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REGULAR ARTICLE

Testing symmetry around a subspace

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Abstract

The article shows how some common measures of association between two random vectors may be used to test multivariate symmetry around a subspace (possibly up to a shift), which also permits testing exchangeability, axial symmetry, halfspace symmetry, and certain goodness-of-fit and equality-of-scale hypotheses. The resulting (parametric, nonparametric, permutation, and asymptotic) tests of the symmetry, consistent in the class of all elliptical distributions, are also illustrated with a few simulation and real data examples.

Keywords Axial symmetry · Halfspace symmetry · Exchangeability · Canonical correlation

Mathematics Subject Classification 62H15 · 62H20

1 Introduction

Various concepts of multivariate symmetry (Serfling 2006) are indispensable for multivariate statistics. The symmetry around a subspace up to a shift has been rather neglected so far, which is why the topic is discussed here in a general context. Recall that a random vector $\boldsymbol{Y}$ is symmetric around a subspace generated by basic vectors $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_q\}$ up to a shift when $\mathcal{L}\{\boldsymbol{Y} - \mathsf{E}\boldsymbol{Y}\} = \mathcal{L}\{\mathbb{R}(\boldsymbol{Y} - \mathsf{E}\boldsymbol{Y})\}$ for the reflection matrix $\mathbb{R} = 2\boldsymbol{u}_1\boldsymbol{u}_1' + \cdots + 2\boldsymbol{u}_q\boldsymbol{u}_q' - \mathbb{I}$, which is the rotational (orthonormal) matrix satisfying $\mathbb{R}\boldsymbol{u}_i = \boldsymbol{u}_i$, $i = 1, \ldots, q$, and $\mathbb{R}\boldsymbol{v} = -\boldsymbol{v}$ for any vector $\boldsymbol{v}$ orthogonal to all vectors $\boldsymbol{u}_i$, $i = 1, \ldots, q$.

It seems that the corpus of published literature lacks such general tests although it contains a kernel-based nonparametric test of multivariate conditional symmetry (Su

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2006) and various tests of axial symmetry, hyperplane symmetry, or exchangeability in the bivariate case; see, e.g., Hollander (1971), Modarres (2008), Rao and Raghunath (2012), Krupskii (2017) and references therein for some nonparametric examples. One should perhaps mention as related even the tests for the first eigenvector of the covariance matrix known from the principal component analysis; see, e.g., Hallin et al. (2010b) and the references given there.

In the broad multidimensional context considered here, there already exists a test of symmetry around a line in a given direction by means of quantile regression (Hudecová and Šiman 2019), inspired by the theory of directional regression quantiles (Hallin et al. 2010a). There also exist some permutation tests of the symmetry around a subspace up to a shift that are based on various scatter matrices (Kalina 2019). Such matrices must have certain elements equal to zero for distributions symmetric around subspaces generated by axial directions under very mild equivariance conditions, see Lemma 2.4 of Dumbgen et al. (2015) or Kalina (2019), which is used in the latter manuscript for testing various hypotheses of symmetry by means of nonparametric combinations of permutation tests regarding the individual scatter matrix coefficients.

This article focuses on the sample covariance matrix and on its use for testing the null hypothesis of symmetry around a subspace (possibly up to a shift) . It also elaborates on the use of (rank) Kendall correlations for such purpose when the parametric tests are not adequate, i.e., when some outliers may be present or when the moment assumptions cannot be relied on. Unlike Hudecová and Šiman (2019), it proposes axial symmetry tests with simple asymptotic distributions that are available in any dimension and include tests with no moment assumptions. Unlike Kalina (2019), it also provides asymptotic and naturally invariant tests of the null hypothesis. It transforms the problem to the testing of association between two vectors which is well known from the literature and addressed there in several different ways, e.g., by means of canonical correlation analysis. The link thus gives rise to various (parametric, nonparametric, permutation, and asymptotic) tests of the null hypothesis that are valid under relatively mild conditions and consistent in the class of all elliptical distributions.

The tests may become useful even in the most general form because such symmetries probably appear in optics, acoustics, astronomy, crystallography or whenever reflections, rotations or mirrors are employed. In particular, such symmetries in the molecular world influence chemical properties of the matter.

In any way, the general testing framework considered here includes many common special cases such as testing symmetry about a particular coordinate axis, axial symmetry, halfspace symmetry, exchangeability, and equality of distributions or their scales. These applications alone justify the tests very well. They are further elaborated and illustrated in the text.

The rest of this article is organized as follows. Theoretical Sect. 2 transforms the testing of symmetry around a subspace up to a shift to the testing of association between two vectors, briefly mentions some tests that are available for the latter problem, elaborates on the use of Kendall's correlations for such purpose, and discusses the case when the shift is known. Empirical Sect. 3 then illustrates the theory with highly representative examples based on simulation and real data and compares the proposed tests to their competitors. Concluding Sect. 4 collects miscellaneous final remarks and comments.

2 Theory

Consider a random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_m)' \in \mathbb{R}^m$ with non-degenerate continuous distribution $\mathcal{L}(\boldsymbol{Y})$. Furthermore, consider the subspace $\mathcal{S}_q$ generated by the column vectors of $\boldsymbol{\Gamma}_A = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_q\} \in \mathbb{R}_{m\times q}$ where $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma}_A|\boldsymbol{\Gamma}_B) \in \mathbb{R}^{m\times m}$ is a regular matrix of column basic vectors.

The null hypothesis of interest is

$$H_0^A : \mathcal{L}(\boldsymbol{Y}) \text{ is symmetric around } \mathcal{S}_q \text{ after a suitable shift } \boldsymbol{s} \in \mathbb{R}^m,$$

which means that $\mathcal{L}((\boldsymbol{\Gamma}'_A\tilde{\boldsymbol{Y}}, \boldsymbol{\Gamma}'_B\tilde{\boldsymbol{Y}})') = \mathcal{L}((\boldsymbol{\Gamma}'_A\tilde{\boldsymbol{Y}}, -\boldsymbol{\Gamma}'_B\tilde{\boldsymbol{Y}})')$ for $\tilde{\boldsymbol{Y}} = \boldsymbol{Y} - \boldsymbol{s}$. If the covariance matrix $\mathsf{var}(\boldsymbol{Y})$ is finite, then H_0^A implies

$$H_0^P : \mathsf{cov}(\boldsymbol{\Gamma}'_A\boldsymbol{Y}, \boldsymbol{\Gamma}'_B\boldsymbol{Y}) = \mathbf{O}$$

that is checked by the parametric tests presented below. Of course, desirable tests should not depend on the particular choice of $\boldsymbol{\Gamma}_A$ or $\boldsymbol{\Gamma}_B$.

If the class of possible distributions $\mathcal{L}(\boldsymbol{Y})$ is fully described by a location parameter and the covariance matrix, then H_0^A is equivalent to H_0^P and the tests of H_0^P consistently test H_0^A. This is the case of continuous elliptical distributions. That is to say that $\boldsymbol{Y}$ is elliptically distributed if and only if $\boldsymbol{Z} = (\boldsymbol{\Gamma}'_A\boldsymbol{Y}, \boldsymbol{\Gamma}'_B\boldsymbol{Y})'$ is elliptically distributed, which means that the density $f(z)$ of $\boldsymbol{Z}$ can be uniquely described by median vector $\boldsymbol{\mu}$, symmetric positive definite matrix $\boldsymbol{\Sigma}$, and positive function g:

$$f(z) = g\big((z - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(z - \boldsymbol{\mu})\big).$$

where $\boldsymbol{\Sigma} = (\rho_{ij})_{i,j=1}^{m,m}$ is proportionate to the covariance matrix of $\boldsymbol{Z}$ if it exists. It makes sense to consider only scale invariant tests. Therefore, $\boldsymbol{\Sigma}$ is assumed to have unit diagonal without any loss of generality.

In any case, H_0^A is equivalent to

$$H_0^E : \Sigma_{AB} \equiv (\rho_{i,j})_{i=1,j=q+1}^{q,m} = \mathbf{O}$$

in the class of elliptical distributions, and even for meta-elliptical distributions (Abdous et al. 2005; Fang et al. 2002). The nonparametric tests presented below, based on Kendall correlations, check H_0^E in these classes of distributions without requiring any moment assumptions.

Recall that H_0^A includes some special cases. Those related to axial symmetry are also mentioned in Hudecová and Šiman (2019).

The test of halfspace symmetry (after a suitable shift) corresponds to $q = m-1$. The test of axial symmetry (after a suitable shift) results from $q = 1$ and includes the test of exchangeability (after a suitable shift) as a special case with $\boldsymbol{\Gamma}_A = \{(1, \ldots, 1)'/\sqrt{m}\}$. That is to say that a multivariate distribution is exchangeable if and only if it is symmetric around the axis of the first orthant. Such a test may be useful even for testing whether m independent univariate distributions are the same up to their location, because only then their joint distribution is exchangeable after a suitable shift. If the only difference among the independent univariate distributions may be in their scale and location, then even the equality-of-scale hypothesis may be tested by means of the exchangeability test.

Furthermore, if one needs to test the equality (after a suitable shift) of k d-dimensional equally-sized samples for some integers k and d, then the null hypothesis implies the exchangeability of all the k-dimensional vectors consisting of all the ith components of the individual samples, $i = 1, \ldots, d$, and it is this compound hypothesis that can be tested.

If the suitable shift is known, then one can proceed as in Sect. 2.4.

Before the general testing framework based on canonical correlations is presented in full detail in Sect. 2.2, it seems reasonable to briefly inspect some special cases and further possibilities.

2.1 Special cases and possibilities

If $m = 2$, then $q = 1$ corresponds to the symmetry about a line in the only direction $\boldsymbol{u}_1$ in $\boldsymbol{\Gamma}_A$. One can use any scale invariant test statistic of the hypothesis that the ordinary correlation coefficient between $\boldsymbol{\Gamma}'_A\boldsymbol{Y}$ and $\boldsymbol{\Gamma}'_B\boldsymbol{Y}$ is zero.

If $m > 2$ and $q = 1$ or $q = m - 1$ (i.e., in the case of axial symmetry or halfspace symmetry), then one can use any scale invariant test statistic of the hypothesis that the multiple correlation coefficient is zero; see, e.g., Croux and Dehon (2003).

If $m > 2$ and $q \in \{2, \ldots, m-2\}$, then the symmetry test can be conducted by using scale invariant test statistics of the hypothesis that the cross-correlation matrix between $\boldsymbol{\Gamma}'_A\boldsymbol{Y}$ and $\boldsymbol{\Gamma}'_B\boldsymbol{Y}$ is zero, e.g., those from (Rencher and Christensen 2012, Sect. 10.7) or from the canonical correlation analysis (Rencher and Christensen 2012, Chap. 11); see Sect. 2.2 for details.

However, one has to keep in mind that the asymptotic null distribution of all the aforementioned test statistics derived under independence cannot be directly applied here.

2.2 Testing procedures based on canonical correlations

Write $\boldsymbol{Z}_A = \boldsymbol{\Gamma}'_A\boldsymbol{Y} \in \mathbb{R}^q$, $\boldsymbol{Z}_B = \boldsymbol{\Gamma}'_B\boldsymbol{Y} \in \mathbb{R}^{m-q}$, and $\boldsymbol{Z} = (\boldsymbol{Z}'_A, \boldsymbol{Z}'_B)'$. Although there are plenty of tools available for testing whether the (conditional) cross-correlation matrix between two random vectors $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_B$ is zero, most of them assume independence of $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_B$ or even joint normality of $\boldsymbol{Z}$, which rather limits their usefulness in the setup considered here. Of course, the corresponding test statistics can still be used in permutation tests, or their distributions can be approximated by means of subsampling. However, these are computationally intensive procedures working satisfactorily only for not too large random samples. Fortunately, the literature also contains a substantial body of work clarifying the behavior of the most popular tests when the original assumptions of normality or independence are violated. Some of the results are summarized below.

Assume $q \leq m - q$ and write $\mathbf{V}_{AA} = \mathsf{var}(\mathbf{Z}_A)$, $\mathbf{V}_{BB} = \mathsf{var}(\mathbf{Z}_B)$, and $\mathbf{V}_{AB} = \mathsf{cov}(\mathbf{Z}_A, \mathbf{Z}_B)$. Popular tests of the cross-correlation matrix $\mathbf{V}_{AB}$ are based on the eigenvalues $r_1^2, \ldots, r_q^2$ of the matrix $\mathbf{V}_{AA}^{-1}\mathbf{V}_{AB}\mathbf{V}_{BB}^{-1}\mathbf{V}'_{AB}$, the so-called squared canonical correlations.

In particular, the Wilks test, the Hotelling-Lawley test, and the Pillai–Bartlett test are respectively based on the following characteristics:

$$T_W = -c_W \sum_{i=1}^{q} \log(1 - \widehat{r}_i^2), \quad T_H = c_H \sum_{i=1}^{q} \frac{\widehat{r}_i^2}{1 - \widehat{r}_i^2}, \quad \text{and} \quad T_P = c_P \sum_{i=1}^{q} \widehat{r}_i^2 \tag{1}$$

or their transforms (Rencher and Christensen 2012) where $\widehat{r}_1^2, \ldots, \widehat{r}_q^2$ are squared sample canonical correlations computed from the sample covariance matrices $\mathbf{S}_{AA}$, $\mathbf{S}_{BB}$, and $\mathbf{S}_{AB}$. Note that all the three tests inherit favorable invariance properties from the squared sample canonical correlations, namely the shift invariance, the scale invariance, and the invariance with respect to the particular bases $\boldsymbol{\Gamma}_A$ and $\boldsymbol{\Gamma}_B$. In other words, it is only the subspaces generated by the bases that really matter.

If $\boldsymbol{Y}$ has a multivariate normal distribution, then each of the test statistics in (1) is known to have the asymptotic null distribution equal to the χ^2 distribution with $q(m-q)$ degrees of freedom for c_W, c_T, and c_P equal to the number of observations n, and some F-approximations are then also available (Rencher and Christensen 2012). Muirhead and Waternaux (1980) showed that this approximation still holds for T_W in certain special cases such as if $\boldsymbol{Z}$ has finite fourth moments and independent marginals. Yuan and Bentler (2000) proved that the results derived under the classical normality assumption also extend to other special settings such as to $\boldsymbol{Z}$ with a pseudo-normal distribution. If $\mathbf{Z}$ comes from an elliptical distribution with excess kurtosis 3κ, then the same asymptotic χ^2 distribution of T_W remains valid for $c_W = n/(1+\kappa)$ (Muirhead and Waternaux 1980). The same result applies also to pseudo-elliptical distributions (Yuan and Bentler 2000), and it can be used for testing with a consistent estimator for κ. Yanagihara et al. (2005) also presented some approximations for the mean and variance of T_W with $c_W = n$ in general situations and provided some results regarding the behavior of such T_W under some kinds of alternatives. Seo et al. (1995) suggested formulas for c_W, c_H, and c_P that improve the χ^2 approximation of corresponding test statistics in finite samples if $\boldsymbol{Z}$ has its sixth moments finite. In particular, if $\boldsymbol{Z}$ has an elliptical distribution with finite sixth moments and with characteristic function $\phi(\mathbf{t}) = \exp(i\mathbf{t}'\boldsymbol{\mu})\psi(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})$, then

$$c_W = \frac{n+1}{1+\kappa} - \frac{1}{2}(3m+11) + \frac{(m+4)(1+\varphi)-1}{(1+\kappa)^2}, \tag{2}$$

$$c_H = \frac{n+1}{1+\kappa} - 2(m+3) + \frac{(m+4)(1+\varphi)-1}{(1+\kappa)^2}, \tag{3}$$

$$c_P = \frac{n+1}{1+\kappa} - (m+5) + \frac{(m+4)(1+\varphi)-1}{(1+\kappa)^2}, \tag{4}$$

for (kurtosis parameter) $\kappa = \left(\frac{\psi^{(2)}(0)}{\psi'(0)}\right)^2 - 1$ and $\varphi = \left(\frac{\psi^{(3)}(0)}{\psi'(0)}\right)^3 - 1$. For a generally distributed $\boldsymbol{Y}$ with finite fourth moments, T_W converges in distribution to a weighted sum of $q(m-q)$ independent χ^2_1 random variables with non-trivial weights that are known functions of the fourth order cumulants of $\boldsymbol{Z}$ (Muirhead and Waternaux 1980).

2.3 Tests based on Kendall correlations

Sample covariances are notorious for being sensitive to outliers, and they are consistent only if the second moments of the underlying distribution are finite. This is why rank methods are so attractive: they are quite robust to outliers and they generally do not need any moment assumption to work. This section, therefore, proposes some ways how to use ranks for testing H_0^A by means of the Kendall rank correlation τ that seems especially promising in this regard because

$$\tau = \frac{2}{\pi} \arcsin(\varrho) \tag{5}$$

for any two bivariate marginals of continuous elliptical or meta-elliptical distributions with the corresponding off-diagonal element of $\boldsymbol{\Sigma}$ equal, say, to ϱ (and to their ordinary correlation if it exists); see Abdous et al. (2005), Fang et al. (2002), and Lindskog et al. (2001). In these classes of distributions, the Kendall τ can thus be used for consistent testing of H_0^A.

Recall that vec denotes the matrix operator stacking matrix columns to a single column vector. Assume that $\boldsymbol{Z} = (\boldsymbol{Z}_A', \boldsymbol{Z}_B')' = (Z^1, \ldots, Z^m)' \in \mathbb{R}^m$, composed of $\boldsymbol{Z}_A \in \mathbb{R}^q$ and $\boldsymbol{Z}_B \in \mathbb{R}^{m-q}$, has a continuous elliptical or meta-elliptical distribution with matrix parameter $\boldsymbol{\Sigma}$, and consider a random sample $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$ from the distribution of $\boldsymbol{Z}$. The Kendall correlation τ_{ij} between Z^i and Z^j, $1 \le i \le q < j \le m$, can then be consistently estimated with $\widehat{\tau}_{ij}$ as follows:

$$\widehat{\tau}_{ij} = \frac{1}{\binom{n}{2}} \sum_{1 \le l < k \le n} \operatorname{sgn}\left\{\left(Z_k^i - Z_l^i\right)\left(Z_k^j - Z_l^j\right)\right\}.$$

Consider also the population and sample Kendall correlation matrices between $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_B$, namely $\mathbf{T} = (\tau_{ij})_{i=1,j=q+1}^{q,m}$ and $\widehat{\mathbf{T}}_n = (\widehat{\tau}_{ij})_{i=1,j=q+1}^{q,m}$, and their vectorized forms $\mathbf{t} = \operatorname{vec}(\boldsymbol{T})$ and $\widehat{\mathbf{t}}_n = \operatorname{vec}(\widehat{\boldsymbol{T}}_n)$. Then $\sqrt{n}(\widehat{\mathbf{t}}_n - \mathbf{t})$ converges in distribution to a zero-mean normal distribution with covariance matrix $\mathbf{U} = (u_{ij,kl})_{i \le q < j, k \le q < l}$,

$$\begin{aligned} u_{ij,kl} &= 4(\tau_{ij,kl} - \tau_{ij}\tau_{kl}) \text{ and} \\ \tau_{ij,kl} &= \mathsf{E}\left[\mathsf{E}\operatorname{sgn}\left\{\left(Z_1^i - Z_2^i\right)\left(Z_1^j - Z_2^j\right)\right\} \middle| \mathbf{Z}_1\right] \mathsf{E}\left[\operatorname{sgn}\left\{\left(Z_1^k - Z_2^k\right)\left(Z_1^l - Z_2^l\right)\right\} \middle| \mathbf{Z}_1\right], \end{aligned}$$

see, e.g., Klüppelberg and Kuhn (2009). Furthermore, Klüppelberg and Kuhn (2009) provided a consistent and asymptotically normal estimator of $\mathbf{U}$, namely $\widehat{\mathbf{U}} = (\widehat{u}_{ij,kl})$, where

$$\widehat{u}_{ij,kl} = 4(\widehat{\tau}_{ij,kl} - \widehat{\tau}_{ij}\widehat{\tau}_{kl}) \text{ and}$$

$$\widehat{\tau}_{ij,kl} = \frac{1}{n(n-1)^2} \sum_{s=1}^{n} \left[\sum_{t=1,t\neq s}^{n} \operatorname{sgn}\left\{ \left(Z_s^i - Z_t^i)(Z_s^j - Z_t^j \right) \right\} \right] \times \left[\sum_{t=1,t\neq s}^{n} \operatorname{sgn}\left\{ \left(Z_s^k - Z_t^k)(Z_s^l - Z_t^l \right) \right\} \right].$$

Due to (5), the null hypothesis H_0^E is equivalent to $\mathbf{T}_{AB} = \mathbf{O}$ where $\mathbf{T}_{AB}$ is the corresponding block of $\mathbf{T}$. If it holds, then

$$T_{K1} := n\widehat{\mathbf{t}}_n' \widehat{\mathbf{U}}^{-1} \widehat{\mathbf{t}}_n \to \chi^2_{q(m-q)} \tag{6}$$

in distribution as $n \to \infty$. In other words, T_{K1} has asymptotically χ^2 distribution with $q(m-q)$ degrees of freedom.

Alternatively, consider the matrix $\boldsymbol{\Sigma}$ defining the distribution of $\boldsymbol{Z}$ and its estimator $\widehat{\boldsymbol{\Sigma}}_n = (\widehat{\rho}_{ij}^{\tau})_{i=1,j=1}^{m,m}$ based on Kendall correlations where $\widehat{\rho}_{ij}^{\tau} = 1$ for $i = j$ and

$$\widehat{\rho}_{ij}^{\tau} = \sin\left(\frac{\pi}{2}\widehat{\tau}_{ij}\right), \quad i \neq j.$$

Let $\boldsymbol{\Sigma}_{AB}$ and $\widehat{\boldsymbol{\Sigma}}_{n,AB}$ be the blocks of $\boldsymbol{\Sigma}$ and $\widehat{\boldsymbol{\Sigma}}_n$ corresponding to the correlation matrix between $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_B$. The delta method then implies that vec $\widehat{\boldsymbol{\Sigma}}_{n,AB}$ is an asymptotically normal consistent estimator of vec $\boldsymbol{\Sigma}_{AB}$ whose asymptotic covariance matrix $\boldsymbol{W}$ is provided in Klüppelberg and Kuhn (2009) together with its consistent estimator $\widehat{\mathbf{W}} = (\widehat{w}_{ij,kl})$, where

$$\widehat{w}_{ij,kl} = \pi^2 \cos\left(\frac{\pi}{2}\widehat{\tau}_{ij}\right) \cos\left(\frac{\pi}{2}\widehat{\tau}_{kl}\right) (\widehat{\tau}_{ij,kl} - \widehat{\tau}_{ij}\widehat{\tau}_{kl}).$$

Under H_0^E,

$$T_{K2} := n(\operatorname{vec} \widehat{\boldsymbol{\Sigma}}_{n,AB})' \widehat{\mathbf{W}}^{-1} \operatorname{vec} \widehat{\boldsymbol{\Sigma}}_{n,AB} \to \chi^2_{q(m-q)} \tag{7}$$

in distribution as $n \to \infty$.

Finally, it is also possible to use the test statistic T_W^{τ} analogous to T_W but with the sample canonical correlations $\widehat{r}_i^2$ computed from $\widehat{\boldsymbol{\Sigma}}_n$. If the hypothesis H_0^E holds and $c_W = n$, then $\widehat{T}_W^{\tau}$ has the same asymptotic distribution as $n(\operatorname{vec} \widehat{\boldsymbol{\Sigma}}_{n,AB})' \mathbf{Q}^{-1} \operatorname{vec} \widehat{\boldsymbol{\Sigma}}_{n,AB}$, where $\mathbf{Q}$ is specified in Proposition 1 below. Consequently, T_W^{τ} has asymptotically the same distribution as the weighted sum of independent χ_1^2 variables $\sum_{i=1}^q \lambda_i \chi_{1,i}^2$, where $\lambda_1, \ldots, \lambda_q$ are eigenvalues of $\mathbf{Q}^{-1}\mathbf{W}$. The corresponding critical values can be approximated by taking $\sum_{i=1}^q \widehat{\lambda}_i \chi_{1,i}^2$ where $\widehat{\lambda}_i$, $i = 1, \ldots, q$, are the eigenvalues of $\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{W}}$, and $\widehat{\mathbf{Q}}^{-1}$ is obtained from (8) by replacing $\boldsymbol{\Sigma}$ with $\widehat{\boldsymbol{\Sigma}}_n$.

Proposition 1 *Consider $q \times m$ matrix $\boldsymbol{A}_A = (\mathbf{I}_q, \mathbf{O})$, $m \times (m-q)$ matrix $\boldsymbol{A}_B = (\mathbf{O}, \mathbf{I}_{m-q})'$, and $m \times m$ matrix $\mathbf{J}_{ij} = (J_{kl})$ with all elements zero but $J_{ij} = 1$. Let $\widehat{r}_i^2$, $i = 1, \ldots, q$, be the canonical correlations computed from $\widehat{\boldsymbol{\Sigma}}_n$. Then $T_W^\tau = -n \sum_{i=1}^q \log(1 - \widehat{r}_i^2)$ has the same asymptotic distribution as $n(\mathrm{vec}\, \widehat{\boldsymbol{\Sigma}}_{n,AB})'\mathbf{Q}^{-1}\mathrm{vec}\, \widehat{\boldsymbol{\Sigma}}_{n,AB}$ where*

$$\mathbf{Q} = \frac{1}{2}\left[\boldsymbol{\Sigma}_{AA} \otimes \boldsymbol{\Sigma}_{BB} + \sum_{i=1}^{q}\sum_{j=q+1}^{m} \boldsymbol{A}_B'\boldsymbol{\Sigma}\mathbf{J}_{ij}\boldsymbol{A}_B \otimes \boldsymbol{A}_A\boldsymbol{\Sigma}\mathbf{J}_{ji}\boldsymbol{A}_A' \right. \tag{8}$$

$$+ \sum_{i=1}^{q}\sum_{j=q+1}^{m} \boldsymbol{A}_B'\mathbf{J}_{ji}\boldsymbol{\Sigma}\boldsymbol{A}_B \otimes \boldsymbol{A}_A\mathbf{J}_{ij}\boldsymbol{\Sigma}\boldsymbol{A}_A'$$

$$\left. + \sum_{i,k=1}^{q}\sum_{j,l=q+1}^{m} \boldsymbol{A}_B'\mathbf{J}_{ji}\boldsymbol{\Sigma}\mathbf{J}_{kl}\boldsymbol{A}_B \otimes \boldsymbol{A}_A\mathbf{J}_{ij}\boldsymbol{\Sigma}\mathbf{J}_{lk}\boldsymbol{A}_A'\right]. \tag{9}$$

Note that $\boldsymbol{A}_B'\boldsymbol{\Sigma}\mathbf{J}_{ij}\boldsymbol{A}_B$ is an $(m-q)\times(m-q)$ matrix which has the ith column of $\boldsymbol{\Sigma}_{BA}$ in its $(j-q)$th column and zeros elsewhere. Similarly, $\boldsymbol{A}_A\boldsymbol{\Sigma}\mathbf{J}_{ji}\boldsymbol{A}_A'$ is a $q \times q$ matrix which has the $(j-q)$th column of $\boldsymbol{\Sigma}_{AB}$ in its ith column and zeros elsewhere. The terms in the second sum are just transposed terms of the first sum. Finally, $\mathbf{J}_{ji}\boldsymbol{\Sigma}\mathbf{J}_{kl}$ takes ρ_{ik} and puts it on position (j, l) in the $m \times m$ zero matrix. Thus, $\boldsymbol{A}_B'\mathbf{J}_{ji}\boldsymbol{\Sigma}\mathbf{J}_{kl}\boldsymbol{A}_B$ is an $(m-q)\times(m-q)$ matrix with ρ_{ik} on position $(j-q, l-q)$ and zeros elsewhere. Analogously, $\boldsymbol{A}_A\mathbf{J}_{ij}\boldsymbol{\Sigma}\mathbf{J}_{lk}\boldsymbol{A}_A'$ is a $q \times q$ matrix with ρ_{jl} on position (i, k) and zeros elsewhere.

Proof Due to the asymptotic normality of $\widehat{\boldsymbol{\Sigma}}_n$, shown in Klüppelberg and Kuhn (2009), the asymptotic equivalence of the two distributions follows from (Tyler 1983, Theorem 2) applied to $H(\boldsymbol{\Sigma}) = \mathrm{vec}\, \boldsymbol{\Sigma}_{AB}$. Note that the function

$$f_n(\widehat{\boldsymbol{\Sigma}}_n, \boldsymbol{\Lambda}) = |\boldsymbol{\Lambda}|^{-n/2}\exp\{-(n/2)\cdot \mathrm{tr}(\boldsymbol{\Lambda}^{-1}\widehat{\boldsymbol{\Sigma}}_n)\} \tag{10}$$

is maximized under the condition $H(\boldsymbol{\Lambda}) = 0$ for $\boldsymbol{\Lambda}_n$ being the block diagonal matrix with blocks $\widehat{\boldsymbol{\Sigma}}_{AA}$ and $\widehat{\boldsymbol{\Sigma}}_{BB}$ on the diagonal and zeros elsewhere. Then

$$L_n \equiv f_n(\widehat{\boldsymbol{\Sigma}}_n, \boldsymbol{\Lambda}_n)/f_n(\widehat{\boldsymbol{\Sigma}}_n, \widehat{\boldsymbol{\Sigma}}_n) = \left(|\widehat{\boldsymbol{\Sigma}}_n|/(|\widehat{\boldsymbol{\Sigma}}_{AA}||\widehat{\boldsymbol{\Sigma}}_{BB}|)\right)^{n/2}$$

and, thus, $-2\log L_n$ equals T_W^τ. The expression for $\mathbf{Q}$ follows from (Tyler 1983, Theorem 2):

$$\mathbf{Q} = 2\widetilde{H}(\boldsymbol{\Sigma})(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})[\widetilde{H}(\boldsymbol{\Sigma})]'$$

where

$$\widetilde{H}(\boldsymbol{\Sigma}) = \frac{1}{2}\frac{d\mathrm{vec}\, \boldsymbol{\Sigma}_{AB}}{d\mathrm{vec}\, \boldsymbol{\Sigma}}\left(\mathbf{I}_{m^2} + \sum_{i=1}^{m}\mathbf{J}_{ii}\otimes\mathbf{J}_{ii}\right)$$

$$= \frac{1}{2}(\boldsymbol{A}_B' \otimes \boldsymbol{A}_A)\frac{d\text{vec}\,\boldsymbol{\Sigma}}{d\text{vec}\,\boldsymbol{\Sigma}}\left(\mathbf{I}_{m^2} + \sum_{i=1}^{m}\mathbf{J}_{ii} \otimes \mathbf{J}_{ii}\right)$$

because $\text{vec}\,(\mathbf{CXB}) = (\mathbf{B}' \otimes \mathbf{C})\text{vec}\,\mathbf{X}$ for any compatible matrices $\mathbf{C}, \mathbf{X}$, and $\mathbf{B}$, and because $\boldsymbol{\Sigma}_{AB} = \boldsymbol{A}_A\boldsymbol{\Sigma}\boldsymbol{A}_B$. Since $\boldsymbol{\Sigma}$ is symmetric, $d\text{vec}\,\boldsymbol{\Sigma}/(d\text{vec}\,\boldsymbol{\Sigma}) = \mathbf{I}_{m^2} + \sum_{i\neq j}\mathbf{J}_{ij} \otimes \mathbf{J}_{ji}$. The rest then follows after some calculations using the properties of the Kronecker product. □

The presentation above is slightly simplified because it assumes that matrices $\widehat{\mathbf{U}}$, $\widehat{\mathbf{W}}$, and $\widehat{\mathbf{Q}}$ are regular, which holds only with probability tending to one for $n \to \infty$. If the singularity problem occurs, it could be addressed exactly as in Klüppelberg and Kuhn (2009). Curious readers could also consult Christensen (2005) for some ways to speed up the computations regarding the Kendall correlations.

2.4 Testing with a known shift

Obviously, no distribution can be symmetric about two different parallel affine spaces. Note also that the suitable shift vector $\boldsymbol{s}$ for testing the symmetry is not defined uniquely because any vector in the form $\boldsymbol{s} = \mathsf{E}\boldsymbol{Y} + \boldsymbol{v}$, $\boldsymbol{v} \in \mathcal{S}_q$, also meets the definition.

Now what if one wants to test the null hypothesis $H_0^A(\boldsymbol{s}_0)$: H_0^A holds with $\boldsymbol{s} = \boldsymbol{s}_0$?

Since $H_0^A(\boldsymbol{s}_0) \subset H_0^A$, one can still test $H_0^A(\boldsymbol{s}_0)$ by means of the tests for H_0^A, which may be reasonable if shift alternatives are to be excluded.

In fact, $H_0^A(\boldsymbol{s}_0)$ is equivalent to H_0^A with further assumption $\mathsf{E}\boldsymbol{\Gamma}_B'(\boldsymbol{Y} - \boldsymbol{s}_0) = \mathbf{0}$. One can thus view $H_0^A(\boldsymbol{s}_0)$ as a compound hypothesis and, therefore, test it with a combination of a test of H_0^A and a test of $\boldsymbol{\Gamma}_B'(\mathsf{E}\boldsymbol{Y} - \boldsymbol{s}_0) = \mathbf{0}$, where the latter test reduces to a standard statistical procedure about the mean vector.

In principle, one could use $\mathsf{E}\boldsymbol{\Gamma}_B'(\boldsymbol{Y} - \boldsymbol{s}_0) = \mathbf{0}$ for estimating $\mathsf{cov}(\boldsymbol{Z}_A, \boldsymbol{Z}_B)$ and $\mathsf{var}(\boldsymbol{Z}_B)$ with the aid of $\boldsymbol{s}_0$ instead of the mean because then

$$\begin{aligned}
\mathsf{cov}(\boldsymbol{Z}_A, \boldsymbol{Z}_B) &= \mathsf{E}\boldsymbol{\Gamma}_A'(\boldsymbol{Y} - \mathsf{E}\boldsymbol{Y})(\boldsymbol{Y} - \mathsf{E}\boldsymbol{Y})'\boldsymbol{\Gamma}_B = \mathsf{E}\boldsymbol{\Gamma}_A'(\boldsymbol{Y} - \boldsymbol{s}_0)(\boldsymbol{Y} - \boldsymbol{s}_0)'\boldsymbol{\Gamma}_B\\
\mathsf{var}(\boldsymbol{Z}_B) &= \mathsf{E}\boldsymbol{\Gamma}_B'(\boldsymbol{Y} - \mathsf{E}\boldsymbol{Y})(\boldsymbol{Y} - \mathsf{E}\boldsymbol{Y})'\boldsymbol{\Gamma}_B = \mathsf{E}\boldsymbol{\Gamma}_B'(\boldsymbol{Y} - \boldsymbol{s}_0)(\boldsymbol{Y} - \boldsymbol{s}_0)'\boldsymbol{\Gamma}_B
\end{aligned}$$

even for the expectation taken with respect to the empirical probability measure. The asymptotic distributions of all the proposed test statistics under $H_0^A(\boldsymbol{s}_0)$ would be the same as under H_0^A even if they were computed from such modified estimators. The same could be said even about the finite sample distributions of all the rank test statistics. Nevertheless, the tests based on the covariance matrix estimator alone could still be insensitive to shift alternatives and still should be complemented with a test of $\boldsymbol{\Gamma}_B'(\mathsf{E}\boldsymbol{Y} - \boldsymbol{s}_0) = \mathbf{0}$.

3 Illustrations

This section illustrates the presented tests of symmetry with a few representative examples. They involve bivariate, trivariate, and multidimensional observations; permutation tests and both parametric and nonparametric asymptotic tests; elliptical and non-elliptical data distributions; small, moderate, and large data sets; and the general tests as well as the special tests of axial symmetry and equality of scale.

All the tests have been implemented in the free software environment for statistical computing and graphics called *R* (R 2018) by means of additional packages *CCP* (Menzel 2012) [permutation and asymptotic tests in the canonical correlation analysis], *mvtnorm* (Genz et al. 2019) [multivariate t RNG], and *LaplacesDemon* (Statisticat 2018) [multivariate Laplace RNG] whose contributions are indicated in the square brackets. The vector of canonical correlations have been computed by the standard function *cancor* from the default *stats* library.

Basically, the first simulation example (Figs. 1 and 2) uses the parametric test statistics T_W, T_H, and T_P of (1) with various approximations, including the default F-approximations (provided by the *CCP* package in *R*) and three χ^2 approximations, differing only in the choice of c_W, c_H, and c_P. The comparison indicates that there is not much difference among the three test statistics if a proper approximation is employed and the number of observations is not too small. Consequently, all the other examples use only the Wilks test statistic, either with the default approximation (because no kurtosis correction is then required) or in a permutation test. In the fourth example, the Wilks test statistic serves only as a benchmark for the comparison of the rank tests T_{K1} and T_{K2} of (6) and (7). The real data examples are also investigated with the rank tests T_{K1} and T_{K2} as well as with the Wilks test statistic, both with the default F-approximation and with the χ^2 approximation by means of the kurtosis correction term c_W of (2). The moderate sample comparison with the test of Rao and Raghunath (2012) uses T_{K1} and T_{K2}, T_W, T_H, and T_P [with correction factors (2), (3), and (4)] and also T_W with $c_W = n$ for an illustration.

The simulation experiments generally compare average empirical p-values. They are preferred because of their independence of testing level and because of their use in Hudecová and Šiman (2019) where the closest test competitor is described. Test powers are reported only in Table 1 for easy comparison with the test of Rao and Raghunath (2012).

The first example deals with general asymptotic tests of symmetry (up to a shift) around a subspace of arbitrary dimension. Figures 1 and 2 compare the performance of three asymptotic tests of symmetry around m-dimensional subspaces (of $\mathbb{R}^d$) that are generated by the first m vectors of the canonical basis, $m \in \{1, \dots, d-1\}$ and $d = 20$. To be specific, the tests based on T_W, T_H, and T_P have been used for this purpose with a kurtosis correction to c_W, c_H, and c_P applied when necessary, i.e., in (c) and (d). The plots show average p-values obtained from 5000 independent samples containing $n = 1000$ observations from four multivariate distributions with zero mean vector, out of which 500 observations were shifted by $\xi(1, \dots, 1)' \in \mathbb{R}^{20}$ where $\xi = 0$, 0.2, 0.3, 0.4, or 0.5. The distributions considered are the (non-elliptical) uniform distribution on $[-0.5, 0.5]^{20}$ and three elliptical distributions: multivariate standard normal distribution, (heavy-tailed) multivariate standard t-distribution with 14 degrees of freedom (and unit diagonal scale matrix) and (light-tailed) multivariate standard Laplace distribution (i.e., the multivariate exponential power distribution of Gómez et al. (1998) with $\beta = 1/2$ and the unit diagonal covariance matrix). The difference between Figs. 1 and 2 lies in the fact that Fig. 1 uses the default F-approximation in (a)

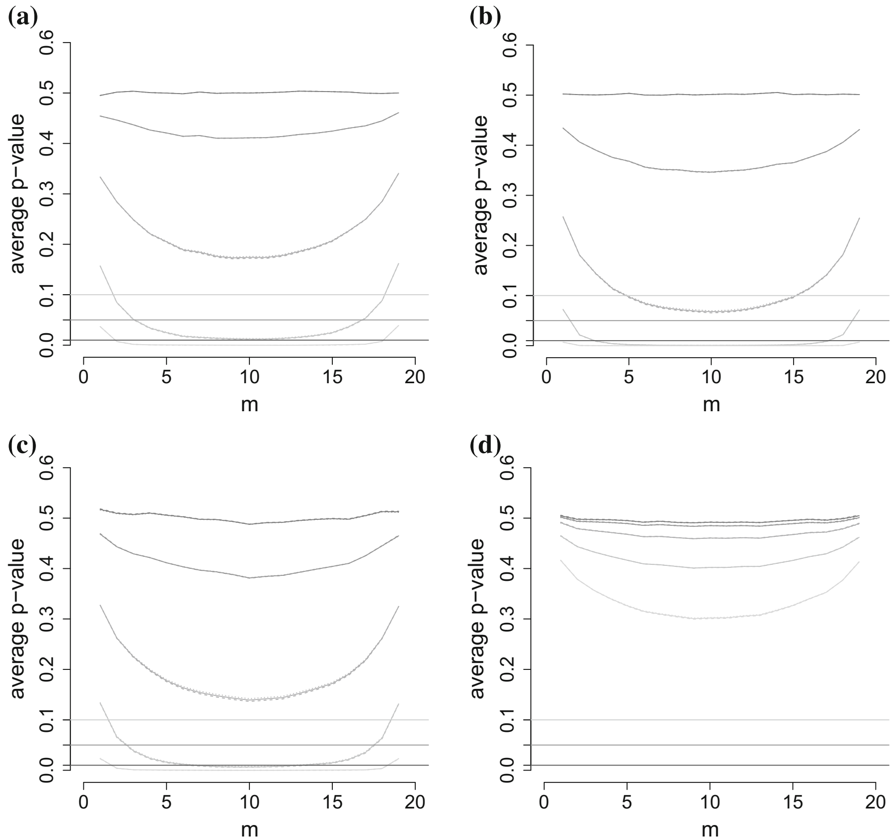


Fig. 1 Testing symmetry (up to a shift) around a subspace by means of asymptotic tests. The figure shows the average of sample p-values from the tests of symmetry around an m-dimensional subspace generated by the first m vectors of the canonical basis in $\mathbb{R}^{20}$. Three asymptotic tests have been used for this purpose: the Wilks test (solid), the Hotelling-Lawley test (dashed) and the Pillai–Bartlett test (dotted), with the default F-approximation applied in (**a**) and (**b**), and with the χ^2 approximation including the kurtosis adjustments (2) to (4) applied in (**c**) and (**d**), i.e., when necessary. The plots have been obtained from 5000 independent samples of $n = 1000$ independent observations that were first generated from (**a**) (non-elliptical) multivariate uniform distribution $\mathcal{U}(-1, 1)^{20}$, (**b**) multivariate standard normal distribution $N(0, 1)^{20}$, (**c**) (heavy-tailed elliptical) standard multivariate student t distribution with 14 degrees of freedom, and (**d**) (light-tailed elliptical) standard multivariate Laplace distribution, and then the last $n/2$ of them were shifted by $\xi(1, \ldots, 1)' \in \mathbb{R}^{20}$ where $\xi = 0$ (black), 0.2 (very dark gray), 0.3 (dark gray), 0.4 (gray), and 0.5 (light gray). The type of line used for plotting is indicated in the parentheses

and (b), and the refined χ^2 approximation with kurtosis corrections (2) to (4) in (c) and (d), while Fig. 2 uses the crude χ^2 approximations with $c_W = c_H = c_P = n$ in (a) and (b), and with $c_W = c_H = c_P = n/(1+\kappa)$ in (c) and (d). Obviously, the approximation matters even for $n = 1000$. The crude approximations work satisfactorily only with the Pillai–Bartlett test T_P. When the refined approximations are used, then the three tests behave almost identically for such large data samples.

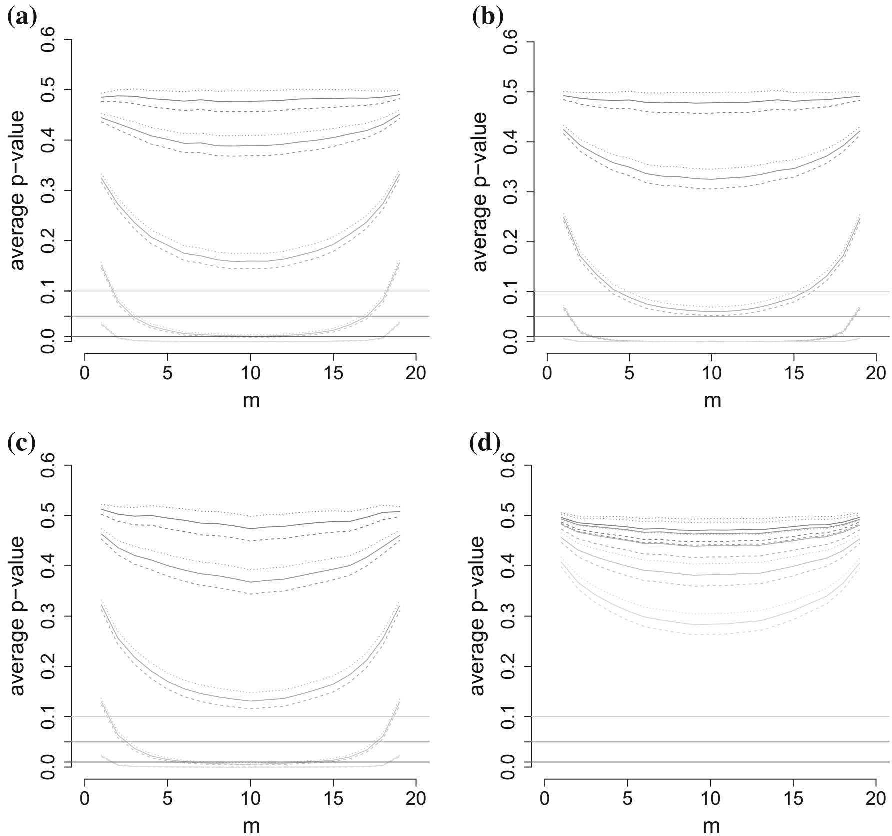
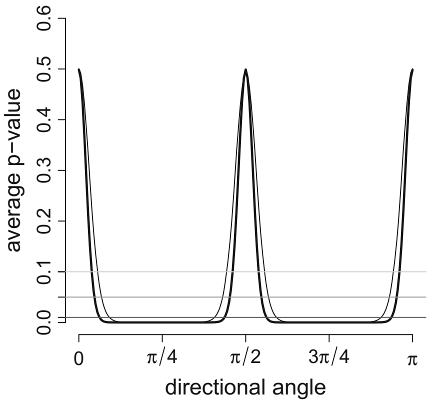


Fig. 2 Testing symmetry (up to a shift) around a subspace by means of asymptotic tests II. The figure settings are the same as in Fig. 1 except for the approximations used for the test statistics. This time the χ^2 approximation is used, with $c_W = c_H = c_P = n$ in (**a**) and (**b**) and with the kurtosis adjustment $c_W = c_H = c_P = n/(1+\kappa)$ in (**c**) and (**d**). The comparison with Fig. 1 indicates that the accuracy of the approximation matters even for $n = 1000$. These crude approximations apparently distort the size of the Pillai–Bartlett test T_P the least

The second example concerns asymptotic testing of axial symmetry. Figure 3 presents average p-values from the test of axial symmetry based on the Wilks statistic (as implemented in the *R* package *CPP*), obtained from 5000 trivariate normal samples of size $n = 100$ or 200 from $N(0,1) \times N(0,4) \times N(0,9)$. The individual p-values have been computed from each sample for several supposed axial directions in the form $\boldsymbol{u} = (\cos(\alpha), \sin(\alpha), 0)'$ for $\alpha \in [0, \pi]$. Their null empirical distribution for the axial directional angle $\alpha = 0$ is reported as well. Compare it to Fig. 2 of Hudecová and Šiman (2019) and note the apparent size inferiority of their test.

The third example illustrates the test of scale equality. Figure 4 shows the average of sample p-values for various scale factors $R \in [1, 2]$ and their empirical distribution functions under the null hypothesis of $R = 1$ for the Wilks test and its permutation variant based on 1000 permutations, as they are implemented in the *CPP* package for



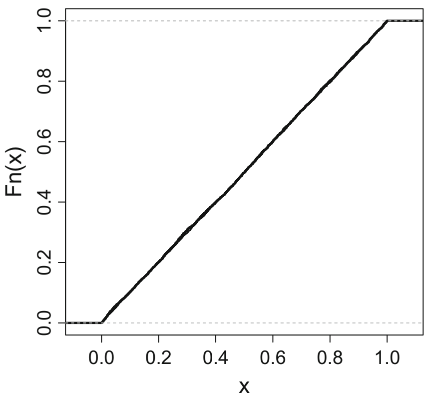


Fig. 3 Testing axial symmetry by means of asymptotic tests. The figure shows the average of sample p-values from the tests of axial symmetry around a line in direction $\boldsymbol{u} = (\cos(\alpha), \sin(\alpha), 0)'$ for $\alpha \in [0, \pi]$ (left) and their empirical distribution function for $\alpha = 0$ (right). The Wilks test has been used for that purpose as implemented in the *R* package *CPP*, i.e., with the default F approximation. The plots have been obtained from 5000 independent samples containing $n = 100$ (thin) or $n = 200$ (thick) independent observations drawn from the true model $(Y_1, Y_2, Y_3)' \sim N(0, 1) \times N(0, 4) \times N(0, 9)$. The type of line used for plotting is indicated in the parentheses

Table 1 The table relates to the problem of testing symmetry of bivariate normal distribution (with zero mean, unit marginal variances and correlation ϱ) around the x-axis

Test comparison for bivariate normal distribution and axial symmetry								
ϱ	$B(k = 2)$	$B(k = 3)$	T_W	T_W^*	T_H^*	T_P^*	T_{K1}	T_{K2}
$n = 60$								
0.0	0.0544	0.0506	0.0514	0.0525	0.0528	0.0513	0.0602	0.0659
0.1	0.0726	0.0660	0.1168	0.1190	0.1209	0.1179	0.1204	0.1292
0.2	0.1346	0.1223	0.3315	0.3321	0.3358	0.3286	0.3207	0.3356
0.3	0.2464	0.2501	0.6539	0.6514	0.6535	0.6492	0.6188	0.6355
0.4	0.4336	0.4451	0.8961	0.8937	0.8949	0.8921	0.8570	0.8672
0.5	0.6478	0.6640	0.9850	0.9828	0.9831	0.9825	0.9734	0.9760
0.6	0.8405	0.8682	0.9997	0.9993	0.9993	0.9993	0.9983	0.9985
0.7	0.9549	0.9737	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999
$n = 75$								
0.0	–	0.0457	0.0507	0.0513	0.0520	0.0509	0.0574	0.0616
0.1	–	0.0660	0.1341	0.1343	0.1352	0.1332	0.1358	0.1438
0.2	–	0.1398	0.4088	0.4097	0.4116	0.4076	0.3871	0.3981
0.3	–	0.2900	0.7586	0.7602	0.7617	0.7585	0.7180	0.7293
0.4	–	0.5305	0.9479	0.9466	0.9473	0.9460	0.9239	0.9288
0.5	–	0.7786	0.9961	0.9956	0.9956	0.9954	0.9920	0.9928
0.6	–	0.9382	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998
0.7	–	0.9928	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 1 continued

Test comparison for bivariate normal distribution and axial symmetry

ϱ	$B(k=2)$	$B(k=3)$	T_W	T_W^*	T_H^*	T_P^*	T_{K1}	T_{K2}
$n=100$								
0.0	0.0498	–	0.0473	0.0487	0.0496	0.0482	0.0529	0.0565
0.1	0.0737	–	0.1651	0.1674	0.1686	0.1669	0.1612	0.1671
0.2	0.1773	–	0.5170	0.5163	0.5186	0.5148	0.4804	0.4917
0.3	0.3705	–	0.8607	0.8594	0.8604	0.8584	0.8272	0.8321
0.4	0.6255	–	0.9869	0.9862	0.9863	0.9861	0.9763	0.9778
0.5	0.8500	–	0.9998	0.9998	0.9998	0.9998	0.9991	0.9991
0.6	0.9707	–	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.7	0.9985	–	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$n=150$								
0.0	0.0548	0.0515	0.0513	0.0521	0.0522	0.0517	0.0529	0.0548
0.1	0.1033	0.0963	0.2305	0.2311	0.2319	0.2303	0.2209	0.2255
0.2	0.2736	0.2878	0.6927	0.6911	0.6921	0.6895	0.6520	0.6595
0.3	0.5686	0.6038	0.9591	0.9580	0.9584	0.9578	0.9404	0.9431
0.4	0.8331	0.8721	0.9998	0.9995	0.9995	0.9995	0.9985	0.9987
0.5	0.9728	0.9833	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.6	0.9977	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

It uses the test of Rao and Raghunath (2012) with the two recommended values of the auxiliary binning parameter k (namely $k=2$ and $k=3$) as a benchmark, denoted as $B(k=2)$ and $B(k=3)$. The benchmark is compared with various tests presented in this article in terms of empirical power (for $\varrho>0$) or size (for $\rho=0$) for testing level $\alpha=0.05$, based on 10,000 simulations of independent samples with $n=60, 75, 100$ or 150 observations. The benchmark empirical sizes and powers were only copied from the original Table 3 of Rao and Raghunath (2012) to minimize the chance of an error. The test T_W uses the default F-approximation and $c_W=n$ while the other tests T_W^*, T_H^* and T_P^* use the test correction factors of (2), (3) and (4) in this order

R. The plots have been obtained from 1000 independent samples containing $n=100$, $n=200$, and $n=400$ observations drawn from the true model $(Y_1, Y_2)'=(\varepsilon_1, R\varepsilon_2)'$ with independent $\varepsilon_1 \sim N(0,1)$ and $\varepsilon_2 \sim N(0,1)$. The figure is analogous to Fig. 3 of Hudecová and Šiman (2019) and clearly shows their test inferior in terms of power.

The fourth example is dedicated to the tests T_{K1} and T_{K2} based on the Kendall rank correlations, and shows their robustness to outliers and ability to work without any moment assumption in Fig. 5. It uses 1000 independent samples containing $n=300$ independent six-dimensional observations and presents the average p-values from the tests of symmetry around the subspace generated by the first three axial directions (up to a shift). The left plot employs the observations generated by the multivariate standard normal distribution where the last three were shifted by $(d,\ldots,d)' \in \mathbb{R}^6$ for $d=0,1,\ldots,6$. The right plot employs the observations generated by the multivariate Cauchy distribution where the last half of them were shifted by $(d,\ldots,d)' \in \mathbb{R}^6$ for $d=0,0.25,\ldots,1.5$. It appears that T_{K1} marginally outperforms T_{K2} in this context, and that both the rank tests are then clearly superior to the asymptotic Wilks parametric

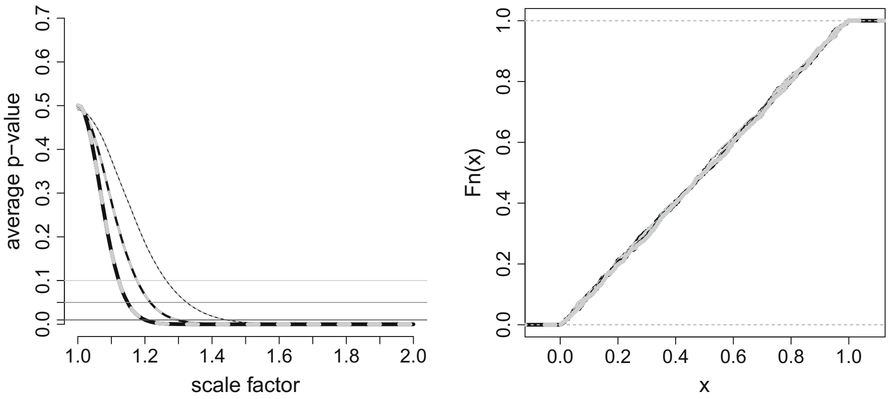


Fig. 4 Testing equality of scale. The figure shows the average of sample p-values for various scale factors $R \in [1, 2]$ (left) and their empirical distribution functions under the null hypothesis of $R = 1$ (right) for the Wilks test (black) and its permutation variant (gray) based on $N = 1000$ permutations, as they are implemented in the *CPP* package for *R*. The plots have been obtained from 1000 independent samples containing $n = 100$ (thin), $n = 200$ (normal), and $n = 400$ (thick) observations drawn from the true model $(Y_1, Y_2)' = (\varepsilon_1, R\varepsilon_2)'$ with independent $\varepsilon_1 \sim N(0, 1)$ and $\varepsilon_2 \sim N(0, 1)$. The type of line used for plotting is indicated in the parentheses

test (as implemented in the *CPP* package for *R*) included for comparison, at least from the two points of view considered.

The moderate sample comparison considers the problem of testing symmetry of a bivariate normal distribution (with zero mean, unit marginal variances and correlation ϱ) around the x-axis. It uses the test of Rao and Raghunath (2012) with the two recommended values of the auxiliary binning parameter k (namely $k = 2$ and $k = 3$) as the benchmark. It is compared with various tests presented here in terms of empirical power (for $\varrho > 0$) or size (for $\rho = 0$), obtained for testing level $\alpha = 0.05$ from 10000 simulated independent random samples with $n = 60, 75, 100$ or 150 observations. The benchmark empirical sizes and powers were copied from the original Table 3 of Rao and Raghunath (2012) for maximum reliability. The tests for comparison include T_{K1} and T_{K2}, T_W, T_H, and T_P [with correction factors (2), (3), and (4)] and also T_W with $c_W = n$.

The tests T_{K1} and T_{K2} appear slightly liberal for such small sample sizes but all the other tests including the benchmark appear correctly sized even for n as low as $n = 75$. However, the benchmark falls badly behind all the other included tests in terms of test power that is otherwise very similar across all the remaining tests, although T_W with $c_W = n$ (i.e., without the kurtosis correction useless in this case) appears marginally the best and rank-based T_{K1} and T_{K2} perform marginally the worst in that respect. It is not surprising that the ad hoc benchmark test loses the comparison to the tests derived by means of the maximum-likelihood approach for the normal distribution.

Next examples use real data. For comparison, some of the data sets employed in Hudecová and Šiman (2019) are also investigated here by means of the Wilks test with the default F-approximation, the Wilks test with the kurtosis correction (2), and

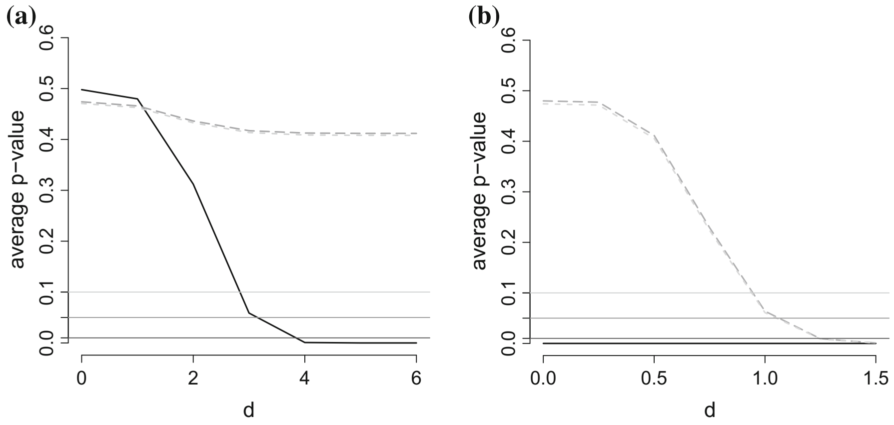


Fig. 5 Testing by means of the Kendall correlations. The figure illustrates two main advantages of the rank tests T_{K1} (dashed dark gray) and T_{K2} (dashed light gray) over the parametric Wilks test T_W with the default F-approximation (solid black), namely their robustness to outliers (**a**) and their ability to work without any moment condition (**b**). Both the pictures report average sample p-values obtained from 1000 independent samples with $n = 300$ six-dimensional observations by means of the tests of symmetry around the subspace generated by the first three axial directions. In (**a**), all the observations come from the multivariate standard normal distribution except for the last three that are moreover shifted by $(d, \dots, d)' \in \mathbb{R}^6$, $d = 0, 1, \dots, 6$. In (**b**), all the observations come from the multivariate Cauchy distribution but the last half of them are shifted by $(d, \dots, d)' \in \mathbb{R}^6$, $d = 0, 0.25, \dots, 1.5$. The Wilks test is then invalid. The type of line used for plotting is indicated in the parentheses

the two rank tests T_{K1} and T_{K2}. Of course, the tests are meaningful only when their assumptions are satisfied.

The first case deals with the famous Fisher's Iris (flower) data set as included in R. It contains 50 observations from each of the three Iris species considered and records four of their characteristics: the length and the width of the sepals and petals (in centimeters). The null hypothesis H_0 that the probability distribution of certain feature is the same for all the three species up to a location shift can be tested by means of the presented tests with $\boldsymbol{\Gamma}_A = \{(1, 1, 1)^\top/\sqrt{3}\}$ in the combined sample. Hudecová and Šiman (2019) thus rejected H_0 for the petal width and petal length. All the tests used here confirm the findings (with p-values less than 10^{-8}) and moreover reject H_0 for the petal length as well (each with p-value less than 0.01).

The second example considers 626 (virtually serially uncorrelated) log-returns of four daily exchange rates (AUD/CZK, CAD/CZK, EUR/CZK, USD/CZK) from 2/5/2017 to 30/10/2019 as a four-variate sample. Then the null hypotheses of exchangeability (up to a shift) and of symmetry around the last coordinate axis (up to a shift) could clearly be rejected by the presented tests (with virtually zero p-values), as in Hudecová and Šiman (2019). The same could be said even for the bivariate sample consisting only of EUR/CZK and USD/CZK.

The last case focuses on the Australian athletes data set ais from the R package DAAG (Maindonald and Braun 2019). Its subsets are employed in Hudecová and Šiman (2019), Kalina (2019) and Henze et al. (2014) for testing various symmetries. In particular, Henze et al. (2014) rejected the spherical symmetry of the joint distribution of the logarithms of the red blood cell count, white blood cell count and hemoglobin concentration regarding 202 athletes. They rejected the null hypothesis, which is not surprising because the tests used here reject the null hypotheses (up to a suitable shift) of exchangeability, axial symmetry around the last coordinate axis, and symmetry around the subspace generated by the last two coordinate axes, each with virtually zero p-value.

4 Concluding discussion

This article presents a coherent approach to testing symmetry of a continuous random vector around a subspace after a suitable shift. The asymptotic parametric tests based on canonical correlations seem suitable for the situations when the moment conditions are satisfied and no outliers are present. The nonparametric asymptotic tests based on Kendall correlations are suitable for the other cases if the underlying distribution is elliptical or meta-elliptical. If the underlying distribution is normal or a shift vector is known, then all the test statistics mentioned or referred to in the text may also be used in permutation tests without any moment assumptions, which can be recommended if there are no more than two to three hundreds of observations.

All the presented examples strongly speak in favor of the proposed tests, demonstrate their usefulness, and confirm their validity in different settings. In the special context of testing axial symmetry considered here, the parametric tests seem sized better and more powerful than the tests of axial symmetry based on the regression rank score process and presented in (Hudecová and Šiman 2019).

Permutation testing could extend the presented methodology even to the linear regression setup [thanks to the test statistics from the partial correlation analysis) or to the simultaneous testing of various symmetries (thanks to the tests if the covariance matrix is block diagonal (Rencher and Christensen 2012, Sect. 7.4)], at least for small to moderate data samples.

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References

Abdous B, Genest C, Rémillard B (2005) Dependence properties of meta-elliptical distributions. In: Duchesne P, Rémillard B (eds) Statistical modeling and analysis for complex data problems. Springer, Boston, pp 1–15

Christensen D (2005) Fast algorithms for the calculation of Kendall's τ. Comput Stat 20:51–62

Croux C, Dehon C (2003) Estimators of the multiple correlation coefficient: local robustness and confidence intervals. Stat Pap 44:315–334

Dümbgen L, Pauly M, Schweizer T (2015) M-functionals of multivariate scatter. Stat Surv 9:32–105

Fang KB, Fang KT, Kotz S (2002) The meta-elliptical distributions with given marginals. J Multivariate Anal 82:1–16 Errata: J Multivariate Anal 94, 222–223

Genz A, Bretz F, Miwa T, Mi X, Leisch F, Scheipl F, Hothorn T (2019) mvtnorm: multivariate Normal and t Distributions. R package version 1.0-7. http://CRAN.R-project.org/package=mvtnorm

Gómez E, Gómez-Villegas MA, Marín JM (1998) A multivariate generalization of the power exponential family of distributions. Commun Stat Theory Methods 27:589–600

Hallin M, Paindaveine D, Šiman M (2010a) Multivariate quantiles and multiple-output regression quantiles: from L_1 optimization to halfspace depth. Ann Stat 38:635–669

Hallin M, Paindaveine D, Verdebout T (2010b) Optimal rank-based testing for principal comonents. Ann Stat 38:3245–3299

Henze N, Hlávka Z, Meintanis SG (2014) Testing for spherical symmetry via the empirical characteristic function. Statistics 48:1282–1296

Hollander M (1971) A nonparametric test for bivariate symmetry. Biometrika 58:203–212

Hudecová Š, Šiman M (2019) Testing axial symmetry and related hypotheses by means of directional regression quantiles. Submitted

Kalina J (2019) Common multivariate estimators of location and scatter capture the symmetry of the underlying distribution. Commun Stat Simul Comput. https://doi.org/10.1080/03610918.2019.1615624

Klüppelberg C, Kuhn G (2009) Copula structure analysis. J R Stat Soc B 71:737–753

Krupskii P (2017) Copula-based measures of reflection and permutation asymmetry and statistical tests. Stat Pap 58:1165–1187

Lindskog F, McNeil A, Schmock U (2001) Kendall's tau for elliptical distributions. In: Bol G, Nakhaeizadeh G, Rachev ST, Ridder T, Vollmer KH (eds) Credit risk. Contributions to economics. Physica, New York

Maindonald JH, Braun WJ (2019) DAAG: data analysis and graphics data and functions. R package version 1(22):1 https://CRAN.R-project.org/package=DAAG

Menzel U (2012) CCP: significance tests for canonical correlation analysis (CCA). R package version 1:1 https://CRAN.R-project.org/package=CCP

Modarres R (2008) Tests of bivariate exchangeability. Int Stat Rev 76:203–213

Muirhead RJ, Waternaux CM (1980) Asymptotic distributions in canonical correlation analysis and other multivariate procedures for nonnormal populations. Biometrika 67:31–43

R Development Core Team (2018) R: a language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. https://www.R-project.org/

Rao KSM, Raghunath M (2012) A simple nonparametric test for bivariate symmetry about a line. J Stat Plann Inference 142:430–444

Rattihalli RN, Rao KSM, Raghunath M (2019) Some count-based nonparametric tests for circular symmetry of a bivariate distribution. Commun Stat Simul Comput 48:922–943

Rencher AC, Christensen WF (2012) Methods of multivariate analysis, 3rd edn. Wiley, New York

Serfling R (2006) Multivariate symmetry and asymmetry. In: Kotz S, Balakrishnan N, Read CB, Vidakovic B (eds) Encylopedia of statistical sciences, vol 8, 2nd edn. Wiley, New York, pp 5338–5345

Seo T, Kanda T, Fujikoshi Y (1995) The effects of nonnormality on tests for dimensionality in canonical correlation and MANOVA models. J Multivariate Anal 52:325–337

Statisticat LLC (2018) LaplacesDemon: complete environment for Bayesian inference. Bayesian-Inference.com. R package version 16.1.1

Su L (2006) A simple test for multivariate conditional symmetry. Econ Lett 93:374–378

Tyler DE (1983) Robustness and efficiency properties of scatter matrices. Biometrika 70:411–420

Yanagihara H, Tonda T, Matsumoto C (2005) The effects of nonnormality on asymptotic distributions of some likelihood ratio criteria for testing covariance structures under normal assumption. J Multivariate Anal 96:237–264

Yuan K-H, Bentler PM (2000) Inferences on correlation coefficients in some classes of nonnormal distributions. J Multivariate Anal 72:230–248