Physica A 588 (2022) 126530

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Heterogeneity in economic relationships: Scale dependence through the multivariate fractal regression

Oussama Tilfani [a], Ladislav Kristoufek [b,c], Paulo Ferreira [d,e,f,*], My Youssef El Boukfaoui [a]

[a] *Faculty of Sciences and Techniques, Cadi Ayyad University, Marrakech, Morocco*
[b] *The Czech Academy of Sciences, Institute of Information Theory and Automation, Czechia*
[c] *Charles University, Faculty of Social Sciences, Institute of Economic Studies, Czechia*
[d] *VALORIZA - Research Center for Endogenous Resource Valorization, Portalegre, Portugal*
[e] *CEFAGE-UE, IIFA, Universidade de Évora, Largo dos Colegiais 2, 7000 Évora, Portugal*
[f] *Instituto Politécnico de Portalegre, Portalegre, Portugal*

ARTICLE INFO

Article history:
Received 19 June 2021
Received in revised form 22 September 2021
Available online 22 October 2021

Keywords:
Detrended fluctuation analysis
Heterogeneity
Multivariate fractal regression
Scale-dependent effects

ABSTRACT

Heterogeneity of effects between economic variables has been a frequently discussed topic for many years now. However, the estimation of such scale-dependent effects has proved challenging. Here, we propose a multivariate multiscale regression approach based on the combination of detrended fluctuation analysis and detrended cross-correlation analysis, but the idea can be easily translated into other time and frequency domain frameworks. As illustrations, we pick two classic economic models – the Taylor's rule and the money demand function for the USA and Japan – and we uncover evident scale-dependence in the individual effects not visible by the simple regression tools. Importantly, the proposed framework can be used in any discipline where studying the effects at various scales is of interest. Further applications are thus certainly at hand.

1. Introduction

The notion of a representative agent as an economic construct (or more precisely a representative firm in the original specification) goes way back to the work of Alfred Marshall in the late 19th century [1] and even though it had been introduced as a rather limited concept, it attracted firm criticism in the early 20th century, most notably by [2]. Nevertheless, through the decades of economics evolution, many concepts have been built on the assumption of a representative agent either explicitly or implicitly assumed [3]. However, the doubts kept growing stronger as insights from other disciplines have been coming steadily into light and helped develop more realistic theories and models based on actual behavior and characteristics of market participants [4,5]. The traditional mainstream economics assumptions of the expected utility, rational expectations and together with them representative agents [6] have been frequently abandoned and replaced by a more realistic bounded rationality assumption [7,8] usually connected with the market agents heterogeneity [9,10]. In financial economics, the heterogeneity of agents has become one of the conditions for a stable market while homogeneity of beliefs and behavior has become the sign of a possible coming disaster [11–13].

Agent-based models [14,15] have become a premier tool of modeling and examining complex economic and financial systems, even though their estimation has proven challenging [16,17]. These models standardly build on a set of microscopic assumptions that lead to observable behavior of market participants [18]. From the opposite side, we may narrow

* Corresponding author.
E-mail addresses: pferreira@ipportalegre.pt, pjsf@uevora.pt (P. Ferreira).

https://doi.org/10.1016/j.physa.2021.126530
0378-4371/

the agents' heterogeneity into differences in the time scale of obtaining, analyzing, and utilizing information [19,20] and study differences in behavior of agents with respect to their representative time scale. This leads to a macroscopic representation of the relationship between variables with respect to time scales (and frequencies). The relationship between two variables in the bivariate setting or among more variables in the multivariate is then characterized by a spectrum of parameters specific for the given scales. For the simplest case when the effect is constant across scales leads to the standard regression techniques whereas the possible spectrum of parameters needs to be estimated with more general techniques. In the frequency domain, the estimators are represented by (fully modified) narrow band least squares [21–23]. In the time domain, the methods are also built on the dynamic properties of long-range dependence processes and lead to fractal regressions [24]. Even though these approaches are quite straightforward and intuitive as they build on bivariate representation of the regression parameter estimation, i.e. the fraction between covariance and variance of the impulse variable, and its substitution with scale or frequency specific counterparts, most economic and financial series are not simply bivariate but multivariate. To utilize the full potential of such methods, one needs to generalize into a multivariate setting. In the present work, we introduce the multivariate generalization of the regression framework building on the notion of long-range dependence and fractality, i.e. in the time domain, and we show its usefulness on a set of traditional economic and financial models.

The paper is structured as follows. In Section 2, we summarize the current literature on scale-dependence of economic and financial relationships and laws. In Section 3, we briefly review the detrended fluctuation analysis and detrended cross-correlation analysis as building stones of the multiscale regression analysis. Its statistical properties and performance on simulated data are shown as well. Sections 4 and 5 provide a brief methodological background of the economic and financial areas with possible utility of the multiscale approach – specifically the Taylor's rule and the money demand function –, describe the datasets, and present the results. Section 6 concludes and presents ideas for future research.

2. From long-range dependence to scale-dependent regressions

The application of scaling analysis is not new in economics and finance. According to Frost and Prechter [25], approaches to find possible patterns in financial data emerging at different time scales goes back to the 1930s. Formally, the inclusion of scaling in models in finance could be given to Osborne [26] building on the Brownian motion proposal of Bachelier [27]. Despite its usefulness and ease of comprehension, a massive literature on deviations from the proposal has been accumulated, summarized in what is now known as the stylized facts of the financial time series see the comprehensive reviews of Cont [28] and Parisi et al. [29]. This led to generalizations of the Brownian motion, like the fractional Brownian or the Lévy motion processes with applications to finance [30–32].

One of the main directions in this field builds on experience connected to scaling laws in complex systems and statistical physics [33,34]. The multiscale regression that we propose to generalize here is based on the detrended fluctuation analysis (DFA) and detrended cross-correlation analysis (DCCA). Both methodologies originally come from physics but they quickly spread to interdisciplinary applications. DFA was proposed by Peng et al. [35] to detect fractal properties and long-range dependence of a time series, specifically to estimate Hurst exponent. The approach was later extended into DCCA [36] to estimate the scaling factor of long-range cross-correlations between two time series. Putting parts of the DFA and DCCA procedures together, Zebende [37] proposed a correlation coefficient allowing to quantify the level of correlation between time series at various scales, i.e. the relationship between series can be described by a spectrum of correlation coefficients rather than a single one. This correlation coefficient has been extensively used in many disciplines and its statistical properties have been studies as well [38–41]. Broadly speeking, DFA and DCCA can be embedded within a more general approach, the multifractal detrended cross correlation analysis MFDCCA [42].

In finance, the methodologies have been widely utilized. Most of the studies applying DFA point to the existence of long-term dependence: Liu et al. [43] for the S&P500, Ausloos et al. [44] and Ausloos [45] for the foreign exchange rate markets, Jaroszewicz et al. [46] for Latin American indices, Di Matteo et al. [47] for emerging stock markets, Alvarez-Ramirez et al. [48] for oil prices, Mariani et al. [49] for the NYSE stocks, Ferreira and Dionísio [50] for the G7 stock indices, Ferreira [51] for sovereign bonds or Zhang et al. [52] and Costa et al. [53] for cryptocurrencies. Long-range dependence of volatility has been extensively studied as well (e.g. [54–56]).

The Zebende [37] correlation coefficient has been vastly utilized in finance as well. Podobnik et al. [57] study the relationship between volume and price, Wang et al. [58] examine oil prices, Wang et al. [59] and Lin et al. [60] analyze commodity markets, Bashir et al. [61] inspect the exchange rate markets, Filho et al. [62] look into the Brazilian gasoline retail market, and Zhang et al. [52] and Costa et al. [53] explore cryptocurrencies. The DCCA correlation coefficient was used to investigate multiscale contagion on stock markets [63–67] and also complementarily to other measures. For example, Zhao and Shang [68] used it in a context of multidimensional analysis of time series jointly with a Principle Component Analysis (PCA) to assess nonstationary time series.

A further step from the scale-dependent correlations was taken by Kristoufek [24,69] who proposed regression frameworks based on DFA and also the detrending moving average analysis (DMA). Building on the ordinary least squares framework and its expression through variances and covariances, a method for estimating an actual effect (not only a correlation) of one variable on another at a specific scale was developed. The method has been further utilized in various applications by the author: to build a scale-specific capital asset pricing model for the DJI components [70] and the Portuguese stock index components [71], and to study the covered and uncovered interest rate parities [72,73]. The methodological steps and caveats between long-range dependence, scale-dependent correlations and regressions, and bivariate long-range dependence are in detail covered by Kristoufek [74].

Tilfani et al. [65,66] used the fractal regression framework to build multiscale efficient frontiers for the US stock market, before, during and after the subprime crisis and found a dominance of short term investors during the crisis period which is supportive of the fractal market hypothesis, which considers heterogeneous expectations of the investors. This framework has found its utility in other topics as well. For example, Wang et al. [75] introduced a bivariate linear regression model to estimate the multiscale dependence to analyze pollution. And Likens et al. [76] investigated the statistical properties of this multiscale frameworks and identified that it works well under several conditions and they also apply the method to a dataset on human posture, once again showing the potential interdisciplinary of the methodology.

3. Methodology

In this section, we introduce the multivariate multiscale regression based on the DFA method. To keep the line from DFA to the final regression procedure clear, we present the building blocks in a necessary detail, yet for further details, please refer to the references therein or in the previous section.

3.1. From detrended fluctuation analysis to multiscale simple regression

The detrended fluctuation analysis (DFA) approach is based on the relationship between the detrended variance function F_{DFA} (s) and the time scale *s*. The DFA procedure is conducted as follows:

Consider a possibly nonstationary time series x_t of length *T:*

i Integrate the original time series, i.e. get the series profile as $X(t) = \sum_{k=1}^{t}(x_t - \bar{x})$, where $\bar{x}$ is the sample mean of the time respective time series.

ii The profile is divided into $T_s = \lfloor T/s \rfloor$ non-overlapping boxes of size *s*, where $\lfloor \ldots \rfloor$ is the sign of lower integer, and *s* is the scale length. Within each box *v*, that starts at *(v-1)s+1* and ends at *vs*, the profile series *X(t)* is detrended, standardly by a linear time trend, but various detrending/filtering approaches can be utilized here.

iii The local detrended variance is calculated in each box *v* as follow:

$$f_X^2(s, v) = \frac{1}{s}\sum_{i=1}^{s}[X_{(v-1)s+i} - \tilde{X}_{(v-1)s+i}]^2 \tag{1}$$

For $v = 1, \ldots, T_s$

iv To obtain the detrended variance, we average over all segments:

$$F_{X,DFA}^2(s) = \frac{1}{T_s}\sum_{v=1}^{T_s} f_X^2(s, v) \tag{2}$$

The procedure then leads to estimation of Hurst exponent *H* as a measure of long-range dependence, but as it is not needed for the final regression that is presented, we do not delve into its details here either.

To perform the detrended cross-correlation analysis (DCCA), there are similar steps as for DFA but applied for two series instead of one. First, we get profiles of x_t and y_t. Second, both series are divided into non-overlapping boxes of equal length *s*. Now we only translate the univariate fluctuation function in each box and the detrended fluctuation function into the bivariate case:

$$f_{XY}^2(s, v) = \frac{1}{s}\sum_{i=1}^{s}[X_{(v-1)s+i} - \tilde{X}_{(v-1)s+i}][Y_{(v-1)s+i} - \tilde{Y}_{(v-1)s+i}] \tag{3}$$

And, the detrend covariance is obtained by averaging over all segments:

$$F_{XY,DCCA}^2(s) = \frac{1}{T_s}\sum_{v=1}^{T_s} f_{XY}^2(s, v) \tag{4}$$

Both DFA and DCCA can be used even for non-stationary time series and such, both can be used to study time series in various disciplines. Building on DFA and DCCA, Zebende [37] developed a measure to quantify the level of cross-correlation between two-time series for a specific scale *s* defining the correlation coefficient as:

$$\rho_{DCCA}(s) = \frac{F_{XY,DCCA}^2(s)}{F_{XDFA}(s)F_{YDFA}(s)} \tag{5}$$

Clearly, this correlation coefficient uses the standard definition of a fraction of covariance and a product of standard deviations while substituting these with their scale-specific counterparts based on DCCA and DFA, respectively. This cross-correlation coefficient is thus scale dependent, identifying differences in behavior according to different time scales. The

DCCA cross-correlation coefficient has the desired property, i.e., $-1 \leq \rho_{DCCA}(s) \leq 1$ with the standard interpretation, only now for the specific scale s. The different properties of the correlation coefficient are described in [38,39,41].

Kristoufek [24] extends the idea of using the DFA and DCCA fluctuation functions as substitutes for variance and covariance, respectively, to construct an estimator for a simple regression model:

$$y_t = \alpha + \beta x_t + \varepsilon_t \tag{6}$$

where y_t is a dependent (response) variable, and x_t is an independent (impulse) variable, and ε_t is an error term, and coefficients α and β are parameters indicating the relationship between x and y. The standard regression framework using the OLS approach estimates α and β though

$$\hat{\beta}^{OLS} = \frac{\sum_{t=1}^{T}(x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{T}(x_t - \bar{x})^2} \sim \frac{\widehat{\sigma_{xy}}}{\widehat{\sigma_x}^2} \tag{7}$$

where $\bar{x} = \frac{1}{T}\sum_{t=1}^{T} x_t$ and $\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t$ and

$$\hat{\alpha}^{OLS} = \bar{y} - \hat{\beta}^{OLS}\bar{x}. \tag{8}$$

In the same logic as in [37], Kristoufek [24] translates the standard variances and covariances in detrended fluctuation measures from DFA and DCCA. The estimators in Eqs. (7) and (8) can be expressed in a scale-dependent form by using the scale-dependent variance and covariance defined in Eqs. (1) and (3) as

$$\hat{\beta}^{\mathrm{DFA}}(s) = \frac{F^2_{xy,DCCA}(s)}{F^2_{x,DFA}(s)} \tag{9}$$

$$\text{and } \hat{\alpha}^{DFA}(s) = \bar{y} - \hat{\beta}^{\mathrm{DFA}}(s)\bar{x}. \tag{10}$$

3.2. Multivariable multiscale regression

The aim of our work is to provide a new framework, generalizing the DFA-based regression into the multivariable setting. To do so, we express the estimators of the beta coefficients into variances/covariances and explore the bridge toward the DFA and DCCA approaches. Let us consider the following multivariate regression equation:

$$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \cdots + \beta_k x_{kt} + \varepsilon_t \quad t = 1, 2, \ldots, T \tag{11}$$

In the matrix form, this can be expressed as $Y = X\beta + \varepsilon$ with $Y = (y_1, y_2, \ldots, y_T)'$, $\beta = (\beta_0, \beta_1, \ldots, \beta_k)'$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T)'$ and

$$X = \begin{pmatrix} 1 & x_{11} & x_{21} & \ldots x_{k1} \\ 1 & x_{12} & x_{22} & \ldots x_{k2} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_{1T} & x_{2T} & \ldots x_{kT} \end{pmatrix}$$

The OLS procedure aims to minimize the sum of squared residuals $S(\hat{\beta}) = \sum_{t=1}^{T} \hat{\varepsilon}_t^2 = \hat{\varepsilon}'\hat{\varepsilon} = (Y - X\beta)'(Y - X\beta)$ to get to the vector estimators

$$\hat{\beta}^{OLS} = (X'X)^{-1}(X'Y) \tag{12}$$

We define two new variables $x'_k = X_k - \tilde{X}_k$ and $y'_k = Y_k - \tilde{Y}_k$, which are residuals of the least square estimation of trends $\tilde{X}_k$ and $\tilde{Y}_k$, by the fact they are zero mean variables, [77], showed that Eqs. (2) and (4) can be expressed as:

$$F^2_{X,DFA}(s) = \frac{1}{s.T_s}\sum_{k=1}^{s.T_s}(X_k - \tilde{X}_k)^2 \tag{13}$$

$$F^2_{X,DFA}(s) = \frac{1}{s.T_s}\sum_{k=1}^{s.T_s} x'^2_k = \mathrm{Var}(x'_k) \tag{14}$$

$$\text{and } F^2_{XY,DCCA}(s) = \frac{1}{s.T_s}\sum_{k=1}^{s.T_s}(X_k - \tilde{X}_k)(Y_k - \tilde{Y}_k) = \mathrm{Cov}(x'_k, y'_k) \tag{15}$$

Furthermore, Zhao and Shang [68] showed that a transformation $\varphi(x_t)$ from x_t to $X_k - \tilde{X}_k$, which consists of calculating profile series and local detrending as specified in the DFA/DCCA procedures, is linear.

If we consider the matrix form of Eq. (11): $Y = X\beta + \varepsilon$ and by applying φ to both sides we obtain $\varphi(Y) = \varphi(X\beta + \varepsilon)$ and using linearity of φ:

$$\varphi(Y) = \varphi(X)\beta + \varphi(\varepsilon) \tag{16}$$

The sum of squared residuals of Eq. (16) is defined as

$$S(\hat{\beta}) = \varphi(\hat{\varepsilon})'\varphi(\hat{\varepsilon}) = (\varphi(Y) - \varphi(X)\beta)'(\varphi(Y) - \varphi(X)\beta) \tag{17}$$

We define $\hat{\beta}$ that minimizes Eq. (17), as

$$\hat{\boldsymbol{\beta}}^{\boldsymbol{DFA}}(\boldsymbol{s}) = (\varphi(X)'\varphi(X))^{-1}(\varphi\,(X)'\,\varphi(Y)) \tag{18}$$

Which can be expressed analytically as:

$$(\varphi(X)'\varphi(X))^{-1} = \begin{pmatrix} s.\,T_s & 0 & \dots & & & & & & 0 \\ 0 & \sum_{t=1}^{s.T_s} & x'^2_{1t} & \sum_{t=1}^{s.T_s} & x'_{1t} & x'_{2t} & \dots & \sum_{t=1}^{s.T_s} & x'_{1t} & x'_{kt} \\ & & & \dots & & & \dots & & & \\ & & & \dots & & & \dots & & & \\ 0 & \sum_{t=1}^{s.T_s} & x'_{kt} & x'_{1t} & \sum_{t=1}^{s.T_s} & x'_{kt} & x'_{2t} & \dots & \sum_{t=1}^{s.T_s} & x'^2_{kt} \end{pmatrix}^{-1} \tag{19}$$

$$(\varphi(X)'\varphi(Y)) = \begin{pmatrix} 0 \\ \sum_{t=1}^{s.T_s} x'_{1t}y'_t \\ \dots \\ \dots \\ \sum_{t=1}^{s.T_s} x'_{kt}y'_t \end{pmatrix} \tag{20}$$

In DFA/DCCA language:

$$(\varphi(X)'\varphi(X))^{-1}(\varphi\,(X)'\,\varphi(Y)) = \begin{pmatrix} 1\ 0 & \dots & 0 \\ 0 & F^2_{X1,DFA}(s) \dots F^2_{X1Xk,DCCA}(s) & \\ & \dots & \\ 0 & \dots F^2_{Xj,DFA}(s) \dots F^2_{XjXk,DCCA}(s) & \\ & \dots & \\ 0 & \dots F^2_{XjXk,DCCA}(s) \dots F^2_{Xk,DFA}(s) & \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ F^2_{X1Y,DCCA}(s) \\ \dots \\ F^2_{XjY,DCCA}(s) \\ \dots \\ F^2_{XkY,DCCA}(s) \end{pmatrix} \tag{21}$$

We write $\hat{\beta}^{DFA} = (\hat{\beta}_0^{DFA}, \hat{\beta}_1^{DFA}, \dots, \hat{\beta}_k^{DFA}) = (\hat{\beta}_0^{DFA}, \hat{\beta}'^{DFA})$ where $\hat{\beta}'^{DFA} = (\hat{\beta}_1^{DFA}, \dots, \hat{\beta}_k^{DFA})$

$$\widehat{\boldsymbol{\beta}}'^{\boldsymbol{DFA}}(\boldsymbol{s}) = \begin{pmatrix} F^2_{x1,DFA}(s) \dots F^2_{x1xk,DCCA}(s) \\ F^2_{x1x2,DCCA}(s) \dots F^2_{x2xk,DCCA}(s) \\ \dots \\ F^2_{x1xj,DCCA}(s) \dots F^2_{xj,DFA}(s) \dots F^2_{xjxk,DCCA}(s) \\ \dots \\ F^2_{x1xk,DCCA}(s) \dots F^2_{xjxk,DCCA}(s) \dots F^2_{xk,DFA}(s) \end{pmatrix}^{-1} \begin{pmatrix} F^2_{x1y,DCCA}(s) \\ F^2_{x2y,DCCA}(s) \\ \dots \\ F^2_{xjy,DCCA}(s) \\ \dots \\ F^2_{xky,DCCA}(s) \end{pmatrix} \tag{22}$$

for $\hat{\beta}_1^{DFA}$ to $\hat{\beta}_k^{DFA}$ and independently

$$\hat{\beta}_0^{DFA}(s) = \overline{y} - \hat{\beta}_1^{DFA}(s)\,\overline{x_1} - \dots - \hat{\beta}_k^{DFA}(s)\,\overline{x_k}. \tag{23}$$

3.3. Statistical inference on estimated parameters and scale dependence

The estimated $\hat{\beta}_j^{DFA}(s)$ are used to describe the effect of the independent variables on the dependent variable at a given time scale s. In the standard regression framework, we define the t-statistic as $t_j = \frac{\widehat{\beta_j} - \beta_j}{\sqrt{var(\widehat{\beta_j})}}$ $(j = 1, \ldots, k)$ with $t_j \sim t_{T-k-1}$. In general, if $|t_j| > t_{1-\alpha/2, T-k-1}$, we should reject the null hypothesis of $\beta_j = 0$, and the dependence between X_j and Y is statistically significant, i.e., statistically different from zero. In our case, we estimate a multiscale $\hat{\beta}_j^{DFA}(s)$. Using the single critical value $t_{1-\alpha/2, T-k-1}$ for all scales may be inappropriate, with another alternative based on generating a critical value $t_c(s)$ for each scale. To do so, we followed the procedure of Podobnik et al. [78] to test the significance of the DCCA correlation coefficients by shuffling the time index of the considered series Y_j and X_j and repeat the DFA regression calculations several times (1000 or more). We let the integral of the probability distribution function (PDF) from $-t_c(s)$ to $t_c(s)$ be equal to $1-\alpha$ (in the empirical applications, we stick to the standard $\alpha = 0.01$ or 0.05). By using $t_c(s)$, we can test whether the dependence between Y and X_j is significant or not at scale s. The scale dependent statistic $t_j^{DFA}(s)$ can be expressed as:

$$t_j^{DFA}(s) = \frac{\hat{\beta}_j^{DFA}(s) - \beta_j}{\sqrt{var(\hat{\beta}_j^{DFA}(s))}} \tag{24}$$

We know that $var(\hat{\beta}_j^{OLS})$ is the jth diagonal component of the matrix σ_ε^2(X'X)-1 with σ_ε^2 as the variance of residuals. In the multiscale case, $var(\hat{\beta}_j^{DFA}(s))$ is the jth diagonal of the matrix $F_\varepsilon^2(s)$A(s), with:

$$A(s) = \begin{pmatrix} F^2_{x1,DFA}(s) \ldots\ldots\ldots\ldots\ldots F^2_{x1xk,DCCA}(s) \\ F^2_{x1x2,DCCA}(s) \ldots\ldots\ldots\ldots F^2_{x2xk,DCCA}(s) \\ \ldots\ldots \\ F^2_{x1xj,DCCA}(s) \ldots F^2_{xj,DFA}(s) \ldots F^2_{xjxk,DCCA}(s) \\ \ldots\ldots \\ F^2_{x1xk,DCCA}(s) \ldots F^2_{xjxk,DCCA}(s) \ldots F^2_{xk,DFA}(s) \end{pmatrix}^{-1} \tag{25}$$

In the similar vein, the goodness of fit can be measured using the scale-specific adjusted $R^2_{adj}(s)$ defined parallelly as

$$R^2_{adj}(s) = 1 - \frac{T-1}{T-k-1} \frac{F_\varepsilon^2(s)}{F_y^2(s)}. \tag{26}$$

The multiscale regression approach proposed above provides a vector of $\hat{\beta}$ for different scales, however it is necessary to identify the statistical significance of βs variability across scales. To do so, we construct a scale-dependent confidence interval and if the estimated $\widehat{\beta^{DFA}(s)}$ falls outside the confidence intervals, we conclude about the heterogeneity of the effect the given independent variable has on the dependent variable. In the procedure, we estimate a multiple linear regression using the standard OLS regression and obtain the estimated parameters and residuals. We create a surrogate series of the residuals and using the phase randomization of Theiler et al. [79], we reconstruct the series using the surrogate residuals and the estimated parameters $\hat{\beta}$. We repeat this procedure 1000 times and to construct the upper (lower) bounds of the 95% confidence intervals, we use the 97.5th quantiles (2.5th quantiles) of the estimated parameters. Please, refer to [70] for a more detailed description of this procedure with a specific application on financial data.

The use of the DFA-based approaches requires time series with a minimum of 500 observations [76]. To satisfy this condition, the following estimation of the Taylor rule and money demand function was based on monthly data. This data frequency has been used in various studies both for Taylor rule [80–82] and for money demand function [83,84].

3.4. Performance of DFA estimators on simulated data

The DFA based multivariable regression provides a scale dependent coefficient, which allows to detect the dependence of a response variable and k dependent variables at different time scales. In order to examine empirically our suggested approach, we perform a numerical test on the following equation:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \varepsilon_t \tag{27}$$

We follow the approach described by [24] and [75] and construct five time series with equal length (N = 10000) to show how the DFA estimators perform under various levels of long-range dependence in series $X_{1t,2t,3t,4t}$ and Y_t. The former series are generated as an AFIRMA(0,d,0) processes with identical fractional integration parameter d and independent Gaussian noises $\xi_i(t)$ $i = 1, 2$ and 3, so that $X_i(t) = \sum_{n=0}^{\infty} a_n(d)\, \xi_i(t-n)$, the quantity $a_n(d)$ is defined by $a_n(d) = \frac{\Gamma(n-d)}{[\Gamma(-d)\Gamma(n+1)]}$, where $\Gamma(.)$ is the Gamma function. The error-term is set as a standard Gaussian noise so that the

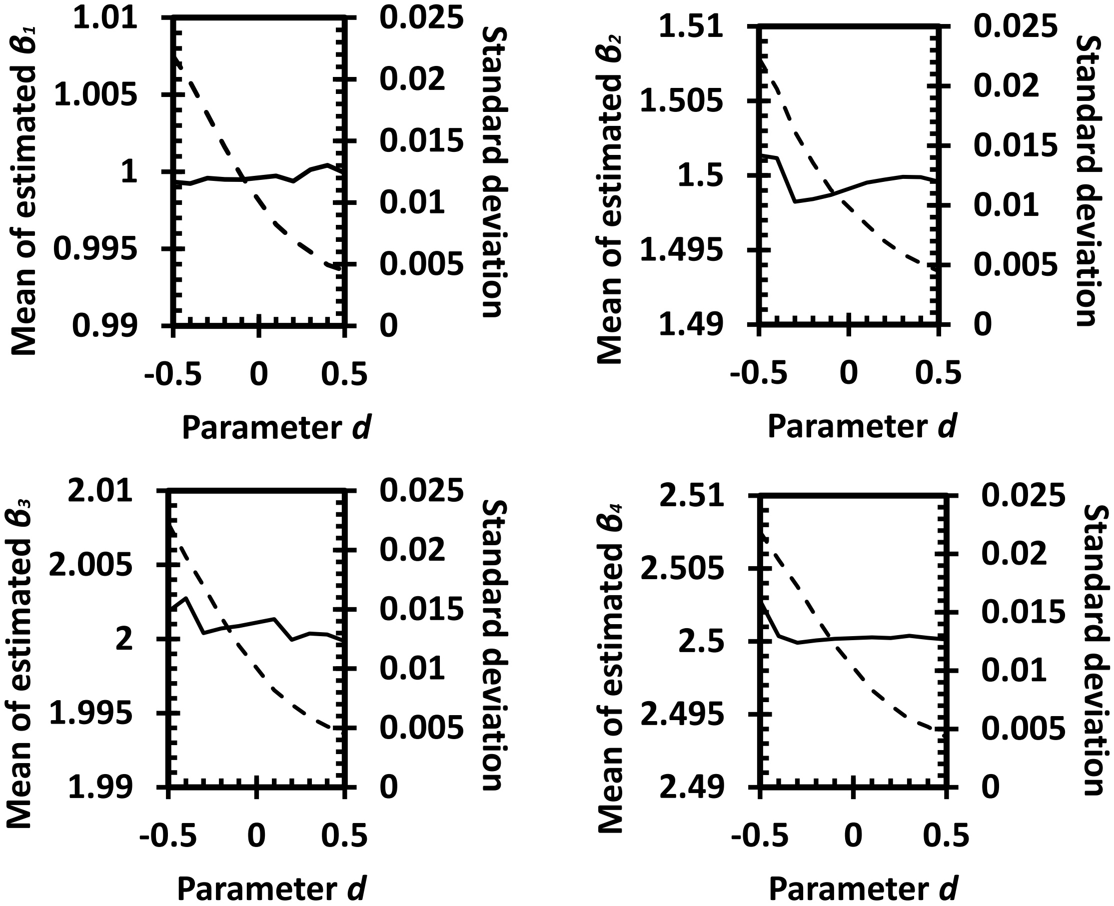

Fig. 1. Simulated results I. Mean values (solid lines, left axis) and standard deviation (dashed lines, right axis) for model $y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \varepsilon_t$, where $\beta_0 = \beta_1 = 1$, $\beta_2 = 1.5$, $\beta_3 = 2$ and $\beta_4 = 2.5$, X_i (i=1,..,4) are 4 independent processes generated by AFRIMA model with the same changing fractional integration parameter (x-axis) and independent Gaussian noise. Each simulation has a length of 10000 and repeated 1000 times.

response variable Y has the same parameter d. The regression coefficients are set as: $\beta_0 = \beta_1 = 1$, $\beta_2 = 1.5$, $\beta_3 = 2$ and $\beta_4 = 2.5$, the parameter d is ranging from -0.5 to 0.5 with a step 0.1. This test, allows to investigate the performance of the DFA estimators under different levels of long-term dependence in $X_{1,2,3,4}$ and Y.

The estimators are averaged over scales between 10 and 1000 with a logarithmic isometric step. Each case is repeated 1000 times to eliminate the noise interference. We report our findings in Fig. 1 where we show that the estimated coefficients are unbiased regardless of the value of d.

Furthermore, the standard deviations of the 4 estimated coefficients decreases with the increasing memory. Second, to investigate the performance of our approach faced with a long-range dependent error term ε, we generate our model as follow: we fix parameter $d = 0.4$ for $X_{1,2,3,4}$, and the ε is generated by an AFRIMA process with d_ε varying from -0.5 to 0.5, the rest of setting remains unchanged. In Fig. 2, we report similar behavior as for the previous case. We observe that the estimated parameters are unbiased and stable, with an increasing standard deviation of estimated parameters, which is expected as the memory of the error term increases leading to an increasing variance of the error term.

4. Example I: Taylor rule for interest rates

4.1. Brief theoretical background and dataset description

The main objective of the modern central banks is to ensure price stability [85] and the Taylor rule [86] is one of their traditional instruments. The rule was firstly estimated for the US economy as a linear representation of connections between interest rate, inflation, and output gap:

$$i_t = \pi_t + \varphi\left(\pi_t - \pi^*\right) + \gamma y_t + R^* \tag{28}$$

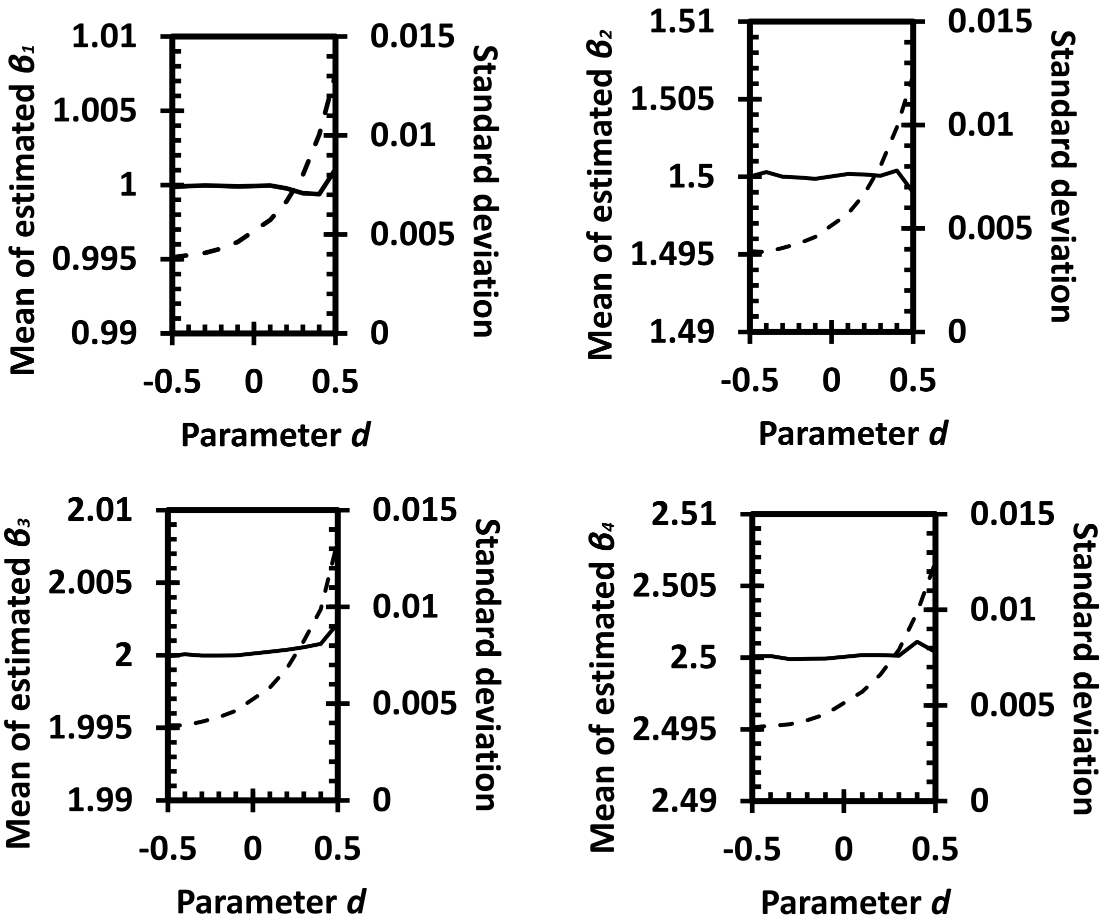

Fig. 2. Simulation results II. Mean values (solid lines, left axis) and standard deviation (dashed lines, right axis) for model $y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t} + \varepsilon_t$, with same β coefficients as in settings I, and X_i (i=1,..,4) are 4 independent processes generated by AFRIMA model with the same fractional integration parameter (d=0.4) and ε_t is an AFRIMA process with changing parameter d_ε (x-axis). Each simulation has a length of 10000 and repeated 1000 times.

where i_t is the target interest rate, π_t the inflation rate, π^* is the target level of inflation, y_t is the output gap (the deviation of actual real GDP from the estimated potential level) and R^* is the equilibrium level of the real interest rate. The simplified and also more popular form [87,88], also known as the modified Taylor rule, is given as

$$i_t = \mu + \alpha\pi_t + \beta y_t. \tag{29}$$

The original empirical formulation of the rule led to the following specification [86,89]:

$$i_t = 1 + 1.5\pi_t + 0.5y_t \tag{30}$$

Standardly, the most interest lays on the values of α higher than one as it suggests amplification of the inflation effect into the interest rates, and similar for β as values above unity suggest overheating.

In this example, we are interested in the scale structure of the effects in the Taylor rule, specifically whether the effects differ in the short run and the long run. We study the economies of the USA and Japan between 1970 and 2019 (600 monthly observations each) using the data from the Federal Reserve of St. Louis. To be able to perform the analysis at the monthly frequency, and thus ensuring enough data points, and also make the datasets comparable, we represent the inflation rate by the year-over-year change in the Consumer Price Index (CPI) for both the USA and Japan, and for the real product, we utilize the Industrial Production Index (IPI). Following [90], we define the output gap as a deviation from the potential output, which is approximated as a quadratic trend of the monthly real product (either GDP or IPI). For the policy rate, we follow [80] and use the effective federal funds rate for the USA and the call money market rate for Japan.

4.2. Results and interpretation

We estimate the model in Eq. (22) with the use of the DFA-based multivariate fractal regression introduced above for the scales of 6, 12, 24, 48, 60, and 120 months, i.e. 0.5, 1, 2, 4, 5, and 10 years to possibly uncover complex dynamics

Table 1
Taylor rule for USA.

	Intercept	α (inflation)	β (output gap)	R^2
scale s = 6	4,42	0,052 (0,051)	0,42*** (0,0279)	29%
scale s = 12	2,79	0,501*** (0,042)	0,288*** (0,0197)	41%
scale s = 24	3,69	0,311*** (0,041)	0,183*** (0,018)	34%
scale s = 48	2,34	0,665*** (0,03)	0,124*** (0,012)	54%
scale s = 60	2,24	0,676 *** (0,023)	0,169*** (0,0095)	80%
scale s = 120	1,76	0,838*** (0,023)	0,03 (0,012)	61%
OLS regression	2,392	0,665*** (0,039)	0,082*** (0,006)	67%

* critical values at 10%, ** critical values at 5%, *** critical values at 1%. Terms between brackets are the standard errors

that is heterogeneous across scales. For statistical validity of this potentially scale-specific effects, we utilize a bootstrap-like mechanics of firstly estimating the model with the OLS estimator, reorganizing the residuals (while keeping the distributional properties and the autocorrelation structure), reconstructing the dependent variable with the estimated effects, to eventually estimate the scale-specific effect via the multivariate fractal regression. This procedure is repeated 1000 times to obtain critical values of the DFA-based estimates under the null hypothesis of no heterogeneity, i.e. the null hypothesis that a constant effect is sufficient for the given scales, rather than a spectrum of effects over the scales.

Table 1 summarizes the results for the USA and Table 2 shows the results for Japan. Starting with the USA, we see that in the short term (6 months), there is no significant effect of inflation on the interest rate and the inverse for the output gap, not significant for long term (10 years). The effect of inflation then increases with the time scale and reaches its maximum at the highest analyzed scale of 10 years at around unity, the effect estimated using the simple OLS procedure corresponds to results obtained for the medium-term horizon (4 and 5 years), i.e., in effect an average effect. However, the dynamics is apparently much richer than a single-parameter estimation would suggest, and it is only in the long run when inflation fully propagates into the interest rates. The effects of 1.5 and 0.5 for inflation and output gap, respectively, as originally suggested by Taylor [86] are thus either not attained either at all (for inflation) or only in the very short run (for output gap). The effect of inflation is thus very far from its baseline levels and the heterogeneity is very high. The average effect of 0.6–0.7, represented by the OLS estimate, is representative only for the medium run between 4 and 5 years. In the long run, the effect is much stronger, whereas for the shorter run, the effect drops. For the output gap, the effect is at its highest for the shortest scales of 6 and 12 months with the effect of 0.4 and 0.3, respectively. Therefore, the effects of the standard strength, i.e. at 0.5, or at least close to it are observed only for the very short run and the effect almost vanishes for the long run (becoming insignificant for the highest analyzed scale). The output deviations from the its potential thus have only a short-lived effect on the interest rate.

The heterogeneity of the effect is further illustrated in Fig. 3 which shows the bootstrap-based critical values for the constant effect. We see that the differences between the short-term effect for inflation and the medium-term effect for output gap are so pronounced that their estimates fall well outside of these critical values (these represent the critical values for the 95% significance level). The results thus add an important perspective to the previous results such as [91,92], and [93].

In the case of Japan, we observe that output gap parameters are not significant for any scale. Moreover, parameters are rather unstable, a finding consistent with [94]. While the inflation coefficient is significant for medium and long term (scale of two years and higher), its magnitude is smaller than one and even smaller than in the case of the USA, which could be related with the low interest rates practiced in Japan [95]. In response to the economic downturn, Japanese monetary authority decided to relax monetary conditions, with the discount rate below 0.5% since the mid-1990s, and at zero since April 2017. Yet again, we find that the effect of inflation mostly grows with the time scale, suggesting that it takes several years for the inflation effects to fully propagate to the economy.

For the scale-dependence significance (Fig. 4), we observe that inflation coefficient is significantly scale dependent for a horizon smaller than four years. For output gap, as the estimated coefficient is not statistically significant and unstable, it seems difficult to judge for scale-dependence significance at all.

Table 2
Estimation of Taylor rule using DFA based regression on Japanese data.

	Intercept	α	β	R^2
scale s = 6	2,49	0,011 (0,018)	0,431 (0,536)	0%
scale s = 12	2,49	−0,007 (0,012)	1,044 (0,598)	0%
scale s = 24	1,97	0,227 (0,015)	0,137 (0,387)	29%
scale s = 48	2,02	0,204*** (0,011)	0,193 (0,526)	44%
scale s = 60	1,69	0,327*** (0,008)	0,254 (0,3334)	76%
scale s = 120	1,95	0,298*** (0,006)	−1,834 (0,436)	80%
OLS regression	1,068	0,412*** (0,012)	5,079*** (0,234)	79%

*** denotes significance with a 1% level; ** denotes significance with a 5% level; * denotes significance with a 10% level. Information in brackets is for the standard errors of the estimated coefficients.

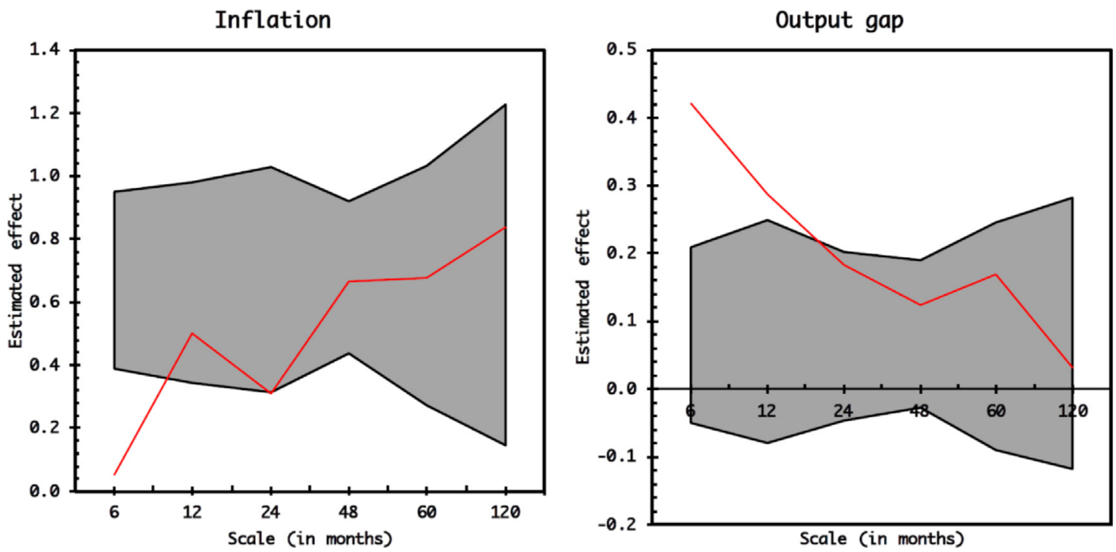

Fig. 3. Statistical inference of the scale dependence for Taylor rule coefficients for US. Red line indicates the estimated coefficients with DFA regression and the shaded area refers to the confidence intervals. In the x-axis, we report time scales.

5. Example II: Money demand

5.1. Brief theoretical background and dataset description

The Quantity Theory of Money [96] is one of the oldest economic laws. Building on a pure accounting tautology, it says that

$$M_s V_T = P_T T \tag{31}$$

where M_s is the quantity of money in circulation, P_T the price level, T the volume of transactions and V_T is the transactions velocity. An alternative given by [97] and [98] relates the quantity of money to the nominal income, and V is determined by the agents' preferences rather than by the payment mechanism [99], which leads to the standard expression of the quantity theory given as

$$MV = Py \tag{32}$$

For the empirical investigation, the equation is given in its combined logarithmic representation as given e.g. by [100,101], and [102]:

$$\ln(M) - \ln(P) = \alpha_0 + \alpha_y \ln(y) + \alpha_\pi \pi + \alpha_i i + \alpha_e e \tag{33}$$

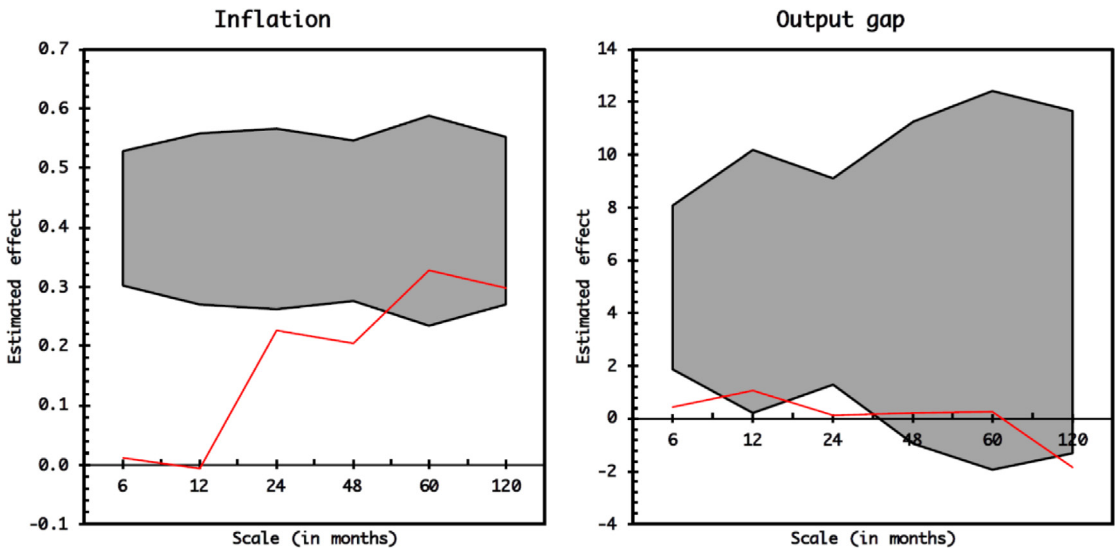

Fig. 4. Statistical inference of the scale dependence for Taylor rule coefficients for US. Red line indicates the estimated coefficients with DFA regression and the shaded area refers to the confidence intervals. In the x-axis, we report time scales.

Here M is the money supply, P the price level, π the inflation rate, y the real output, i the interest rate and e the exchange rate. This representation thus extends the original equation by additional variables important for the monetary system [103].

In addition to the variables used in the Taylor rule example, we use the monetary aggregate M2 in USD for the USA and in JPY for Japan, and for the real effective exchange rate, we use the CPI-based narrow index [104]. The money demand equation is examined for monthly data between 1970 and 2019 under the same setting as the Taylor rule inspection in the previous section.

5.2. Results and interpretation

The results suggest that the main driver for the money demand is the output, i.e. the more productive the economy is the more economic agents tend to spend. This is not a surprising result, but as it is usually the case, the details tell a more complete story. For the USA (Table 3), we see that the effect of the output on money demand is scale-dependent and positive, in accordance with economic theory. Regarding the inflation, we observe that coefficients are significant in short term (from 6 months to two years) with negative coefficients, highlighting that goods are reasonable substitute to holding domestic currency. However, even though the effect of inflation is statistically significant, inspecting both the scale-dependent and the OLS estimates (albeit with a different sign) uncovers that the economic significance is quite weak. The specification given in Eq. (33) makes inflation coefficient a semi-elasticity. This makes, e.g., the OLS estimate translate into the change in inflation by 1%, i.e., a strong effect, causing 1% increase in money demand, which is a weak effect. For interest rates, coefficients are significant and negative across all scales. The sign is supportive of the Keynesian theory of money demand highlighting the speculative motive [104,105]. Yet again, the economic significance is not high as an increase in interest rates by 1%, i.e., again a large swing, leads to a decrease in the real money demand in small units of percentages, lower than 1% in the short run. The real exchange rate effect is not significant, except for horizons of two and ten years, with a negative sign for the former and positive one for the later.

The heterogeneity is illustrated in Fig. 5 and we can see that it is less pronounced than for the Taylor's rule shown above. Apparently, the confidence intervals are made wider by the higher number of independent variables in the system so that, combined with the rather low number of observations, most of the heterogeneity remains hidden within the confidence intervals. Even though the intervals for inflation and REER suggest scale-dependence of the effects, we see that the effects are rather flat. Yet the effects for the real output and interest rate show a more relevant story. Interestingly, the period after which the effect dynamics flattens for the real output is the same or at least similar here as for the Taylor's rule — between 4 and 5 years. The transmission time for the real output thus seems to be in this interval.

In the case of Japan, we observe similar patterns, however some differences should be highlighted (Table 4). The effect of the real output is greater than unity in the long term and the sign of inflation is negative in the short run, positive and greater in magnitude in the long run (for 60 and 120 months). For interest rates, estimates are significant for all scales except in the short term (6 months), and they show a negative sign. For REER, only estimates for scales of 12 and 24 months are significant and positive. Regarding the scale-dependence significance, we observe that the coefficients of real output, inflation and interest rates fall outside the limits of the confidence intervals for short and medium term while the

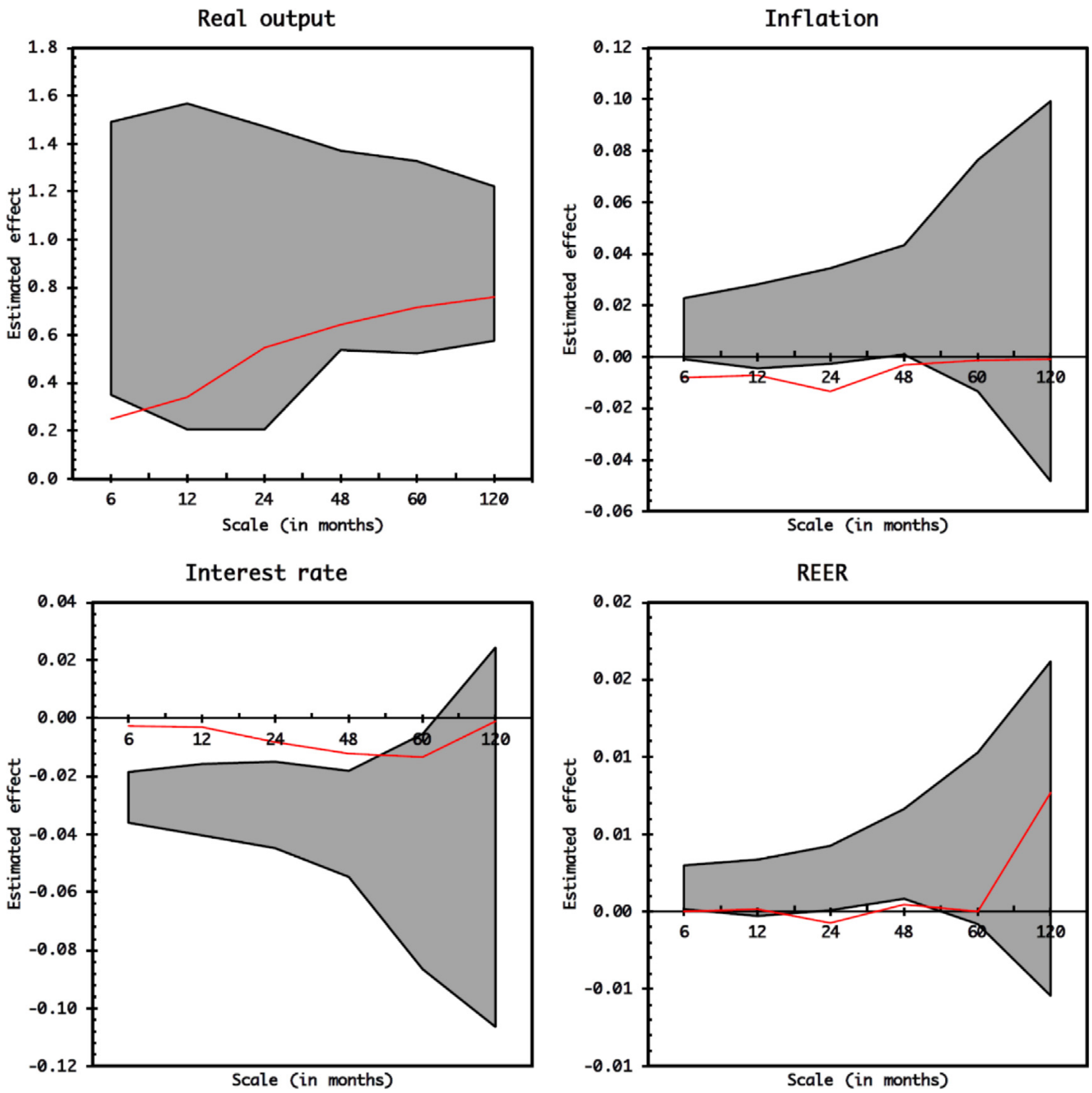

Fig. 5. Statistical inference of the scale dependence for money demand coefficients for US. Red line indicates the estimated coefficients with DFA regression and the shaded area refers to the confidence intervals. In the *x*-axis, we report time scales.

coefficients for REER mostly remain within the confidence bands. It is again the real output and interest rate, as in the USA case, that show more pronounced dynamics and scale dependence (Fig. 6).

6. Concluding remarks

Heterogeneity and temporal distribution of effects between economic and financial variables have been frequently discussed topics for many years now. However, the estimation of such scale-dependent effects has been challenging. Even though the research into scale-dependent correlations as well as scale-dependent regressions between a pair of variables have already been quite widely explored, both in the time and the frequency domain, the final steps toward a multivariate regression have been long missing even though it is obvious that studying not only economic and financial variables in only pairs has limited implications for interpretation and possible policy utilization of the results. Here, we propose a multivariate multiscale regression approach based on the combination of detrended fluctuation analysis and detrended cross-correlation analysis, but the idea can be easily translated into other time domain frameworks. Even more, the multiscale regression can be generalized for the frequency domain estimators as well.

Even though we illustrate the framework on the set of traditional economic models, it can be used in any discipline where studying the effects at various scales is of interest. In fact, the methodology will certainly benefit from multivariate

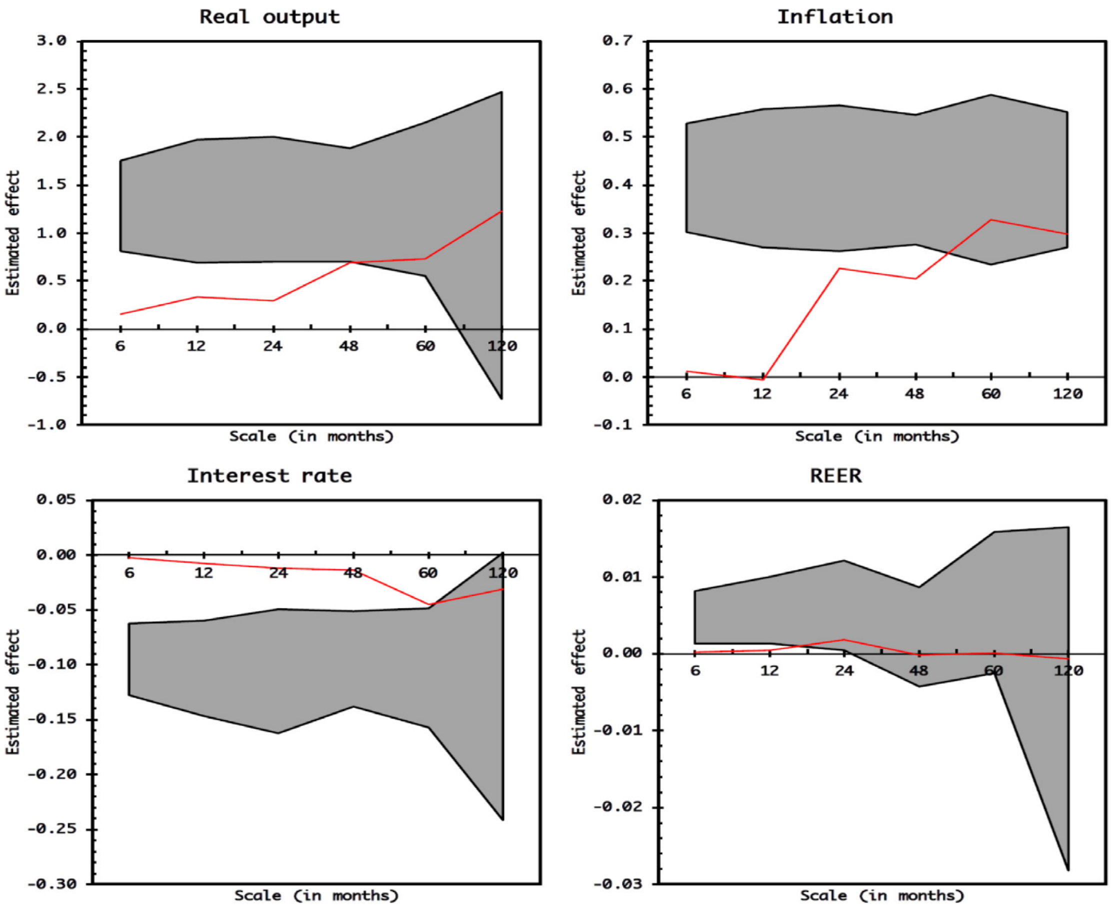

Fig. 6. Statistical inference of the scale dependence for money demand coefficients for Japan. Red line indicates the estimated coefficients with DFA regression and the shaded area refers to the confidence intervals. In the *x*-axis, we report time scales.

Table 3
Estimation of money demand using DFA regression on US data.

	Intercept	α_{output}	$\alpha_{inflation}$	$\alpha_{interest}$	α_{REER}	R^2
scale s = 6	2,28	0,252*** (0,03)	−0,008*** (5,12E−04)	−0,0028*** (3,94E−04)	−5,38E−06 (5,92-05)	35%
scale s = 12	1,89	0,341 *** (0,035)	−0,0073*** (8,04E−04)	−0,0032*** (6,64E−04)	1,75E−04 (8,62E−05)	29%
scale s = 24	1,078	0,547*** (0,032)	−0,0132*** (9,45E−04)	−0,0082*** (7,44E−04)	−7,4E04*** (1,06E−04)	41%
scale s = 48	0,64	0,645*** (0,032)	−0,0032 (0,0012)	−0,0121*** (0,0011)	4,64E−04 (1,72E−04)	50%
scale s = 60	0,33	0,717*** (0,029)	−0,0013 (0,0016)	−0,0134*** (0,0017)	1,75E−05 (2,09E−04)	52%
scale s = 120	0,08	0,758*** (0,028)	−6,42E−04 (0,0024)	−0,001*** (0,0024)	0,0077*** (4,33E−04)	65%
OLS regression	−0,203	0,846 *** (0,026)	0,0098 *** (0,003)	−0,026*** (0,002)	0,0009* (5,72E−04)	85%

*** denotes significance with a 1% level; ** denotes significance with a 5% level; * denotes significance with a 10% level. Information in brackets is for the standard errors of the estimated coefficients.

setting where more data is available so that the results are more clear-cut. Still, our two examples – the Taylor's rule and the money demand function for the USA and Japan – uncovered evident scale dependence in the individual effects not visible by the simple regression tools. Further applications are certainly at hand as well as further developments in the

Table 4
Estimation of money demand using DFA regression on Japanese data.

	Intercept	α_{output}	$\alpha_{inflation}$	$\alpha_{interest}$	α_{REER}	R^2
scale s = 6	28,51	0,159*** (0,021)	−0,0045*** (5,81E−04)	−0,0023 (0,0015)	2,61E−04 (1,32E−04)	18%
scale s = 12	27,74	0,334*** (0,026)	−0,0028* (0,0011)	−0,0076** (0,0023)	4,83E−04* (1,74E−04)	26%
scale s = 24	27,92	0,294*** (0,032)	−6,64E−04 (8,88E−04)	−0,0117** (0,0026)	0,0019*** (2,86E−04)	17%
scale s = 48	26,14	0,694*** (0,036)	−0,001 (0,0013)	−0,0133* (0,0029)	−2,069E−04 (4,22E−04)	42%
scale s = 60	26,03	0,727*** (0,035)	0,0144*** (0,0013)	−0,045*** (0,003)	7,69E−04 (4,88E−04)	65%
scale s = 120	24,76	1,231*** (0,032)	0,0094** (0,0011)	−0,0312*** (0,002)	−6,53E−05 (4,36E−04)	86%
OLS regression	23,55	1,313*** (0,041)	0,02*** (0,002)	−0,107*** (0,004)	0,006*** (9,8E−04)	92%

*** denotes significance with a 1% level; ** denotes significance with a 5% level; * denotes significance with a 10% level. Information in brackets is for the standard errors of the estimated coefficients.

methodological and statistical aspects such as treating autocorrelation and heteroskedasticity in error terms or different types of distributions and their influence on estimators' performance.

Funding

Paulo Ferreira acknowledges the financial support of Fundação para a Ciência e a Tecnologia (grants UIDB/05064/2020 and UIDB/04007/2020). Ladislav Kristoufek acknowledges financial support from the Czech Science Foundation under project No. 17-12386Y and the PRIMUS program of Charles University under project No. PRIMUS/19/HUM/17.

Code availability

Code will be supplied under request.

CRediT authorship contribution statement

Oussama Tilfani: Methodology, Software, Data curation, Writing – original draft, Writing – review & editing. **Ladislav Kristoufek:** Methodology, Formal analysis, Writing – original draft, Writing – review & editing. **Paulo Ferreira:** Conceptualization, Methodology, Formal analysis, Data curation; Writing – original draft, Writing – review & editing. **My Youssef El Boukfaoui:** Supervision, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

References

[1] A. Marshall, Principles of Economics, Macmillan, London, 1890.
[2] L. Robbins, The representative firm, Econ. J. 38 (151) (1928) 387–404.
[3] J. Hartley, The origins of the representative agent, J. Econ. Perspect. 10 (2) (1996) 169–177.
[4] D. Kahneman, A. Tversky, Prospect theory: An analysis of decision under risk, Econometrica 47 (2) (1979) 263–291.
[5] X. Gabaix, Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance, Q. J. Econ. 172 (2) (2012) 645–700.
[6] G. Akerlof, R. Shiller, Animal Spirits: How Human Psychology Drives the Economy, and Why It Matters for Global Capitalism, Princeton University Press, 2010.
[7] H. Simon, Models of Man, Wiley, New York, 1957.
[8] T. Sargent, Bounded Rationality in Macroeconomics, Claredon Press, Oxford, 1993.
[9] J. Farmer, D. Foley, The economy needs agent-based modelling, Nature 460 (7256) (2009) 685.

[10] S. Battiston, J. Farmer, A. Flache, D. Garlaschelli, A. Haldane, H. Heesterbeek, M. Scheffer, Complexity theory and financial regulation, Science 351 (6275) (2016) 818–819.
[11] E. Peters, Fractal structure in the capital markets, Financial Anal. J. 45 (4) (1989) 32–37.
[12] A. Lo, The adaptive markets hypothesis, J. Portf. Manag. 30 (5) (2004) 15–29.
[13] A. Lo, Reconciling efficient markets with behavioral finance: The adaptive market hypothesis, J. Investm. Cons. 7 (2) (2005) 21–44.
[14] A. Chakraborti, I. Toke, M. Patriarca, F. Abergel, Econophysics review: II, Agent-Based Models. Quant. Finance 11 (7) (2011) 1013–1041.
[15] D. Sornette, Physics and financial economics (1776-2014): Puzzles, Ising and agent-based models, Rep. Prog. Phys. 77 (2014) 062001.
[16] J. Kukacka, J. Barunik, Estimation of financial agent-based models with simulated maximum likelihood, J. Econom. Dynam. Control 85 (2017) 21–45.
[17] J. Grazzini, M. Richiardi, M. Tsionas, BayesIan estimation of agent-based models, J. Econom. Dynam. Control 77 (2017) 26–47.
[18] S. Chen, C. Chang, Y. Du, Agent-based economic models and econometrics, Knowl. Eng. Rev. 27 (2) (2012) 187–219.
[19] L. Calvet, A. Fisher, Multifractality in asset returns: Theory and evidence, Rev. Econ. Stat. 84 (3) (2002) 381–406.
[20] L. Calvet, M. Fearnley, A. Fisher, M. Leippold, What is beneath the surface? Option pricing with multifrequency latent states, J. Econom. 187 (2015) 498–511.
[21] D. Marinucci, P. Robinson, Semiparametric fractional cointegration analysis, J. Econometrics 105 (1) (2001) 225–247.
[22] B. Christensen, M. Nielsen, Asymptotic normality of narrow-band least squares in the stationary fractional cointegration model and volatility forecasting, J. Econom. 133 (1) (2006) 343–371.
[23] M. Nielsen, P. Frederiksen, Fully modified narrow-band least squares estimation of weak fractional cointegration, Econom. J. 14 (2011) 77–120.
[24] L. Kristoufek, Detrended fluctuation analysis as a regression framework: Estimating dependence at different scales, Phys. Rev. E 91 (2015) 022802.
[25] A. Frost, J. Prechter, Elliot Wave Principle: Key To Market Behavior, New Classics Library, 1978.
[26] M. Osborne, Brownian motion in the stock market, Oper. Res. 7 (1959) 145–173.
[27] L. Bachelier, Theory of Speculation in the Random Character of Stock Market Prices, MIT, Cambridge, MA, 1900.
[28] R. Cont, Empirical properties of asset returns: stylized facts and statistical issues, Quant. Finance I (2001) 223–236.
[29] D. Parisi, D. Sornette, D. Helbing, Financial price dynamics and pedestrian counterflows: a comparison of statistical stylized facts, Phys. Rev. E 87 (2013) 012804.
[30] B. Mandelbrot, J. Van Ness, Fractional Brownian motions, Fractional Noises and Applications. SIAM Rev 10 (4) (1968) 422–437.
[31] J. Beran, Statistics for Long-Memory Processes, Chapman & Hall, 1994.
[32] G. Samorodnitsky, M. Taqqu, Stable Non-Gaussian Random Processes, Chapman & Hall, 1994.
[33] T. Di Matteo, Multi-scaling in finance, Quant. Finance 7 (1) (2007) 21–36.
[34] T. Lux, S. Alfarano, Financial power laws: Empirical evidence, models, and mechanisms, Chaos Solitons Fractals 88 (2016) 3–18.
[35] C. Peng, S. Buldyrev, S. Havlin, M. Simons, H. Stanley, A. Goldberger, Mosaic organization of DNA nucleotides, Phys. Rev. E 49 (1994) 1685–1689.
[36] B. Podobnik, H.E. Stanley, Detrended cross-correlation analysis: a new method for analyzing two nonstationary time series, Phys. Rev. Lett. 100 (2008) 084102.
[37] G. Zebende, DCCA cross-correlation coefficient: quantifying a level of cross-correlation, Physica A 390 (4) (2011) 1643–1662.
[38] L. Kristoufek, Measuring correlations between non-stationary series with DCCA coefficient, Physica A 402 (2014) 291–298.
[39] L. Kristoufek, Detrending moving-average cross-correlation coefficient: Measuring cross-correlations between non-stationary series, Physica A 406 (2014) 169–175.
[40] L. Piao, Z. Fu, Quantifying distinct associations on different temporal scales: comparison of DCCA and Pearson methods, Sci. Rep. 6 (36759) (2016).
[41] X. Zhao, P. Shang, J. Huang, Several fundamental properties of DCCA cross-correlation coefficient, Fractals 25 (2) (2017) 1750017.
[42] Z. Jiang, W. Xie, W. Zhou, D. Sornette, Multifractal analysis of financial markets: a review, in: Reports on Progress in Physics, Physical Society, 2019.
[43] Y. Liu, P. Cizeau, C. Meyer M. Peng, H. Stanley, Correlations in economic time series, Physica A 245 (1997) 437–440.
[44] M. Ausloos, N. Vandewalle, P. Boveroux, A. Minguet, K. Ivanova, Applications of statistical physics to economic and financial topics, Physica A 274 (1999) 229–240.
[45] M. Ausloos, Statistical physics in foreign exchange currency and stock markets, Physica A 285 (2000) 48–65.
[46] S. Jaroszewicz, M. Mariani, M. Ferraro, Long correlations and truncated Levy walks applied to the study Latin-American market indices, Physica A 355 (2005) 461–474.
[47] T. Di Matteo, T. Aste, M. Dacorogna, Long-term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development, J. Bank. Financ. 29 (4) (2005) 827–851.
[48] J. Alvarez-Ramirez, J. Alvarez, E. Rodriguez, G. Fernandez-Anaya, Time-varying hurst exponent for US stock markets, Physica A 387 (2008) 6159–6169.
[49] M. Mariani, I. Florescu, E. Ncheuguim, Long correlations and Levy models applied to the study of memory effects in high frequency (tick) data, Physica A 388 (2009) 1659–1664.
[50] P. Ferreira, A. Dionísio, Revisiting serial dependence in the stock markets of the G7 countries, Portugal, Spain and Greece, Appl. Financial Econ. 24 (5) (2014) 319–331.
[51] P. Ferreira, Efficiency or speculation? A time-varying analysis of European sovereign debt, Physica A 490 (2018) 1295–1308.
[52] Y. Zhang, S. Chan, J. Chu, S. Nadarajah, Stylised facts for high frequency cryptocurrency data, Physica A 513 (2019) 598–612.
[53] N. Costa, C. Silva, P. Ferreira, Long-range behaviour and correlation in cryptocurrencies DFA and DCCA analysis, Int. J. Financial Stud. 7 (51) (2019).
[54] T. Bollerslev, D. Jubinski, Equity trading volume and volatility: latent information arrivals and common long-run dependencies, J. Bus. Econ. Stat. 17 (1999) 9–21.
[55] M. Karanasos, A. Kartsaklas, Dual long-memory, structural breaks and the link between turnover and the range-based volatility, J. Empir. Finance 16 (2009) 838–851.
[56] D. Cajueiro, B. Tabak, Are the crude oil markets becoming weakly efficient over time? A test for time-varying long-range dependence in prices and volatility, Energy Econ. 29 (2007) 28–36.
[57] B. Podobnik, D. Horvatic, A. Petersen, H. Stanley, Cross-correlations between volume change and price change, Proc. Natl. Acad. Sci. USA 106 (52) (2009) 22079–22084.
[58] G. Wang, C. Xie, Y. Chen, S. Chen, Statistical properties of the foreign exchange network at different time scales: Evidence from detrended cross-correlation coefficient and minimum spanning tree, Entropy 15 (5) (2013) 1643–1662.
[59] Y. Wang, C. Wu, L. Yang, Oil price shocks and stock market activities: Evidence from oil-importing and oil-exporting countries, J. Complementary Econ. 41 (2013) 1220–1239.
[60] M. Lin, G. Wang, C. Xie, H. Stanley, Cross-correlations and influence in world gold markets, Physica A 490 (2018) 504–512.

[61] U. Bashir, Y. Yu, M. Hussain, G. Zebende, Do foreign exchange and equity markets co-move in Latin American region? Detrended cross-correlation approach, Physica A 462 (2016) 889–897.
[62] A. Filho, E. Pereira, P. Ferreira, T. Murari, M. Moret, Cross-correlation analysis on Brazilian gasoline retail market, Physica A 505 (2018) 550–557.
[63] A. Lin, P. Shang, H. Zhou, Cross-correlations and structures of stock markets based on multiscale MF-DXA and PCA, Nonlinear Dyn. 78 (2014) 485–494.
[64] M. Silva, É. Pereira, A. Filho, A. Castro, J. Miranda, G. Zebende, Quantifying the contagion effect of the 2008 financial crisis between the G7 countries (by GDP nominal), Physica A 453 (2016) 1–8.
[65] O. Tilfani, P. Ferreira, Y. El Boukfaoui, Dynamic cross-correlation and dynamic contagion of stock markets: a sliding windows approach with the DCCA correlation coefficient, Emp. Econ. (2019).
[66] O. Tilfani, P. Ferreira, Y. El Boukfaoui, Building multi-scale portfolios and efficient market frontiers using fractal regressions, Physica A 532 (2019) 121758.
[67] W. Mohti, A. Dionísio, I. Vieira, P. Ferreira, Financial contagion analysis in frontier markets: Evidence from the US subprime and the Eurozone debt crises, Physica A 525 (2019) 1388–1398.
[68] X. Zhao, P. Shang, Principal component analysis for non-stationary time series based on detrended cross-correlation analysis, Nonlinear Dyn. 84 (2016) 1033–1044.
[69] L. Kristoufek, Scaling of dependence between foreign exchange rates and stock markets in Central Europe, Acta Phys. Pol A 129 (2016) 908–912.
[70] L. Kristoufek, Fractality in market risk structure: Dow Jones Industrial components case, Chaos Solitons Fractals 110 (2018) 69–75.
[71] L. Kristoufek, P. Ferreira, Capital asset pricing model in Portugal: Evidence from fractal regressions, Port. Econ. J. 17 (3) (2018) 173–183.
[72] P. Ferreira, L. Kristoufek, What is new about covered interest parity condition in the European union? Evidence from fractal cross-correlation regressions, Physica A 486 (2017) 554–566.
[73] P. Ferreira, L. Kristoufek, Uncovered interest rate parity through the lens of fractal methods: Evidence from the European union, Physica A 545 (2020) 123803.
[74] L. Kristoufek, Power-law cross-correlations: Issues, solutions and future challenges, in: D. Grech, J. Miskiewicz (Eds.), Simplicity and Complexity in Economic and Social Systems, 2021, pp. 43–54.
[75] F. Wang, L. Wang, Y. Chen, A DFA-based bivariate regression model for estimating the dependence of PM2.5 among neighbouring cities, Sci. Rep. 8 (7475) (2018).
[76] A. Likens, P. Amazeen, S. West, C. Gibbons, Statistical properties of Multiscale Regression Analysis: Simulation and application to human postural control, Physica A 532 (2019) 121580.
[77] X. Zhao, P. Shang, A. Lin, Distribution of eigenvalues of detrended cross-correlation matrix, Europhys. Lett. 107 (40008) (2014).
[78] B. Podobnik, Z. Jiang, W. Zhou, H. Stanley, Statistical tests for power-law cross-correlated processes, Phys. Rev. E 84 (2011) 066118.
[79] J. Theiler, S. Eubank, A. Longtin, B. Galdrikan, J. Farmer, Testing for nonlinearity in time series: The method of surrogate data, Physica D 58 (1992) 77–94.
[80] R. Clarida, J. Gali, M. Gertler, Monetary policy rules in practice. Some international evidence, Eur. Econ. Rev. 42 (1998) 1033–1067.
[81] V. Teles, M. Zaidan, Taylor principle and inflation stability in emerging market countries, J. Dev. Econ. 91 (1) (2010) 180–183.
[82] S. Papadamou, M. Sidiropoulos, A. Vidra, A Taylor rule for EU members. Does one rule fit to all EU member needs? J. Econ. Asymmetries (18) (2018) e00104.
[83] D.L. Hoffman, R.H. Rasche, Long-run income and interest elasticities of money demand in the United States, Rev. Econom. Statist. 73 (1991) 665–674.
[84] H. Fujiki, Japanese money demand: evidence from regional monthly data, Jpn. World Econ. 11 (3) (1999) 375–393.
[85] N. Loyaza, L. Schmidt-Hebbel, Monetary Policy Functions and Transmission Mechanisms: An Overview, Central Bank of Chile, Santiago, 2002.
[86] J. Taylor, Discretion versus policy rules in practice, Carnegie-Rochester Conf. Ser. Public Policy 39 (1993) 195–214.
[87] G. Rudebusch, The fed's exit strategy for monetary policy, Feder. Reserve Bank San Franc. Econ. Lett. (2010) June 14.
[88] J. Yellen, Perspectives on monetary policy, speech at Boston Economic Club Dinner, June 6, 2012.
[89] M. Woodford, Interest and Prices, Princeton University Press, 2003.
[90] C. Murray, A. Nikolsko-Rzhevskyy, D. Papell, Markov switching and the Taylor principle, Macroecon Dyn. 19 (4) (2015) 913–930.
[91] A. Nikolsko-Rzhevskyy, D. Papell, R. Prodan, The Taylor principles, J. Macroecon. 62 (2019) 103159.
[92] R. Clarida, J. Gali, M. Gertler, Monetary policy rules and macroeconomic stability: evidence and some theory, Q. J. Econ. 115 (1) (2000) 147.
[93] J. Boivin, Has U.S. monetary policy changed? Evidence from drifting coefficients and real time-data, J. Money Credit Bank 38 (2006) 1149–1179.
[94] J. Chen, M. Kashiwagi, The Japanese Taylor rule estimated using censored quantile regressions, Empir. Econ. 52 (2017) 357–371.
[95] Stephen D. Morris, Is the Taylor principle still valid when rates are low? J. Macroeconom. 64 (2020) 103192.
[96] I. Fisher, The Purchasing Power of Money, Macmillan, New York, 1911.
[97] A. Pigou, The value of money, Q. J. Econ. 37 (1917) 38–65.
[98] A. Marshall, Money, Credit and Commerce, Macmillan, London, 1923.
[99] K. Cuthbertson, D. Barlow, Disequilibrium, buffer stocks and consumers' expenditure on non-durables, Rev. Econ. Stat. 73 (4) (1991) 643–653.
[100] J. Frenkel, The forward exchange rate, expectations, and the demand for money: The Germna hyperinflation, Am. Econ. Rev. 67 (4) (1979) 653–670.
[101] J. Stock, M. Watson, A simple estimator of cointegrating vectors in higher order integrated systems, Econometrica 61 (1993) 783–820.
[102] D. Hoffman, R. Rasche, Assessing forecast performance in a cointegrated system, J. Appl. Econom. 11 (5) (1996) 495–517.
[103] R. McKinnon, Currency substitution and instability in the world dollar standard, Am. Econ. Rev. 72 (3) (1982) 320–333.
[104] F. Jawadi, R. Sousa, Modelling money demand: further evidence from an international comparison, Appl. Econ. Lett. 20 (11) (2013) 1052–1055.
[105] R. Sousa, Wealth, asset portfolio, money demand and policy rule, Bull. Econ. Res. 66 (1) (2014) 95–111.