Available online at www.sciencedirect.com
ScienceDirect
IFAC PapersOnLine 54-14 (2021) 60–65

Chain of four integrators as a possible essence of the under-actuated planar walking ⋆

Sergej Čelikovský * Milan Anderle *

* The Czech Academy of Sciences
Institute of Information Theory and Automation
182 00 Prague 8, Czech Republic, celikovs@utia.cas.cz

Abstract: This paper continues the previously published research showing that for the three-link there always exist collocated virtual holonomic constraints for the torso and mechanical parameters readjustment such that the resulting restricted dynamics is state and feedback equivalent to the chain of four integrators. In such a way, the role of the torso in walking patterns design achieves nice and clear control-theoretic interpretation. To underline a potential of that interesting feature, a thorough numerical optimization based design of suitable trajectories for the integrator chains is performed in this paper. Successful solution is then used to design hybrid stable multi-step walking for the three-link demonstrated by the simulations as well.

Keywords: Underactuated walking, feedback linearization, collocated virtual holonomic constraints.

1. INTRODUCTION

The aim of this paper is to further develop the idea that the chain of four integrators is state and feedback equivalent to the restricted dynamics of various planar underactuated walking-like configurations imposed by conveniently designed collocated virtual holonomic constraints for those links (*i.e.* "torso", or "hands") that are not directly related to the walking support, swing, impact *etc.*. This idea was first introduced in Čelikovský and Anderle [2019] for the so-called three-link where it required the specific readjustment of the values of mechanical inertial parameters of these links as well. In such a way, the intrinsic concept for the torso and the hands movement strategy was thereby postulated which otherwise is not so intuitively straightforward. Indeed, the virtual constraints for the "knees" follow intuitively clear necessity to bend them in order not to hit the ground during the swing. At the same time, torso, or hands are expected have rather indirect, "ballancing-like" role in order to facilitate the walking movement. Yet, it is not visibly clear what should *e.g.* the torso do to fulfill that expectation. In this context, the exact linearizable restricted dynamics appeared to be a challenging option.

The crucial tool enabling doing so is the so-called **collocated virtual holonomic constraints (CVHC)**. Detailed exposition on CVHC is available in Čelikovský and Anderle [2017]. Briefly, CVHC impose fixed relations between all directly actuated variables only, while general **virtual holonomic constraints (VHC)** may include the unactuated generalized coordinates components as well. Recall, that the mechanical system variable (*i.e.* the component of generalized coordinates) is called **cyclic** if the respective kinetic energy does not depend on that variable. Unlike general VHC, the CVHC always preserve the cyclic variable property of the unactuated angle of the resulting restricted dynamics being thereby 2-DOF underactuated Lagrangian system with one actuator and cyclic property of the unactuated variable. Such a system is well-known by Olfati-Saber [2002] to be state and feedback equivalent to an almost linear four-dimensional normal form. Its only nonlinearity depends on the selected CVHC and robot inertial parameters. In Čelikovský and Anderle [2018] the CVHC for the three-link were designed based on some intuitive ideas, but giving a limited performance. Choosing properly CVHC for the three-link and tuning inertial parameters of its torso enabled then in Čelikovský and Anderle [2019] to design a single step which is state and feedback equivalent to the chain of four integrators.

To make advantage of such a favorable property the long existing problem of the steering the integrators chain is revisited. Indeed, this problem seems to be theoretically well understood and straightforward, yet, it is quite complex when practically viable solutions are needed, *e.g.* meeting the simple and natural requirement of the inputs to be bounded. Reader is referred here to seminal paper Teel [1992] launching the extensive research on that topic and survey parts of Zhou and Yang [2016], Zhou and Lam [2017] for a more detailed list of references, including other related topic, like input delays, or more general forms of linear systems. Čelikovský and Anderle [2019] provided a simple and even analytical closed form solution for a single step trajectory design for the three-link based on the integrators chain steering. Yet, it failed to design a reasonable hybrid cyclic multi-step walking trajectory. Indeed, the impact invariance requirements lead to special initial and terminal states of the three-link generating thereby quite peculiar initial and terminal conditions for the respective integrators chain.

In view of that, the contribution of the current paper is the thorough and comprehensive numerical optimization based design taking into the account even other mechanical parameters of the three-link, in addition to the step pattern parameters (its velocity *etc.*) will be presented here. Its success will be demonstrated by simulations of the multi-step hybrid stable walking.

The rest of the paper is organized as follows. Section 2 gives some preliminaries and recalls the exact feedback linearizable constrained dynamics for the three-link while the detailed

⋆ Supported by the Czech Science Foundation - the grant No. 21-03689S.

2405-8963
10.1016/j.ifacol.2021.10.329

three-link walking design is given in Section 3. Section 4 collects simulation results and Section 5 concludes the paper.

Notation. For a smooth function $\phi(q)$, $q \in \mathbb{R}^n$, differential $\mathrm{d}\phi = \partial\phi(q)/\partial q$ is the row vector of the partial derivatives, $0_{m\times p}, m, p \in \mathbb{Z}$, stands for $(m \times p)$ zero matrix, I_m for $(m \times m)$ the identity matrix. COM stands for the centre of mass, MI for moment of inertia and DOF for degree(s) of freedom.

2. PRELIMINARY DEFINITIONS AND RESULTS

Consider n-DOF underactuated Lagrangian system (LS):

$$\frac{d}{dt}\left[\frac{\partial \mathscr{L}}{\partial \dot q}\right]^\top - \left[\frac{\partial \mathscr{L}}{\partial q}\right]^\top = \begin{bmatrix} 0_k \\ u \end{bmatrix} = [0_k, u_{k+1}, \ldots, u_n]^\top, \quad (1)$$

$$\mathscr{L}(q,\dot q) = K(q,\dot q) - V(q), \;\; K(q,\dot q) = \frac{1}{2}\dot q^T D(q)\dot q, \quad (2)$$

where $q = (q_1,\ldots,q_n)^\top$, $\dot q = (\dot q_1,\ldots,\dot q_n)^\top$ are the generalized coordinates and velocities, $D(q) = D(q)^\top > 0$ is the inertia (aka mass) matrix, while K, V are the system kinetic and potential energy. Coordinate q_i is called **cyclic variable** if D does not depend on q_i. Integer $k \geq 1$ is called as the degree of the underactuation, while $u_{k+1},\ldots,u_k$ are the input actuators (control inputs). The coordinates $q_{k+1},\ldots,q_n$ are called **(directly) actuated** while $q_1,\ldots,q_k$ **unactuated**. The equations (1) give

$$D(q)\ddot q + C(q,\dot q)\dot q + G(q) = [0_k, u_{k+1}, \ldots, u_n]^\top, \quad (3)$$

$$G^\top(q) = \frac{\partial V(q)}{\partial q}, \;\; C(q,\dot q)\dot q = \left[\sum_{i=1}^n \frac{\partial D(q)}{\partial q_i}\dot q_i\right]\dot q - C_s(q,\dot q), \; (4)$$

$$C_s^\top(q,\dot q) = [C_{s1}(q,\dot q),\ldots,C_{sn}(q,\dot q)] = \frac{\partial K(q,\dot q)}{\partial q}, \quad (5)$$

$$C_{si}(q,\dot q) = \frac{1}{2}\dot q^\top\left[\frac{\partial D(q)}{\partial q_i}\right]\dot q, \; i = 1,\ldots,n. \quad (6)$$

Here, $G(q)$ is the gravity vector while the Coriolis terms $C(q,\dot q)\dot q$ are expressed in (4)-(6) in a bit unusual way which better suits our exposition later on.

Definition 2.1. The **three-link** is the Lagrangian mechanical system modelled by (1), (2) and (3) with

$$\begin{gathered}
D = [d_{ij}], \; i,j = 1,2,3, \; D^\top = D > 0, \;\; G = [G_1, G_2, G_3]^\top, \\
d_{11} = I_1 + I_2 + I_3 + l_1^2 m_2 + l_1^2 m_3 + l_{c1}^2 m_1 + \\
l_{c2}^2 m_2 + l_{c3}^2 m_3 + 2l_1 l_{c2} m_2 \cos q_2 + 2l_1 l_{c3} m_3 \cos q_3, \\
d_{12}(q_2) = m_2 l_{c2}^2 + l_1 m_2 \cos q_2 l_{c2} + I_2 \\
d_{13}(q_3) = m_3 l_{c3}^2 + l_1 m_3 \cos q_3 l_{c3} + I_3, \; d_{23} = 0, \\
d_{22}(q_2,q_3) = m_2 l_{c2}^2 + I_2, \; d_{33}(q_2,q_3) = m_3 l_{c3}^2 + I_3, \\
G_1 = -g\,(l_1 m_2 \sin q_1 + l_1 m_3 \sin q_1 + l_{c1} m_1 \sin q_1 + \\
l_{c2} m_2 \sin q_1 + q_2 + l_{c3} m_3 \sin q_1 + q_3), \\
G_2 = -g l_{c2} m_2 \sin q_1 + q_2, \quad G_3 = -g l_{c3} m_3 \sin q_1 + q_3.
\end{gathered}$$

Possible planar mechanical realization of the three-link is depicted in Fig. 1. The angle q_1 is not directly actuated while the torques u_2 and u_3 directly actuate q_2 and q_3, respectively. Since $D = D(q_2,q_3)$, the unactuated angle q_1 is the cyclic variable.

Definition 2.2. VHC for the system (1-3) are given by

$$\varphi_i(q) = 0, \; [\mathrm{d}\varphi_i(q)]\dot q = 0, \;\; i = 1,\ldots,l, \quad (7)$$

where $\varphi_1,\ldots,\varphi_l$ are smooth functions of the generalized coordinates having $\forall q \in \mathbb{R}^n$ satisfying (7) linearly independent differentials $\mathrm{d}\varphi_i(q), i = 1,\ldots,l$. The VHC are called global if the functions $\varphi_i(q), i = 1,\ldots,l$ in (7) can be completed to a

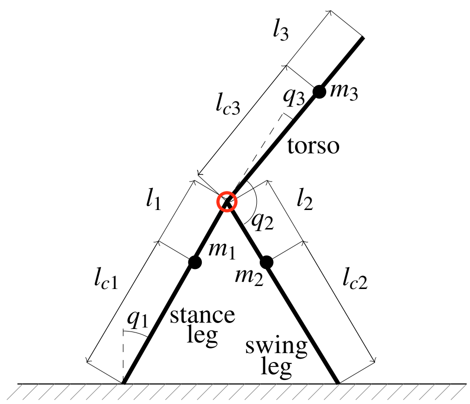

Fig. 1. **The three-link.** The i-th link ($i = 1,2,3$) is a homogeneous rod of mass μ_i with attached point mass M_i, $m_i = \mu_i + M_i$, black bold point is the location of the link COM, and the link MI with respect to COM is I_i.

diffeomorphism of $\mathbb{R}^n$. Furthermore, the VHC are called locally regular for (1-3) at some q^0, if it holds

$$\operatorname{rank}\left[\frac{\partial \varphi_1}{\partial q},\ldots\frac{\partial \varphi_l}{\partial q}\right]^\top (q^0)D^{-1}(q^0)\begin{bmatrix} 0_{k\times n} \\ I_{n-k}\end{bmatrix} = l. \quad (8)$$

The global VHC are called globally regular on some subset of $\mathbb{R}^n$ if they are locally regular at each its point. The VHC are called flat if there $\exists l$ mutually distinct integers $j_1,\ldots,j_l \in \{1,\ldots,n\}$ with $\varphi_1 \equiv q_{j_1},\ldots,\varphi_l \equiv q_{j_l}$. Finally, the VHC (7) are called CVHC if they depend only on actuated coordinates.

Remark 2.3. The regular VHC can be imposed by static state feedback since by Definition 2.2 they form a set of virtual outputs having the relative degree 2 with respect to a suitable selected set of inputs (recall that there may be more inputs than the constraints). The CVHC (7) are always regular and can be enforced *e.g.* by Algorithm 1 of Čelikovský [2015], or alternative Hamiltonian representation and backstepping based procedure in Čelikovský and Anderle [2017], while the resulting restricted dynamics is the underactuated LS, having $(n-l)$ DOF and $n-k-l$ inputs. Finally, if any q_i, $i = 1,\ldots,k$ is a cyclic variable of (1)-(3), it is cyclic for the respective CVHC restricted dynamics as well. In particular, the following proposition is a direct consequence of results of Čelikovský [2015].

Proposition 2.4. Consider the three-link given by Definition 2.2 and for two times continuously differentiable ϕ_3 the CVHC

$$\varphi_1(q_2,q_3) := q_3 - \phi_3(q_2) = 0, \; \dot q_3 - \phi_3'(q_2)\dot q_2 = 0. \quad (9)$$

Then the restricted dynamics is the LS (1)-(3) with generalized coordinates q_1, q_2, the input $\overline{u}_2 \in \mathbb{R}$, the inertia matrix $\overline{D}(q_2)$, the potential energy $\overline{V}(q_1,q_2)$ and the right hand side $[0,\overline{u}_2]^\top$:

$$\overline{D}(q_2) = \Phi_3^\top D(q_2,\phi_3(q_2))\Phi_3, \; \Phi_3 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \phi_3'(q_2)\end{bmatrix}, \quad (10)$$

$$\overline{V}(q_1,q_2) = V(q_1,q_2,\phi_3(q_2)), \;\; u_2 = \alpha + \overline{u}_2\beta, \quad (11)$$

where α, β are known pre-computed functions of $(q_1,q_2,\dot q_1,\dot q_2)$.

Since the restricted dynamics of the three-link has the cyclic unactuated variable q_1, it is state and feedback equivalent to the following normal form, as shown by Olfati-Saber [2002]:

$$\dot\xi_1 = \psi(\xi_1,\xi_3)\xi_2, \;\; \dot\xi_2 = \xi_3, \;\; \dot\xi_3 = \xi_4, \;\; \dot\xi_4 = w \quad (12)$$

having the states $\xi = (\xi_1,\xi_2,\xi_3,\xi_4)^\top$ and the input w. The respective state and feedback diffeomorphic transformations

$$(q_1,q_2,\dot q_1,\dot q_2,\overline{u}) \mapsto (\xi_1,\xi_2,\xi_3,\xi_4,w) \quad (13)$$

can be found in Čelikovský and Anderle [2019] and are omitted here for the space reasons. Moreover, by Olfati-Saber [2002]:

$$\psi(\xi_1,\xi_3) = 1/\overline{d}_{11}(q_2(\xi_1,\xi_3)).$$

The latter is the only nonlinearity in (12) and it is, in general, quite complex as $q_2(\xi_1,\xi_3)$ is given implicitely by transformations (13) inversion. Yet, if $\overline{d}_{11}(q_2)$ is constant with respect to q_2, so is $\psi(\xi_1,\xi_3)$ regardless how complex $q_2(\xi_1,\xi_3)$ is. Recall, that $\overline{d}_{11}(q_2)$ depends on choice of $\phi_3(q_2)$:

$$\begin{aligned}&\overline{d}_{11}(q_2) = d_{11}(q_2,\phi_3(q_2)),\\ &d_{11}(q_2,q_3) = I_1+I_2+I_3+l_1^2(m_2+m_3)+l_{c_1}^2 m_1+\\ &l_{c_2}^2 m_2+l_{c_3}^2 m_3+2l_1 l_{c_2} m_2\cos q_2+2l_1 l_{c_3} m_3 \cos q_3.\end{aligned} \tag{14}$$

This motivates the following proposition stated and proved in Čelikovský and Anderle [2019].

Proposition 2.5. Consider $d_{11}(q_2,q_3)$ given by (14) and let any $q_2^0 \in (\pi, 2\pi)$ be given. Then the following equality

$$d_{11}(q_2,\phi_3(q_2)) = d_{11}(q_2^0,\phi_3(q_2^0)),\ \forall q_2 \in [q_2^0, 2\pi - q_2^0], \tag{15}$$

holds if and only if the following equalities hold

$$l_{c_3} m_3 = l_{c_2} m_2 (1+\cos q_2^0)(1-\sin(q_2^0/2))^{-1}, \tag{16}$$

$$\begin{aligned}&\phi_3(q_2^0) = (q_2^0-\pi)/2,\ \ \phi_3(2\pi - q_2^0) = (\pi - q_2^0)/2,\\ &\phi_3(q_2) = \operatorname{sign}(q_2-\pi)\operatorname{acos}(\overline{\varphi}(q_2)),\ \operatorname{sign}(0) := 0,\\ &\overline{\varphi}(q_2) = \frac{\sin(q_2^0/2)(\cos q_2+1)+\cos q_2^0-\cos q_2)}{\cos q_2^0+1}.\end{aligned} \tag{17}$$

The function $\phi_3(q_2)$ given by (17) is twice differentiable with

$$\phi_3'(q_2) = \left|\sin\left(\frac{q_2}{2}\right)\right| \frac{\sqrt{2-2\sin(q_2^0/2)}}{\sqrt{\eta(q_2,q_2^0)}},$$

$$\eta := 1+2\cos q_2^0+\cos q_2(\sin(q_2^0/2)-1)+\sin(q_2^0/2), \tag{18}$$

$$\begin{aligned}&\phi_3''(q_2) = -(\sqrt{2}/2)\sqrt{1-\sin(q_2^0/2)}\times\\ &\left(\eta^{-1/2}\cos\left(\frac{q_2}{2}\right)+\eta^{-3/2}\sin q_2 \sin\left(\frac{q_2}{2}\right)\sin(q_2^0/2)\right).\end{aligned} \tag{19}$$

Remark 2.6. Proposition 2.5 can be used to generate the CVHC (9) with the exact linearizable restricted dynamics. Actually, for every q_2^0 and the three-link in double support, the angle q_1^0 is uniquely defined, moreover, $q_3^0 = \phi_3(q_2^0) = (q_2^0-\pi)/2$ is also given (it is easy to see that it corresponds to the torso being strictly vertical). Based on q_2^0 the parameter $l_{c_3} m_3$ is adjusted in such a way that the equality $l_{c_3} m_3 = l_{c_2} m_2 (1+\cos q_2^0)(1-\sin(q_2^0/2))^{-1}$ holds. Such a tasks is actually practically realizable by the movable mass on the torso, adjusting l_{c_3}. As (17) ensure that $d_{11}(q_2,\phi_3(q_2)) = d_{11}(q_2^0,\phi_3(q_2^0))\ \forall q_2 \in [q_2^0, 2\pi - q_2^0]$, the function $\psi(\xi_1,\xi_3)$ becomes constant equal to $d_{11}(q_2^0,\phi_3(q_2^0))\ \forall \xi_1,\xi_3$ and the system (12) becomes the linear one. Simple re-scaling of ξ_1 then gives

$$\dot{\xi}_1 = \xi_2,\ \ \dot{\xi}_2 = \xi_3,\ \ \dot{\xi}_3 = \xi_4,\ \ \dot{\xi}_4 = w. \tag{20}$$

Summarizing, the chain of four integrators (20) can be used to design both target single step walking-like trajectory and its stable tracking provided CVHC (9), (17) holds. The respective feedback law for w is then re-computed first to $\overline{u}_2$ using (13) and then to the real input torque u_2 using (11). Combining it with the feedback enforcement of CVHC (9), (17) using the feedback for u_3 provided by Algorithm 1 of Čelikovský [2015], or alternative Hamiltonian representation and backstepping based procedure in Čelikovský and Anderle [2017], then provides the **single-step walking design** for the three-link, *i.e.* the target walking trajectory feedforward and the exponentially stable tracking feedback of this feedforward.

Multi-step hybrid-cyclic walking design requires that the so-called **impact map** composed with legs relabelling that occurs at the double support phase maps velocities at the end of the step into the same velocities as they were at its beginning. The resulting target trajectory is then called as **hybrid-cyclic**. This composition is obtained by the specific modelling, Westervelt et al. [2007], and has the form $\dot{q}^+ = \Phi^{imp}(q^-)\dot{q}^-$, where

$$\Phi^{imp}(q) = [\phi_{ij}(q_1,q_2,q_3)],\ i,j = 1,2,3, \tag{21}$$

is the so-called **impact matrix** and the upper indices "-/+" mean "just before/after the impact and relabeling". To use the approach described in Remark 2.6, the CVHC (9), (17) has to be **hybrid-invariant**, *i.e.* for $q = (q_1,q_2,q_3)^\top$, $\dot{q} = (\dot{q}_1,\dot{q}_2,\dot{q}_3)^\top$:

$$\dot{q}_3^- = \phi_3'(q_2^-)\dot{q}_2^- \Rightarrow \dot{q}_3^+ = \phi_3'(q_2^+)\dot{q}_2^+,\ \dot{q}^+ = \Phi^{imp}(q^-)\dot{q}^-. \tag{22}$$

Proposition 2.7. Denote by $q^-, \dot{q}^-, q^+, \dot{q}^+$ coordinates and velocities "just before" and "just after" impact, respectively, and let the impact matrix (21) and CVHC (9) be given. Assume that

$$\Phi^{impM} = \begin{bmatrix}\phi_{11} & \phi_{12}+\phi_{13}\phi_3'(q_2^-) & 0\\ \phi_{21} & \phi_{22}+\phi_{23}\phi_3'(q_2^-) & -1\\ \phi_{31} & \phi_{32}+\phi_{33}\phi_3'(q_2^-) & -\phi_3'(q_2^+)\end{bmatrix}. \tag{23}$$

is invertible. Denote Q_1, Q_2 as

$$Q_1 := \frac{(\phi_{22}+\phi_{23}\phi_3'(q_2^-))\phi_3'(q_2^+) - \phi_{32} - \phi_{33}\phi_3'(q_2^-)}{\phi_{31}-\phi_{21}\phi_3'(q_2^+)}, \tag{24}$$

$$Q_2 := \frac{\phi_{21}Q_1+\phi_{22}+\phi_{23}\phi_3'(q_2^-)}{\phi_{11}Q_1+\phi_{12}+\phi_{13}\phi_3'(q_2^-)}. \tag{25}$$

Then the hybrid invariance (22) is satisfied for every $\dot{q}_1^-$ and:

$$\dot{q}_1^- = Q_1\dot{q}_2^-,\ \ \dot{q}_2^+ = Q_2\dot{q}_1^+, \tag{26}$$

$$\dot{q}_2^- = \dot{q}_1^+[\phi_{11}Q_1+\phi_{12}+\phi_{13}\phi_3'(q_2^-)]^{-1}, \tag{27}$$

$$\dot{q}_3^+ = \phi_3'(q_2^+)\dot{q}_2^+,\ \dot{q}_3^- = \phi_3'(q_2^-)\dot{q}_2^-. \tag{28}$$

Proof of Proposition 2.7 is given in Čelikovský and Anderle [2019]. Proposition 2.7 describes constructively how to choose the trajectories belonging during the whole swing phase to the restricted dynamics given by the selected CVHC and mapped by the impact inside that restricted dynamics again. One can simply choose arbitrarily the initial velocity $\dot{q}_1^+$ and then to adjust based on it all remaining velocities at the beginning of the step and end of the step using (26) - (28).

3. THE THREE-LINK MULTI-STEP WALKING DESIGN

3.1 Steering the chain of integrators

Motivated by (20), consider the following problem:

$$\dot{\xi} = J\xi + bw,\ \xi(0) = \xi_{in},\ \xi(T) = \xi_{fin}, \tag{29}$$

$$J = \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\ b = \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\ \xi := \begin{bmatrix}\xi_1\\\xi_2\\\xi_3\\\xi_4\end{bmatrix}, \tag{30}$$

where $\xi_{in} \in \mathbb{R}^4$, $\xi_{fin} \in \mathbb{R}^4$ and time interval $[0,T]$ are given and $w(t)$ is to be found. Introduce the following notation

$$\Delta T = T/k,\ \ k \in \mathbb{Z},\ k \geq 4,$$

$$\overline{A} = \exp(J\Delta T) = \begin{bmatrix} 1 & \Delta T & (\Delta T)^2/2 & (\Delta T)^3/6 \\ 0 & 1 & \Delta T & (\Delta T)^2/2 \\ 0 & 0 & 1 & \Delta T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (31)$$

$$A = \overline{A}^k = \exp(JT), \ \overline{b} = \left[\frac{\Delta T^4}{24}, \frac{\Delta T^3}{6}, \frac{\Delta T^2}{2}, \Delta T\right]^\top \quad (32)$$

$$\widetilde{A} := \left[\overline{A}^{k-1}\overline{b}|\overline{A}^{k-2}\overline{b}|\ldots|\overline{A}\overline{b}|\overline{b}\right]. \quad (33)$$

For the convenience, note that $\forall i = 0,1,2,\ldots$ it holds

$$\overline{A}^i = \exp(iJ\Delta T) = \begin{bmatrix} 1 & i\Delta T & (i\Delta T)^2/2 & (i\Delta T)^3/6 \\ 0 & 1 & i\Delta T & (i\Delta T)^2/2 \\ 0 & 0 & 1 & i\Delta T \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next, choose some $k \geq 4$ and consider the class of the following piecewise constant solutions $w(t)$ of the problem (29-30):

$$w(t) = w_{i-1}, t \in [(i-1)\Delta T, i\Delta T], \ i = 1,\ldots,k, \quad (34)$$

$$\overline{w} := (w_0,\ldots,w_{k-1})^\top \in \mathbb{R}^k. \quad (35)$$

Lemma 3.1. Recall (31), (32), then (29-30) is solved by

$$\xi^{i+1} = \overline{A}\xi^i + \overline{b}w_i, \ \xi^0 = \xi_{in}, \ \xi^k = \xi_{fin}, \quad (36)$$

$$\xi^i := \xi(i\Delta T) \in \mathbb{R}^4, \ i = 0,\ldots,k, \quad (37)$$

$$\overline{\xi} := ((\xi^1)^\top,\ldots,(\xi^{k-1})^\top)^\top \in \mathbb{R}^{4(k-1)}. \quad (38)$$

Proof. Solving (29-30) gives $\forall i = 0,1,\ldots,k-1$:

$$\xi((i+1)\Delta T) = \exp(J\Delta T)\left[\xi(i\Delta T) + w_i\int_0^{\Delta T}\exp(-Js)b\mathrm{d}s\right] =$$

$$\overline{A}\xi(i\Delta T) + \overline{b}w_i, \ \text{since} \ \exp(J\Delta T) = \overline{A},$$

$$\overline{A}\int_0^{\Delta T}\exp(-Js)b\mathrm{d}s = \overline{A}\int_0^{\Delta T}(-s^3/6, s^2/2, -s, 1)^\top\mathrm{d}s = \overline{b},$$

giving by (37), (32) the claim to be proved. □

Clearly, the problem (36) has no general solution for $k < 4$, it has a unique solution for $k = 4$ and infinite many solutions for $k > 4$. Specific methods to solve (36) are given next.

Input least squares only. Eliminating $\overline{\xi}$, the problem (36) can be straightforwardly reformulated as follows

$$\xi_{fin} = \overline{A}^k\xi(0) + \overline{A}^{k-1}\overline{b}w_0 + \ldots + \overline{A}\overline{b}w_{k-2} + \overline{b}w_{k-1} \implies$$

$$\xi_{fin} - A\xi_{in} = \widetilde{A}\overline{w}.$$

The well-known solution of the above linear equations is *e.g.*

$$\overline{w} = \widetilde{A}^\top[\widetilde{A}\widetilde{A}^\top]^{-1}(\xi_{fin} - A\xi_{in}), \quad (39)$$

being the unique solution to the optimization problem

$$\|\overline{w}\|^2 \to \min, \ \text{subject to} \ \widetilde{A}\overline{w} = (\xi_{fin} - A\xi_{in}),$$

as well. In such a way, (39) provides $\forall k \geq 4$ the **piecewise input** having the minimal quadratic norm $\|\overline{w}\|^2$ and steering the integrator chain (29-30).

Both input and intermediate states least squares. Keeping both $\overline{\xi}$ and $\overline{w}$ as equally free variables, the problem (36) can be straightforwardly reformulated as follows (here, 0_j stands for j-dimensional zero column vector):

$$\hat{A}\begin{bmatrix}\overline{\xi} \\ \overline{w}\end{bmatrix} = \overline{\xi}^{finin} := \begin{bmatrix}\overline{A}\xi_{in} \\ 0_{4k-8} \\ -\xi_{fin}\end{bmatrix} \in \mathbb{R}^{4k}, \quad (40)$$

$$\hat{A} := \left[\tilde{\tilde{A}}|\tilde{B}\right], \quad (41)$$

$$\tilde{\tilde{A}} = \begin{bmatrix} I_4 & 0 & \cdots & 0 & 0 \\ -\overline{A} & I_4 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & 0 \\ \vdots & 0 & -\overline{A} & I_4 & 0 \\ 0 & \cdots & 0 & -\overline{A} & I_4 \\ 0 & \cdots & 0 & 0 & -\overline{A} \end{bmatrix}, \quad (42)$$

$$\tilde{B} = \begin{bmatrix} -\overline{b} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\overline{b} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\overline{b} & 0 & 0 \\ 0 & \cdots & 0 & 0 & -\overline{b} & 0 \\ 0 & \cdots & 0 & 0 & 0 & -\overline{b} \end{bmatrix}, \quad (43)$$

where I_4 stands for the (4×4) identity matrix and $\hat{A}$ is the $(4k)\times(5k-4)$ dimensional matrix and $\overline{b}$ is given by (32). The vector $\overline{\xi}^{finin} \in \mathbb{R}^{4k}$ comprises in a compact way the initial and terminal conditions in (29), (36) which explains its upper index. The equivalent of (29), (36) is (40) having a unique solution

$$\begin{bmatrix}\overline{\xi} \\ \overline{w}\end{bmatrix} = W^{-1}\hat{A}^\top[\hat{A}W^{-1}\hat{A}^\top]^{-1}\overline{\xi}^{finin}, \quad (44)$$

W being arbitrary $(5k-4)\times(5k-4)$ matrix $W = W^\top > 0$.

Remark 3.2. Clearly, (44) solves the optimization problem

$$\left[\overline{\xi}^\top, \overline{w}^\top\right]W\begin{bmatrix}\overline{\xi} \\ \overline{w}\end{bmatrix} \to \min \ \text{ subject to} \ \hat{A}\begin{bmatrix}\overline{\xi} \\ \overline{w}\end{bmatrix} = \overline{\xi}^{finin}. \quad (45)$$

Matrix W in (44) can be chosen as $W = \mathrm{diag}[P,R]$, where P is $4(k-1)$-dimensional square matrix weighting the state, while P is k-dimensional square matrix weighting the input. In computations later on, just diagonal W is being used.

3.2 Step and mechanical parameters optimization

As it will be explained in the sequel, the optimization problem (45) depends implicitly and nonlinearly on further system parameters since $\xi(0) = \xi_{in}, \xi(T) = \xi_{fin}$ depend on these parameters. To make the optimization more feasible, rather than optimizing (45) itself, the solution (44) will be used for some simple diagonal pre-selected W and $k \in \mathbb{Z}$. Optimization criterion for numerical optimization later on is then formed as $\sum_{i=0}^{k-1}|\overline{w}_i|$, where $\overline{w}$ is given by (44). More specifically, the boundary conditions $\xi(0) = \xi_{in}, \xi(T) = \xi_{fin}$ are the images obtained by diffeomorphism (13) applied to the initial step states $q(0), \dot{q}(0)$ and terminal step states $q(T), \dot{q}(T)$ given by:

(i) Double support configuration at the beginning and the end of the step $q(0), q(T)$. It is uniquely given by the length of the step, which by a simple triangulation (cf. Fig 1) determines $q_1(0), q_2(0), q_1(T), q_2(T)$ and by the suggested CVHC $q_3(0) = \phi_3(q_2(0))$, $q_3(T) = \phi_3(q_2(T))$, ϕ_3 given by (17).

(ii) Initial and terminal velocities $\dot{q}(0), \dot{q}(T)$, that should again comply both with CVHC and impact conditions as postulated by Proposition 2.7 giving one free parameter only. Indeed, the value $\dot{q}_1(0) = \dot{q}_1^+$ can be freely selected and the remaining velocities are then given by $\dot{q}_2(0) = \dot{q}_2^+, \dot{q}_3(0) = \dot{q}_3^+, \dot{q}_1(T) = \dot{q}_1^-, \dot{q}_2(T) = \dot{q}_2^-, \dot{q}_3(T) = \dot{q}_3^-$ using (26)-(28).

(iii) The step duration $T > 0$ **time length.** This parameter is assumed to be a given and fixed design requirement.

Therefore only above **(i), (ii)** suggest some further free parameters available to optimize step. While **(i)** indicates a single clear parameter - the distance between legs at double support uniquely given by q_2^0, **(ii)** is more involved and suggests numerous free parameters. First of all, the value $\dot{q}_1(0)$. Yet, attempts to use only this value did not provide any reasonable results. Fortunately, **(ii)** contains implicitely **7 more independent mechanical parameters** (cf. Fig 1): $l_{c1} = l_{c2}$, $l_1 = l_2$, $I_1 = I_2$, $m_1 = m_2$, l_3, I_3, m_3. Indeed, both legs should be equal and l_{c3} should fulfill torso adjustment law (16) (inevitable for the exact feedback linearizability of the restricted dynamics). These parameters enter not only to a swing phase Lagrange model, but also to the impact model derived properly according theory in Westervelt et al. [2007]. Summarizing, there are **9 independent parameters to be optimized**, further denoted as the vector x. Remaining ingredients of the optimization procedure are:

Performance index (criterion). Using (44), one can compute for any selection of 9 parameters x the vector $\overline{w}$ and based on them the scalar value $\sum_{i=0}^{k-1} |\overline{w}_i|$, which is the criterion to optimize, detoted in the sequel *costFun*(x). Note, that crucial goal is to find at least some feasible set of parameters, but some criterion is needed to use *FMINCON* later on.

Constraints. Linear constraints on $x \in \mathbb{R}^9$ are:

$$l_{1,2,3} \geq l^-_{1,2,3}, \ \ l_{1,2} \geq l_{c1,c2} \geq l^-_{c1,c2}, \ \ I_{1,2,3} \geq 0,$$

$$m_{1,2,3} \geq m^-_{1,2,3}, \ \dot{q}_1 \in [v_{min}, v_{max}], \ q_2^0 \in [q^0_{2min}, q^0_{2max}],$$

where $l^-_{1,2,3} > 0, m^-_{1,2,3} > 0, v_{min}, v_{max}, q^0_{2min}, q^0_{2max} \in \mathbb{R}$ are some practically reasonable selected margins. Nonlinear constraints are ($l^-_{c3} > 0$ is some minimal pre-selected margin for l_{c3}):

$$l^-_{c3} \leq (1/m_3) l_{c_2} m_2 (1 + \cos q_2^0)(1 - \sin(q_2^0/2))^{-1} \leq l_3,$$

due to the fact that l_{c3} is not free parameter, but is expressed via the other parameters using (16). There are three more nonlinear constraint due to the practical feasibility of the impact (some tip leg velocity and contact force constraints, see Westervelt et al. [2007] for details), that are not given here explicitly. Summarizing, we have in total **17 linear and 5 nonlinear inequality constraints.**

To attenuate all these aspects and find parameters x, the nonlinear programming solver *FMINCON* was used as follows

x = fmincon(costFun,x0,A,b,[ ],[ ],lb,ub,nonlCon),

where $costfun(x)$ was described above, $x0$ is some initial parameters guess, A, b define linear type constraints $Ax \leq b$, while *nonlCon* the above mentioned nonlinear constraints and lb, ub lower and upper bounds type constraints $lb \leq x0 \leq ub$.

4. SIMULATIONS

Due to the non-convex character of the optimization problem the optimal values of the three-link parameters may not be uniquely determined or even may not exist. Yet, a locally optimal feasible vector of parameters values x was successfully computed using *FMINCON*. Obtained set of parameters is listed in Tab. 1. Their importance is not in immediate use in laboratory experiments but rather demonstrating that walking with exact feedback linearizable restricted dynamics **is possible** for the conveniently built three-link. This encourages a future investigation trying to reduce number of optimized parameters fixing some of them at values of real model built in advance while others adjustable in a similar way as the torso position at Fig. 1. Nevertheless, it is not sufficient just to have feasible outcome of the parameters numerical optimization, but one has to demonstrate that these parameters provide sustainable hybrid-stable walking.

Indeed, the current section will show that the parameters given in Tab. 1 guarantee that the step in $q, \dot{q}$-coordinates was successfully performed in requested time. Since the impact conditions are guaranteed theoretically by the conditions incorporated in various *FMINCON* constraints, hybrid cyclic character of the multi-step trajectory is even theoretically justified, and double-checked by simulations. Moreover, using the integrators chain enable not only to design the target hybrid cyclic trajectory (say **feedforward** part of the controller), but even more simply its **feedback** part guaranteeing exponential tracking of that target trajectory during each swing phase. This then results in overall hybrid stability, despite the impact effect.

To describe these simulations in detail, let us start with Fig. 2 showing target one-step trajectory in the ξ-coordinates of the integrator chain (20). Again, it is guaranteed even theoretically that their sampled values fulfill all conditions implemented in *FMINCON* constraints. Actually, solutions in terms of ξ-coordinates were always easy to achieve even without complex numerical computations. The real challenge emerges when going back to $q, \dot{q}$-coordinates. Linearizing transformations posses natural singularities at some awkward three-link configurations and only implementing the control strategy in $q, \dot{q}$-coordinates shows if these singularities were avoided, or not. During reasonable naturally looking walking step pattern no singularities are present, so transformations failure indicates that the designed step would not be anyway reasonable (*e.g.* forcing the swinging legs to perform a few 2π rotation).

In this context we are happy to say that our efforts lead to successful target trajectory tracking in $q, \dot{q}$-coordinates depicted in Figs. 3, 4. The dotted lines represent the reference values from the reference step whereas the solid lines represent $q, \dot{q}$ values of the controller model of the three-link. Only three steps are shown, yet clearly demonstrating both hybrid cyclic target trajectory and its stable tracking. Animations of these courses are shown in Fig. 5 where the reference is dashed and the controlled "real" three-link is depicted by the solid line.

Table 1. Computed parameters of the three-link

l_1, l_2	length of both legs	0.68	$[m]$
l_3	length of the torso	1.02	$[m]$
l_{c1}, l_{c2}	COM location of both legs	0.23	$[m]$
l_{c3}	COM location of the torso	1.02	$[m]$
m_1, m_2	mass of both legs	0.97	$[Kg]$
m_3	mass of the torso	1.71	$[Kg]$
I_1, I_2	moment of inertia of both legs	0.01	$[m^2 Kg]$
I_3	moment of inertia of the torso	0.60	$[m^2 Kg]$

5. CONCLUSIONS AND OUTLOOKS

This paper demonstrated that one can, in principle, construct walking-like mechanism possessing hybrid stable target walking trajectory that is between impacts state and feedback equivalent to the linear chain of four integrators.

The ongoing and future research focus to extending that concept to more realistic settings, both in term of partly *a priori* fixed mechanical parameters and configurations. We expect that adding the knees, or even hands, may provide further naturally selectable mechanical parameters, as well as virtual constrains making realistic feasible solutions easier to be achieved.

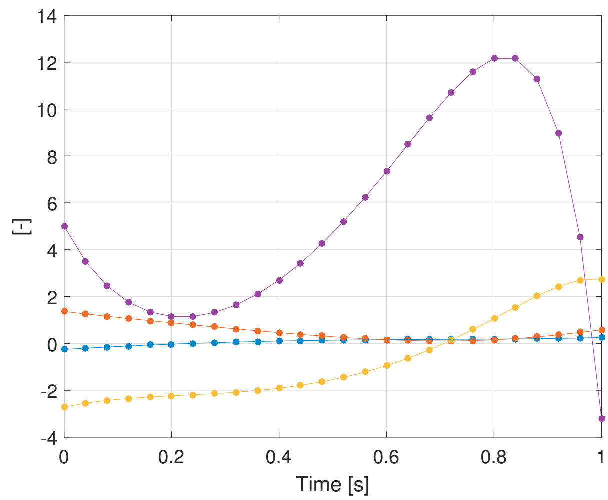

Fig. 2. Linear coordinates ξ. The blue, red, yellow and violet line corresponds to ξ_1, ξ_2, ξ_3, and ξ_4, respectively.

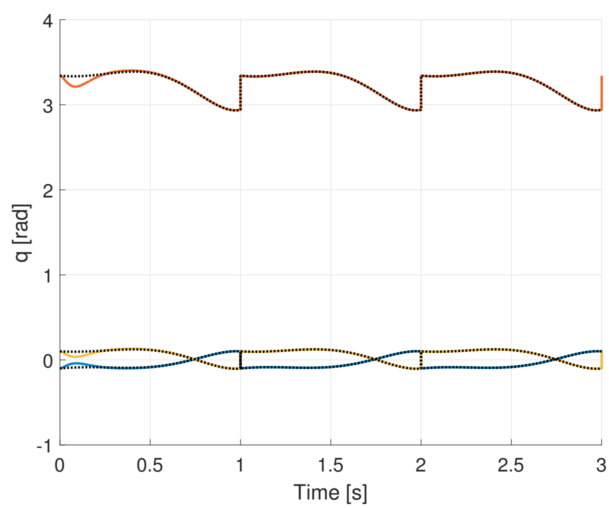

Fig. 3. The angular positions of the 3link. The blue, red and yellow lines correspond to q_1, q_2, and q_3, respectively.

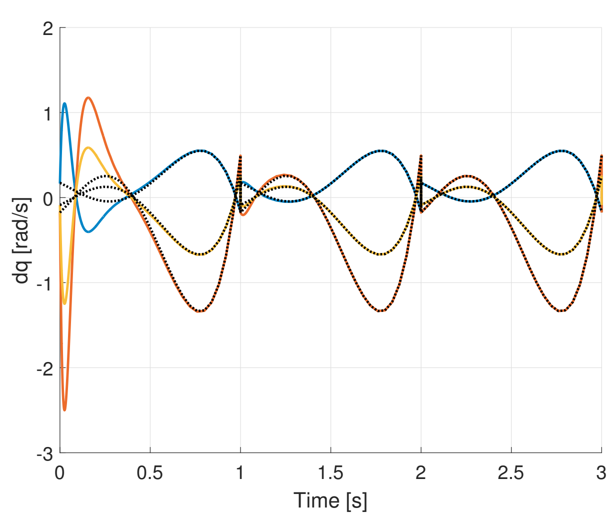

Fig. 4. The angular velocities of the 3link. The blue, red and yellow lines correspond to dq_1, dq_2, and dq_3, respectively.

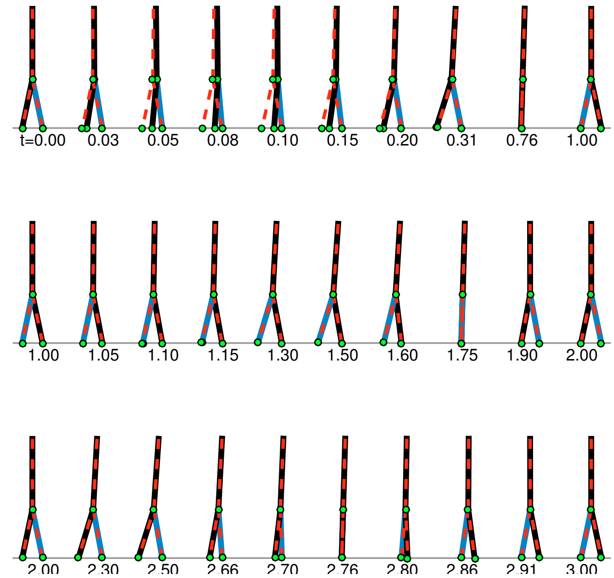

Fig. 5. The animation of the three-link hybrid stable walking.

REFERENCES

S. Čelikovský. Flatness and realization of virtual holonomic constraints. In *Proceedings of the 5th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control (2015)*, pages 25–30, Lyon, France, 2015.

S. Čelikovský and M. Anderle. On the Hamiltonian approach to the collocated virtual holonomic constraints in the underactuated mechanical systems. In *Proceedings of the 4th International Conference on Advanced Engineering Theory and Applications (AETA)*, Ho Chi Min City, Vietnam, 2017.

S. Čelikovský and M. Anderle. Stable walking gaits for a three-link planar biped robot with two actuators based on the collocated virtual holonomic constraints and the cyclic unactuated variable. *IFAC-PapersOnLine*, 51(22):378–385, 2018. 12th IFAC SYROCO.

S. Čelikovský and M. Anderle. Exact feedback linearization of the collocated constrained dynamics of the three-link with adjustable torso and its application in the underactuated planar walking. In *2019 IEEE 15th International Conference on Control and Automation (ICCA)*, pages 1289–1295, Edinburg, Scotland, 2019.

R. Olfati-Saber. Normal forms for underactuated mechanical systems with symmetry. *IEEE Transactions on Automatic Control*, 47(2):305–308, Feb. 2002.

A. R. Teel. Global stabiliation and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18:165–171, 1992.

E. Westervelt, J. Grizzle, Ch. Chevallereau, J. Choi, and B. Morris. *Feedback control of dynamic bipedal robot locomotion*. CRC Press, 2007.

B. Zhou and J. Lam. Global stabilization of linearized spacecraft rendezvous system by saturated linear feedback. *IEEE Transactions on Control Systems Technology*, 25(6):2185–2193, 2017.

B. Zhou and X. Yang. Global stabilization of the multiple integrators system by delayed and bounded controls. *IEEE Transactions on Automatic Control*, 61(12):4222–4228, 2016.