Nonlinear Luenberger observer for systems with quantized and delayed measurements

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Abstract— **A Luenberger-like observer for nonlinear systems with quantized measurements is proposed. The observer design is based on the solution of a certain partial differential equation that is solved numerically. Then, stability of this observer is proved even in presence of quantized measurements and delayed measurements. The results are illustrated by an example.**

I. Introduction

In practice, it is often impossible to measure all states of the controlled dynamical system. Hence their values must be estimated, an observer must be designed for this task. If the controlled system is linear time invariant and no noise is present, the Luenberger observer is the available option to solve this issue. However, in the case of nonlinear systems, the situation is much more involved. Several different approaches for observer construction of nonlinear systems have been proposed so far. The first approach is the simple application of the linear robust observer, as presented in [28]. The crucial idea is to handle the nonlinearity as an uncertainty. As a second way, let us mention the nowadays well-developed theory of the high gain observers, thoroughly presented in e.g. [11]. Many practical applications use this observer. However, a high sensitivity to measurement noise is a serious issue for this type of observers.

As shown in [9], it is possible to derive a nonlinear counterpart of the Luenberger observer that was derived in the theory of linear systems. This theory was further extended to time delay systems in [10]. In both cases, the key part of the observer design is to derive an equation corresponds to the Sylvester equation used in the case of linear Luenberger observers. This equation is a linear first-order partial differential equation (PDE) but with non-constant coefficients. The original way to find an approximation of its solution using the Taylor polynomials is described in the original paper [9]. Let us briefly introduce this method: the coefficients and the right-hand side are expanded into Taylor polynomials first. The solution is also sought in form of a Taylor polynomial.

To prove existence of the approximations by the Taylor polynomials, the so-called Lyapunov auxiliary theorem is applied. The downside is that this theorem has rather restrictive assumptions. Namely, all eigenvalues of the linearization of the original observed system around the origin are assumed to have the same sign of the real parts. This means, systems with purely imaginary eigenvalues are not covered. Relaxation of this assumption is presented in [24]. This is done by proposing an iterative method to solve the PDE. It is based on an iterative method originally developed for computation of the stable, center-stable etc. manifolds, see [23]. Note that this method was successfully applied in [26].

The aforementioned method for center-manifold computation shows also similarities to the regulator equation introduced in the nonlinear output regulation problem. Here, this equation can be numerically approximated by the finite-element method (FEM) as shown in [19], [20], [21], [27]. An advantage is that existence of an L^2 solution of this PDE domain can be proved, see [15] where results concerning numerical solution using FEM described. These were successfully extended to the observer problem presented in [9] in the paper [18], hence this approach is used in this paper as well.

The data from the sensors are usually transmitted via a transmission channel that has a limited bandwith. To cope with this restriction, one has to realize that it is not possible to transmit all values. Rather, only some discrete values of the signal can be transmitted. This process is called quantization. Quantized signals are investigated in a large number of papers, such as [12] or [6]. In these papers, the reader can find a well-elaborated introduction to quantizationas well as presentation of linear and logarithmic quantizers. Description of the control loop with quantized signals is presented in e.g. [25]. Naturally, one has to design observers that reeive quantized signals from the sensors. Such an observer is proposed in [17].

Recently, attention has moved to the quantizers with finitely many levels as this class of observers is practically important [4], [13]. The quantized control of a nonlinear system is described in [5], [8].

Transmission of signals through the communication networks is typically connected with another feature: the delay due to packet dropouts etc. Moreover, in some applications it is not possible to obtain measurements immediately but the measurement processes time consuming, this holds for e.g. in chemical or biological processes, among others. Thus the need for an observer that is capable to deal with delayed measurements is obvious. As a vast number of results exists, we mention a few here.

Besides the aforementioned paper [17] where also analysis of imprecisely known time delay is presented. Moreover, an observer for a special case of nonlinear time delay systems - namely for polynomial systems with time-delay is derived in [16]. The Luenberger observer for systems with delays is

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This work was supported by the Czech Science Foundation through the grant No. GA CR 19-07635S.

presented in [10]. As this paper directly extends the approach introduced in [9] to the case of systems with delayed output measurements, it is not suprising that the design is again based on a solution of a PDE. This PDE is numerically solved in [18] using FEM. Results from this paper are a base for the research described here.

The encouraging fact is that this approach can be practically used, for instance for state reconstruction of a biological systems, as in [22].

A. Purpose of this paper

To derive a Luenberger-like observer for nonlinear systems with time delays and quantized measurements. This is an important problem as both these phenomena occur in the networked control systems, however as far as we know, a nonlinear Luenberger observer with these features has not been investigated before.

B. Notation

- the symbol $\mathbb{R}$ denotes the real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space endowed with the quadratic norm which is denoted by $\|.\|$,
- if P is a matrix then inequality $P > 0$ means that matrix P is symmetric positive definite,
- for symmetric matrices, the terms below the diagonal are replaced by an asterisk: $\left(\begin{smallmatrix} a & b \\ * & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ b^T & c \end{smallmatrix}\right)$,
- the symbol I_n denotes the n-dimensional identity matrix,
- if $v : \mathbb{R} \to \mathbb{R}^n$ is a function of time then the argument can be omitted: symbols v and $v(t)$ have the same meaning; if the argument is different from t or if this omission could cause confusion or compromise clarity of the presentation, the argument is always written,
- the subscript τ denotes the time delay: $v_\tau = v_\tau(t) = v(t-\tau)$.

II. Logarithmic quantizer

In this paper, the logarithmic quantizer q is used. Its detailed description can be found e.g. in [6]. Assume the constants $\rho \in (0,1)$, $\sigma > 0$ are given. Then, the quantization levels attain values $\{\sigma\rho^j\} \cup \{-\sigma\rho^j\} \cup \{0\}$, $j \in N$. The scalar σ is called the quantizer range. From these considerations, one can derive the maximal quantization error as

$$|\sigma q(\frac{x}{\sigma}) - x| \leq \delta|x|, \quad \delta = \frac{1-\rho}{1+\rho}. \tag{1}$$

In the following text, it will be assumed that the quantization range σ is sufficiently large so that amplitude of the quantized signal does not exceed this range.

III. Problem Setting

Let $\tau \geq 0$ is a positive constant. This is the delay that elapses from measurement to the arrival to the observer. It is assumed this delay is constant and known for the observer design.

Notation: The symbol x_τ denotes the delayed state x: $x_\tau(t) = x(t-\tau)$.

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth functions satisfying $f(0) = 0$, $h(0) = 0$. Let also the approximate linearization of this system at the origin be observable. Consider system

$$\dot{x} = f(x), \ x(0) = x_0, \tag{2}$$

$$y = h(q(x_\tau)). \tag{3}$$

The limited bandwith of the transmission channel does not allow to transmit a continuous range of measured values. Hence, the quantized is used. It is connected immediately after the sensor, in front of the transmission channel.

The scheme of the observer with quantized measurements is shown in Fig. 1. The transmitted value of the output of the quantizer is transmitted through the channel.

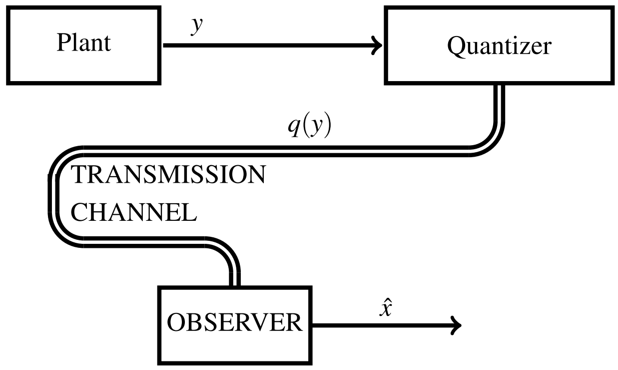

Fig. 1. Observer with quantized measurements.

Assumption 1. *The state of the of system* (2) *that is to be reconstructed is contained in a pre-defined bounded domain* Ω *containing the origin.*

This assumption is not too restrictive as in practice, all trajectories to be observed are usually bounded. Moreover, some region where these trajectories live is also usually known. Moreover, the numerical methods (necessary for a solution of a PDE) also require a bounded domain where the solution is approximated.

IV. Nonlinear Luenberger observer without quantization

Let us consider system (2) with output

$$y = h(x_\tau). \tag{4}$$

Results derived using this observer will be useful for the quantized observer design.

At the beginning of the observer design, choose matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ so that the following relation is satisfied:

$$\max \ \mathrm{Re} \ \mathrm{eig}(\tilde{A}) < \min\Big(\min \ \mathrm{Re} \ \mathrm{eig}(\frac{\partial f}{\partial x}(0)), 0\Big). \tag{5}$$

Besides of that, we select a vector $b \in \mathbb{R}^n$ so that the controllability conditions of the pair $(\tilde{A}, b)$ is satisfied.

Along the lines of [18], we solve the following equation

$$\frac{\partial \Phi}{\partial x} f(x) = \tilde{A}\Phi(x) + bh(x_\tau), \ \Phi(0) = 0. \tag{6}$$

with unknown function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$.

Having done that, we are ready to define the observer gain as

$$L(x') = \left(\frac{\partial \Phi}{\partial x}(x')\right)^{-1} b. \quad (7)$$

Finally, using this gain L, the observer can be defined as follows

$$\hat{x} = f(\hat{x}) + L(\hat{x})(h(x_\tau) - h(\hat{x_\tau}). \quad (8)$$

[9] shows that $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$.

The noteworthy fact is that in the linear case, the PDE (6) is converted to the Sylvester equation. As shown in e.g. [2], its solution is a nonsingular matrix.

There exists a neighborhood of the origin on which function Φ has a non-singular Jacobi matrix, hence the observer gain is correctly defined on this neighborhood. The following assumption can thus be made:

Assumption 2. *On the domain Ω, function L is correctly defined.*

Let us follow the proof given in [10], repeated also in [18]. Let $z = \Phi(x_\tau)$, $\hat{z} = \Phi(\hat{x}_\tau)$. Then

$$\begin{aligned}
&\dot{z} - \dot{\hat{z}} \\
&= \frac{\partial \Phi}{\partial x}(x)(f(x)) - \frac{\partial \Phi}{\partial x}(\hat{x})\Big(f(\hat{x}) + L(\hat{x})(h(x_\tau) - h(\hat{x}_\tau))\Big) \\
&= \tilde{A}(\Phi(x) - \Phi(\hat{x})) = \tilde{A}(z - \hat{z}).
\end{aligned}$$

Since matrix $\tilde{A}$ was chosen to be Hurwitz, one can, for a given $c > 0$ find a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ so that

$$\tilde{A}^T P + P\tilde{A} = -cI_n. \quad (9)$$

Using matrix P we define Lyapunov function

$$V = (z - \hat{z})^T P(z - \hat{z}) \quad (10)$$

which leads to

$$\dot{V} \leq -c(z - \hat{z})^T (z - \hat{z}). \quad (11)$$

This implies $\dot{V} < 0$ for all $t \geq 0$. In the non-quantized case, the observation error converges to zero.

V. QUANTIZED OBSERVER

In this section, the observer that reconstructs the state from quantized measurements is designed. The output equation is considered in the form (3). The results from the previous section will be useful.

Note that, using the Taylor expansion in $\mathbb{R}^n$, there exists $\xi \in \Omega$ so that

$$h(q(x_\tau)) - h(x_\tau) = \nabla h(\xi)(q(x_\tau) - x_\tau). \quad (12)$$

Since Ω is bounded and function h was supposed to be smooth in $\mathbb{R}^n$, one has there exists a constant $\varkappa > 0$ so that $\sup_{\xi\in\Omega}(|\nabla h(\xi)| = \varkappa < +\infty$.

Let us consider the derivative of $z - \hat{z}$ if the quantized observer is applied. Then, using the results from the previous section,

$$\begin{aligned}
&\dot{z} - \dot{\hat{z}} \\
&= \frac{\partial \Phi}{\partial x}(x)(f(x)) - \frac{\partial \Phi}{\partial x}(\hat{x})\Big(f(\hat{x}) + L(\hat{x})(h(q(x_\tau)) - h(\hat{x}_\tau))\Big) \\
&= \frac{\partial \Phi}{\partial x}(x)(f(x)) - \frac{\partial \Phi}{\partial x}(\hat{x})\Big(f(\hat{x}) + L(\hat{x})(h(x_\tau) - h(\hat{x}_\tau) \\
&\quad + h(q(x_\tau)) - h(x_\tau))\Big) \\
&= \tilde{A}\Big(\Phi(x) - \Phi(\hat{x})\Big) - \frac{\partial \Phi}{\partial x}(\hat{x})\Big(h(q(x_\tau)) - h(x_\tau)\Big) \\
&= \tilde{A}(z - \hat{z}) - \frac{\partial \Phi}{\partial x}(\hat{x})\Big(h(q(x_\tau)) - h(x_\tau)\Big).
\end{aligned}$$

Since the observer gain L is well defined on $\overline{\Omega}$, there exists a constant $k > 0$ so that

$$\frac{\partial \Phi}{\partial x}(\hat{x}) < k. \quad (13)$$

Then, using (13) and definition of constant $\varkappa$ and using the logarithmic quantizer (1) with δ satisfying

$$|q(x_\tau) - x_\tau| \leq \delta |x_\tau| \quad (14)$$

that

$$\left|\frac{\partial \Phi}{\partial x}(\hat{x})(h(q(x_\tau)) - h(x_\tau))\right| \leq k\varkappa\delta|x_\tau - \hat{x}_\tau|. \quad (15)$$

Choose now a constant $\gamma \in (0, c)$. Using (15), one gets for the derivative of Lyapunov function (10)

$$\begin{aligned}
\dot{V} &\leq -c(z - \hat{z})^T - 2(z - \hat{z})^T P - \frac{\partial \Phi}{\partial x}(\hat{x})\Big(h(q(x_\tau)) - h(x_\tau)\Big) \\
&\leq -(c - \gamma)(z - \hat{z})^T (z - \hat{z}) + \frac{1}{\gamma}k^2\varkappa^2\delta^2|x_\tau - \hat{x}_\tau|^2. \quad (16)
\end{aligned}$$

If $\tau = 0$ we have the following result:

Theorem 3. *Consider system* (2) *and* (3) *with $\tau = 0$. Then there exists a constant $\delta > 0$ so that, if the quantizer satisfies* (1)*, then*

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0. \quad (17)$$

Proof: First, note that, since mapping Φ is a diffeomorphism, there exists a constant $k' > 0$ so that (16) satisfies

$$\dot{V} \leq -(c - \gamma)(z - \hat{z})^T (z - \hat{z}) + k'\delta^2|z_\tau - \hat{z}_\tau|^2. \quad (18)$$

Choosing $\delta < \sqrt{\frac{c-\gamma}{k'}}$ yields that $\dot{V} < 0$ everywhere in Ω except the origin, hence $\|z - \hat{z}\| \to 0$ as $t \to \infty$. Using the fact that Φ is a diffeomorphism once more we get that

$$\lim_{t\to\infty} \|z - \hat{z}\| = 0. \quad (19)$$

□

If the time delay is present, this simple approach cannot be used. To guarantee convergence of the observation error to zero one can make use of the Razumikhin theorem, see e.g. [7]. The version presented here is from [17].

Lemma 4. *Let $\bar{\tau} > 0$ and let $\tau \in [0, \bar{\tau}]$. Consider system $\dot{\xi} = \mathscr{F}(\xi, \xi_\tau)$. Assume there exists matrix $\Pi > 0$ and a scalar*

$\omega > 0$ *so that for all* $t \geq 0$ *holds the following implication (with* $\mathscr{V}(t) = \xi^T(t)\Pi\xi(t)$*):*

$$\sup_{s\in[t-\bar{\tau},t]} \mathscr{V}(s) < \omega\mathscr{V}(t) \Rightarrow \frac{d}{dt}\mathscr{V}(t) < 0. \quad (20)$$

Then $\lim_{t\to\infty}\|\xi(t)\| = 0$.

This lemma will be applied to the observer with delay. From the fact that mapping Φ is a diffeomorphism follows existence of a constant $M > 0$ such that (15) can be reformulated as

$$\left|\frac{\partial\Phi}{\partial x}(\hat{x})(h(q(x_\tau)) - h(x_\tau))\right| \leq M\delta|z_\tau - \hat{z}_\tau|. \quad (21)$$

Then, there exists an (unknown) measurable function $F : \mathbb{R}^n \to \mathbb{R}^{\ltimes}$ satisfying $\|F(t)\| \leq 1$ for all $t \geq 0$ and

$$\dot{z} - \dot{\hat{z}} = \tilde{A}(z - \hat{z}) + M\delta F(t)(z_\tau - \hat{z}_\tau). \quad (22)$$

Theorem 5. *Consider system* (2) *and* (3) *with the time delay* $\tau > 0$. *Let there exist* $n\times n$*-dimensional matrices* $P > 0$, $Q > 0$ *and a scalar* $h > 0$ *so that the following matrix inequalities hold:*

$$\tilde{A}^T P + P\tilde{A} + \tau\Big(2Q + (M\delta)^2 P + (M\delta)^4 P\Big) <0, \quad (23)$$

$$h\tilde{A}^T\tilde{A} <P, \quad (24)$$

$$hI_n <P, \quad (25)$$

$$\begin{pmatrix} Q & P \\ * & hI_n \end{pmatrix} >0, \quad (26)$$

Then $\lim_{t\to\infty}\|x(t) - \hat{x}(t)\| = 0$.

Before the proof is presented, let us state the following proposition that will be useful. Its proof can be found in [3].

Proposition 6. *(Schur Complement) Let* $\mathscr{P} \in \mathbb{R}^{\nu\times\nu}$, $\mathscr{P} > 0$, $\mathscr{R} \in \mathbb{R}^{\mu\times\nu}$, $\mathscr{S} \in \mathbb{R}^{\mu\times\mu}$, $\mathscr{S} > 0$. *Then the following equivalence holds:*

$$\begin{pmatrix} \mathscr{P} & \mathscr{R}^T \\ * & \mathscr{S} \end{pmatrix} > 0 \Longleftrightarrow \mathscr{P} - \mathscr{R}^T\mathscr{S}^{-1}\mathscr{R} > 0 \quad (27)$$

Proof:

Denote $e_z = z - \hat{z}$. Let the Lyapunov function V be defined as in (10). Then

$$\dot{V} = 2e_z P\dot{e}_z = 2e_z P(\tilde{A}e_z + M\delta F(t)e_{z,\tau}). \quad (28)$$

The goal is to prove validity of the following implication: if there exists $\omega > 0$ so that for any $t \geq 0$ where the inequality

$$\sup_{s\in[-2\tau+t,t]} V(s) < \omega V(t). \quad (29)$$

holds, then $\dot{V}(t) < 0$. This is the assumption of the Razumikhin theorem. Then, convergence of the observation error will be proved.

To prove the above mentioned implication, we proceed as in [1] or [14]. Note that

$$e_{z,\tau} = -\int_{t-\tau}^{t} \tilde{A}e_z(s) + M\delta F(s)e_{z,\tau}(s)ds.$$

Substituting this relation into (28) together with (9) yields

$$\begin{aligned}\dot{V} = \;& e_z^T(\tilde{A}^T P + P\tilde{A})e_z \\ & -2e_z PM\delta F(t)\textstyle\int_{t-\tau}^{t}\tilde{A}e_z(s) + M\delta F(s)e_{z,\tau}(s)ds.\end{aligned} \quad (30)$$

First, the term $2e_z PM\delta\int_{t-\tau}^{t}\tilde{A}e_z(s)ds$ is treated. First, note that relation (26) in connection with the Schur complement yields

$$Q > \frac{1}{h}P^2. \quad (31)$$

In the following inequalities, relation (24) is used in the third row. The fourth row uses (31) together with condition (29):

$$\begin{aligned}& 2e_z PMF(t)\delta\int_{t-\tau}^{t}\tilde{A}e_z(s)ds \\ \leq & \int_{t-\tau}^{t}\frac{1}{h}e_z^T P^2 e_z + (M\delta)^2 h e_z^T(s)\tilde{A}^T F^T(t)F(t)\tilde{A}e_z(s)ds \\ \leq & \frac{\tau}{h}e_z P^2 e_z + (M\delta)^2\int_{t-\tau}^{t}e_z^T(s)Pe_z(s)ds \\ \leq & \tau e_z^T Q e_z + (M\delta)^2\tau\omega e_z^T P e_z.\end{aligned}$$

The term $e_z PM\delta F(t)\int_{t-\tau}^{t}2M\delta F(s)e_{z,\tau}(s)ds$ is treated analogously; in the last row, ineq. (25) is used.

$$\begin{aligned}& e_z PM\delta F(t)\int_{t-\tau}^{t}2M\delta F(s)e_{z,\tau}(s)ds \\ \leq & \int_{t-\tau}^{t}\frac{1}{h}e_z^T P^2 e_z + (M\delta)^4 h e_{z,\tau}^T(s)e_{z,\tau}(s)ds \\ \leq & \tau e_z^T Q e_z + (M\delta)^4\tau\omega e_z^T P e_z.\end{aligned}$$

To sum up, one obtained:

$$\begin{aligned}\dot{V} \leq \;& e_z^T(\tilde{A}^T P + P\tilde{A})e_z \\ & +\tau e_z^T\Big(2Q + (M\delta)^2\omega P + (M\delta)^4 P\omega\Big)e_z.\end{aligned} \quad (32)$$

Note that, since (23) is a sharp inequality, from (23) follows existence of an $\omega > 1$ so that

$$\tilde{A}^T P + P\tilde{A} + \tau\Big(2Q + \omega(M\delta)^2 P + \omega(M\delta)^4 P\Big) < 0.$$

This implies validity of (32).

Hence, under the assumptions of this theorem follows existence of $\omega > 1$ such that, if for all $s \in [t - 2\tau, t]$ holds $V(s) < \omega V(t)$ then $\dot{V}(t) < 0$. Validity of this implication is the assumption of the Razumikhin theorem. Hence $\lim_{t\to\infty}\|e_z(t)\| = 0$ and, consequently, this implies $\lim_{t\to\infty}\|x(t) - \hat{x}(t)\| = 0$. □

Remark 7. *Note that feasibility of the system of the linear matrix inequalities (23-26) must be solved in order to verify conditions guaranteeing that the observation error converges to zero. However, unfortunately, this set of linear matrix inequalities cannot be used to design some of the parameters - be it the parameter of the quantizer* δ *or the time delay* τ. *Concerning the parameter* δ, *this parameter appears in the set of inequalities in the second and fourth power, hence non-linearly which prevents this parameter from inclusion into the variables that are to be found by the linear matrix inequalities solver. In the case of the time delay, although inequality* (23) *depends linearly on this parameter, this parameter is also present during solution of equation* (6), *hence the observer gain L - and consequently, the constant M - also depend on this parameter. Finding conditions linear in the quantizer parameter δ is a matter of future research.*

VI. Example

As an example system, we choose the system described from the Example 1 in the paper [18]. The system is governed by the following equations:

$$\begin{aligned}\dot{x}_1 =& x_2,\\ \dot{x}_2 =& -(x_1+x_1^3)e^{x_1}-0.1x_2,\\ y =& x_{1,\tau}\end{aligned}$$

with $\tau = 0.2s$. It is chosen

$$\tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Function Φ is determined as a solution of Eq. (6). In practice, this is conducted with help of a numerical software. To obtain the solution in this example, the finite-element method was used, see [18] for details. From these definitions follows that function Φ is determined as

$$\Phi(x_1,x_2) = \begin{pmatrix} 0.2368x_1 - 0.2632x_2 - 0.1657x_1^2 \\ +0.0975x_1x_2 - 0.1218x_2^2 - 0.1590x_1^3 \\ +0.1383x_1^2x_2 - 0.0544x_1x_2^2 + 0.0777x_2^3 \\ \\ 0.1979x_1 - 0.1042x_2 - 0.0391x_1^2 \\ +0.0260x_1x_2 - 0.0144x_2^2 - 0.0467x_1^3 \\ +0.0368x_1^2x_2 - 0.0207x_1x_2^2 + 0.0122x_2^3 \end{pmatrix}$$

and finally

$$L(x_1',x_2') = \begin{pmatrix} 0.0264 + 0.2339x_1' - 0.1461x_2' - 0.3387x_1'^2 \\ +0.1678x_1'x_2' + 0.1787x_2'^2 \\ \\ 0.0937 - 0.0522x_1' + 0.0548x_2' - 0.1033x_1'^2 \\ +0.0322x_1'x_2' + 0.0159x_2'^2 \end{pmatrix}.$$

Let us define the set $\Omega = \{(x',y') \in \mathbb{R}^2 \mid x'^2 + y'^2 < 4\}$.

With this choice, one can take $M = 2.2$. The set of linear matrix inequalities (23-26) has a feasible solution with the quantizer parameter $\delta = 0.01$.

In Fig. 2, one can see the states of the observer and the observed system. The dash-dot line represents the state x_1 of the observed system, the dashed line is the estimate of this state. The dash-double dot line illustrates the state x_2, the solid line is the estimate of this state. The norm of the estimation error is depicted in Fig. 3. Finally, the quantized signal is shown in Fig. 4.

VII. Conclusions

An observer design for nonlinear systems with delayed outputs based on the Luenberger design approach was combined with a logarithmic quantizer. It was proved that, if the quantization constant is small, the observation error converges to zero. To prove this, the vanishing nonlinearity theorem was used if the measurements are not delayed. In the case when the measurement dellays are present, the conergence of the observation error to zero was shown via

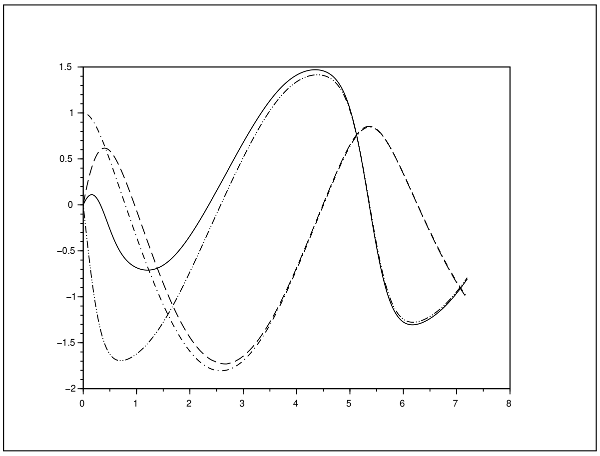

Fig. 2. State of the observed system and the observer

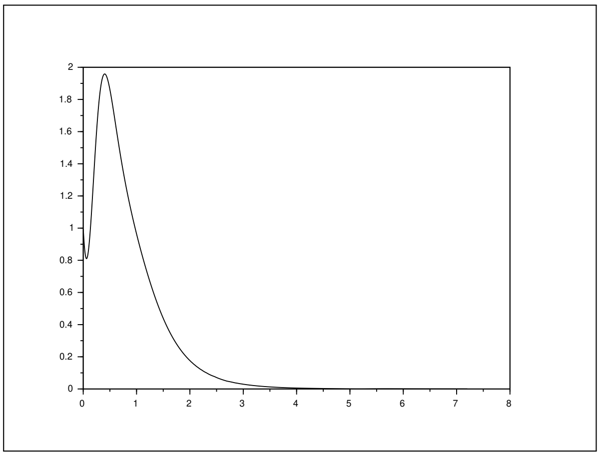

Fig. 3. Norm of error

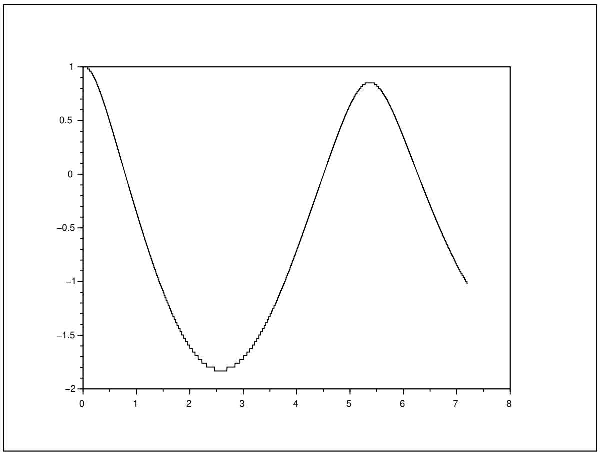

Fig. 4. Quantized output

the Razumikhin theorem, the proof of convergence is based on linear matrix inequalities. The results are illustrated by an example.

REFERENCES

[1] L. Bakule, M. Papík, and B. Rehák. Decentralized H-infinity control of complex systems with delayed feedback. Automatica, 67:127 – 131, 2016.

[2] G. Birkhoff and S.M. Lane. A Survey of Modern Algebra. CRC Press, 2017.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia, 1994.

[4] G. Cruz Campos, J. M. Gomes da Silva Jr., S. Tarbouriech, and C. E. Pereira. Stabilisation of discrete-time systems with finite-level uniform and logarithmic quantisers. IET Control Theory & Applications, 12:1125–1132(7), May 2018.

[5] C. De Persis. On feedback stabilization of nonlinear systems under quantization. In Proceedings of the 44th IEEE Conference on Decision and Control, pages 7698–7703, Dec 2005.

[6] M. Fu. Quantization for feedback control and estimation. In Proc. of the 27th Chinese Control Conference, Kunming, China, 2008.

[7] J.K. Hale and S.M. Verdyun-Lunel. Introduction to Functional Differential Equations. Springer, New York, 1993.

[8] Z.-P. Jiang and T.-F. Liu. Quantized nonlinear control — a survey. Acta Automatica Sinica, 39(11):1820 – 1830, 2013.

[9] N. Kazantzis and C. Kravaris. Nonlinear observer design using Lyapunov's auxiliary theorem. Systems and Control Letters, 34:241 – 247, 1998.

[10] N. Kazantzis and R. Wright. Nonlinear observer design in the presence of delayed output measurements. Systems and Control Letters, 54:877 – 886, 2005.

[11] H. Khalil. Nonlinear systems. Prentice Hall, New Jersey, 2001.

[12] D. Liberzon and D. Nešić. Input-to-state stabilization of linear systems with quantized state measurements. IEEE Transactions on Automatic Control, 52:767 – 781, 2007.

[13] R. Maestrelli, D. Coutinho, and C. E. de Souza. Input and output finite-level quantized linear control systems: Stability analysis and quantizer design. Journal of Control, Automation and Electrical Systems, 26(2):105–114, Apr 2015.

[25] E. Tian, D. Yue, and C. Peng. Quantized output feedback control for networked control systems. Information Sciences, 178:2734 – 2749, 2008.

[14] C. Peng and Y.-C. Tian. Networked hinf control of linear systems with state quantization. Information Sciences, 177:5763 – 5774, 2007.

[15] B. Rehák. Alternative method of solution of the regulator equation: L2 -space approach. Asian Journal of Control, 14:1150 – 1154, 2011.

[16] B. Rehák. Sum-of-squares based observer design for polynomial systems with a known fixed time delay. Kybernetika, 51:858 – 873, 2015.

[17] B. Rehák. Observer design for a time delay system via the Razumikhin approach. Asian Journal of Control, 19(6):2226–2231, 2017.

[18] B. Rehák. Finite element-based observer design for nonlinear systems with delayed measurements. Kybernetika, pages 1050–1069, 2019.

[19] B. Rehák, J. Orozco-Mora, S. Čelikovský, and J. Ruiz-León. Real-time error-feedback output regulation of nonhyperbolically nonminimum phase system. In 2007 American Control Conference, pages 3789–3794, July 2007.

[20] B. Rehák and S. Čelikovský. Numerical method for the solution of the regulator equation with application to nonlinear tracking. Automatica, 44(5):1358 – 1365, 2008.

[21] B. Rehák, S. Čelikovský, J. Ruiz-León, and J. Orozco-Mora. A comparison of two Fem-based methods for the solution of the nonlinear output regulation problem. Kybernetika, 45:427 – 444, 2009.

[22] Lynnyk V. Rehák, B. Design of a nonlinear observer using the finite element method with application to a biological system. Cybernetics and Physics, 8:292 – 297, 2019.

[23] N. Sakamoto and B. Rehák. Iterative methods to compute center and center-stable manifolds with application to the optimal output regulation problem. In Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), Orlando, USA, 2011.

[24] N. Sakamoto, B. Rehák, and K. Ueno. Nonlinear Luenberger observer design via invariant manifold computation. In Proceedings of the 19th IFAC World Congress, 2014, Cape Town, South Africa, 2014.

[26] A. T. Tran, S. Suzuki, and N. Sakamoto. Nonlinear optimal control design considering a class of system constraints with validation on a magnetic levitation system. IEEE Control Systems Letters, 1(2):418–423, Oct 2017.

[27] S. Čelikovský and B. Rehák. Output regulation problem with nonhyperbolic zero dynamics: Femlab-based approach. IFAC Proceedings Volumes, 37(21):651 – 656, 2004. 2nd IFAC Symposium on System Structure and Control, Oaxaca, Mexico, December 8-10, 2004.

[28] Y. Yu and Y. Shen. Robust sampled-data observer design for Lipschitz nonlinear systems. Kybernetika, pages 699–717, 09 2018.