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A functional equation-based computational method for the discrete-time nonlinear observer ⋆

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Abstract: To solve the discrete nonlinear observer problem, it is necessary to find a solution of a certain functional equation. The existence conditions of this functional equation have already been well established, nevertheless, they are rather restrictive. Moreover, less attention was paid to the design of numerical methods to find its solution. In this paper, the approximation of the solution using the finite difference method is presented. From the theoretical point of view, this method has milder assumptions. The algorithm is thoroughly described and attention is paid to numerical aspects. The method is illustrated by an example.

Keywords: Nonlinear discrete-time system, nonlinear observer, functional equation.

1. INTRODUCTION

Observer problem for nonlinear systems is an important problem in the modern control theory. This is since in many practical applications, all quantities necessary for regulation are not directly measurable.

While the observer problem for linear systems has been successfully solved in the past, observers for nonlinear systems are still subject to investigation. There are several approaches: one can approximate the nonlinearity by a linear function, then handle the remaining (higher order) terms as uncertainty, as e.g. in Yu and Shen (2018). For polynomial systems, one can use also the design of the observer using the sum-of-squares theory Rehák (2015). This approximation leads to rather conservative results. On the other hand, one can easily design an observer for systems with time delays which is also important in practice, e.g. in the control of chemical or biological processes, see e.g. Rehák (2017).

Another approach is represented by the so-called high-gain observers Khalil (2001). Here, the gain of a linear observer is adjusted in order to achieve the convergence of the observation error to zero. The well-known drawback of this kind of observers is the high sensitivity to noise. Yet, they find many applications, e.g. in biology: see Čelikovský et al. (2018) or others.

In the late 1990s, another way for the observer design for nonlinear system emerged, for continuous systems first Kazantzis and Kravaris (1998), modified for time delay systems in Kazantzis and Wright (2005). It is based on solution of a partial differential equation that is a direct counterpart of the Sylvester equation known from the theory of linear systems. Then, the observer is constructed using the solution of this equation. Originally, the assumptions for existence of this partial differential equations were rather restrictive - it was required that the linearization of the observed system is either stable or all its eigenvalues have positive real parts. This assumption is much more strict than the assumption guaranteeing existence of a solution of the Sylvester equation in the linear case. Moreover, a method based on expansion into the Taylor series was proposed to find the solution numerically, even though this method is difficult to algorithmize and, on top of that, it is difficult to determine the domain where the results are valid. Fortunately, both these drawbacks were overcome by proposing a numerical method based on finding certain center manifold in Sakamoto et al. (2014). Later, approximation of the aforementioned partial differential equation by the finite element method was investigated in Rehák (2019) where also an alternative proof of existence of a solution of the partial differential equation was given together with definition of a domain where the solution is guaranteed to exist. For application of these observers to the problem of state reconstruction of a biological systems see e.g. Lynnyk and Rehák (2019). This method is based on successful solution of the regulator equation originating from the output regulation theory by the center manifolds (see e.g. Sakamoto and Rehák (2011)) or finite elements as in Rehák (2011).

Situation in discrete-time systems is less extensively elaborated. A "discrete-time counterpart" of the pioneering paper Kazantzis and Kravaris (1998) is Kazantzis and Kravaris (2001). Here, it is shown that the observer for a discrete-time nonlinear system can be constructed using a

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solution of a functional equation (in contrast to the partial differential equation in the continuous case). Existence conditions for the solution were also given, nevertheless, as in the continuous-time case, they are rather restrictive. Again, it is assumed that the linearization of the observed system must be either stable or all its eigenvalues must lie outside of the unit circle. Moreover, the proposed method for approximative solution of the functional equation is again based on Taylor expansions. Paper Brivadis et al. (2019) investigates other qualitative properties of this observer like existence or uniqueness.

Purpose of the paper: To deliver an easy-to-implement method for approximation of the solution of the aforementioned functional equation. This method is based on the finite-difference method known from the theory of numerical solution of partial differential equations. It gives an opportunity to obtain an approximation of the solution on a predefined domain. Moreover, the assumption about the eigenvalues of the linearization of the observed system is relaxed.

2. NONLINEAR DISCRETE OBSERVER PROBLEM

Theory of the nonlinear discrete-time observers is described in detail in e.g. Kazantzis and Kravaris (2001). Hence only the most important facts are repeated here.

Consider the discrete-time system

$$x(k+1) = F(x(k)),\ x(0) = x_0, \tag{1}$$
$$y(k) = h(x(k)) \tag{2}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ is the state, $y : \mathbb{N} \to \mathbb{R}$ is the output, $F : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

The following assumption is quite natural:

Assumption 1. Assume the linearization of (1,2) around he origin is observable.

The observer of the discrete-time system (1,2) is defined in Kazantzis and Kravaris (2001) as follows:

Definition 1. Let the mapping $\Psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be such that the dynamical system

$$z(k+1) = \Psi(z(k), y(k)) \tag{3}$$

where $y(k)$ is given by (2), admits a map $T : \mathbb{R}^n \to \mathbb{R}^n$ that is defined and locally invertible on a neighborhood U of the origin, $T(0) = 0$ and the following holds: if $x(0) \in U$ and $z(0) = T(x(0))$, then

$$z(k) = T(x(k)) \tag{4}$$

for all $k \in \mathbb{N}$.

Remark 1. The identity observer is obtained with $T = Id$.

Remark 2. Note that the estimate of the state of the observed system can be obtained as $\hat{x}(k) = T^{-1}(z(k))$. The inversion exists thanks to Assumption 1. Hence one can see that system (3) carries the same information as the original equation (1).

Substituting (4) into (3) yields

$$T(F(x)) = \Psi(x, h(x)),\ T(0) = 0. \tag{5}$$

Note that (5) is a functional equation.

In what follows, a linear system is used as the observer: eq. (3) is supposed to attain the special form as

$$z(k+1) = Az(k) + by(k) \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix with all eigenvalues in the interior of the unit circle in the complex plane and $b \in \mathbb{R}^n$ is a vector chosen so that the following assumption is satisfied.

Assumption 2. The pair (A, b) is controllable.

In this case, the observer is stable. Moreover, eq. (5) boils down to

$$T(F(x)) = Ax + bh(x),\ T(0) = 0. \tag{7}$$

Remark 3. Paper Kazantzis and Kravaris (2001) presents a method for the solution of eq. (7) by means of the power series: the nonlinear functions F and h as well as the unknown function T are expanded into Taylor polynomials. Then, comparing the coefficients, one obtains the approximation of the function T.

For the numerical method based on the finite differences, the solution of eq. (7) for linearization of the system (1,2) will be useful. Note that, in this case, this equation reduces to the Sylvester equation (with $\bar{F}$ being the Jacobian of F at the origin, C being the Jacobian of h at the origin):

$$\bar{T}\bar{F} = A\bar{T} + bC \tag{8}$$

for an unknown matrix $\bar{T} \in \mathbb{R}^{n\times n}$. This equation has a solution if and only if there are no common eigenvalues of matrices $\bar{F}$ and A, see e.g. Birkhoff and Lane (2017). This assumption is not a problem as matrix A is a design parameter, hence its eigenvalues can be determined so that this requirement is satisfied. Note also that this assumption is much milder that the conditions for existence of a solution of (5).

It is possible to seek the entire solution of (7) by the finite difference method. However, since the first-order terms can be computed exactly via the Sylvester equation (8), we will decompose the function T into the first-order terms and the remaining terms denoted by τ as follows:

$$T(x) = \bar{T}x + \tau(x) \tag{9}$$

where function τ vanishes at the origin together with its derivatives. This approach combining the exactly computed solution of a Sylvester equation with numerically computed function containing the higher-order terms has already been successfully applied for computing the nonlinear observer in the continuous-time case in Rehák (2019). To find the functional equation that function τ obeys, we make the following notation:

$$F(x) = \bar{F}x + f(x),\ h(x) = Cx + \gamma(x) \tag{10}$$

where function f and γ vanish at the origin together with their derivatives. We will also assume eq. (8) has a solution $\bar{T}$.

Substituting both (9) and (10) into (7) yields

$$\bar{T}(\bar{F}x + f(x)) + \tau(F(x)) = A\bar{T}x + A\tau(x) + bCx + b\gamma(x). \tag{11}$$

Taking (8) into account one arrives to equation

$$\tau(F(x)) = A\tau(x) + b\gamma(x) - \bar{T}f(x). \tag{12}$$

This is the functional equation to be solved here using the finite differences.

For the sake of the following section, let us note that mapping τ is composed of functions $\tau_k : \mathbb{R}^n \to \mathbb{R}$ with $k = 1, \ldots, n$.

3. SOLUTION OF THE FUNCTIONAL EQUATION USING FINITE DIFFERENCES

The finite difference method is a well-known method that has long tradition for solution of partial differential equations or to solve the boundary problems of ordinary differential equations. This method is based on approximating the function values on a grid, that is usually equidistant.

Due to the space reasons, the topic of this paper is solely to introduce the algorithms, the proof of the convergence of the approximations to the solution of (12) is left for a future publication.

3.1 Algorithm 1

If approximating a partial or ordinary differential equation, the derivatives are approximated by difference quotients. This is not the problem we face here since equation (12) contains no derivatives. On the other hand, one has to deal with the problem of defining the function values in the points given as $F(x_d)$ for any point x_d from the original grid.

Let $\Omega \subset \mathbb{R}^n$ be a domain such that $0 \in \Omega$. Define also a grid $\Omega_d \subset \Omega$ which is a finite set.

For every point $x_d \in \Omega_d$ compute the value $F(x_d)$. Then, for a fixed $x_d \in \Omega_d$ one can solve the equation

$$t(F(x_d)) = At(x_d) + b\gamma(x_d) - \bar{T}f(x_d) \tag{13}$$

for the unknown values $t(x_d)$ which are approximations of the function values of T. This is a set of n algebraic equations. However, the value $F(x_d)$ does not, in general, belong to the set Ω_d. Then, one has to interpolate the value $F(x_d)$ on the grid: one has to find M points $x_d^1, \dots, x_d^M$ so that $F(x_d) \in \mathrm{conv}(x_d^1, \dots, x_d^M)$ and if a point $x_d' \in \Omega_d$ satisfies $x_d' \in \mathrm{conv}(x_d^1, \dots, x_d^M)$ then $x_d' \in \{x_d^1, \dots, x_d^M\}$. (Roughly speaking, the point $F(x_d)$ is a convex combination of the points $x_d^1, \dots, x_d^M$ and, moreover, among all the points of the grid, points $x_d^1, \dots, x_d^M$ are closest points to $F(x_d)$. Assume $F(x_d) = \alpha_1 x_d^1 + \dots + \alpha_M x_d^M$. Then, one can approximate the value $t(F(x_d))$ by

$$t(F(x_d)) \approx \alpha_1 t(x_d^1) + \dots + \alpha_M t(x_d^M) \tag{14}$$

which, in fact, corresponds to the linear approximation of the function τ on the set $\mathrm{conv}(x_d^1, \dots, x_d^M)$. One obtains the set of equations

$$\alpha_1 t(x_d^1) + \dots + \alpha_M t(x_d^M) = At(x_d) + b\gamma(x_d) - \bar{T}f(x_d). \tag{15}$$

The method, as described above, suffers from several issues. First, it is difficult to find the points $x_d^1, \dots, x_d^M$ for every x_d. Then, the resulting set of algebraic equations (15) is large and it is not obvious whether it is solvable. Hence an improved algorithm is proposed.

3.2 Algorithm 2

The problem of interpolating the values on the grid points is circumvented in this algorithm. The crucial idea is to approximate the mapping τ by n-tuple of n-variate polynomials T_d^k, $k = 1, \dots, n$ such that the polynomial T_d^k approximates the function τ_k. Assume their degree is $\delta \geq 2$ (if $\delta = 1$, one has the linear approximation of T which is determined by matrix $\bar{T}$). Polynomials T_d^i are a linear combination of a finite number of monomials $x_1^{\iota_1} \dots x_n^{\iota_n}$ where ι_i are positive integers; let us denote the total number of these monomials by $\tilde{\nu}$. Then, one can introduce an ordering of these monomials: $m_i(x)$ for $i = 1, \dots, \tilde{\nu}$ where $m_i(x)$ is the ith monomial in this chosen ordering.

With this notation,

$$T_d^k(x) = \sum_{i=2}^{\tilde{\nu}} t_d^{k,i} m_i(x), \ k = 1, \dots, n \tag{16}$$

where $t_d^{k,i} \in \mathbb{R}$ are the coefficients to be determined. Then, for every point $x \in \Omega$ one can construct the n-tuple of equations (with $b = (b_1, \dots, b_n)^T$ and $f(x) = (f_1(x), \dots, f_n(x))^T$):

$$\begin{aligned} &\sum_{i=1}^{\tilde{\nu}} t_d^{k,i} m_i(F(x)) \\ &= \sum_{j=1}^{n} A_{k,j} \sum_{i=1}^{\tilde{\nu}} t_d^{j,i} m_i(x) \\ &\quad + b_k \gamma((x_1, \dots, x_n)^T) - \sum_{j=1}^{n} \bar{T}_{kj} f_j(x_1, \dots, x_n). \end{aligned} \tag{17}$$

This is a set of equations that are linear in all $t_d^{k,i}$. Denote also the sequence of all coefficients in the approximation of the polynomial τ_k by $\widetilde{\tau}_k = (t_d^{k,i})$, $i = 1, \dots, \tilde{\nu}$. Finally, denote by $\widetilde{\tau}$ the matrix whose elements are all coefficients $t_d^{k,i}$: $\widetilde{\tau} = (t_d^{k,i})_{k=1,\dots,n;\ i=1,\dots,\tilde{\nu}}$.

Define the following matrix function: $\Gamma : \mathbb{R}^n \to \mathbb{R}^{\tilde{\nu}}$ given so that for every $x \in \Omega_d$ holds:

$$\Gamma(x) = (m_1(x), \dots, m_{\tilde{\nu}}(x))^T \tag{18}$$

and function $\Delta : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\Delta(x) = \Big(b\gamma(x) - \bar{T}f(x)\Big). \tag{19}$$

Then, equation (7) is approximated by

$$\widetilde{\tau}\Gamma(F(x)) = A\widetilde{\tau}\Gamma(x) + \Delta(x). \tag{20}$$

This equation resembles the so-called non-homogeneous generalized Sylvester equation, see Bouhamidi and Jbilou (2008); Kaabi (2014); Wu et al. (2008) or others. Moreover, its solvability for over-determined systems of equations is a matter of research. Hence we are going to show existence of an approximation of its solution (in the sense of mean squares) under an additional assumption.

Assumption 3. Matrix A is diagonal:
$A = \mathrm{diag}(a_1, \dots, a_n)$.

This assumption is not restrictive as this matrix is the design parameter. However, note that one has still to ensure validity of Assumption 2.

Under this assumption, the set of equations (17) splits into n sets of equations, each set governs the $\widetilde{\tau}_k$ for $k = 1, \dots, n$. Namely (with $(\Delta(x))_k$ denoting the kth element of the vector $\Delta(x)$)

$$\widetilde{\tau}_k\Gamma(F(x)) = \widetilde{\tau}_k a_k \Gamma(x) + (\Delta(x))_k. \tag{21}$$

Eq. (21) cannot be evaluated at all $x \in \Omega$ but only on all $x_d \in \Omega_d$. Let

$$\Xi_k = \Big(\Gamma(F(x_d^1)), \dots, \Gamma(F(x_d^N))\Big) - a_k\Big(\Gamma(x_d^1), \dots, \Gamma(x_d^N)\Big), \tag{22}$$

$$\zeta_k = \Big((\Delta(x_d^1))_k, \dots, (\Delta(x_d^N))_k\Big). \tag{23}$$

Then, we can rewrite (21) into

$$\widetilde{\tau}_k \Xi_k = \zeta_k. \tag{24}$$

The number of equations in (24) is equal to the number of points in the grid, hence it is usually much greater than the number of variables to be determined. Hence equation (24) cannot be solved precisely, but, using the least squares method, one can use this approximation:

$$\widetilde{\tau}_k = \zeta_k \Xi_k^- \tag{25}$$

where Ξ_k^- is the right pseudoinverse of the matrix Ξ_k. Moreover, if desirable, using the weighted least squares, one can achieve a better fitting in some of the grid points. This can typically be the case of points close to the origin where one can obtain not only the function values of the polynomials T_d but also its derivatives to be close to zero. Note that a suitable choice of the polynomials m_i and of the parameters a_k can ensure existence of the pseudoinverse Ξ_k^-.

Let us describe the computation algorithm in detail:

Algorithm:

(1) Compute $F(\Omega_d)$.
(2) Choose the order of polynomials T_d^i approximating the mapping T. Denote by $\tilde{\nu}$ the number of monomials used to construct the polynomials T_d^i.
(3) For $k = 1, \dots, n$:
(4) Compose the matrix $\Xi_k \in \mathbb{R}^{\tilde{\nu} \times N}$ by (22).
(5) Define the row vector $\zeta_k \in \mathbb{R}^N$ by (23).
(6) Solve the least-squares problem (25).
(7) construct the polynomial $T_d^k(x)$ by (16).

To reconstruct the states of the observed system, one has to find the inverse transformation of the function T. However, as shown in Brivadis et al. (2019), a neighborhood of the origin where function T is injective exists. Hence this question can be positively answered. Therefore this problem does not lie in the focus of this paper.

Remark 4. Note that, in order to achieve the approximation of mapping T, the following is required besides Assumptions 1 and 2: first, the linear Sylvester equation (8) must be solvable. Moreover, pseudoinverse Ξ^- must exist. However, unlike Kazantzis and Kravaris (2001) it is not necessary to require that the eigenvalues of the linearization of the observed system lie all inside or all outside the unit circle.

Remark 5. The choice of the grid requires also some attention. As noted above, when solving partial differential equations using the finite difference method, a rectangular grid with equidistant points is usually (but not necessarily) used as such grid facilitates the discretization of the partial derivatives. This problem is not encountered in the case of functional equations. Thus, one can make use of a nonuniform grid without any precautions. To be specific, the grid should be dense around the origin to precisely capture the fact that the derivatives of the function τ vanish at the origin. This is rather important since in the opposite case, convergence of the observation error to zero might be violated.

4. EXAMPLE

Consider the system

$$\begin{aligned} x_1(k+1) &= x_2(k), \\ x_2(k+1) &= -x_1(k) - \frac{1}{2}\frac{x_1^3(k)}{1+x_1^2(k)}, \\ y(k) &= x_1(k). \end{aligned}$$

The initial conditions are $x_1(0) = x_{1,0}, x_2(0) = x_{2,0}$. Note that linearization of this system has eigenvalues lying on the unit circle.

To define the observer, the following choices were made (note that here, the only requirements to be satisfied are controllability of the pair (A, b) and the restriction on the position of eigenvalues of matrix A - the maximum of their real parts must be smaller than the minimum of the real parts of the Jacobian of the linearization of the original system; this is satisfied with this choice):

$$A = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{26}$$

Note that function $\gamma = 0$ in this example as the output function is linear. Equation (12) attains the form

$$\begin{aligned} &\tau\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2}\frac{x_1^3}{1+x_1^2} \end{pmatrix}\Big) = A\tau(x) \\ &-\bar{T}\begin{pmatrix} 0 \\ -\frac{1}{2}\frac{x_1^3}{1+x_1^2} \end{pmatrix}. \end{aligned} \tag{27}$$

This equation is solved numerically.

First, the matrix $\bar{T}$ is computed using the linear equation (8). The result is

$$\bar{T} = \begin{pmatrix} -0.1923 & -0.9615 \\ -0.0990 & -0.990 \end{pmatrix}.$$

The functional equation was discretized on the square-shaped domain $\Omega = [-1, 1] \times [-1, 1]$ with the nonuniform grid defined as follows: let

$$\begin{aligned} \overline{\Omega} = &\{-1, -0.98, -0.96, \dots, -0.22, -0.2\} \\ &\cup \{-0.18, -0.175, \dots, -0.025, -0.02\} \\ &\cup \{-0.015, -0.014, \dots, 0.014, 0.015\} \\ &\cup \{0.02, 0.025, \dots, 0.175, 0.18\} \\ &\cup \{0.2, 0.22, \dots, 0.98, 1\} \end{aligned}$$

and

$$\Omega_d = \overline{\Omega} \times \overline{\Omega}.$$

Using the method presented in the previous section, one can obtain that approximation of the function $\tau = (\tau_1, \tau_2)^T$ solving (27) by polynomials of third order:

$$\begin{aligned} \tau_1(x_1, x_2) &= -0.0016775x_1^3 - 0.0221048x_1x_2^2, \\ \tau_2(x_1, x_2) &= -0.0008425x_1^3 - 0.0116150x_1x_2^2. \end{aligned}$$

The function τ_1 is depicted in Fig. 1 while Fig. 2 shows function τ_2.

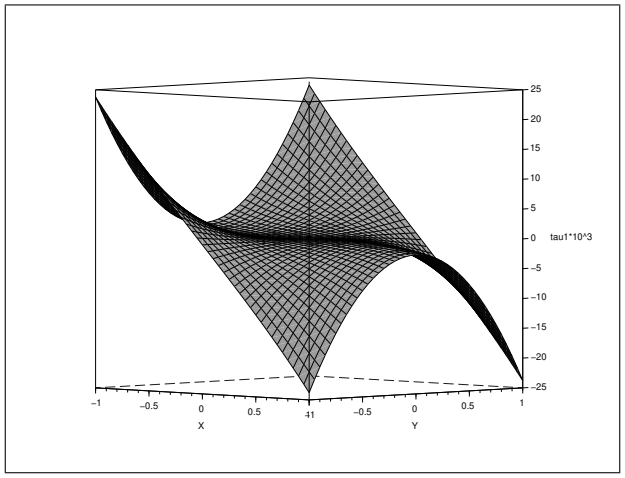

Fig. 1. Function τ_1

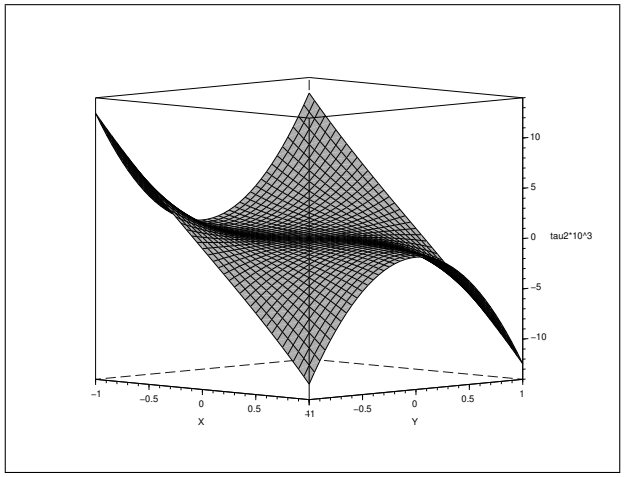

Fig. 2. Function τ_2

The simulation results are seen in Fig. 4. The symbol "+", for better clarity of the presentation interpolated by the dashed straight line, represents the state of the observed system with initial conditions $(0.9, 0)$, the diamonds interpolated by the solid line illustrate the observer that was started with initial condition $(0, 0)$. Note that the figure is in the transformed coordinates (that means, the z-coordinates; see (4)). One can see that the observation error converges to zero. Since the transformation T defined in (5) is a diffeomorphism, the same holds for the observation error in the original coordinates.

Remark 6. The order of the approximating polynomials must be chosen high enough to capture the behavior of the mapping τ. On the other hand, too high order of these polynomials requires many grid points. Moreover, the computation can be too demanding on the computer resources, both memory and time.

Remark 7. Paper Kazantzis and Kravaris (2001) (and, in the continuous-time case, Kazantzis and Kravaris (1998) and Kazantzis and Wright (2005)) use expansions to Taylor polynomials to find the solution of (5). Even though this is probably the simplest method for presentation since this method is easy to explain and understand, it suffers from serious drawbacks. To mention at first, let us mention that the result is only "local" in nature: this means, convergence is generally guaranteed only in an unknown

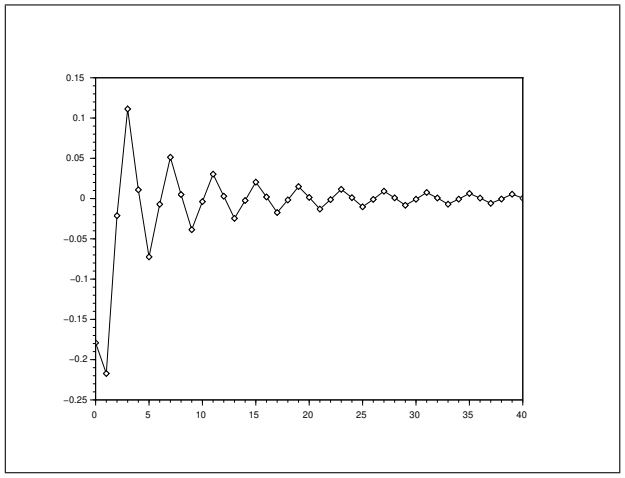

Fig. 3. Observation error in the transformed coordinates

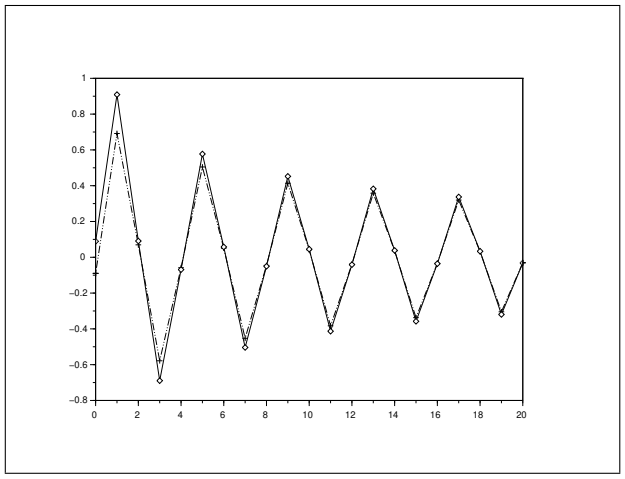

Fig. 4. Observer system and observer

neighborhood of the origin. Moreover, precision of the approximation decreases with distance from the origin. In contrast with the Taylor polynomial-based method, the approach presented here gives results whose precision on the set Ω (which was chosen a-priori) is guaranteed by the finite difference solver. It is also important to notice that computation of the Taylor polynomials usually requires lengthy calculations, even for the polynomials of low order (e.g. third order). Hence the results are difficult to obtain without the software for symbolic computations.

The algorithm presented here requires some knowledge of the finite difference method. However, unlike other methods stemming from the numerical methods for partial differential equations (like FEM), this method is also quite easy to understand and can be implemented without the use of any specialized software.

5. CONCLUSIONS

An algorithm for a numerical approximation of a solution of the functional equation originating from the discrete nonlinear observer problem was presented. This method is based on approximating the function by finite differences. This allowed to relax some assumption and also to deliver a computationally tractable algorithm. An example was provided to illustrate the properties of the method.

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