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Complete and Incomplete Sets of Invariants

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Abstract
The paper shows that the moment invariants proposed recently in this journal by Hjouji et al. (J Math Imaging Vis 62:606–624, 2020) are incomplete, which leads to a limited discriminability. We prove this by means of circular projection of the image. In a broader context, we demonstrate that completeness of the invariants leads to a better recognition power.

Keywords Moment invariants · Complex moments · Completeness · Discriminability · Circular projection

1 Introduction

Moments and moment invariants have established a popular and widely used category of "handcrafted" features for object description and recognition (see [1] for a comprehensive survey). A large amount of effort has been spent to study moment invariants with respect to various image degradations such as rotation, affine and projective transformations and blurring that has led to a huge number of papers on moments and moment invariants indexed in Scopus. (We can find there about 13,000 relevant papers published in the period 2015–2020 and more than 30,000 in total.) Regretfully, most authors have focused primarily on the invariance property of the moments while overlooking other properties of the same practical importance, such as *discriminability* and *completeness*. They have believed that the invariance is the principle theoretical challenge (which is true) and that the other properties are either of low importance or can be assured just by increasing the order of the moments. No one of these two assumptions is, however, true. Discriminability and completeness of the features are in pattern recognition as important as the invariance. Ignoring this fact resulted in many systems of invariants, which exhibit theoretical invariance but are almost useless for applications because their ability to classify objects/patterns is very limited.

The foundations of a systematic study of completeness and independence of moment invariants were laid by Flusser [2,3] in connection with rotation invariants and by Suk and Flusser [4] in case of affine invariants. Thanks to their work, the researchers started to understand the practical importance of these properties. The paper by Hjouji et al. [5] that has appeared quite recently in this journal belongs to those that still ignore this issue. In this short paper, we take the opportunity not only to show the link between [5] and some earlier papers but also to expose in this example that discriminability and completeness of any features should be studied along with their invariance if we want to get practically useful set of descriptors.

2 Basic Terms

In this section, we introduce a few basic terms, which are used throughout the paper. Rotation $(x, y) \mapsto (x', y')$ by angle θ is a coordinate transformation given as

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta, \\ y' &= x\sin\theta + y\cos\theta. \end{aligned} \tag{1}$$

Let $\{\pi_{pq}(x, y)\}$ be a polynomial basis of the image space. Moment $M_{pq}(x, y)$ of image $f(x, y)$ is defined as

$$M_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \pi_{pq}(x, y) f(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{2}$$

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that can be understood as a projection of $f(x, y)$ onto $\pi_{pq}(x, y)$ (provided that the integral converges). If $f(x, y)$ is compactly supported and piecewise continuous, then it can be precisely reconstructed from the set of all its moments.[1] Numerous moment families have been used in pattern recognition, each of them providing specific pros and cons. The choice of $\pi_{pq}(x, y) = x^p y^q$ yields *geometric moments* μ_{pq} [6] and $\pi_{pq}(x, y) = (x + iy)^p (x - iy)^q$ generates *complex moments* c_{pq} [2,7]. Other popular choices are Legendre moments [8], Chebyshev moments [9], Hermite and Gaussian–Hermite moments [10,11], Krawtchouk moments [12] and Gegenbauer moments [13] in Cartesian domain, and in polar coordinates Zernike moments [14,15], Fourier–Mellin moments [16,17], Jacobi–Fourier moments [18] and Chebyshev–Fourier moments [19]. Although numerical properties of individual moment families are different, they are all theoretically equivalent thanks to the mapping between any two polynomial bases.

Moment invariant to rotation is a function F of moments such that $F(M_{k,j}|\ k,j = 0,\ldots,r) = F(M'_{k,j}|k,j = 0,\ldots,r)$, where $M'_{k,j}$ stand for the moments of the rotated image. A set of invariants $\mathcal{F} = \{F_1, F_2, \ldots\}$ is called *complete*, if it allows an exact recovery of all moments $\{M_{pq}\}$ except one, which can be chosen almost arbitrary.[2] It means that a complete set of invariants allows an exact reconstruction of the original image (provided it is compactly supported and piece-wise continuous) up to a particular orientation. This is a single degree of freedom, because the orientation intrinsically cannot be captured by rotation invariants. That is why we may freely choose the value of a single moment, by which we actually select the orientation of the reconstructed image. An example of the moment recovery from a complete set of invariants is described in [3].

Using a language of pattern recognition, invariance means that all images belonging to the same class should have the same value of F. An opposite notion is the *discriminability*, sometimes also called the *recognition power*. We say F is discriminable if its values on images from different classes are different from each other. In the terminology of the group theory, where the classes are *orbits* of the rotation action, F is constant along each orbit (the invariance) while never gets the same value between the orbits (the discriminability).

One should understand the difference between discriminability on one hand and invariance and completeness on the other hand. While the first term depends on the particular set of images (image classes) we are working with in the given application and it is meaningless without specifying the data, the latter two terms are pure theoretical properties of F, totally independent of the data. There is of course a link between completeness and discriminability. If set of invariants $\mathcal{F}$ is complete, then it discriminates any data. If $\mathcal{F}$ discriminates some given dataset, it does not imply its completeness, it only means the $\mathcal{F}$ provides enough recognition power to resolve this task. If $\mathcal{F}$ is not complete, then there always exist two image classes which cannot be distinguished.

Flusser [2] proposed a general method which allows to construct a complete set of rotation invariants from geometric and complex moments. In case of other moments, the same idea can be adopted [20] or an indirect method of image normalization into a canonical position can be used [21]. In case of moments defined in polar coordinates, where the construction of rotation invariants is much simpler, a complete set was proposed by Wallin [15]. So, current literature offers enough solutions to resolve the completeness problem in full for any kind of rotation moment invariants regardless of the particular polynomial basis used for moment calculation.

3 Recalling the Invariants by Hjouji et al.

In [5], Hjouji et al. proposed the following set of rotation moment invariants

$$\Phi_n = \sum_{k=0}^{n} \binom{n}{k} \mu_{2k,2n-2k}, \quad n = 0, 1, \ldots \tag{3}$$

[check Eq. (5) of [5]]. Then, they applied Gram–Schmidt-like orthogonalization and showed that certain linear combinations of $\{\Phi_n\}$ can be viewed as moments with respect to an OG basis of polynomials. The orthogonalization was applied in order to make the computation faster and more stable thanks to recurrent relations and to eliminate information redundancy among the individual invariants. It has no influence on invariance, completeness and discriminability because $\{\Phi_n\}$ and the OG invariants generate exactly the same subspace of the moment space. From this point of view, the definition of invariants $\{\Phi_n\}$ performs the key result of [5].

Hjouji et al. concluded their paper saying that they "... introduced invariants of infinite order .... while Hu [6] derived only seven invariants of a finite order". This conclusion might give an impression that their invariants have a very good discriminability. In the next section, we show that the opposite is true because the set $\{\Phi_n\}$ is highly incomplete.

4 Exposing the Problems

First of all, let us point out that the invariants $\{\Phi_n\}$ are the same as complex moments. Particularly, $\Phi_n = c_{nn}$, as can

[1] This statement is known as Moment Uniqueness Theorem, see [1] for more details.

[2] For some images, certain moments may be constrained to be zero or non-zero, so the choice may not be totally free.

Fig. 1 Original image of the radar tower (**a**) and its circular projection Pf (**b**)

be seen directly from the definition

$$
\begin{aligned}
c_{nn} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^n (x-iy)^n f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)^n f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=0}^{n}\binom{n}{k} x^{2k} y^{2n-2k} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{k=0}^{n}\binom{n}{k}\mu_{2k,2n-2k} = \Phi_n.
\end{aligned}
\tag{4}
$$

So, the set of invariants $\{\Phi_n\}$ is incomplete, because instead of a full moment matrix $\{c_{kj}\}$ (which is a complete feature set) it contains its main diagonal only. Thanks to Eq. (4), we can also immediately recognize that the set $\{\Phi_n\}$ is just a small subset of (incomplete) invariants proposed by Mostafa [7] and of the complete system by Flusser [2].

Expressing c_{nn} in polar coordinates

$$
c_{nn} = \int_{0}^{\infty}\int_{0}^{2\pi} r^{2n} f(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta
$$

provides us with the insight what image features are actually captured by $\{\Phi_n\}$. Let us define *circular projection* Pf of image function f as

$$
(Pf)(r) = \frac{1}{2\pi}\int_{0}^{2\pi} f(r,\theta)\,\mathrm{d}\theta. \tag{5}
$$

This projection maps the image space onto the space of all circularly symmetric functions (see Fig. 1). Operator P is linear and idempotent as $P(Pf) = Pf$. Obviously, it is not a one-to-one mapping as many different images may be mapped onto the same symmetric function. It is important to note that P is a "moment preserving/vanishing" operation because $c_{mn}^{(Pf)} = \delta_{mn} c_{mn}^{(f)}$, where δ_{mn} is the Kronecker symbol.

(Detailed discussion about moments vanishing on symmetric patterns can be found in [22].) It means the invariants $\{\Phi_n\}$ can be understood as complex moments of Pf. On one hand, this clearly shows their invariance (Pf itself is a rotation-invariant pattern, so all its moments must be invariant, too) but on the other hand it explains their low discriminability. The invariants $\{\Phi_n\}$ are able to discriminate only the images having distinct circular projections. If we have $Pf = Pg$, then $\Phi_n(f) = \Phi_n(g)$ for any n and f and g are not discriminable, even if they are visually very different (see Fig. 2 for some examples). In other words, knowing an (infinite) sequence $\{\Phi_n(f)\}$, we can reconstruct the projection Pf but not the image f itself.

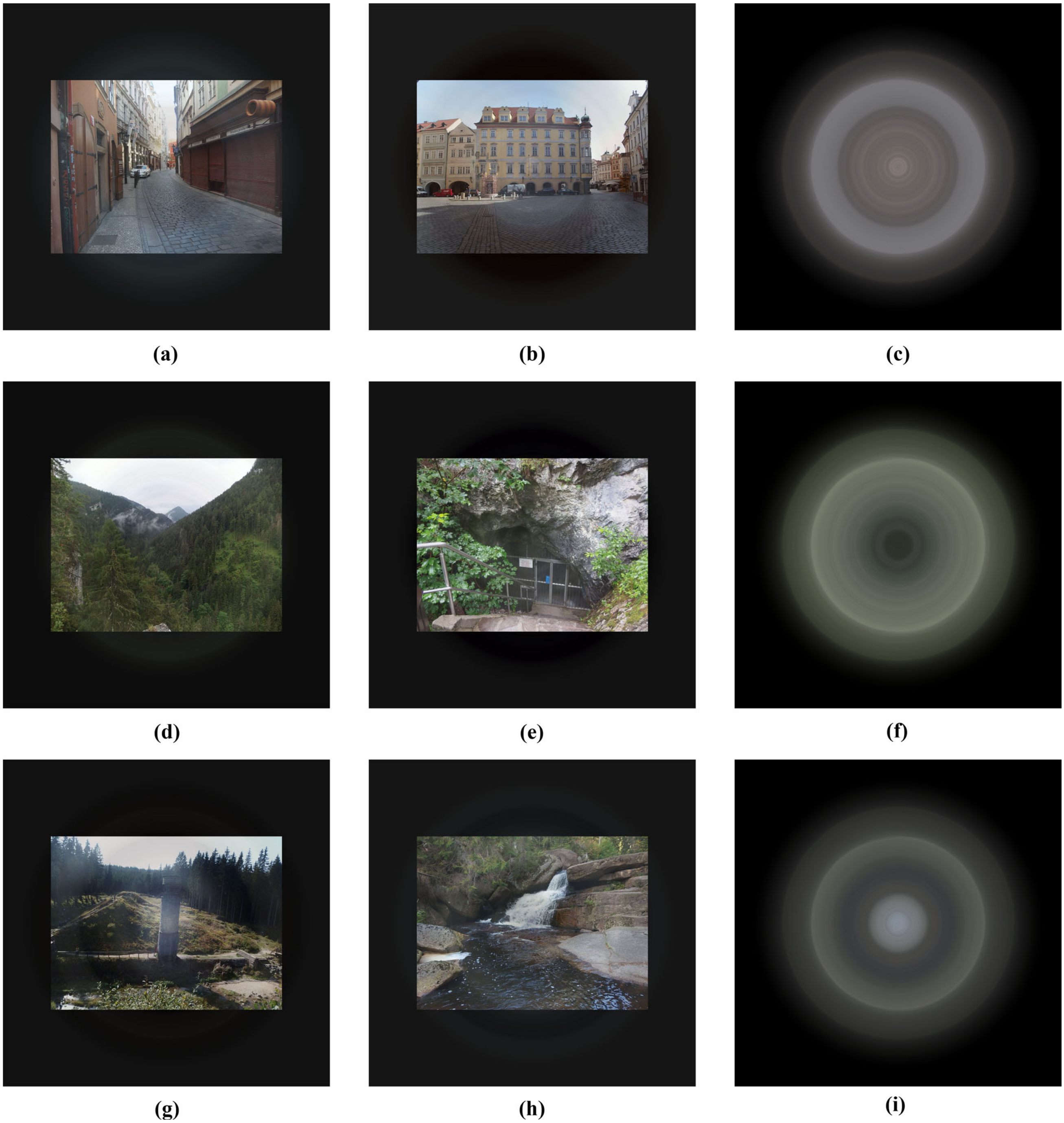

Fig. 2 Three sample pairs of images having the same circular projection, which is shown on the right. These images cannot be distinguished from each other by invariants $\{\Phi_n\}$ even if n is taken arbitrary high. **a**, **b** The city of Prague, Czechia; **d**, **e** Low Tatra Mountains, Slovakia; **g**, **h** Jizera Mountains, Czechia

5 Conclusion

In this paper, we showed that the invariants proposed in [5] form a highly incomplete system, which leads to a low recognition power. We explained this phenomenon by means of circular projection of the image, the complex moments of which are equal to the invariants from [5]. Overall, we demonstrated the necessity of studying completeness of any invariants one wants to use in practice, because only complete invariant sets maximize the recognition power.

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