.$$

In particular, completely immersed, rigid solids, namely those given by $y(X) = QX + v$ with $Q \in \mathrm{SO}(3)$, $v \in \mathbb{R}^3$, and $\sup y_3 \le 0$, realize $E = 0$ and are thus global minimizers. In case $W = 0$ solely on $\mathrm{SO}(3)$, these are actually the unique global minimizers.

The existence of floating or barely floating solids minimizing the energy can be proved even in the case of slight compressibility, as long as $\overline{J}^y$ is bounded away from zero, say

$$\overline{J}^y \ge \tau > 0. \tag{15}$$

In this case, one can still recover the existence of floating solids minimizing the energy whenever $\rho_f\tau \ge \rho_s$. A lower bound on $\overline{J}^y$ of the form of (15) would follow in case W was constrained to be $W(F) = \infty$ for $\det F < \tau$ (a closed condition with respect to the weak $W^{1,p}$ topology). Alternatively, one could replace the term $c_1(J^y)^{-s}$ by $c_1(J^y - \tau)^{-s}$ in the coercivity condition (6) in order to restrict to deformations with $J^y > \tau$ almost everywhere.

The findings of this section for incompressible solids can be summarized as follows:

- If $\rho_s > \rho_f$ no local minimizer exists and the energy is not bounded from below.
- If $\rho_s = \rho_f$ solids minimizing the energy exist and are completely immersed.
- If $\rho_s < \rho_f$ solids minimizing the energy exist and are floating.

By possibly resorting to a nonhomogeneous density $\rho_s \colon \Omega \to [0,\infty)$ one could discuss the case of a vessel having load or flotation tanks. This is for instance the case of a submarine, see Figure 2, where buoyancy is controlled by allowing water to fill the ballast tanks or by expelling water from the ballast tanks by means of a compressed-air reserve. In order to model the floating of a submarine, one

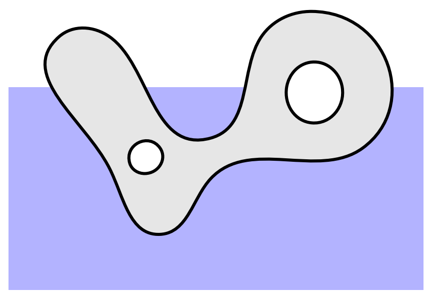

FIGURE 2. The submarine setting.

assumes Ω to be disjointly partitioned as $\Omega = \Omega_h \cup \Omega_b$, where Ω_h is the reference configuration of the hull of the submarine (where density $\rho_s = \rho_h$ is assumed to be constant) and Ω_b represents the ballast tanks with density $\rho_s = \rho_b$ depending on the air-water ratio. A possible form for the energy in this case is

$$E(y) = \int_{\Omega_h} W(\nabla y(X))\,\mathrm{d}X + \int_{\Omega^y} g\rho_f x_3^-\,\mathrm{d}x + \int_{\Omega_h} g\rho_h y_3(X)\,\mathrm{d}X + \int_{\Omega_b} g\rho_b y_3(X)\,\mathrm{d}X. \tag{16}$$

In order to determine the correct air-water balance in the ballast tanks keeping the (incompressible) submarine completely immersed and neutrally floating, one has to simply reconsider the discussion of (14) to find the density ρ_b (assumed constant for simplicity) fulfilling $\rho_f|\Omega| = \rho_h|\Omega_h| + \rho_b|\Omega_b|$. (Here, neutrally floating means that the energy is independent from $\overline{y}_3$ as long as the solid is fully immersed.)

6. **Compressible solids.** Let us now turn to the case of a free-floating compressible solid. In this case, regardless of the values of ρ_f and ρ_s, the energy can be proved to be unbounded from below, so that no global minimizer exists. Indeed, for all ρ_f, $\rho_s > 0$, one can consider a deformation y with $\rho_f J^y < \rho_s$ almost everywhere, define $y_k = y - ke_3$ for $k \geq \mathrm{diam}(\Omega^y)$, so that Ω^{y_k} is completely immersed, and compute

$$\begin{aligned} E(y_k) &= \int_\Omega W(\nabla y(X))\,\mathrm{d}X + \int_\Omega g(-\rho_f J^y(X) + \rho_s)(y_3(X) - k)\,\mathrm{d}X \\ &\leq \int_\Omega W(\nabla y(X))\,\mathrm{d}X + g(-\rho_f \overline{J}^y + \rho_s)|\Omega|(\sup y_3 - k). \end{aligned} \tag{17}$$

By taking $k \to \infty$ one has that $E(y_k) \to -\infty$. Note that the energy is unbounded from below for any choice of the densities, regardless of the fact that ρ_s could be smaller than ρ_f (and hence the solid would float, were it incompressible).

As global minimizers do not exist, we now turn to consider local minimizers instead. In case $\rho_s > \rho_f$, given any y with $J^y = 1$ almost everywhere, one has that $\alpha \mapsto E(y + \alpha e_3)$ increases on $\{\alpha < 0\}$. In particular, E admits no local energy minimizer.

We hence turn to the case $\rho_s < \rho_f$ for the remainder of this section. The argument in (17) is based on considering deformations with $J^y(X) < \rho_s/\rho_f$. In fact, $J^y(X)$ can be made arbitrarily small, still keeping the energy finite. On the other hand, the coercivity (6) entails that extreme compressions have high energy. Indeed, the higher the value of the parameter c_1, the higher is the energy needed to obtain $J^y(X) < \rho_s/\rho_f$. It is hence conceivable that, for materials having a very large c_1 and for $\rho_s < \rho_f$, local minimizer of the energy among floating solids may exist. We devote the following discussion to check this fact, by assuming a slightly more specific form of the energy, namely,

$$\begin{aligned} E(y) = \int_\Omega W(\nabla y(X))\,\mathrm{d}X + c_1 \int_\Omega (J^y(X))^{-s}\,\mathrm{d}X \\ + \int_{\Omega^y} g\rho_f x_3^-\,\mathrm{d}x + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X \end{aligned} \tag{18}$$

for $W\colon \mathrm{GL}_+(3) \to [0,\infty]$ with $W(I) = 0$ fulfilling (6)-(8) (with $c_1 = 0$ there). In particular, we have highlighted the coercive part on $(J^y)^{-s}$ by separating it from the elastic energy.

Before moving on, we need to refine the bound (10) for a floating solid, i.e., for deformations with $\sup y_3 > 0$. By (6) (for $c_1 = 0$) we have that

$$\begin{aligned} \mathrm{diam}(\Omega^y) &\le c\|\nabla y\|_{L^p(\Omega)} \le c'\left(E(y) + 1/c_0 - \int_{\Omega^y} g\rho_f x_3^-\,\mathrm{d}x - \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X\right)^{1/p} \\ &\le c'\big(E(y) + 1/c_0 + c''\mathrm{diam}(\Omega^y)\big)^{1/p}. \end{aligned}$$

From this we get that

$$\mathrm{diam}(\Omega^y) \le c\,(E(y))^{1/p} + c. \tag{19}$$

Let us start by restricting ourselves to a specific sublevel of the energy. Define $\hat{y} := \mathrm{id} + \alpha e_3$, where $\alpha \in \mathbb{R}$ is chosen in such a way that $\sup \hat{y}_3 = 0$. In this case, it holds that $W(\nabla\hat{y}(X)) = 0$ and $J^{\hat{y}}(X) = 1$ for all $X \in \Omega$ by $W(I) = 0$. Moreover, we have $x_3^- = -x_3$ for all $x \in \Omega^{\hat{y}}$. Then, we readily compute that

$$E(\hat{y}) = c_1|\Omega| + g(\rho_s - \rho_f)(\overline{X}_3 + \alpha)|\Omega| \tag{20}$$

where $\overline{X}_3 := \int_\Omega X_3 \mathrm{d}X/|\Omega|$. As $\rho_s < \rho_f$, such $\hat{y}$ cannot be a minimizer of the energy, for the energy decreases by increasing α. As a first step, we fix $1 > \tau > \rho_s/\rho_f$ and we check that the energy E can be minimized on the set

$$A = \{y \in W^{1,p}(\Omega;\mathbb{R}^3) \ \text{ fulfilling } \overline{J}^y \ge \tau\}.$$

In fact, for minimizers we can restrict our considerations to the sublevel $\{E \le E(\hat{y})\}$. All such solids are necessarily floating, see (15) in Section 5. Thus, in particular, the energy E is coercive on $A \cap \{E \le E(\hat{y})\}$ and A is closed with respect to the weak $W^{1,p}$-topology. This implies that there exists a minimizer $y^* \in A$ of E on A (with Ω^{y^*} floating).

Our key step is to check that, if $c_1 > 0$ is sufficiently large, one can find $r > 0$ small such that

$$\big(\|y_3 - y_3^*\|_{L^\infty(\Omega)} < r,\ E(y) \le E(\hat{y})\big) \ \Rightarrow\ \overline{J}^y > \tau. \tag{21}$$

This then proves that y^* is a local (in L^∞) minimizer of the energy. In fact, given y with $\|y_3 - y_3^*\|_{L^\infty(\Omega)} < r$, one either has $E(y) > E(\hat{y}) \geq E(y^*)$ or $E(y) \leq E(\hat{y})$. In the latter case, implication (21) entails that $y \in A$ and therefore $E(y) \geq E(y^*)$.

In order to check (21), we argue by contradiction and assume to be given a $y \in W^{1,p}(\Omega;\mathbb{R}^3)$ with $\|y_3 - y_3^*\|_{L^\infty(\Omega)} < r$ and $E(y) \leq E(\hat{y})$ such that

$$\overline{J}^y \leq \tau. \tag{22}$$

By letting $\mu = g(\rho_s - \rho_f)(\overline{X}_3 + \alpha)$ and recalling that $W \geq 0$, we then have

$$\begin{aligned}
c_1|\Omega| + \mu|\Omega| &\overset{(20)}{=} E(\hat{y}) \geq E(y) \\
&\overset{(18)}{\geq} c_1 \int_\Omega (J^y(X))^{-s}\,\mathrm{d}X + \int_{\Omega^y} g\rho_f x_3^- + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X \\
&\geq c_1|\Omega|\big(\overline{J}^y\big)^{-s} + \int_{\Omega^{y^*}} g\rho_f x_3^- + \int_\Omega g\rho_s y_3^*(X)\,\mathrm{d}X - c\|y_3 - y_3^*\|_{L^\infty(\Omega)} \\
&\overset{(22)}{\geq} c_1|\Omega|\tau^{-s} - g\rho_s|\Omega|\mathrm{diam}(\Omega^{y^*}) - cr \\
&\overset{(19)}{\geq} c_1|\Omega|\tau^{-s} - c(E(y^*))^{1/p} - c - cr \\
&\geq c_1|\Omega|\tau^{-s} - c(E(\hat{y}))^{1/p} - c - cr \\
&\overset{(20)}{=} c_1|\Omega|\tau^{-s} - c(c_1|\Omega| + \mu|\Omega|)^{1/p} - c - cr.
\end{aligned}$$

Here, we have also used Jensen's inequality in the third inequality and $\sup y_3^* > 0$ in the fourth inequality. In the latter computation and up to the end of this section, the generic constant c is always independent of c_1 and r as well. We have checked that

$$c_1 + \mu \geq c_1\tau^{-s} - cc_1^{1/p} - c - cr. \tag{23}$$

Since $\tau < 1$, given any $r > 0$ the latter does not hold in case $c_1 > 0$ is sufficiently large. This leads to a contradiction, proving (21).

In conclusion, we have checked that, for all $\rho_s < \rho_f$ and all $r > 0$, there exists a $c_1 > 0$ such that a minimizer of E in A is a local minimizer of E (in a L^∞ ball of radius r).

Note that, given any c_1, the argument of (23) fails for r large enough. This corresponds to the former observation that the energy is de facto unbounded from below. On the other hand, for small values of c_1, (23) does not allow to conclude for the existence of r such that a local minimizer in the L^∞ ball of radius r exists. In fact, in the limiting case $c_1 = 0$ and $W = 0$, one can check that the energy E has not even local minimizers.

The discussion on the buoyancy of a submarine from Section 5, see Figure 2, can be extended to the case where the air-water balance of the ballast tanks is compressible (but, for simplicity, the hull of the submarine is not). In this case, we could specify the energy from (18) by following (16), namely,

$$\begin{aligned}
E(y) = &\int_{\Omega_h} W(\nabla y(X))\,\mathrm{d}X + c_1\int_{\Omega_b} (J(X))^{-s}\mathrm{d}X + \int_{\Omega^y} g\rho_f x_3^-\,\mathrm{d}x \\
&+ \int_{\Omega_h} g\rho_h y_3(X)\,\mathrm{d}X + \int_{\Omega_b} g\rho_b y_3(X)\,\mathrm{d}X.
\end{aligned}$$

The argument above can be adjusted to the case of the latter energy. In case c_1 and $|\Omega_b|$ are sufficiently large, one can find neutrally floating solids locally minimizing the energy.

7. **Solids at anchor.** Independently of compressibility, the coercivity of the energy in L^∞ (and hence in the weak topology of $W^{1,p}$) can be obtained by prescribing some form of *anchoring* of the solid. A classical choice in this sense consists in assuming a prescribed deformation $y = y_D$ at some point in Ω (or on some portion ω of Ω or some portion Γ of $\partial\Omega$), see Figure 3 left. In this case, one can compute

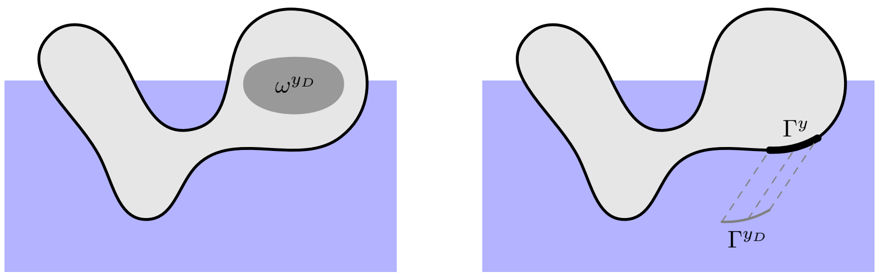

FIGURE 3. Two anchored situations: prescribed deformation on $\omega \subset \Omega$ (left) and elastic boundary conditions on $\Gamma \subset \partial\Omega$ (right).

$$\begin{aligned}|\Omega|/c_0 + E(y) &\geq c_0\|\nabla y\|^p_{L^p(\Omega)} + \int_{\Omega^y} g\rho_f x_3^- \,\mathrm{d}x + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X \\ &\geq c_0\|\nabla y\|^p_{L^p(\Omega)} - g\rho_s(\mathrm{diam}(\Omega^y) + |y_D|) \\ &\overset{(10)}{\geq} c_0\|\nabla y\|^p_{L^p(\Omega)} - c\|\nabla y\|_{L^p(\Omega)} - c \geq \frac{c_0}{2}\|\nabla y\|^p_{L^p(\Omega)} - c'. \end{aligned} \tag{24}$$

In particular, this entails that

$$\|y_3\|_{L^\infty(\Omega)} \leq \mathrm{diam}(\Omega^y) + |y_D| \overset{(10)}{\leq} c\|\nabla y\|_{L^p(\Omega)} + |y_D| \overset{(24)}{\leq} c(E(y))^{1/p} + c' + |y_D|$$

so that the energy is coercive in L^∞. However, one has to mention that prescribing some specific deformation at some point or portion of Ω or $\partial\Omega$ could be practically not realizable, especially in three space dimensions, see again Figure 3 left.

An alternative choice could be that of considering so-called *elastic* boundary conditions, again to be imposed on a portion of $\partial\Omega$ (or, alternatively, on a portion of Ω). In order to give an example in this direction, we fix $\Gamma \subset \partial\Omega$ open in the relative topology of $\partial\Omega$ with $\mathcal{H}^2(\Gamma) > 0$, let $y_D\colon \Gamma \to \mathbb{R}^3$ be continuous, and augment the energy by the term

$$c_3 \int_\Gamma |y(X) - y_D(X)|^r \,\mathrm{d}\mathcal{H}^2(X). \tag{25}$$

Here, $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure on Γ, $c_3 > 0$, and $r > 1$. See Figure 3 right. By taking into account the Poincaré inequality (see, e.g., [13, Lemma 3.3]), we obtain

$$\|y\|_{W^{1,p}(\Omega)} \leq c\|\nabla y\|_{L^p(\Omega)} + c\|y - \mathrm{id}\|_{L^1(\Gamma)} \quad \forall\, y \in W^{1,p}(\Omega;\mathbb{R}^3). \tag{26}$$

We can adapt the argument in (24) to this situation: first, we get

$$|\Omega|/c_0 + E(y) \geq c_0\|\nabla y\|^p_{L^p(\Omega)} + c_3\|y - y_D\|^r_{L^r(\Gamma)} + \int_{\Omega^y} g\rho_f x_3^- \,\mathrm{d}x + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X$$
$$\geq c_0\|\nabla y\|^p_{L^p(\Omega)} + c_3\|y - \mathrm{id}\|^r_{L^r(\Gamma)} - c'' - g\rho_s\|y\|_{L^\infty(\Omega)}.$$

Then, due to (26) and the embedding $W^{1,p} \subset L^\infty$, we obtain

$$|\Omega|/c_0 + E(y) \geq c\|y\|^{\min\{p,r\}}_{W^{1,p}(\Omega)} - c'\|y\|_{W^{1,p}(\Omega)} - c''$$
$$\geq c\|y\|^{\min\{p,r\}}_{W^{1,p}(\Omega)} - c, \tag{27}$$

where in the last step we used that $cz^q - c'z \geq -c_*$ for all $z \in (0,\infty)$, provided $q > 1$ and c_* is sufficiently large. Once again, the energy is coercive in L^∞.

A further sophistication could be that of assuming that the elastic response of the boundary condition is inactive before a given critical elongation $|y - y_D| = \lambda > 0$ is reached. This would indeed correspond to the case of a buoy or a floating vessel anchored via an elastic cable of length λ at rest. In this case, the boundary condition term would be modified as

$$c_3 \int_\Gamma \max\{0, |y(X) - y_D(X)|^r - \lambda^r\}\,\mathrm{d}\mathcal{H}^2(X).$$

The argument in (27) can be adapted to this case by simply replacing the term $c_3\|y - y_D\|^r_{L^r(\Gamma)}$ by $c_3\|y - y_D\|^r_{L^r(\Gamma)} - c_3\mathcal{H}^2(\Gamma)\lambda^r$.

Eventually, one could consider the case of an inextensible anchoring of length $\lambda > 0$. This would be modeled by imposing the constraint

$$\|y - y_D\|_{L^\infty} \leq \lambda$$

which would directly entail coercivity, simply by replacing $|y_D|$ by $|y_D| + \lambda$ in the chain of inequalities (24).

8. **Bounded reservoir.** In this section we discuss the case of a compressible solid floating in a *bounded* fluid reservoir. We assume the container to be large enough so that the solid does not touch its walls, see Figure 4. For the sake of definiteness, we

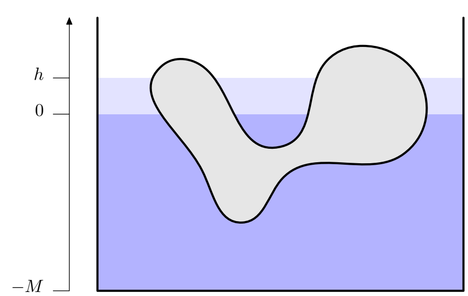

FIGURE 4. The bounded-reservoir setting.

assume the reservoir to be of cylindrical shape $S \times [-M, \infty)$ with $S \subset \mathbb{R}^2$ compact and $M > 0$ sufficiently large. The issue is here that the immersed portion of the solid lets the water level rise to some level h, so that the energy E from (1) has to be modified as

$$E(y,h) = \int_\Omega W(\nabla y(X))\,\mathrm{d}X + \int_{\Omega^y} g\rho_f(x_3 - h)^- \,\mathrm{d}x + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X. \tag{28}$$

Note that the energy now depends on h as well, and this is a priori unknown.

In case the solid is free-floating, see Sections 5-6, the treatment of the extra variable h is straightforward: letting y_* be a (either local or global) minimizer of the original energy from (1), one has that $y := y_* + he_3$ minimizes E from (28), where h is a posteriori determined by solving

$$M\mathcal{H}^2(S) + |\Omega^y \cap \{x_3 \le h\}| = (M+h)\mathcal{H}^2(S). \tag{29}$$

This equation is nothing but the expression of the conservation of fluid content. On the left is the sum of the original and the displaced fluid volume. On the right is the volume of the portion of the reservoir under water level.

The problem is more involved if the solid is anchored, see Section 7. In this case, the value of h cannot be computed a posteriori and one has to minimize E from (28) directly, under the additional constraint (29). This is however possible, as we now check. Assume for definiteness that the solid is clamped, as in the left of Figure 3 (other cases, including (25), can be treated as well). Let (y_k, h_k) be an infimizing sequence for (28) under the constraint (29). The values h_k are surely bounded from below by 0, for the water level can only raise as effect of the immersed solid. In a similar fashion, the values h_k are bounded from above by $|\Omega|\overline{J}^{y_k}/\mathcal{H}^2(S)$, which in turn is bounded by the energy itself. Up to subsequences (not relabeled), we hence have that $y_k \rightharpoonup y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $h_k \to h$ in $\mathbb{R}$. In particular, this entails that $g\rho_f(x_3-h_k)^- \to g\rho_f(x_3-h)^-$ uniformly and $|\Omega^{y_k} \triangle \Omega^y| \to 0$ [8, Lemma 5.2], where $\triangle$ denotes the symmetric difference of sets. The latter implies that $\chi_{\Omega^{y_k}} \to \chi_{\Omega^y}$ strongly in L^1 where $\chi_E(x)$ is the characteristic function of the measurable set $E \subset \mathbb{R}^3$. We can hence pass to the limit in the hydrostatic terms and obtain

$$\begin{aligned}\int_{\Omega^{y_k}} g\rho_f(x_3-h_k)^-\,\mathrm{d}x &= \int_{\mathbb{R}^3} g\rho_f(x_3-h_k)^-\,\chi_{\Omega^{y_k}}(x)\,\mathrm{d}x\\ \to \int_{\mathbb{R}^3} g\rho_f(x_3-h)^-\,\chi_{\Omega^y}(x)\,\mathrm{d}x &= \int_{\Omega^y} g\rho_f(x_3-h)^-\,\mathrm{d}x.\end{aligned}$$

This convergence entails the lower semicontinuity of the energy E from (28). In addition, $|\Omega^{y_k} \triangle \Omega^y| \to 0$ and $h_k \to h$ entail that

$$|\Omega^{y_k} \cap \{x_3 \le h_k\}| \to |\Omega^y \cap \{x_3 \le h\}|.$$

One can hence pass to the limit in equation (29), written for y_k, in order to check that the limiting pair (y,h) fulfills (29) as well.

9. **The ship problem.** Let us now go back to the case of an infinitely extended fluid reservoir. All discussions of the previous sections have been based on the assumption that all subsets of $\{x_3 \le 0\} \setminus \Omega^y$ are actually filled with fluid, see Figure 1. This is nonrestrictive in case Ω^y is convex (a stable property with respect to weak $W^{1,p}$ convergence of deformations). Still, the case of a nonconvex Ω^y is of major applicative relevance, for it corresponds to the idealized situation of a floating vessel, see Figure 5. In this case, the set $\{x_3 \le 0\}\setminus\Omega^y$ may have different connected components, one of which is unbounded (since Ω^y is necessarily bounded). We shall use the notation

$$D^y := \text{union of all bounded connected components of } \{x_3 \le 0\} \setminus \overline{\Omega^y}.$$

The former choice in (1) corresponds to the assumption that D^y is filled by the fluid. On the other hand, one could assume that some of the connected components of D^y do not contain fluid, or are partially filled with fluid. Among all options, we

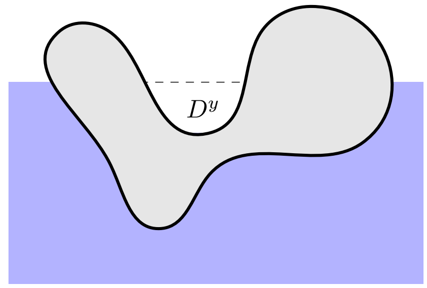

FIGURE 5. The ship setting.

consider here the case in which D^y contains no fluid at all, see Figure 5. In this setting, the energy E is redefined as

$$E(y) = \int_\Omega W(\nabla y(X))\,\mathrm{d}X + \int_{\Omega^y \cup D^y} g\rho_f x_3^-\,\mathrm{d}x + \int_\Omega g\rho_s y_3(X)\,\mathrm{d}X. \qquad (30)$$

In particular, the set D^y contributes to the hydrostatic term of the energy, for the fluid is displaced out of D^y, although D^y is not a subset of Ω^y. For a justification in terms of the Euler-Lagrange equations we refer to Section 2, where now (5) holds with $\partial(\Omega^y \cup D^y)$ in place of $\partial\Omega^y$.

The effect of keeping track of the set D^y in the integral of the hydrostatic term allows the solid to float, even for $\rho_s > \rho_f$, as it commonly happens for usual vessels. From now on, we restrict our considerations to the case $\rho_s > \rho_f$ and suppose that the solid is incompressible, i.e., $J^y = 1$ almost everywhere. Note that then the energy is again unbounded from below since the energy decreases as the solid sinks, as soon as $\sup y_3 < 0$ (and hence $D^y = \emptyset$). This is indeed the mechanism responsible for all shipwrecks.

As no global minimizers can be expected to exist, we address the existence of floating solids locally minimizing the energy E (30). Local minimizers can only be expected if $|D^y|$ is sufficiently large. Therefore, we impose

$$|D^y| \geq \eta \quad \text{for} \quad \eta = \frac{\rho_s - \rho_f}{\rho_f}|\Omega|. \qquad (31)$$

The specific choice of η is indeed tailored to let $|D^y| = \eta$ exactly correspond to the case of barely floating solids. In fact, one can extend the discussion leading to (14) to the specific case of energy E from (30) in order to derive that a critical point of E fulfills

$$g\rho_f|(\Omega^y \cap \{x_3 \leq 0\}) \cup D^y| = g\rho_f|\Omega^y \cap \{x_3 \leq 0\}| + g\rho_f|D^y| = g\rho_s|\Omega|.$$

In case Ω^y is barely floating, since $|\Omega^y| = |\Omega|$, one has that

$$\rho_f|\Omega| + \rho_f|D^y| = \rho_s|\Omega|$$

and therefore $|D^y| = \eta$. On the other hand, if $|D^y| = \eta$, we get

$$|\Omega^y \cap \{x_3 \leq 0\}| = (\rho_s/\rho_f)|\Omega| - \eta = |\Omega| = |\Omega^y|,$$

which implies that Ω^y is barely floating. We have hence proved that in case of $|D^y| > \eta$ the solid necessarily floats.

As in the previous sections, our goal is to show the existence of (local) minimizers by the direct method. The presence of the set D^y in (30), however, is posing lower semicontinuity problems: let Ω^y be the barely floating deformation depicted on the left of Figure 6 and consider the sequence $y_k = y - e_3/k$. As $|D^y| > 0$ but $|D^{y_k}| = 0$, one has that

$$\liminf_{k\to\infty} \int_{\Omega^{y_k}\cup D^{y_k}} g\rho_f x_3^- \,\mathrm{d}x = \liminf_{k\to\infty} \int_{\Omega^{y_k}} g\rho_f x_3^- \,\mathrm{d}x$$
$$= \int_{\Omega^y} g\rho_f x_3^- \,\mathrm{d}x < \int_{\Omega^y\cup D^y} g\rho_f x_3^- \,\mathrm{d}x$$

and lower semicontinuity fails. The case of a barely floating solid is, however, of

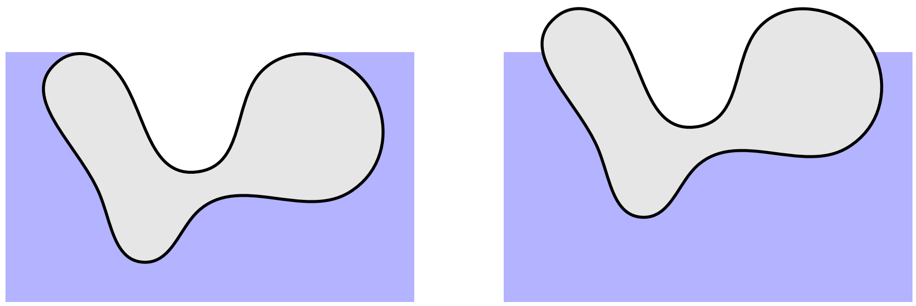

FIGURE 6. A barely floating solid (left) and an admissible $y \in A$ with A from (33) (right).

limited practical interest.

In order to restore lower semicontinuity, we strengthen the requirements by seeking local minimizers such that the map $y \mapsto |D^y|$ is continuous in a specific way, namely we ask y to satisfy for all $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon_0 > 0$ fixed)

$$\overline{y} \in W^{1,p}(\Omega;\mathbb{R}^3): \quad \|y_3 - \overline{y}_3\|_{L^\infty(\Omega)} < \varepsilon \;\Rightarrow\; |D^{\overline{y}} \triangle D^y| \le \varepsilon^{1/2}. \tag{32}$$

We now define the set of admissible deformations as

$$A = \{y \in W^{1,p}(\Omega;\mathbb{R}^3) \;\text{ fulfilling (31) and (32)}\}. \tag{33}$$

Note that A is not empty, for any reference configuration Ω can be deformed into a thin spherical half-shell of arbitrarily given internal radius, possibly at a large elastic energy cost. In this case, (31) and (32) can be indeed verified.

As all solids Ω^y with $y \in A$ are floating or barely floating, i.e., $\sup y_3 \ge 0$, the energy E from (30) is coercive on A by (10)–(11). Moreover, it is elementary to check that A is closed under uniform convergence. In order to check lower semicontinuity, we see that we can pass to the limit in the hydrostatic terms by (32). This yields the existence of a minimizer in A.

Let $y^* \in A$ be a minimizer of E in A. We have that Ω^{y^*} is floating but not barely floating, see Figure 6 right. Indeed, assume Ω^{y^*} to be barely floating. Then, $\overline{y} = y^* - (\varepsilon_0/2)e_3$ would have $D^{\overline{y}} = \emptyset$ and $|D^{\overline{y}} \triangle D^{y^*}| = |D^{y^*}| \ge \eta$ would entail that y^* does not satisfy (32), provided ε_0 is small with respect to η. This is a contradiction, and we have hence proved that $|D^{y^*}| > \eta$.

Moving from this, one has that y^* is a true local minimizer of E if there exists $\varepsilon_0 > 0$ such that for all y with $\|y_3^* - y_3\|_{L^\infty(\Omega)} < \varepsilon < \varepsilon_0$ one has that $|D^y \triangle D^{y^*}| < \varepsilon^{1/2}$. An example in this direction is in Figure 6 right.

Note that the case of y^* not being a local minimizer, for all $\varepsilon_0 > 0$, is of no real applicative interest. In fact, this would happen if one could find a sequence $\varepsilon_n \to 0$ and deformations y^n such that $\|y_3^* - y_3^n\|_{L^\infty(\Omega)} < \varepsilon_n$ but $|D^{y^n} \triangle D^{y^*}| \ge \varepsilon_n^{1/2}$. We see, in this case, that small changes in the deformation cause large variations in the volume of the symmetric difference and this means that Ω^{y^*} is either barely floating, see Figure 6 left, or that a part of the ship is barely floating, i.e., the picture applies only to an open subset $\omega^{y^*} \subset \Omega^{y^*}$ for which $\sup_{\omega^{y^*}} y_3^* = 0$ although $\sup_{\Omega^{y^*}} y_3^* > 0$, see Figure 7.

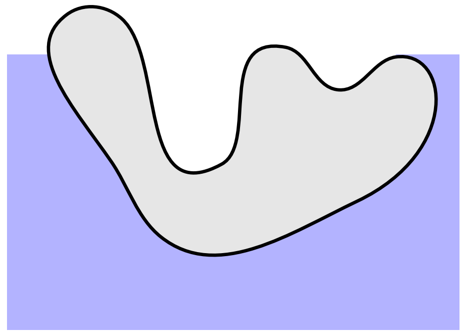

FIGURE 7. A deformation with $\omega^{y^*} \subset \Omega^{y^*}$ with $\sup_{\omega^{y^*}} y_3^* = 0$ and $\sup_{\Omega^{y^*}} y_3^* > 0$

Let us conclude by explicitly remarking that we are not in the position of establishing a priori if y^* is indeed a local minimizer (for some $\varepsilon_0 > 0$) or not. Indeed, this depends on the data of the problem, most notably on Ω and W. A positive example would be given by a spherical half-shell Ω with sufficiently large radius under an energy density W very much penalizing deformations far from identity. On the other hand, the same W in case of Ω being a ball can be expected to provide a negative example.

10. **Outlook.** We conclude our discussion by mentioning some possible future developments. A number of interesting extensions are indeed within easy reach. One could for instance consider the case of non homogeneous densities $\rho_s : \Omega \to (0,\infty)$ and $\rho_\ell : \mathbb{R}^3 \to (0,\infty)$ (this second one possibly depending on x_3 only). Also the case of a nonhomogeneous gravity $g : \mathbb{R}^3 \to (0\infty)$ could be easily handled.

For a porous solid, the analysis can be readily extended to the capillarity case, if the height of the wet part of the body over seapage is a priori fixed. This can be expected to be quite common for relatively large systems under moderate deformations (a wooden raft, for instance). For smaller systems or large strains (a soft sponge), capillarity depends on the deformation, adding a challenging coupling effect to the picture. The interplay between capillarity and anisotropy may also enter the picture, making the analysis even more involved.

Still in the stationary regime (evolution is out of the scope of the paper), one could consider the fluid to be viscous, incompressible, and in stationary motion. This amounts to couple the minimization of the energy with the stationary Navier-Stokes or with the Stokes system. In this case, the hydrostatic pressure term has to be augmented by the normal component of the normal stress. To the best of our knowledge, an existence result in this direction is still missing. The reader is however referred to [2], where the dynamic setting is tackled.

The equilibrium problem for the immersed body can be combined with other physical effects, in a multiphysics setting. Thermal and electromagnetic effects can be taken into the picture, for instance. A particularly relevant prospect would be to allow for contact with other hyperleastic bodies or with the bottom of the reservoir. Here, one would be asked to limit the set of admissible configurations to those avoiding material interpenetration, an option which should be amenable, possibly under stricter conditions [21].

Another relevant issue is the stability of local minimizers (imagine a loaded ship at see). This issue is classical in the case of a rigid convex body [9, 15, 24] and, to our knowledge, completely open in the case of a deformable body. Note once again that our analysis of the ship situation from Section 9 left open the question on how to check local minimality of the configuration y^*, given information on Ω and W.

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