2021 29th Mediterranean Conference on Control and Automation (MED)
June 22-25, 2021. Bari, Puglia, Italy

Non-fragile control of complex network composed of identical systems with delayed inputs - application of the descriptor approach

Volodymyr Lynnyk and Branislav Rehák

Abstract— **The non-fragile control of complex network with delayed controls is proposed. The control is designed using the linear matrix inequalities with application of the descriptor approach. The system is stabilized even in presence of perturbations of the control gain. Moreover, the time delay of the inputs can be varying, the derivative of the time delay is not bounded. This makes the proposed algorithm applicable to the networked control of large-scale systems. The results are illustrated by example.**

I. Introduction

The problem of controlling networks of interconnected systems is often met in practice. As such networks cannot be controlled in a centralized manner, decentralized control laws were proposed, see e.g. [1], [2]. This paper is focused on the control problem of complex networks composed of identical subsystems; these networks are also often encountered ([3], [4] or others). The goal is to design a control law that is identical for each subsystem and for any particular subsystem, the controller uses information about state of this subsystem only, the control does not depend on the state of other subsystems. Hence the control law must be robust enough to mitigate the effects of the interconnections as explained e.g. in [5] or [6], among others. Usually, the algorithms are formulated using linear matrix inequalities (LMI).

However, this is not the only effect making the control design challenging. Control of large-scale networks usually required to utilize communication networks for transmitting signals from sensors to the controller or from the controller to the actuators. This allows for a cost-effective implementation of the control strategy but also poses some difficulties. The transmitted signals may be delayed, e.g. due to the packet dropouts, hence the control law must be able to stabilize the system under fast-varying time-varying time delay, see e.g. [7], [8] or [9] where the Razumikhin functional is used to derive the control law. In the recent time, application of the descriptor approach (e.g. [10]) delivered promising results, such as in [11] for nonlinear networks of interconnected systems. Application of the sampled control of large-scale systems is studied in [12]. Other factor to be taken into account is quantization of signals as discussed in [13], among others.

If a large number of controllers are to be used, it is natural to assume that not all of them have exactly the same properties. Also, some variations in the controller might happen. Hence the need for a control law able to tolerate certain changes in the controller gain. One of the possibilities is to use additive perturbations, e.g. in [14], [15] or multiplicative ones as in [16], [17]; these are also considered in this paper.

Purpose of this paper

- An algorithm for stabilization of an interconnected network composed of identical systems with delayed controls is presented,
- Non-fragile control law is proposed allowing for controller gain changes and its perturbations,
- The time delay changes can be rapid, thus the algorithm is applicable for systems with sampled control.

Notation:

1) If P is a square symmetric positive definite matrix, then we write $P > 0$.
2) In symmetric matrices, the elements below the diagonal are not written explicitly, they are replaced by an asterix: $\begin{pmatrix} a & b \\ b^T & c \end{pmatrix} = \begin{pmatrix} a & b \\ * & c \end{pmatrix}$.
3) Where no confusion can arise, the time argument t may be omitted for brevity, the time delay is written using subscript: $x = x(t)$, $x(t-\tau) = x_\tau(t) = x_\tau$. If the time argument is different from t, it is written in full.
4) The symbol I_m denotes the m-dimensional identity matrix.
5) The symbol $\|.\|$ denotes the Euclidean norm (even for matrices).
6) The symbol $\otimes$ stands for the Kronecker product.

II. Problem setting

In this paper, we deal with a complex network (large-scale system) composed of N identical subsystems (nodes). This network is defined as follows: First, let $A, \tilde{A} \in R^{n\times n}$, $B \in R^{n\times m}$, $G \in R^{n\times q}$.

For every $i = 1, \dots, N$ the ith subsystem is described as

$$\dot{x}_i = Ax_i + Bu_i + \sum_{j=1}^{N} l_{ij}\tilde{A}x_j + Gw_i, \tag{1}$$

$$x_i(0) = x_{i,0}. \tag{2}$$

Here, $x_i(t) \in R^n$ is the state of the ith subsystem, the disturbance is denoted by $w_i(t) \in R^q$. The matrix $L = (l_{ij}) \in R^{N\times N}$ is called interconnection matrix. Its elements are defined as follows: $l_{ij=1}$ if there is a connection directly from the jth to the ith subsystem, otherwise $l_{ij} = 0$.

Remark II.1. *There is some similarity of the large-scale systems control to the synchronization of multi-agent systems.*

V. Lynnyk and B. Rehák are with the Czech Academy of Sciences, Institute of Information Theory and Automation, Pod vodárenskou věží 4, 182 00 Praha 8, Czech Republic, {voldemar, rehakb}@utia.cas.cz

This work was supported by the Czech Science Foundation through the grant No. GA19-05872S.

It uses analogous mathematical tools methods and notations. Still, the problems are not identical. The control of large-scale systems has this characteristic feature: the control action of one particular subsystem is computed only from the state of this subsystem. In contrast, for synchronization of complex networks, the control is computed from the sum of differences between the state of this node and the neighboring nodes. The coupling of the subsystems in the large-scale interconnected system is represented by matrix $\tilde{A}$ which has no counterpart in the case of complex networks. The notion of the disagreement vector has no sense for the large-scale systems. On top of that, the couplings in the large-scale interconnected system are representations of physical interconnections of the subsystems. More can be found in [18]. Let us also not an important distinctive feature between the multi-agent systems and interconnected networks with time delays. If the time delays are not equal, then, in general, the synchronization error in the multi-agent system may not converge to zero ([19], [20], [21], [22]) while, as shown e.g. in [11], stabilization of the interconnected network can still be achieved if delays in the subsystems are not equal.

Assumption II.2. *The interconnection matrix L is symmetric. Moreover, we assume that $l_{ii} = 0$ for each $i = 1,\ldots,N$.*

This assumption implies that matrix L has N real eigenvalues with multiplicity 1 and its eigenvectors are orthogonal: there exists an orthogonal matrix $T \in R^{N\times N}$ and a diagonal matrix D so that

$$L = T^T DT. \tag{3}$$

As eigenvalues of matrix L are real, we can write without loss of generality $D = \mathrm{diag}(d_1,\ldots,d_N)$ and $d_1 \le \cdots \le d_N$, $d_i \in \mathbb{R}$.

Assumption II.3. *There exists a constant $\bar{\tau} > 0$ and measurable functions $\tau_i : [0,\infty) \to [0,\bar{\tau}]$.*

These functions are used to express the time delays.

Assume there are matrices $D_K \in R^{m\times\mu}$, $E_K \in R^{\mu\times n}$ and an N-tuple of measurable matrix-valued functions $F_{i,K} : [0,\infty) \to R^{\mu\times\mu}$, $i = 1,\ldots,N$ so that, for every $t \in [0,\infty)$ and every $i = 1,\ldots,N$ holds $\|F_i(t)\| \le 1$.

The goal is to find a matrix $K \in R^{m\times n}$ so that the control laws given by

$$u_i = (I_m + D_K F_{i,K} E_K)K x_{\tau_i} \tag{4}$$

stabilizes the overall system for any (unknown) set of functions F_i. Let $x = (x_1^T,\ldots,x_N^T)^T$, $w = (w_1^T,\ldots,w_N^T)^T$, $\tilde{x} = (x_{1,\tau_1}^T,\ldots,x_{N,\tau_N}^T)^T$ and $u = (u_1^T,\ldots,u_N^T)^T$. Then the N-tuple of subsystems (1) can be written compactly as

$$\dot{x} =(I_N \otimes A)x + (L\otimes\tilde{A})x + (I_N \otimes B)u + (I_N \otimes G)w. \tag{5}$$

System (5) is called the overall system. With the control law (4), one can write

$$\begin{aligned}\dot{x} =&(I_N \otimes (A+BK))x + (L\otimes\tilde{A})x + (I_N \otimes BK)\tilde{x}\\ &+ (I_N \otimes BD_K)\mathrm{diag}(F_{1,K}(t),\ldots,F_{N,K}(t))(I_N \otimes E_K K)\tilde{x}\\ &+ (I_N \otimes G)w.\end{aligned} \tag{6}$$

III. H_∞ CONTROL AND NON-FRAGILE CONTROL FOR DELAYED SYSTEMS

Our investigations are based on the following result (for its proof, see [23], Proposition 5.3)

Theorem III.1. *Consider the system*

$$\dot{\zeta} = \mathscr{A}\zeta + \mathscr{B}\mathscr{K}\zeta_\tau + \mathscr{G}\omega \tag{7}$$

where $\mathscr{A} \in R^{\mathbf{n}\times\mathbf{n}}$, $\mathscr{B} \in R^{\mathbf{n}\times\mathbf{m}}$, $\mathscr{K} \in R^{\mathbf{m}\times\mathbf{n}}$, $\mathscr{G} \in R^{\mathbf{n}\times\mathbf{q}}$, the time delay τ satisfies Assumption II.3. Let there exist $\mathbf{n}\times\mathbf{n}$-dimensional matrices $\mathscr{P} > 0$, $\mathscr{Q}$ nonsingular, $\mathscr{R} > 0$, $\mathscr{S} > 0$ and $\mathscr{M}$, a $\mathbf{m}\times\mathbf{n}$-dimensional matrix $\mathscr{Y}$ and scalars $\gamma > 0$ and $\varepsilon > 0$. Assume also matrix Ω given as

$$\begin{aligned}\Omega_{11} =&\mathscr{A}^T\mathscr{Q} + \mathscr{Q}^T\mathscr{A} - \mathscr{R} + \mathscr{S},\\ \Omega_{12} =&\mathscr{P} - \mathscr{Q}^T + \varepsilon\mathscr{A}^T\mathscr{Q}^T,\\ \Omega_{22} =&-\varepsilon(\mathscr{Q}+\mathscr{Q}^T) + \bar{\tau}^2\mathscr{R},\\ \Omega_{33} =&-(\mathscr{R}+\mathscr{S}),\\ \Omega_{14} =&\mathscr{B}\mathscr{Y} + \mathscr{R} - \mathscr{M},\\ \Omega_{44} =&-2\mathscr{R} + \mathscr{M} + \mathscr{M}^T,\end{aligned}$$

$$\Omega = \begin{pmatrix}\Omega_{11} & \Omega_{12} & \mathscr{M} & \Omega_{14} & \mathscr{G} & \mathscr{Q}^T\\ * & \Omega_{22} & 0 & \varepsilon\mathscr{B}\mathscr{Y} & \varepsilon\mathscr{G} & 0\\ * & * & \Omega_{33} & \mathscr{R}-\mathscr{M}^T & 0 & 0\\ * & * & * & \Omega_{44} & 0 & 0\\ * & * & * & * & -\gamma I_{\mathbf{q}} & 0\\ * & * & * & * & * & -I_{\mathbf{n}}\end{pmatrix}$$

satisfies

$$\Omega < 0 \tag{8}$$

and, moreover, let

$$\begin{pmatrix}-\mathscr{R} & \mathscr{M}\\ * & -\mathscr{R}\end{pmatrix} < 0. \tag{9}$$

Let also

$$\mathscr{K} = \mathscr{Y}\mathscr{Q}^{-1}. \tag{10}$$

Then, system (7) is H_∞-stable.

Remark III.2. *The proof is given in [23] as Proposition 5.3. The cited version corresponds precisely to this proposition. Note that there is only one delay τ. As the topic of this paper are multidimensional systems, a MIMO version of this theorem is necessary. However, inspection of the proof in [23] reveals that this generalization is quite straightforward. Hence it is not presented here for the space reasons.*

Remark III.3. *Under H_∞-stability of system (7), the following is understood: If $w = 0$ for all $t \in [0,\infty)$ then (7) is asymptotically stable. If, on the other hand, if $\zeta(t) = 0$ for all $t \in [-\bar{\tau},0]$, then $\|\zeta\| \le \gamma\|w\|$.*

Now let us focus attention on the case when the control gain matrix is subject to perturbations. To be specific, assume there are matrices $\mathscr{D}_K \in R^{\mathbf{n}\times\mathbf{p}}$ and $\mathscr{E}_K \in R^{\mathbf{p}\times\mathbf{n}}$ and a measurable matrix-valued function $\mathscr{F} : [0,\infty) \to R^{\mathbf{p}\times\mathbf{p}}$ satisfying $\|\mathscr{F}(t)\| \le 1$ for all $t \in [0,\infty)$. Finally, the control gain matrix can be expressed as the sum of a constant term (called nominal control gain, denoted by $\mathscr{K}_0$) and a perturbation, hence it is time-dependent. The multiplicative perturbation will be used here:

$$\mathscr{K}(t) = (I_\mathbf{m} + \mathscr{D}_K \mathscr{F}(t) \mathscr{E}_K)\mathscr{K}_0. \tag{11}$$

This implies that matrix $\mathscr{Y}$ is also time-dependent. From (10) follows that

$$\mathscr{Y}(t) = (I_\mathbf{m} + \mathscr{D}_K \mathscr{F}(t) \mathscr{E}_K)\mathscr{Y}_0 \tag{12}$$

where $\mathscr{Y}_0 = \mathscr{K}_0 \mathscr{Q}^{-1}$. Substituting this into the definition of matrix Ω leads to the definition of a matrix valued function $\Omega'(t)$. To be specific, the time varying terms appear on the positions $(1,4)$ and $(2,4)$. Let

$$\begin{aligned}
\Omega'_{14}(t) =& \mathscr{B}\mathscr{Y}_0 + \mathscr{B}\mathscr{D}_K\mathscr{F}(t)\mathscr{E}_K\mathscr{Y}_0 + \mathscr{R} - \mathscr{M},\\
\Omega'_{24}(t) =& \varepsilon\mathscr{B}\mathscr{Y}_0 + \varepsilon\mathscr{B}\mathscr{D}_K\mathscr{F}(t)\mathscr{E}_K\mathscr{Y}_0,
\end{aligned}$$

$$\Omega'(t) = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \mathscr{M} & \Omega'_{14}(t) & \mathscr{G} & \mathscr{Q}^T \\ * & \Omega_{22} & 0 & \Omega'_{24}(t) & \varepsilon\mathscr{G} & 0 \\ * & * & \Omega_{33} & \mathscr{R} - \mathscr{M}^T & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 \\ * & * & * & * & -\gamma I_\mathbf{n} & 0 \\ * & * & * & * & * & -I_\mathbf{n} \end{pmatrix}$$

Define the following matrices:

$$\begin{aligned}
\Omega'_{11} =& \Omega_{11} + \lambda\mathscr{B}\mathscr{D}_K\mathscr{D}_K^T\mathscr{B}^T,\\
\Omega'_{12} =& \Omega_{12} + \lambda\varepsilon\mathscr{B}\mathscr{D}_K\mathscr{D}_K^T\mathscr{B}^T,\\
\Omega'_{14} =& \mathscr{B}\mathscr{Y}_0 + \mathscr{R} - \mathscr{M},\\
\Omega'_{22} =& \Omega_{22} + \lambda\varepsilon^2\mathscr{B}\mathscr{D}_K\mathscr{D}_K^T\mathscr{B}^T,\\
\Omega'_{44} =& \Omega_{44} + \frac{1}{\lambda}\mathscr{Y}_0^T\mathscr{E}_K^T\mathscr{E}_K\mathscr{Y}_0.
\end{aligned}$$

Proposition III.4. *With matrices $\mathscr{B}$, $\mathscr{D}_K$ etc. defined above, the following LMI holds for every $\lambda > 0$:*

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega'_{14}(t) \\ * & \Omega_{22} & \Omega'_{24}(t) \\ * & * & \Omega_{44} \end{pmatrix} \le \begin{pmatrix} \Omega'_{11} & \Omega'_{12} & \Omega'_{14} \\ * & \Omega'_{22} & \varepsilon\mathscr{B}\mathscr{Y}_0 \\ * & * & \Omega'_{44} \end{pmatrix}$$

Proof. Simple application of the Young inequality. □

The following corollary of this proposition will be useful: first, matrix Ω'' as:

$$\Omega'' = \begin{pmatrix} \Omega'_{11} & \Omega'_{12} & \mathscr{B}\mathscr{Y}_0 + \mathscr{R} - \mathscr{M} & 0 \\ * & \Omega'_{22} & \varepsilon\mathscr{B}\mathscr{Y}_0 & 0 \\ * & * & \Omega'_{44} & \mathscr{Y}_0^T\mathscr{E}_K^T \\ * & * & * & -\lambda I_\mathbf{p} \end{pmatrix}$$

Corollary III.5. *Under the above assumptions, the following implication holds for all $t \in [0,\infty)$:*

$$\Omega'' < 0 \Rightarrow \Omega'(t) < 0 \tag{13}$$

Proof. Application of the Schur complement to the right-hand side of the LMI in Proposition III.4. □

Define now the following matrix:

$$\Omega''' = \begin{pmatrix} \Omega'_{11} & \Omega'_{12} & \mathscr{M} & \Omega'_{14} & \mathscr{G} & \mathscr{Q}^T & 0 \\ * & \Omega'_{22} & 0 & \varepsilon\mathscr{B}\mathscr{Y}_0 & \varepsilon\mathscr{G} & 0 & 0 \\ * & * & \Omega_{33} & \mathscr{R} - \mathscr{M}^T & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & \mathscr{Y}_0^T\mathscr{E}_K^T \\ * & * & * & * & -\gamma I_\mathbf{q} & 0 & 0 \\ * & * & * & * & * & -I_\mathbf{n} & 0 \\ * & * & * & * & * & * & -\lambda I_\mathbf{p} \end{pmatrix}.$$

Then, combining Theorem III.1 with Corollary III.5, one obtains the following lemma:

Lemma III.6. *Consider the system* (7) *where $\mathscr{A} \in R^{\mathbf{n}\times\mathbf{n}}$, $\mathscr{B} \in R^{\mathbf{n}\times\mathbf{m}}$, $\mathscr{K} \in R^{\mathbf{m}\times\mathbf{n}}$, $\mathscr{G} \in R^{\mathbf{n}\times\mathbf{q}}$, the time delay τ satisfies Assumption II.3. Let there exist $\mathbf{n}\times\mathbf{n}$-dimensional matrices $\mathscr{P} > 0$, $\mathscr{Q}$ nonsingular, $\mathscr{R} > 0$, $\mathscr{S} > 0$ and $\mathscr{M}$, a $\mathbf{m}\times\mathbf{n}$-dimensional matrix $\mathscr{Y}$ and scalars $\gamma > 0$, $\lambda > 0$ and $\varepsilon > 0$. Assume there exist matrices $\mathscr{D}_K \in R^{\mathbf{n}\times\mathbf{p}}$ and $\mathscr{E}_K \in R^{\mathbf{p}\times\mathbf{m}}$ such that the control gain satisfies* (4) *with some matrix-valued function $\mathscr{F}$ satisfying $\|\mathscr{F}(t)\| \le 1$ for all $t \in [0,\infty)$. Denote also $\mathscr{Y}_0 = \mathscr{K}_0\mathscr{Q}^{-1}$. Let also*

$$\Omega''' < 0, \tag{14}$$

$$\begin{pmatrix} -\mathscr{R} & \mathscr{M} \\ * & -\mathscr{R} \end{pmatrix} < 0. \tag{15}$$

Then system (7) *is H_∞-stable.*

IV. NON-FRAGILE CONTROL OF LARGE-SCALE INTERCONNECTED SYSTEMS

Results of the previous section will be applied to the overall system (6). Note that, up to the term $L \otimes \tilde{A}$ and the uncertainty in the control gain, all other matrices can be expressed as the Kronecker product of the identity matrix I_N with another matrix. Hence matrices $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{S}$, $\mathscr{M}$ and $\mathscr{Y}_0$ will also be supposed to attain this form.

Assume there exist $n \times n$-dimensional matrices P, Q, R, S and M and a $m \times n$-dimensional matrix Y_0. With these matrices, define matrix Γ as follows: in matrix Ω''', replace matrix $\mathscr{A}$ by the matrix $I_N \otimes A + L \otimes \tilde{A}$, $\mathscr{B}$ by $I_N \otimes B$, $\mathscr{G}$ by $I_N \otimes G$, $\mathscr{D}_K$ by $I_N \otimes D_K$, $\mathscr{E}_K$ by $I_N \otimes E_K$, $\mathscr{P}$ by $I_N \otimes P$, $\mathscr{Q}$ by $I_N \otimes Q$, $\mathscr{R}$ by $I_N \otimes R$, $\mathscr{S}$ by $I_N \otimes S$, $\mathscr{M}$ by $I_N \otimes M$ and finally $\mathscr{Y}_0$ by $I_N \otimes Y_0$. Substitute also nN for $\mathbf{n}$ and $N\mu$ for $\mathbf{p}$.

Then, the immediate consequence of Lemma III.6 is

Lemma IV.1. *Consider system* (6) *satisfying Assumption II.3. Assume there exist $n \times n$-dimensional matrices $P > 0$, $R > 0$, $S > 0$, Q nonsingular and M, a $m \times n$-dimensional matrix Y_0 and scalars $\lambda > 0$, $\gamma > 0$ and $\varepsilon > 0$ so that*

$$\Gamma < 0, \tag{16}$$

$$\begin{pmatrix} -(I_N \otimes R) & (I_N \otimes M) \\ * & -(I_N \otimes R) \end{pmatrix} < 0. \tag{17}$$

Then system (6) *is H_∞ stable with control gain $K = Y_0 Q$.*

Proof. This is a reformulation of Lemma III.6 for the interconnected system (6), the terms $I_N \otimes BK$ and $I_N \otimes E_K K$ can be rewritten as $(I_N \otimes B)(I_N \otimes K)$ and $(I_N \otimes E_K)(I_N \otimes K)$, respectively. Hence the control gain matrix in (6) is $I_N \otimes K$ which equals $(I_N \otimes Y_0)(I_N \otimes Q) = I_N \otimes Y_0 Q$. $\square$

The LMI problem formulated in Lemma IV.1 can be used for obtaining the stabilizing control. However, the size of this set of LMIs is rather large: it is proportional to the number of subsystems. This number will be reduced in the sequel.

For this reduction, two cases of interconnections will be distinguished. First, a rather special case is the "symmetrically interconnected system" - this means, every subsystem is connected with every other subsystem. In this case, as [1], Chapter 12 shows that, with matrix T' defined as

$$T' = \frac{1}{N}\begin{pmatrix} N-1 & -1 & \dots & -1 & -1 \\ -1 & N-1 & \dots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \dots & N-1 & -1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}, \tag{18}$$

the following relation holds for the interconnection matrix L:

$$L = T'^{-1}\operatorname{diag}(\underbrace{-1,\dots,-1}_{N-1 \text{ times}}, N-1)T' \tag{19}$$

Thanks to the properties of the Kronecker product, one sees that

$$(T' \otimes I_n)^{-1}(I_N \otimes A + L \otimes \tilde{A})(T' \otimes I_n) = I_N \otimes A + \operatorname{diag}(\underbrace{-1,\dots,-1}_{N-1 \text{ times}}, N-1) \otimes \tilde{A}. \tag{20}$$

Hence the term $I_N \otimes A + L \otimes \tilde{A}$ is transformed into a block-diagonal matrix.

A similar procedure can be conducted if the interconnections satisfy (3) but are not, in general, symmetrically connected. Then

$$(T \otimes I_n)^{-1}(I_N \otimes A + L \otimes \tilde{A})(T \otimes I_n) = I_N \otimes A + D \otimes \tilde{A}. \tag{21}$$

Again, this procedure converts the state matrix into a block-diagonal form.

Remark IV.2. *Note that the symmetrically interconnected systems also satisfy condition* (3). *However, it is worth to point out that a special case of the transformation matrix T is applicable in this case. First reason is that using the transformation matrix in form* (18) *avoids the need to compute the eigenvalues of matrix L. The second one is that an alternative formulation can exhibit different properties in terms of feasibility of the resulting set of LMI, yields a different value of the constant γ, etc. Hence, it can be useful to consider both options for a symmetrically interconnected system.*

V. Reducing the dimension of the LMI optimization problem

As noted in the previous section, there are two ways how to diagonalize matrix L, one being applicable only for the symmetrically interconnected systems. Denote as $\overline{D} = D$ if the more general way based on eigenvalues computations of matrix L is chosen; if the system is symmetrically interconnected and this special structure is utilized, then let $\overline{D} = \operatorname{diag}(\underbrace{-1,\dots,-1}_{N-1 \text{ times}}, N-1)$. Analogously, define $\overline{T}$ as $\overline{T} = T$ in the first case and $\overline{T} = T'$ if one uses the symmetrical interconnection structure.

Either way, one has

$$(\overline{T}^{-1} \otimes I_n)(L \otimes \tilde{A})(\overline{T} \otimes I_n) = (\overline{D} \otimes I_n). \tag{22}$$

Define matrix $\overline{\Gamma}$ from matrix Ω''' as follows: replace matrix $\mathscr{A}$ by $I_N \otimes A + \overline{D} \otimes \tilde{A}$, the other substitutions are the same as in the definition of matrix Γ. Define also matrix $\mathbb{T}$ as $\mathbb{T} = \operatorname{diag}(\overline{T} \otimes I_n, \overline{T} \otimes I_n, \overline{T} \otimes I_n, \overline{T} \otimes I_n, \overline{T} \otimes I_q, \overline{T} \otimes I_n, \overline{T} \otimes I_\mu)$. Then

$$\mathbb{T}^{-1}\Gamma\mathbb{T} = \overline{\Gamma}. \tag{23}$$

To proceed further, let us define the following matrix-valued functions:

$$\Lambda_{11}(d) = (A + d\tilde{A})^T Q + Q^T (A + d\tilde{A}) - R + S + \lambda B D_K D_K^T B^T,$$
$$\Lambda_{12}(d) = P - Q^T + \varepsilon(A + d\tilde{A})^T Q^T + \lambda \varepsilon B D_K D_K^T B^T,$$

and matrices

$$\Lambda_{14} = BY_0 + R - M,$$
$$\Lambda_{22} = -\varepsilon(Q + Q^T) + \bar{\tau}^2 R + \lambda \varepsilon^2 B D_K D_K^T B^T,$$
$$\Lambda_{33} = -(R + S),$$
$$\Lambda_{44} = -2R + M + M^T$$

and finally the matrix-valued function Λ by

$$\Lambda(d) = \begin{pmatrix} \Lambda_{11}(d) & \Lambda_{12}(d) & M & \Lambda_{14} & G & Q^T & 0 \\ * & \Lambda_{22} & 0 & \varepsilon BY_0 & \varepsilon G & 0 & 0 \\ * & * & \Lambda_{33} & R - M^T & 0 & 0 & 0 \\ * & * & * & \Lambda_{44} & 0 & 0 & Y_0^T E_K^T \\ * & * & * & * & -\gamma I_q & 0 & 0 \\ * & * & * & * & * & -I_n & 0 \\ * & * & * & * & * & * & -\lambda I_\mu \end{pmatrix}.$$

Consider now the case when $\overline{D} = D$. Then there exists a permutation matrix Π so that

$$\Pi^T \overline{\Gamma} \Pi = \operatorname{diag}(\Lambda(d_1), \dots, \Lambda(d_N)). \tag{24}$$

If the special structure of the symmetrically connected systems is used, then a permutation matrix Π' exists so that

$$\Pi'^T \overline{\Gamma} \Pi' = \operatorname{diag}(\underbrace{\Lambda(-1), \dots, \Lambda(-1)}_{N-1 \text{ times}}, \Lambda(N-1)). \tag{25}$$

Note that function $\Lambda(d)$ is convex in d. Thus, either way, eqs. (24) and (25) show that it is sufficient to verify negative definiteness of $\Lambda(d)$ for only two values of d: for $d = d_1$ and $d = d_N$ in the "general" case while it is sufficient to investigate for $\Lambda(-1)$ and $\Lambda(N-1)$ if one uses the symmetrical interconnection structure.

For completeness, note that validity of (9) is guaranteed by the LMI

$$\begin{pmatrix} -R & M \\ * & -R \end{pmatrix} < 0. \tag{26}$$

To sum up, the results can be formulated in form of the following theorem:

Theorem V.1. *Consider system* (6) *satisfying Assumption II.3. Let also Assumption II.2 holds. If there exist* $n \times n$*-dimensional matrices* $P > 0$, $R > 0$, $S > 0$, Q *nonsingular and* M, *a* $m \times n$*-dimensional matrix* Y_0 *and scalars* $\lambda > 0$, $\gamma > 0$ *and* $\varepsilon > 0$ *so that at least one of the following sets of conditions is valid:*

1)

$$0 > \Lambda(d_1), \quad (27)$$

$$0 > \Lambda(d_N), \quad (28)$$

$$0 > \begin{pmatrix} -R & M \\ * & -R \end{pmatrix}, \quad (29)$$

2) *the system is symmetrically interconnected and*

$$0 > \Lambda(-1), \quad (30)$$

$$0 > \Lambda(N-1), \quad (31)$$

$$0 > \begin{pmatrix} -R & M \\ * & -R \end{pmatrix}. \quad (32)$$

Then the system is H_∞*-stable if every subsystem is fed by the control* (4) *with* $K = Y_0 Q^{-1}$ *even in presence of multiplicative perturbations of the control gain given by the term* $D_K F_{i,K} E_K$.

Remark V.2. *Usually, it is required that influence of the disturbance* w *on the controlled system is minimized, hence one can add the optimization objective*

$$\text{minimize } \gamma \quad (33)$$

to the set of LMIs (27 - 29), resp. (30 - 32) in Theorem V.1.

Remark V.3. *As noted above, the proof of this theorem relies upon a MIMO version of Proposition 5.3 of [23]. Since this generalization of the aforementioned Proposition is straightforward we do not present its proof here. The main change is to reformulate the Lyapunov-Krasovskii functionals as a sum of N "sub-functionals" each of which is acting on one subsystem only. Then, one can easily see that unequal delays in different subsystems do not matter.*

VI. Example

Consider the large-scale network composed of 10 identical subsystems (nodes). Every subsystem is a linear oscillator, the subsystems are interconnected:

$$\dot{x}_{1,i} = x_{2,i}, \ \dot{x}_{2,i} = -x_{1,i} + u_i + w_i + \mathscr{I}_i$$

where w_i denotes the perturbation and the term $\mathscr{I}_i$ describes the interconnections. This term is defined as follows:

$$\mathscr{I}_i = 0.1(x_{1,i-1} + x_{1,i+1}) \text{ for } i = 2, \ldots, 9,$$

$$\mathscr{I}_1 = 0.1(x_{1,2} + x_{1,10}) \text{ for } i = 1,$$

$$\mathscr{I}_{10} = 0.1(x_{1,1} + x_{1,9}) \text{ for } i = 10.$$

Hence the subsystems are interconnected in a circular (ring) topology, see Fig. VI. Ring topology is one of the possible topology of complex network [24]; synchronization of complex networks with ring topology is studied rigorously by authors in [25], [26], [27], [28]. From this interconnection

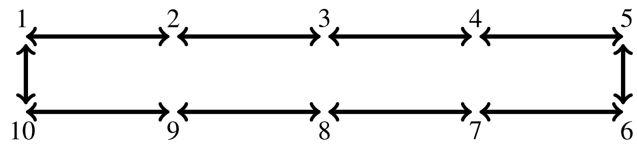

Fig. 1. Connection of subsystems (nodes).

follows that the interconnection matrix L has eigenvalues in the interval $[-2, 2]$. Without control, the overall interconnected network is unstable.

The control is delayed, the delays at the subsystems are time-varying, with $\tau_i \in [0, 0.3]s$, the changes of the time delay can be rapid (there is no limit on the derivative of the time delay). In the example, all the time delays have the sawtooth shape with derivative of the delay being 1 up to the points where the time delay is discontinuous. Moreover, the control gain matrices are perturbed. To be specific, the nominal control gain is designed so that stabilization of the plant is guaranteed if it is perturbed up to 20% of its nominal value. Hence, in (4), we set $D_K = 1$, $E_K = 0.2$.

LMIs (27)-(29) from Theorem V.1 with inclusion of the minimization objective (33) yields the solution of the LMI optimization problem as $K_0 = (0.1770, \ -1.301)$ while for the disturbance attenuation constant holds $\gamma = 244$.

The initial conditions of the first state were randomly selected within the interval $[-1, 1]$, the second state was zero in all subsystems at the beginning of the simulations. The control gain multiplicative perturbations were also random within the allowed bounds.

Fig. 2 shows the state x_1 of the first subsystem (solid line), third subsystem (longdash-shortdash line), sixth subsystem (dashed line) and ninth subsystem (dash-dot line). The control signals for this selection of subsystems is depicted in Fig. 3, meaning the line types remains unchanged. One can see oscillations in the control signals. They are caused by time varying disturbances in the control gain. The norm of vector x of the controlled network is shown in Fig. 4.

VII. Conclusions

An algorithm for stabilization of a network composed of identical subsystems was presented. The controls of each subsystem can be delayed, the delays are not equal in every subsystem but they have to obey a common bound. The derivative of the time delay is not supposed to be bounded, hence this algorithm is capable of dealing with networked control. Moreover, the control gain can be perturbed within given bounds, hence the proposed control law is non-fragile. This was achieved by adapting the descriptor approach to the case of a large network.

References

[1] J. Lunze, Feedback Control of Large-Scale Systems. Hemel Hempstead: Prentice Hall, 1992.

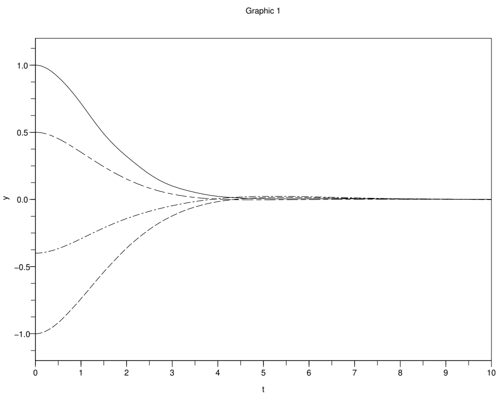

Fig. 2. Comparison of $x_{1,1}$, $x_{3,1}$, $x_{6,1}$ and $x_{9,1}$.

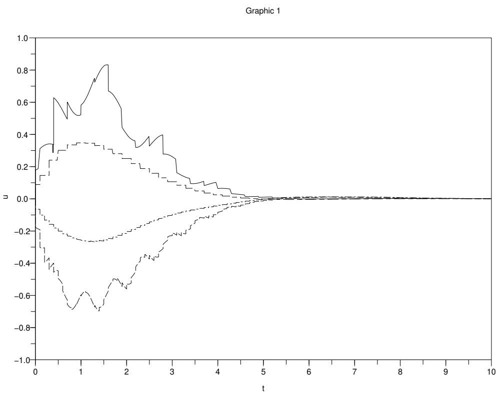

Fig. 3. Comparison of u_1, u_3, u_6 and u_9.

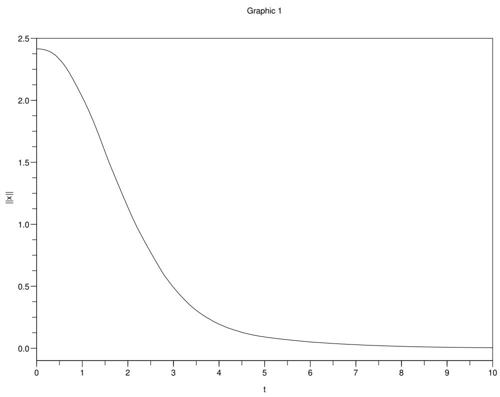

Fig. 4. Norm of vector x.

[2] L. Bakule, "Decentralized control: Status and outlook," Annual Reviews in Control, vol. 38, pp. 71 – 80, 2014.

[3] L. Bakule and J. Lunze, "Decentralized design of feedback control for large-scale systems," Kybernetika, vol. 24, pp. 1 – 100, 1988.

[4] D. L. Córdoba and R. Ortega, "An observer-based scheme for decentralized stabilizaation of large-scale systems with application to power systems," Asian Journal of Control, vol. 17, pp. 124 – 132, 2015.

[5] G. Antonelli, "Interconnected dynamic systems: An overview on distributed control," IEEE Control Systems, vol. 33, no. 1, pp. 76–88, 2013.

[6] O. Demir, J. Lunze, and P. Otte, "Analogies in the controller design for multi-agent and physically interconnected systems with identical subsystems," in 2012 20th Mediterranean Conference on Control Automation (MED), 2012, pp. 1037–1042.

[7] L. Bakule, M. de la Sen, M. Papík, and B. Rehák, "Decentralized stabilization of complex systems with delayed feedback," IFAC Proceedings Volumes, vol. 46, no. 13, pp. 31–36, 2013.

[8] L. Bakule, M. De la Sen, M. Papik, and B. Rehak, "Decentralized stabilization of symmetric systems with delayed observer-based feedback," 06 2013, pp. 6679–6684.

[9] L. Bakule, M. Papík, and B. Rehák, "Decentralized H-infinity control of complex systems with delayed feedback," Automatica, vol. 67, pp. 127 – 131, 2016.

[10] E. Fridman, "Tutorial on Lyapunov-based methods for time-delay systems," European Journal of Control, vol. 20, pp. 271 – 283, 2014.

[11] B. Rehák and V. Lynnyk, "Network-based control of nonlinear large-scale systems composed of identical subsystems," Journal of the Franklin Institute, vol. 356, no. 2, pp. 1088 – 1112, 2019.

[12] B. Rehák, "Wirtinger inequality-based control design for an interconnected large-scale system with sampled controls," in 2019 Chinese Control Conference (CCC), July 2019, pp. 1009–1014.

[13] B. Rehák and V. Lynnyk, "Decentralized networked stabilization of a nonlinear large system under quantization," IFAC-PapersOnLine, vol. 52, no. 20, pp. 49–54, 2019.

[14] L. Bakule and M. de la Sen, "Non-fragile controllers for a class of time-delay nonlinear systems," Kybernetika, vol. 45, pp. 15 – 32, 2009.

[15] J. Chen, R. Ling, and D. Zhang, "Distributed non-fragile stabilization of large-scale systems with random controller failure," Neurocomputing, vol. 173, pp. 2033 – 2038, 2016.

[16] G.-H. Yang and J. L. Wang, "Non-fragile H∞ control for linear systems with multiplicative controller gain variations," Automatica, vol. 37, no. 5, pp. 727–737, 2001.

[17] Z. Li, Z. Wang, D. Ding, and H. Shu, "Non-fragile H∞ control with randomly occurring gain variations, distributed delays and channel fadings," IET Control Theory & Applications, vol. 9, no. 2, pp. 222–231, 2015.

[18] O. Demir and J. Lunze, "Optimal and event-based networked control of physically interconnected systems and multi-agent systems," International Journal on Control, vol. 87, pp. 169 – 185, 2014.

[19] B. Rehák and V. Lynnyk, "Leader-following synchronization of a multi-agent system with heterogeneous delays," Frontiers of Information Technology & Electronic Engineering, vol. 22, no. 1, pp. 97–106, 2021.

[20] ——, "Consensus of a multi-agent systems with heterogeneous delays," Kybernetika, vol. 56, pp. 363–381, 2020.

[21] B. Rehák and V. Lynnyk, "Synchronization of multi-agent systems with directed topologies and heterogeneous delays," in 2020 European Control Conference (ECC). IEEE, 2020, pp. 1671–1676.

[22] B. Rehák and V. Lynnyk, "Synchronization of symmetric complex networks with heterogeneous time delays," in 2019 22nd International Conference on Process Control (PC19), June 2019, pp. 68–73.

[23] E. Fridman, Introduction to Time-Delay Systems. Basel: Birkhäuser, 2015.

[24] S. Čelikovský, V. Lynnyk, and G. Chen, "Robust synchronization of a class of chaotic networks," Journal of the Franklin Institute, vol. 350, no. 10, pp. 2936–2948, 2013.

[25] V. Lynnyk, B. Rehák, and S. Čelikovský, "On detection of generalized synchronization in the complex network with ring topology via the duplicated systems approach," in 2019 8th International Conference on Systems and Control (ICSC). IEEE, 2019, pp. 251–256.

[26] V. Lynnyk, B. Rehák, and S. Čelikovský, "On applicability of auxiliary system approach in complex network with ring topology," Cybernetics and Physics, vol. 8, no. 3, pp. 143–152, 2019.

[27] V. Lynnyk, B. Rehák, and S. Čelikovský, "On the detection of the generalized synchronization in the complex network with ring topology," in 2020 European Control Conference (ECC), 2020, pp. 1955–1960.

[28] V. Lynnyk, B. Rehák, and S. Čelikovský, "On estimation of the parameters in the complex network via the adaptive observer," in 2020 28th Mediterranean Conference on Control and Automation (MED). IEEE, 2020.