International Journal of Approximate Reasoning 148 (2022) 162–173

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Rectangles-based discrete universal fuzzy integrals

Radko Mesiar [a,b,*], Anna Kolesárová [c]

[a] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovakia
[b] Institute of Information Theory and Automation, Czech Academy of Sciences, Pod Vodárenskou věží 4, CZ-180 00, Prague 8, Czech Republic
[c] Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology, Radlinského 9, 812 37 Bratislava, Slovakia

ARTICLE INFO

Article history:
Received 16 February 2022
Received in revised form 19 May 2022
Accepted 9 June 2022
Available online 18 June 2022

Keywords:
Aggregation function
Choquet integral
Discrete fuzzy integral
Hypergraph of survival function
Rectangle mapping

ABSTRACT

Using hypergraphs of survival functions, we propose a rather general method for the construction of discrete fuzzy integrals. Our method is based on various rectangle decompositions of hypergraphs and on rectangle mappings suitably evaluating the rectangles of the considered decompositions. By means of appropriate binary aggregation functions we define two types of rectangle mappings and four types of discrete fuzzy integral constructions, and we also investigate the properties of the introduced integrals and the relationships between them. All the introduced methods based on non-overlapping rectangles coincide in the case of the product aggregation function, and then the related integral is the Choquet integral. Several examples are given.

1. Introduction

Discrete fuzzy integrals play an important role in all domains of fuzzy set theory dealing with fuzzy measures. Discrete fuzzy integrals are applied in multicriteria decision support, preference modeling, fuzzy rule-based systems, image processing, classification problems, etc. Motivated by universal fuzzy integrals introduced by Klement et al. in [14] and decomposition integrals proposed by Even and Lehrer [9], we introduce a new approach how to build discrete fuzzy integrals based on some appropriate aggregation functions. As a new tool for building such integrals we use rectangle mappings evaluating the considered rectangles in various non-standard ways. In addition to that, we exploit rectangle decompositions of hypergraphs related to the survival functions $h_{m,f}$, m being a fuzzy measure and f an integrated function. In some special cases, our concept covers some well-known fuzzy integrals as, for example, copula-based integrals leading to the Choquet integral (when the product copula Π is considered) or to the Sugeno integral (when the comonotonicity copula $M = \min$ is applied). Another class of integrals covered by our approach is the class of the smallest universal integrals related to semicopulas, see [14]. In general, newly proposed integrals need not be monotone. We focus on this problem and characterize some types of the introduced integrals which are weakly monotone.

The rest of the paper is organized as follows. In the next section, we give some preliminary notions concerning fuzzy integrals and aggregation functions. In Section 3, the graphs of survival functions are discussed and then four types of rectangle decompositions of the related hypergraphs are introduced. In Section 4, the notion of rectangle mappings is defined and two types of rectangle mappings based on appropriate binary aggregation functions are provided. Section 5 is

* Corresponding author.
E-mail addresses: radko.mesiar@stuba.sk (R. Mesiar), anna.kolesarova@stuba.sk (A. Kolesárová).

https://doi.org/10.1016/j.ijar.2022.06.003
0888-613X/

devoted to introducing rectangles-based discrete fuzzy integrals related to the introduced hypergraph decompositions and rectangle mappings, as well as to the study of their properties, in particular their symmetry, idempotency, monotonicity and weak monotonicity. In Section 6, some particular discrete fuzzy integrals are discussed. Finally, Section 7 contains several concluding remarks.[1]

2. Preliminaries

We begin by recalling several basic definitions which will be needed throughout the paper. In this paper, we will work with a finite universe $[n] = \{1, \dots, n\}$, n being any (but fixed) element in $\mathbb{N}$.

Definition 2.1. A fuzzy measure m on $[n]$ is a monotone set function $m\colon 2^{[n]} \to [0,1]$ such that $m(\emptyset) = 0$ and $m([n]) = 1$. The set of all fuzzy measures on $[n]$ will be denoted by $\mathcal{M}_n$.

The notion of fuzzy measures on a general measurable space $(\Omega, \mathcal{S})$ was introduced in [22], including some continuity properties. In the case where $(\Omega, \mathcal{S}) = ([n], 2^{[n]})$, i.e., for a finite σ-algebra $\mathcal{S} = 2^{[n]}$, these properties are automatically satisfied, therefore we do not deal with them. More properties of fuzzy measures can be found, e.g., in [23]. Here, we only recall that a fuzzy measure m on $[n]$ is symmetric if for all $E, F \subseteq [n]$, $m(E) = m(F)$ whenever $\mathrm{card}(E) = \mathrm{card}(F)$. Equivalently, m is symmetric on $[n]$ if for any $E \subseteq [n]$ and any permutation $\sigma\colon [n] \to [n]$ we have $m(E) = m(\sigma(E))$, where $\sigma(E) = \{\sigma(i) \mid i \in E\}$.

In what follows, we will use the symbol $\mathcal{F}_n$ to denote the set of all functions $f\colon [n] \to [0,1]$. To avoid any misunderstanding, we stress that the letter n in symbols $\mathcal{M}_n$ and $\mathcal{F}_n$ indicates that we work on a universe $[n] = \{1, \dots, n\}$, we consider functions $f\colon [n] \to [0,1]$ and fuzzy measures $m\colon 2^{[n]} \to [0,1]$. Moreover, we point to the fact that $\mathcal{F}_n$ can be seen as the set of all fuzzy subsets of $[n]$, and it can be identified with the set of all n-tuples from $[0,1]^n$. There is a one-to-one correspondence between $f \in \mathcal{F}_n$ and $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$, given by $f(i) = x_i$, $i = 1, \dots, n$. For example, a constant function $f(i) = c$ for each $i \in [n]$ will be denoted by $\mathbf{c} = (c, \dots, c)$, in particular $\mathbf{0} = (0, \dots, 0)$ denotes the function identically equal to zero, and similarly $\mathbf{1} = (1, \dots, 1)$ is a constant function whose value always equals 1.

Making use of this correspondence, the definition of n-ary aggregation functions, which are standardly defined as increasing mappings $A\colon [0,1]^n \to [0,1]$ satisfying the boundary conditions $A(0, \dots, 0) = 0$ and $A(1, \dots, 1) = 1$, see, e.g., [1,3,11], can be expressed as follows.

Definition 2.2. An (n-ary) aggregation function is a mapping $A\colon \mathcal{F}_n \to [0,1]$ that is increasing and satisfies the boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$.

Note that the increasingness of A can be characterized by the inequality $A(f + g) \geq A(f)$ required for all $f, g \in \mathcal{F}_n$ such that also $f + g \in \mathcal{F}_n$.

Let us still recall the notion of aggregation functions with zero annihilator which will play an important role in the next parts of the paper. The element $\alpha = 0$ is the zero annihilator of an aggregation function $A\colon [0,1]^n \to [0,1]$ if and only if for all inputs $(x_1, \dots, x_n) \in [0,1]^n$ such that $0 \in \{x_1, \dots, x_n\}$, we have $A(x_1, \dots, x_n) = 0$. Equivalently said, an aggregation function $A\colon \mathcal{F}_n \to [0,1]$ has zero annihilator if and only if for each function $f \in \mathcal{F}_n$ such that $f(i) = 0$ for some $i \in [n]$, we have $A(f) = 0$.

Recently, several generalizations of aggregation functions have been proposed and successfully applied. We only recall the notion of weak aggregation functions [2]. A general notion of weakly monotone functions was originally introduced in [25].

Definition 2.3. A weak aggregation function is a mapping $B\colon \mathcal{F}_n \to [0,1]$ that is weakly increasing, i.e., $B(f + \mathbf{c}) \geq B(f)$ for all f and $\mathbf{c} = (c, \dots, c)$ in $\mathcal{F}_n$ such that $f + \mathbf{c} \in \mathcal{F}_n$, and satisfies the boundary conditions $B(\mathbf{0}) = 0$, $B(\mathbf{1}) = 1$.

Among a huge number of discrete fuzzy integrals we recall the following three well-known integrals:

- the Choquet integral [4] $Ch\colon \mathcal{M}_n \times \mathcal{F}_n \to [0,1]$, defined by

$$Ch(m, f) = \sum_{i=1}^{n} \left(f(\sigma(i)) - f(\sigma(i-1))\right) \cdot m\left(\{\sigma(i), \dots, \sigma(n)\}\right),$$

  where $\sigma\colon [n] \to [n]$ is an arbitrary permutation such that $f(\sigma(1)) \leq \dots \leq f(\sigma(n))$, and $f(\sigma(0)) = 0$, by convention;

[1] This paper is a significantly extended version of the contribution presented at the Conference ESCIM 2021 in Budapest.

- the Sugeno integral [22] $Su\colon \mathcal{M}_n \times \mathcal{F}_n \to [0,1]$, defined by

$$Su(m, f) = \bigvee_{i=1}^{n} \left(f(\sigma(i)) \wedge m(\{\sigma(i), \dots, \sigma(n)\}) \right);$$

- the Shilkret integral [21] $Sh\colon \mathcal{M}_n \times \mathcal{F}_n \to [0,1]$, defined by

$$Sh(m, f) = \bigvee_{i=1}^{n} f(\sigma(i)) \cdot m(\{\sigma(i), \dots, \sigma(n)\}).$$

All these three integrals are idempotent aggregation functions [11], and they belong to the class of universal fuzzy integrals proposed by Klement et al. in [14]. Discrete universal fuzzy integrals have been built with the aid of semicopulas [7]. Recall that a semicopula $\otimes\colon [0,1]^2 \to [0,1]$ is a binary aggregation function with neutral element $e = 1$, i.e., satisfying $x \otimes 1 = 1 \otimes x = x$ for each $x \in [0,1]$.

Definition 2.4. A discrete universal fuzzy integral I is a mapping

$$I\colon \bigcup_{k \in \mathbb{N}} \mathcal{M}_k \times \mathcal{F}_k \to [0,1],$$

where, for each $k \in \mathbb{N}$, $\mathcal{M}_k$ is the set of all fuzzy measures on $[k] = \{1, \dots, k\}$, and $\mathcal{F}_k$ is the set of all functions $f\colon [k] \to [0,1]$, such that

(i) I is increasing in both variables;
(ii) there is a semicopula $\otimes\colon [0,1]^2 \to [0,1]$ such that

$$I(m, c \cdot \mathbf{1}_E) = c \otimes m(E)$$

for each $m \in \mathcal{M}_k$ and $E \subseteq [k]$, $k \in \mathbb{N}$, and each $c \in [0,1]$;
(iii) for all $k_1, k_2 \in \mathbb{N}$, and all $(m_i, f_i) \in \mathcal{M}_{k_i} \times \mathcal{F}_{k_i}$, $i = 1, 2$, such that

$$m_1(\{j \in \{1, \dots, k_1\} \mid f_1(j) \geq t\}) = m_2(\{j \in \{1, \dots, k_2\} \mid f_2(j) \geq t\})$$

for all $t \in [0,1]$, we have

$$I(m_1, f_1) = I(m_2, f_2).$$

Note that for a given subset $E \subseteq [k]$, the characteristic function $\mathbf{1}_E\colon [k] \to [0,1]$ is given by $\mathbf{1}_E(i) = 1$ if $i \in E$, and otherwise, it equals zero.

An important property of discrete universal fuzzy integrals is expressed by axiom (iii) in Definition 2.4, which can be interpreted as follows:

For a fixed $m \in \mathcal{M}_n$ and $f \in \mathcal{F}_n$, the value $I(m, f)$ of a discrete universal integral I only depends on the survival function $h_{m,f}\colon [0,1] \to [0,1]$, given by

$$h_{m,f}(t) = m(\{j \in [n] \mid f(j) \geq t\}) = m(\{j \in [n] \mid x_j \geq t\}). \tag{1}$$

Note that when $([n], 2^{[n]}, m)$ is a probability space and $\xi \in \mathcal{F}_n$ is a random variable then $h_{m,\xi} = 1 - F_{m,\xi}$ is the complement of the distribution function $F_{m,\xi}$, and for the expected value $E_m(\xi)$ we have

$$E_m(\xi) = \int_{[0,1]} \xi \, dm = \int_0^1 h_{m,\xi}(t) dt = Ch(m, \xi).$$

Let us stress an important fact, namely, that the hypergraph

$$H_{m,f} = \{(t, u) \in [0,1]^2 \mid u \leq h_{m,f}(t)\}$$

is in a one-to-one correspondence with the survival function $h_{m,f}$, and it consists of at most n non-overlaping non-compatible rectangles of the type $[a, b] \times [c, d]$, and, possibly, of two segments. Note that two rectangles are non-compatible if their union is not a rectangle. Then the Choquet integral evaluates just the area of $H_{m,f}$, and is equal to the sum of the (standardly evaluated) areas of the related rectangles. This fact motivated our investigation and has given rise to this paper in which we will deal with various ways of possible non-standard evaluations of the related rectangles.

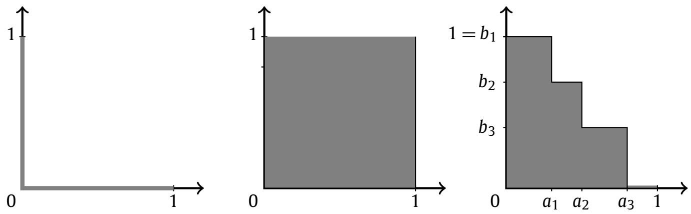

Fig. 1. Illustrations of 3 extremal hypergraphs $H_{m,f}$: minimal hypergraph (left); maximal hypergraph (middle); the hypergraph of a survival function given by (2) for $n = 3$ (right).

3. Hypergraph decompositions

To understand well the geometry of hypergraphs $H_{m,f}$, it is necessary to describe the graphs of survival functions $h_{m,f}$. Evidently, each $h_{m,f}$ is a step-wise, left-continuous, decreasing function on $[0, 1]$ such that $h_{m,f}(0) = 1$. The smallest survival function $h_*\colon [0, 1] \to [0, 1]$ is given by

$$h_*(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $h_{m,f} = h_*$ if and only if $m(\{f > 0\}) = 0$. The greatest survival function $h^*\colon [0, 1] \to [0, 1]$ is the constant function $h^*(t) = 1$ for each $t \in [0, 1]$, and $h_{m,f} = h^*$ if and only if $m(\{f = 1\}) = 1$. The most complex situation occurs when there are no ties among the values $0, f(1), \ldots, f(n), 1$, as well as among the values $1 = m(\{\sigma(1), \ldots, \sigma(n)\}), \ldots, m(\{\sigma(n)\}), 0$. Then, putting

$$a_i = f(\sigma(i)) \text{ and } b_i = m(\{\sigma(i), \ldots, \sigma(n)\}),\ i \in [n],$$

we get $a_0 = 0 < a_1 < \cdots < a_n$ and $1 = b_1 > b_2 > \cdots > b_n > 0 = b_{n+1}$, and

$$h_{m,f}(t) = \begin{cases} b_1 = 1 & \text{if } t \in [0, a_1], \\ b_2 & \text{if } t \in]a_1, a_2], \\ \vdots & \\ b_n & \text{if } t \in]a_{n-1}, a_n], \\ b_{n+1} = 0 & \text{if } t \in]a_n, 1]. \end{cases} \tag{2}$$

The above discussed survival functions h_*, h^* and $h_{m,f}$ given by (2) if $n = 3$, lead to hypergraphs depicted in Fig. 1. In all other cases when $h_{m,f} \notin \{h_*, h^*\}$, there is a $k \in \{1, \ldots, n\}$ and some values $a_0 = 0 < a_1 < \cdots < a_k \leq 1$ and $1 \geq b_1 > b_2 > \cdots > b_k > 0$ such that

$$h_{m,f}(t) = \begin{cases} 1 & \text{if } t = 0, \\ b_1 & \text{if } t \in]0, a_1], \\ b_2 & \text{if } t \in]a_1, a_2], \\ \vdots & \\ b_k & \text{if } t \in]a_{k-1}, a_k], \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

From the mentioned properties of survival functions $h_{m,f}$ it follows that for each $(m, f) \in \mathcal{M}_n \times \mathcal{F}_n$ the related hypergraph $H_{m,f}$ consists of k pairwise non-compatible non-overlapping neighboring rectangles, $k \in \{0, 1, \ldots, n\}$ and r segments, $r \in \{0, 1, 2\}$.

Now, we introduce some decompositions of hypergraphs $H_{m,f}$ into rectangles, possibly neglecting some segments contained in $H_{m,f}$.

Definition 3.1. Let $(m, f) \in \mathcal{M}_n \times \mathcal{F}_n$ and let the corresponding survival function $h_{m,f}$ be given by (3). Let $H_{m,f}$ denote the hypergraph corresponding to $h_{m,f}$.

(i) A vertical rectangle decomposition of $H_{m,f}$, denoted by $D_{m,f}$, is given by

$$D_{m,f} = \{[a_{i-1}, a_i] \times [0, b_i] \mid i \in \{1, \ldots, k\}\},\ \ k \in \{1, \ldots, n\}. \tag{4}$$

(ii) A horizontal rectangle decomposition of $H_{m,f}$, denoted by $G_{m,f}$, is given by

$$G_{m,f} = \{[0, a_i] \times [b_{i+1}, b_i] \mid i \in \{1, \dots, k\}\}, \quad k \in \{1, \dots, n\}, \tag{5}$$

and the convention $b_{k+1} = 0$.

(iii) Let $K_{m,f}$ denote a decomposition of $H_{m,f}$, refining both $D_{m,f}$ and $G_{m,f}$, which is given by

$$K_{m,f} = \{[a_{i-1}, a_i] \times [b_{j+1}, b_j] \mid i, j \in \{1, \dots, k\},\ i \le j\},\ k \in \{1, \dots, n\}, \tag{6}$$

and the convention $b_{k+1} = 0$.

The introduced decompositions $D_{m,f}, G_{m,f}$ and $K_{m,f}$ consist of non-overlapping rectangles and these ways of decomposition are similar to the Riemann approach to integration based on divisions, which are also non-overlapping but not disjoint as it is considered in the case of the Lebesgue integral. Another approach to rectangle decompositions of hypergraphs $H_{m,f}$ (cumulative) considered, e.g., in the case of the Sugeno integral, is introduced in the following definition.

Definition 3.2. Let $(m, f) \in \mathcal{M}_n \times \mathcal{F}_n$ and let the corresponding survival function $h_{m,f}$ be given by (3). Let $H_{m,f}$ denote the hypergraph corresponding to $h_{m,f}$. A cumulative decomposition $P_{m,f}$ of the hypergraph $H_{m,f}$ is given by

$$P_{m,f} = \{[0, a_i] \times [0, b_i] \mid i \in \{1, \dots, k\}\}. \tag{7}$$

Finally, for all the four introduced decompositions, if $h_{m,f} = h_*$, we put $D_{m,f} = G_{m,f} = K_{m,f} = P_{m,f} = \emptyset$.

4. Rectangle mappings based on aggregation functions

To evaluate the information contained in a couple $(m, f) \in \mathcal{M}_n \times \mathcal{F}_n$, i.e., to evaluate some discrete fuzzy integral of f with respect to m, we need to evaluate suitably the above considered rectangles.

Definition 4.1. Consider all $a, b, c, d \in [0, 1]$, $a \le b$, $c \le d$. Let $\mathcal{R}$ be the set of all rectangles $R = [a, b] \times [c, d]$, i.e., $\mathcal{R} = \{R = [a, b] \times [c, d] \mid R \subseteq [0, 1]^2\}$. A mapping $S\colon \mathcal{R} \to [0, 1]$ will be called a rectangle mapping if

(i) S is monotone, i.e., $S(R_1) \le S(R_2)$ for any $R_1, R_2 \in \mathcal{R}$, $R_1 \subseteq R_2$,
(ii) $S(R) = 0$ whenever $R \in \mathcal{R}$ is a degenerated rectangle,
(iii) $S([0, 1]^2) = 1$.

Note that R is a degenerated rectangle if it is either a segment or a single point.

Observe that rectangle mappings can be seen as fuzzy measures on $\mathcal{R}$.

For a fixed rectangle mapping S, we can introduce several integrals on $\mathcal{M}_n \times \mathcal{F}_n$. In what follows, for any S, we define four types of integrals corresponding to the above mentioned four decompositions of hypergraphs. Moreover, we propose two approaches how to define rectangle mappings with the aid of aggregation functions.

The first approach to the construction of rectangle mappings is independent of the location of the considered rectangles $R = [a, b] \times [c, d] \subseteq [0, 1]^2$, it only depends on their size, i.e., on the values $b - a$ and $d - c$. Let $A\colon [0, 1]^2 \to [0, 1]$ be a binary aggregation function. We define a mapping $S_A\colon \mathcal{R} \to [0, 1]$ by

$$S_A(R) = A(b - a, d - c). \tag{8}$$

Obviously, for an arbitrary A, S_A is a monotone increasing mapping and $S_A([0, 1]^2) = 1$. For proving that S_A is a rectangle mapping, it is necessary to ensure that $S_A(R) = 0$ for any degenerated rectangle R. There are two possibilities how to guarantee this requirement: either it can be covered by a convention, i.e., item (8) can be modified by imposing the condition $S_A(R) = 0$ whenever R is a degenerated rectangle, or one can require A to satisfy the properties $A(0, d - c) = A(b - a, 0) = 0$ for any $a, b, c, d \in [0, 1]$, $a \le b$ and $c \le d$, which is satisfied whenever A has zero annihilator.

Theorem 4.1. *Let $A\colon [0, 1]^2 \to [0, 1]$ be a binary aggregation function. Then the mapping $S_A\colon \mathcal{R} \to [0, 1]$ given by*

$$S_A([a, b] \times [c, d]) = \begin{cases} 0 & \text{if } a = b \text{ or } c = d, \\ A(b - a, d - c) & \text{otherwise,} \end{cases} \tag{9}$$

is a rectangle mapping.

Proof. The proof of the claim has already been done in the previous discussion. □

Evidently, if A is the product aggregation function, $S_A(R)$ is equal to the standard area of a rectangle R.

Another approach to the construction of rectangle mappings is based on A-volumes of rectangles generated by a binary aggregation function $A\colon [0,1]^2 \to [0,1]$. For a rectangle $R = [a,b] \times [c,d] \subseteq [0,1]^2$, its A-volume $V_A(R)$ is given by

$$V_A(R) = A(b,d) + A(a,c) - A(a,d) - A(b,c). \tag{10}$$

It is not difficult to check that for any two non-overlapping rectangles $R_1, R_2 \in \mathcal{R}$ whose union $R = R_1 \cup R_2$ also belongs to $\mathcal{R}$, we have

$$V_A(R) = V_A(R_1) + V_A(R_2). \tag{11}$$

Now, for any binary aggregation function A, let us introduce a mapping $S^A\colon \mathcal{R} \to]-\infty,\infty[$, defined as

$$S^A(R) = V_A(R). \tag{12}$$

Due to (10), $S^A(R) = A(b,d) + A(a,c) - A(a,d) - A(b,c)$.

Obviously, a necessary condition for $S^A\colon \mathcal{R} \to]-\infty,\infty[$ to be a rectangle mapping is the requirement $S^A(R) \geq 0$ for each $R \in \mathcal{R}$. It will be satisfied if the A-volume $V_A(R)$ of each rectangle $R \in \mathcal{R}$ is non-negative. Thus we restrict our considerations to 2-increasing aggregation functions A which are just characterized by the property

$$V_A(R) = A(b,d) + A(a,c) - A(a,d) - A(b,c) \geq 0$$

for each rectangle R, see, e.g., [8,11,20]. Finally, applying (12) and (10), we obtain

$$S^A([a,a] \times [c,d]) = A(a,d) + A(a,c) - A(a,d) - A(a,c) = 0,$$

and also

$$S^A([a,b] \times [c,c]) = A(b,c) + A(a,c) - A(a,c) - A(b,c) = 0,$$

i.e., the values of S^A for degenerated rectangles are vanishing. These facts lead to the following result.

Theorem 4.2. *Let $A\colon [0,1]^2 \to [0,1]$ be a 2-increasing aggregation function with zero annihilator. Then the mapping $S^A\colon \mathcal{R} \to]-\infty,\infty[$ given by (12) is a rectangle mapping.*

Proof. For proving the claim it is only needed to show that S^A is an increasing mapping, and that for any $R \in \mathcal{R}$, $S^A(R) \in [0,1]$. Obviously, as zero is an annihilator of A,

$$S^A([0,1]^2) = A(1,1) + A(0,0) - A(0,1) - A(1,0) = 1.$$

Now, suppose that $R_1, R_2 \in \mathcal{R}$, $R_i = [a_i,b_i] \times [c_i,d_i]$, $i = 1,2$, and $R_1 \subseteq R_2$. These assumptions yield $a_1 \geq a_2$, $b_1 \leq b_2$, $c_1 \geq c_2$ and $d_1 \leq d_2$. Then R_2 can be written as

$$\begin{aligned} R_2 = {} & R_1 \cup [a_2,b_2] \times [c_2,c_1] \cup [a_2,b_2] \times [d_1,d_2] \cup [a_2,a_1] \times [c_1,d_1] \\ & \cup [b_1,b_2] \times [c_1,d_1], \end{aligned}$$

i.e., R_2 is a union of 5 non-overlapping rectangles. Due to (12), (11) and the 2-increasing property of A, we obtain

$$\begin{aligned} S^A(R_2) &= V_A(R_2) = V_A(R_1) + V_A([a_2,b_2] \times [c_2,c_1]) + \cdots \\ &\quad + V_A([b_1,b_2] \times [c_1,d_1]) \geq V_A(R_1) = S^A(R_1), \end{aligned}$$

which proves the increasing monotonicity of S^A. Also, for any $R \in \mathcal{R}$, we have $R \subseteq [0,1]^2$ and thus,

$$0 \leq S^A(R) \leq S^A([0,1]^2) = 1,$$

which completes the proof. □

Note that the requirement of the zero annihilator of an aggregation function A in Theorem 4.2 cannot be omitted, because it is equivalent to the fulfilling of the conditions $A(1,0) = A(0,1) = 0$ as well as the boundary condition $S^A([0,1]^2) = 1$. We also recall that in the framework of binary aggregation functions, the 2-increasing property of an aggregation function $A\colon [0,1]^2 \to [0,1]$ is equivalent to the supermodularity of A, i.e., the property

$$A((x_1,y_1) \vee (x_2,y_2)) + A((x_1,y_1) \wedge (x_2,x_2)) \geq A(x_1,y_1) + A(x_2,y_2)$$

valid for all couples $(x_1,y_1),(x_2,y_2) \in [0,1]^2$, see [8,11]. Thus, Theorem 4.2 can be modified considering supermodular aggregation functions with zero annihilator.

5. Discrete fuzzy integrals based on aggregation functions

In this section, we first define discrete fuzzy integrals related to the introduced decompositions $D_{m,f}, G_{m,f}, K_{m,f}$ and $P_{m,f}$ and an arbitrary rectangle mapping S.

Definition 5.1. Let $S\colon \mathcal{R} \to [0,1]$ be a rectangle mapping. We define four mappings $DS, GS, KS, PS\colon \mathcal{M}_n \times \mathcal{F}_n \to [0,\infty[$ as follows:

(i) if $h_{m,f} = h_*$ then $DS(m,f) = GS(m,f) = KS(m,f) = PS(m,f) = 0$,
(ii) if $h_{m,f}$ is given by (3) then

$$DS(m,f) = \sum_{R \in D_{m,f}} S(R) = \sum_{i=1}^{k} S([a_{i-1}, a_i] \times [0, b_i]), \tag{13}$$

$$GS(m,f) = \sum_{R \in G_{m,f}} S(R) = \sum_{i=1}^{k} S([0, a_i] \times [b_{i+1}, b_i]), \tag{14}$$

$$KS(m,f) = \sum_{R \in K_{m,f}} S(R) = \sum_{i=1}^{k} \sum_{j=i}^{k} S([a_{i-1}, a_i] \times [b_{j+1}, b_j]), \tag{15}$$

$$PS(m,f) = \bigvee_{R \in P_{m,f}} S(R) = \bigvee_{i=1}^{k} S([0, a_i] \times [0, b_i]), \tag{16}$$

$D_{m,f}, G_{m,f}, K_{m,f}, P_{m,f}$, being rectangle decompositions described in (4), (5), (6) and (7), respectively.

Formally, based on the above considerations, one can introduce eight types of discrete fuzzy integrals. Four of them, namely DS_A, GS_A, KS_A and PS_A can be built by (13) - (16) and rectangle mappings of the type $S = S_A$, and the other four integrals, denoted by DS^A, GS^A, DS^A and PS^A, can be defined in the case when $S = S^A$ for an appropriate A. However, in general, some of these functionals coincide.

Theorem 5.1. *Let $A\colon [0,1]^2 \to [0,1]$ be a 2-increasing binary aggregation function with zero annihilator. Then for each $(m,f) \in \mathcal{M}_n \times \mathcal{F}_n$ we have $DS^A(m,f) = GS^A(m,f) = KS^A(m,f)$, i.e., the integrals DS^A, GS^A and KS^A coincide.*

Proof. For a given A, consider $S = S^A$. Then, using (13) and (12), for any $(m,f) \in \mathcal{M}_n \times \mathcal{F}_n$ we get

$$\begin{aligned} DS^A(m,f) &= \sum_{i=1}^{k} V_A\left([a_{i-1}, a_i] \times [0, b_i]\right) = \sum_{i=1}^{k} (A(a_i, b_i) + A(a_{i-1}, 0) \\ &\quad - A(a_{i-1}, b_i) - A(a_i, 0)) = \sum_{i=1}^{k} \left(A(a_i, b_i) - A(a_{i-1}, b_i)\right). \end{aligned}$$

Similarly, by (14) and (12), it can be shown that, for each $(m,f) \in \mathcal{M}_n \times \mathcal{F}_n$,

$$GS^A(m,f) = \sum_{i=1}^{k} \left(A(a_i, b_i) - A(a_i, b_{i+1})\right),$$

which, by means of the convention $b_{k+1} = 0$ and the fact that zero is an annihilator of A, can be rewritten as follows:

$$\begin{aligned} GS^A(m,f) &= \sum_{i=1}^{k} A(a_i, b_i) - \sum_{i=1}^{k-1} A(a_i, b_{i+1}) \\ &= \sum_{i=1}^{k} A(a_i, b_i) - \sum_{i=2}^{k} A(a_{i-1}, b_i) = \sum_{i=1}^{k} \left(A(a_i, b_i) - A(a_{i-1}, b_i)\right). \end{aligned}$$

As regards KS^A, whenever $h_{m,f} \neq h_*$, we have

$$KS^A(m,f) = \sum_{i=1}^{k}\sum_{j=i}^{k} V_A([a_{i-1},a_i]\times[b_{j+1},b_j]) = \sum_{i=1}^{k} V_A([a_{i-1},a_i]\times[0,b_i])$$
$$= DS^A(m,f),$$

where the second equality follows from the additivity of volumes,

$$V_A([a_{i-1},a_i]\times[0,b_i]) = \sum_{j=i}^{k} V_A([a_{i-1},a_i]\times[b_{j+1},b_j]).$$

Thus $GS^A(m,f) = DS^A(m,f) = KS^A(m,f)$ for each $(m,f)\in\mathcal{M}_n\times\mathcal{F}_n$, hence $GS^A = DS^A = KS^A$. $\square$

Now, for any fixed fuzzy measure m, we define the function $DS^A_m\colon \mathcal{F}_n\to[0,1]$,

$$DS^A_m(f) = DS^A(m,f),$$

and similarly, for any $f\in\mathcal{F}_n$, let $GS^A_m(f) = GS^A(m,f)$ and $KS^A_m(f) = KS^A(m,f)$. Evidently, due to Theorem 5.1, for each $f\in\mathcal{F}_n$, we have $DS^A_m(f) = GS^A_m(f) = KS^A_m(f)$.

Theorem 5.2. *Let $A\colon[0,1]^2\to[0,1]$ be a 2-increasing binary aggregation function with zero annihilator. Then for any fixed fuzzy measure $m\in\mathcal{M}_n$, the function $DS^A_m\colon\mathcal{F}_n\to[0,1]$ is an n-ary aggregation function satisfying the properties:*

(i) *DS^A_m is symmetric if and only if m is symmetric;*
(ii) *for any $m\in\mathcal{M}_n$, DS^A_m is idempotent and gives back the measure m, i.e., $DS^A_m(\mathbf{1}_E) = m(E)$ for each $E\subseteq[n]$, if and only if A has neutral element $e=1$.*

Proof. For proving that DS^A_m is an aggregation function we need to verify its monotonicity and the boundary conditions for aggregation functions. As already applied in the proof of Theorem 5.1, due to the additivity of A-volumes, see (11), the rectangle mapping S^A is also additive. The hypergraph $H_{m,f}$ can be split into a finite number of non-overlapping rectangles and then $DS^A_m(f)$ is just the sum of A-volumes of these rectangles.
Further, observe that for any functions $f,g\in\mathcal{F}_n$ such that $f\le g$, evidently, for any $j\in[n]$ and $t\in[0,1]$, if $f(j)\ge t$ then also $g(j)\ge t$. Thus, for any fuzzy measure $m\in\mathcal{M}_n$, due to the monotonicity of m and because of (1), for the survival functions we have $h_{m,f}\le h_{m,g}$ and consequently, for the related hypergraphs we also get $H_{m,f}\subseteq H_{m,g}$. This fact enables us to split the hypergraph $H_{m,g}$ into non-overlapping rectangles R_λ, for $\lambda\in L_g$, L_g being an index set, so that for some $L_f\subseteq L_g$, $H_{m,f}$ can be written as

$$H_{m,f} = \bigcup_{\lambda\in L_f} R_\lambda.$$

Then we get

$$DS^A_m(f) = \sum_{\lambda\in L_f} S^A(R_\lambda) \le \sum_{\lambda\in L_g} S^A(R_\lambda) = DS^A_m(g),$$

which proves the monotonicity of DS^A_m. It is easy to see that $DS^A_m(\mathbf{0}) = 0$ and $DS^A_m(\mathbf{1}) = 1$, thus DS^A_m is an aggregation function.

Further, we prove the stated properties of DS^A_m.
(i) Consider a permutation $\sigma\colon[n]\to[n]$. For a function $f\in\mathcal{F}_n$, let $f_\sigma(i) = f(\sigma(i))$, and for a set $E\subseteq[n]$, let $E_\sigma = \{\sigma(i)\mid i\in E\}$. Then the claim (i) follows from the fact that $h_{m,f_\sigma} = h_{m,f}$ holds for any permutation σ and any $f\in\mathcal{F}_n$ if and only if m is a symmetric fuzzy measure, i.e., a fuzzy measure satisfying the property $m(E_\sigma) = m(E)$ for any σ and $E\subseteq[n]$.
(ii) The claim follows from the facts that, for any $c\in[0,1]$, $E\subseteq[n]$ and $m\in\mathcal{M}_n$, we have

$$D_{m,\mathbf{c}} = \{[0,c]\times[0,1]\}, \text{ and } D_{m,\mathbf{1}_E} = \{[0,1]\times[0,m(E)]\}.$$

Then $c = DS^A_m(\mathbf{c}) = A(c,1)$ for each $c\in[0,1]$ if and only if 1 is a right neutral element of A, and $m(E) = DS^A_m(\mathbf{1}_E) = A(1,m(E))$ for each $E\subseteq[n]$ if and only if 1 is a left neutral element of A. Summarizing, both these properties are satisfied if and only if $e=1$ is a neutral element of A. $\square$

Observe that due to Theorem 5.1, Theorem 5.2 also holds for GS^A_m and KS^A_m.

Remark 5.1. It is known that a binary aggregation function A is 2-increasing with zero annihilator and neutral element $e=1$ if and only if A is a copula [20]. In the case of copulas, fuzzy integrals $DS^A = GS^A = KS^A$ were studied in [15], see also [16,19].

In general, the integrals DS_A, GS_A and KS_A are not equal. For any fixed $m \in \mathcal{M}_n$, $(DS_A)_m$, $(GS_A)_m$ and $(KS_A)_m$ always satisfy the boundary conditions, i.e., $(DS_A)_m(\mathbf{0}) = (GS_A)_m(\mathbf{0}) = (KS_A)_m(\mathbf{0}) = 0$, $(DS_A)_m(\mathbf{1}) = (GS_A)_m(\mathbf{1}) = (KS_A)_m(\mathbf{1}) = 1$, but they may violate other properties required for aggregation functions. For example, they need not be monotone, and they can even attain the values exceeding 1. In general, these integrals are weakly increasing.

Theorem 5.3. *Let $A\colon [0,1]^2 \to [0,1]$ be a binary aggregation function with zero annihilator. Then for each $m \in \mathcal{M}_n$, the functions $(DS_A)_m, (GS_A)_m, (KS_A)_m\colon \mathcal{F}_n \to [0,\infty[$ given by*

$$(DS_A)_m(f) = \sum_{i=1}^{k} A(a_i - a_{i-1}, b_i),$$

$$(GS_A)_m(f) = \sum_{i=1}^{k} A(a_i, b_i - b_{i+1}),$$

$$(KS_A)_m(f) = \sum_{i=1}^{k} \sum_{j=i}^{k} A(a_i - a_{i-1}, b_j - b_{j+1}),$$

respectively, are weakly increasing.

Proof. The form of the stated functions follows from (13)-(15) a (9). Let M be an arbitrary fuzzy measure in $\mathcal{M}_n$. For proving the weak monotonicity of $(DS_A)_m$ we have to show that the inequality $(DS_A)_m(f) \le (DS_A)_m(f+\mathbf{c})$ holds for any f and $\mathbf{c}$ in $\mathcal{F}_n$ such that also $f+\mathbf{c} \in \mathcal{F}_n$, and analogously for $(GS_A)_m$ and $(KS_A)_m$. There are two possible cases, either $f((1)) > 0$ or $f((1)) = 0$. If $f((1)) > 0$ (i.e., $\min\{f(i) \mid i \in [n]\} > 0$) and $h_{m,f}$ is given by (3), i.e., it is characterized by some values $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, then $h_{m,f+\mathbf{c}}$ is characterized by the values $a_1 + c, \ldots, a_k + c$ and $b_1, \ldots, b_k$. Thus

$$\begin{aligned}(DS_A)_m(f+\mathbf{c}) &= A(a_1 + c, b_1) + \sum_{i=2}^{k} A(a_i - a_{i-1}, b_i)\\ &= (DS_A)_m(f) + A(a_1+c, b_1) - A(a_1, b_1) \ge (DS_A)_m(f).\end{aligned}$$

In the considered case, the inequalities $(GS_A)_m(f) \le (GS_A)_m(f+\mathbf{c})$ and $(KS_A)_m(f) \le (KS_A)_m(f+\mathbf{c})$ ensuring the weak monotonicity of the integrals $(GS_A)_m$ and $(KS_A)_m$ can be proved similarly.
In the latter case, i.e., if $f((1)) = 0$ (which means that $f(i) = 0$ for some $i \in [n]$), and $h_{m,f}$ is characterized by some values $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, then the survival function $h_{m,f+\mathbf{c}}$ is characterized either by the values $c, a_1 + c, \ldots, a_k + c$ and $b_1, \ldots, b_k$ (when $b_1 < 1$), or by the values $a_1 + c, \ldots, a_k + c$ and $b_1, \ldots, b_k$ (if $b_1 = 1$). In both situations, these observations are instrumental in proving the inequalities ensuring the weak monotonicity of all three discussed integrals. □

Observe that despite a function $Z\colon \mathcal{F}_n \to \mathbb{R}$ is weakly increasing and satisfies the properties $Z(\mathbf{0}) = 0$ and $Z(\mathbf{1}) = 1$, there can be an $f \in \mathcal{F}_n$ such that $Z(f) \notin [0,1]$. This may also happen to our discrete integrals $(DS_A)_m, (GS_A)_m, (KS_A)_m$. Under the assumptions of Theorem 5.3, they are weakly increasing, and it is obvious that for any $f \in \mathcal{F}_n$ the values $(DS_A)_m(f), (GS_A)_m(f)$ and $(KS_A)_m(f)$ are non-negative. However, to ensure that these integrals are weak aggregation functions we have to guarantee that their values do not exceed the value 1.

Theorem 5.4. *Let $A\colon [0,1]^2 \to [0,1]$ be a binary aggregation function with zero annihilator. Then for each $m \in \mathcal{M}_n$ we have:*

(i) *if $A(x,1) \le x$ for each $x \in [0,1]$ then $(DS_A)_m$ is a weak aggregation function;*
(ii) *if $A(1,x) \le x$ for each $x \in [0,1]$ then $(GS_A)_m$ is a weak aggregation function;*
(iii) *if $A(x,y) \le xy$ for each $x \in [0,1]$ then $(KS_A)_m$ is a weak aggregation function.*

Proof. (i) Let $A(x,1) \le x$ for each $x \in [0,1]$. Then

$$\begin{aligned}(DS_A)_m(f) &= \sum_{i=1}^{k} A(a_i - a_{i-1}, b_i) \le \sum_{i=1}^{k} A(a_i - a_{i-1}, 1) \le \sum_{i=1}^{k} (a_i - a_{i-1})\\ &= a_k - a_0 = a_k \le 1.\end{aligned}$$

The claims (ii) and (iii) can be proved in a similar way. □

As mentioned above, any semicopula $\otimes\colon [0,1]^2 \to [0,1]$ is a binary aggregation function with zero annihilator, satisfying $1 \otimes x = x \otimes 1 = x$ for each $x \in [0,1]$. Therefore, due to Theorem 5.4, if $A = \otimes$ then $(DS_\otimes)_m$ and $(GS_\otimes)_m$ are weak aggregation functions for each $m \in \mathcal{M}_n$. Similarly to the case described in Theorem 5.2, for an arbitrary semicopula $\otimes$, the discrete fuzzy integrals $(DS_\otimes)_m$ and $(GS_\otimes)_m$ are idempotent and give back the measure m. This claim also holds for the integral $(KS_\otimes)_m$. In general, the symmetry of a fuzzy measure $m \in \mathcal{M}_n$ ensures the symmetry of all three types of integrals $(DS_A)_m, (GS_A)_m, (KS_A)_m$, independently of the considered aggregation function A (with zero annihilator).

Finally, recall that a conjunctive aggregation function A is characterized by the condition $A \le \min$ (see [6]) which in the binary case is equivalent to the validity of the inequalities $A(x,1) \le x$ and $A(1,x) \le x$ for each $x \in [0,1]$. Hence, if A is a conjunctive binary aggregation function then for any $m \in \mathcal{M}_n$, the discrete fuzzy integrals $(DS_A)_m$ and $(GS_A)_m$ are weak aggregation functions.

6. Some discrete fuzzy integrals

It can be shown that the only binary aggregation function A such that $DS_A = GS_A = KS_A = DS^A = GS^A = KS^A$ is the product $A(x,y) = xy$, and the related fuzzy integral is the Choquet integral. The Sugeno integral is related to the greatest copula M, $M(x,y) = \min\{x,y\}$, and then $Su = DS^M = GS^M = KS^M$. The opposite Sugeno integral ${}^{op}Su$ introduced by Imaoka [12] is related to the smallest copula W, $W(x,y) = \max\{0, x+y-1\}$, and ${}^{op}Su = DS^W = GS^W = KS^W$. As already stated in Theorem 5.2, all these integrals are, for each $m \in \mathcal{M}_n$, idempotent aggregation functions giving back the fuzzy measure m.

Due to [8], $A\colon [0,1]^2 \to [0,1]$ is a 2-increasing aggregation function if and only if for all $x, y \in [0,1]$,

$$A(x,y) = C(\varphi(x), \eta(y)),$$

for some copula $C\colon [0,1]^2 \to [0,1]$ and increasing functions $\varphi, \eta\colon [0,1] \to [0,1]$ satisfying the conditions $C(\varphi(0), \eta(0)) = 0$ and $\varphi(1) = \eta(1) = 1$. It can be shown that if A also has zero annihilator then necessarily $\varphi(0) = \eta(0) = 0$, i.e., φ and η are unary aggregation functions. Hence, A is a 2-increasing aggregation function with zero annihilator if and only if there is a bivariate copula C and increasing functions $\varphi, \eta\colon [0,1] \to [0,1]$ satisfying $\varphi(0) = \eta(0) = 0$ and $\varphi(1) = \eta(1) = 1$ such that $A(x,y) = C(\varphi(x), \eta(y))$ for all $(x,y) \in [0,1]^2$.

In that case, for any $(m, f) \in \mathcal{M}_n \times \mathcal{F}_n$ we have

$$DS^A(m,f) = GS^A(m,f) = KS^A(m,f) = DS^C(\eta \circ m, \varphi \circ f).$$

Corollary 6.1. *Under the assumptions of Theorem 5.1, the discrete fuzzy integral DS^A can be seen as a transform of a copula-based integral DS^C for some copula C, transforming $m \in \mathcal{M}_n$ into $\eta \circ m \in \mathcal{M}_n$, and $f \in \mathcal{F}_n$ into $\varphi \circ f \in \mathcal{F}_n$.*

Example 6.1. Let A be a binary aggregation function given by $A(x,y) = x^2 y^3$.
(i) Then for any $(m,f) \in \mathcal{M}_n \times \mathcal{F}_n$ we have

$$DS^A(m,f) = \sum_{i=1}^{k} (a_i^2 b_i^3 - a_i^2 b_{i+1}^3) = Ch(m^3, f^2).$$

(ii) Consider $n = 2$. Let $m(\{2\}) = \frac{2}{3}$, $f(1) = \frac{1}{2}$ and $f(2) = 1$. Then

$$Ch(m,f) = \frac{1}{2} + \left(1 - \frac{1}{2}\right) \cdot \frac{2}{3} = \frac{5}{6},$$

$$DS^A(m,f) = \left(\frac{1}{2}\right)^2 + \left(1 - \left(\frac{1}{2}\right)^2\right)\left(\frac{2}{3}\right)^3 = \frac{17}{36} = GS^A(m,f) = KS^A(m,f),$$

$$DS_A(m,f) = \left(\frac{1}{2}\right)^2 + \left(1 - \frac{1}{2}\right)^2 \left(\frac{2}{3}\right)^3 = \frac{35}{108},$$

$$GS_A(m,f) = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2 \left(1 - \frac{2}{3}\right)^3 = \frac{33}{108},$$

$$KS_A(m,f) = \frac{17}{108},$$

$$PS_A(m,f) = \frac{1}{4}.$$

(iii) If we take the aggregation function $B(x,y) = \sqrt{xy}$ instead of A, and m and f again as stated in (ii), then

$$DS_B(m, f) = \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{12}} = 0.996,\ GS_B(m, f) = \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{6}} = 1.225.$$

Note that B does not satisfy the constraints of Theorem 5.3, e.g., $B(x, 1) = \sqrt{x} > x$ for each $x \in]0, 1[$, and thus, in general, DS_B and GS_B need not be bounded from above by 1. However, for some particular fuzzy measures they can be bounded, for example, considering the bottom element m_* of $\mathcal{M}_n$, i.e., $m_*(E) = 0$ for each proper subset of $[n]$, we obtain $(DS_B)_{m_*}(f) = \min\{f(i) \mid i \in [n]\} \le 1$.

Now, let us turn the attention to the discrete integrals $(PS^A)_m$ and $(PS_A)_m$, $m \in \mathcal{M}_n$. If A is a 2-increasing aggregation with zero annihilator then evidently $(PS^A)_m = (PS_A)_m$, and therefore, in such a case, it is enough to deal with the integrals $(PS_A)_m$. We omit simple proofs of their monotonicity and the fact they meet the boundary conditions, we only stress that for any $m \in \mathcal{M}_n$ and any binary 2-increasing aggregation function A with zero annihilator, $(PS_A)_m$ is an n-ary aggregation function. It generalizes the approach of Klement et al. introduced in [14], where semicopulas $\otimes$ were considered to guarantee the idempotency of the considered integral and its ability to give back the fuzzy measure m. Finally, we note that $PS_{\min}$ is the well-known Sugeno integral [22], PS_{prod} is the Shilkret integral [21], and PS_T, T being a strict t-norm [13], is the Weber integral [24].

7. Concluding remarks

Based on representation of hypergraphs $H_{m,f}$ of survival functions $h_{m,f}$, we have proposed a general approach to introducing discrete fuzzy integrals, exploiting some rectangle mappings S which evaluate rectangles $[a, b] \times [c, d] \subseteq [0, 1]^2$ of the considered hypergraph decompositions in some non-standard ways. Note that for some special choices of S the resulting discrete fuzzy integrals have already been introduced and applied. This is, e.g., the case of copula-based integrals $DS^C = GS^C = KS^C$ studied in [15,16,19]. Also, some special cases of discrete fuzzy integrals DS_A, GS_A and PS_A have already been proposed and discussed, mostly in the framework of generalizations of the discrete Choquet and Sugeno integrals, see, e.g., [10]. The main contribution of our paper lies in a common view at several constructions of discrete fuzzy integrals, with a large freedom in the choice of rectangle mappings S. For the future research, some alternative views at rectangle mappings can be considered. As regards some applications, in the case of new types of fuzzy integrals which are monotone, we expect their applications in multicriteria decision support, and in the case of integrals which are weak aggregation functions, we expect their applications primarily in any domain where weak aggregation functions have already been successfully applied, in particular in image processing and fuzzy rule-based classification systems, see, e.g., [5,16–18].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

Both authors kindly acknowledge the support of the project APVV-18-0052. A. Kolesárová was also supported by the grant VEGA 1/0267/21 and R. Mesiar by the grant VEGA 1/0006/19. The authors are also grateful to the referees for their valuable comments.

References

[1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer-Verlag, Berlin Heidelberg, 2007.
[2] G. Beliakov, S. James, A. Kolesárová, R. Mesiar, Cardinality-limiting extended pre-aggregation functions, Inf. Fusion 76 (2021) 66–74.
[3] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators, Physica-Verlag, Heidelberg, 2002, pp. 3–107.
[4] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953) 131–295.
[5] C.M. Detchart, G.P. Dimuro, M. Sesma-Sara, A. Castillo-Lopez, J. Fernández, H. Bustince, Consensus image feature extraction with ordered directionally monotone functions, in: NAFIPS 2018, 2018, pp. 155–166.
[6] D. Dubois, H. Prade, On the use of aggregation operations in information fusion processes, Fuzzy Sets Syst. 142 (1) (2004) 143–161.
[7] F. Durante, C. Sempi, Semicopulae, Kybernetika 41 (3) (2005) 315–328.
[8] F. Durante, R. Mesiar, P.L. Papini, C. Sempi, 2-Increasing binary aggregation operators, Inf. Sci. 177 (1) (2007) 111–129.
[9] Y. Even, E. Lehrer, Decomposition integral: unifying Choquet and the concave integrals, Econ. Theory 56 (2014) 33–58.
[10] Ľ. Horanská, A. Šipošová, A generalization of the discrete Choquet and Sugeno integrals based on a fusion function, Inf. Sci. 451–452 (2018) 83–99.
[11] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, 2009.
[12] H. Imaoka, Opposite Sugeno integral, J. Japan Soc. Fuzzy Theor. Syst. 10 (2) (1998) 200–205.
[13] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[14] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (1) (2010) 178–187.
[15] E.P. Klement, R. Mesiar, F. Spizzichino, A. Stupňanová, Universal integrals based on copulas, Fuzzy Optim. Decis. Mak. 13 (3) (2014) 273–286.
[16] G. Lucca, J.A. Sanz, G.P. Dimuro, B. Bedregal, M.J. Aisian, M. Elkano, H. Bustince, CC-integrals: Choquet-like copula-based aggregation functions and its application in fuzzy rule-based classification systems, Knowl.-Based Syst. 119 (2017) 32–43.

[17] G. Lucca, J.A. Sanz, G.P. Dimuro, B.R.C. Bedregal, H. Bustince, R. Mesiar, C_F-integrals: a new family of pre-aggregation functions with application to fuzzy rule-based classification systems, Inf. Sci. 435 (2018) 94–110.
[18] G. Lucca, G.P. Dimuro, J. Fernández, H. Bustince, B.R.C. Bedregal, J.A. Sanz, Improving the performance of fuzzy rule-based classification systems based on a nonaveraging generalization of CC-integrals named CF1F2-integrals, IEEE Trans. Fuzzy Syst. 27 (1) (2019) 124–134.
[19] R. Mesiar, A. Stupňanová, A note on CC-integral, Fuzzy Sets Syst. 355 (2019) 106–109.
[20] R.B. Nelsen, An Introduction to Copulas, 2nd edition, Springer, New York, 2006.
[21] N. Shilkret, Maxitive measure and integration, Indag. Math. 74 (1971) 109–116.
[22] M. Sugeno, Theory of Fuzzy Integrals and Applications, Ph.D. Thesis, Tokyo Institute of Technology, Tokyo, 1974.
[23] Z. Wang, G.J. Klir, Generalized Measure Theory, Springer US, 2009.
[24] S. Weber, Two integrals and some modified versions: critical remarks, Fuzzy Sets Syst. 20 (1) (1986) 97–105.
[25] T. Wilkin, G. Beliakov, Weakly monotonic averaging functions, Int. J. Intell. Syst. 30 (2) (2015) 144–169.