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A new class of decomposition integrals on finite spaces

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ABSTRACT

A new type of decomposition integral is introduced by using a family of decomposition integrals based on the collections relating to partitions and maximal chains of sets. This new integral extends the Lebesgue integral, and it is different from those well-known decomposition integrals, such as the Choquet, concave, pan-, Shilkret integrals and PC-integral. In the structure of a lattice on the class of decomposition integrals, the introduced decomposition integral is between the Choquet integral and the concave integral, and also between the pan-integral and the concave integral, and it is a lower bound of PC-integral. The coincidences among several well-known integrals and this new integral are also shown.

1. Introduction

The decomposition integral proposed by Even and Lehrer [5] provides a common framework for the concave integral, the Choquet integral and the pan-integral, etc. As it is known, a decomposition integral is based on the system of collections (also called as decomposition system). In general, two different decomposition systems induce different decomposition integrals, but not necessarily. Recall three important decomposition integrals, the Choquet integral, the concave integral and the pan-integral, they are based on all (maximal) finite chains of sets, arbitrary finite set systems and all finite partitions, respectively. All these integrals extend the Lebesgue integral, i.e., for σ-additive measures, each of them coincides with the Lebesgue integral. For a general monotone measure, they are significantly different from each other.

These three integrals and the Shilkret integral form a diamond lattice, see Fig. 1. The concave integral is the top element of this lattice and it can be strictly greater than the Choquet and pan-integrals simultaneously. From a lattice viewpoint, it is of interest to find a new element locating between these three elements, i.e., find a new decomposition integral which is smaller than the concave integral but greater than both the Choquet and pan-integrals.

The first step of this issue has been done by Stupňanová. In [23], she introduced a new type of decomposition integral, PC-integrals, based on the so-called PC-decomposition system in which the collection includes pairwise disjoint sets and chains of sets. The PC-integral locates between the concave integral and the Choquet integral, and also between the concave integral and the pan-integral. Since the PC-integral can be strictly greater than the Choquet and pan-integrals simultaneously, it is also interesting to find a new element locating between the PC-integral, Choquet integral and pan-integrals.

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In this paper, we propose a new type of decomposition integral, which is different from the above mentioned decomposition integrals. A brief summarization of the idea for this integral is as follows:

Suppose that X is a finite set and let $(X, \mathcal{A}, \mu)$ be a monotone measure space. For a given partition $\wp$ of X, $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$ (where $\hat{\mathcal{P}}$ is the class of all partitions of X), we take a maximal chain in each $A_i, i = 1, 2, \ldots, k$, respectively. The union of these maximal chains constitutes a collection. Consider the set of all these collections, denoted by $\mathcal{H}^c_{(\wp)}$, i.e., a decomposition system related to the partition $\wp$ and maximal chains. These decomposition systems $\{\mathcal{H}^c_{(\wp)}\}_{\wp\in\hat{\mathcal{P}}}$ determine a family $\{I_{\mathcal{H}^c_{(\wp)}}(\mu, \cdot)\}_{\wp\in\hat{\mathcal{P}}}$ of decomposition integrals. Then the new type of decomposition integral, denoted by $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, \cdot)$, is defined as the supremum of the family $\{I_{\mathcal{H}^c_{(\wp)}}(\mu, \cdot)\}_{\wp\in\hat{\mathcal{P}}}$ of decomposition integrals. Such the decomposition integrals $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, \cdot)$, including the integrals $I_{\mathcal{H}^c_{(\wp)}}(\mu, \cdot)(\wp \in \hat{\mathcal{P}})$ have some new characteristics. The first, they all extend the Lebesgue integral, and for any $\wp \in \hat{\mathcal{P}}$, $\mathcal{H}^c_{(\wp)}$ is a minimal element in the upper semilattice $\mathbb{L}$ consisting of Lebesgue decomposition systems with the inclusion relation "$\subseteq$". The second, if we consider the structure of a lattice on the class of decomposition integrals, then, of all the decomposition integrals, the concave integral is the top and the Shilkret integral is the bottom of this lattice (see Fig. 1). The PC-integral is a lower bound of the concave integral and, the newly introduced integral $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, \cdot)$ is below the PC-integral, and is between the Choquet integral and the concave integral, and also between the pan-integral and the concave integral (see Fig. 2).

The framework of this article is as follows: Section 2 collects some known results of decomposition integral, including its definition and some basic properties of four specific decomposition integrals (namely, the concave, Choquet, pan- and Shilkret integrals). Section 3 is devoted to investigating the Lebesgue decomposition system, including several illustrative examples. Section 4, the main part of the paper, begins with a review of the PC-integral and then introduces a new type of decomposition integral. Section 5 discusses the coincidences of this new integral and the concave, Choquet, pan- and PC-integrals. In this discussion, the characteristics of monotone measures, such as subadditivity, supermodularity, (M)-property and minimal atoms, play important roles. Finally, Section 6 ends this paper.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X, and μ be a monotone measure on $(X, \mathcal{A})$, i.e., $\mu : \mathcal{A} \to [0, +\infty]$ satisfies the following conditions: (1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$; (2) $\mu(Q) \le \mu(R)$ whenever $Q \subseteq R$ and $Q, R \in \mathcal{A}$.

The set of all monotone measures on $(X, \mathcal{A})$ will be denoted by $\mathcal{M}$ and the class of all σ-additive measures on $(X, \mathcal{A})$ will be denoted by $\mathcal{M}_a$. Then $\mathcal{M}_a \subset \mathcal{M}$. $\mathcal{F}^+$ denotes the set of all finite $\mathcal{A}$-measurable functions $f : X \to [0, +\infty[$. For $A \in \mathcal{A}$, $\chi_A : X \to \{0, 1\}$ denotes its characteristic function, i.e., $\chi_A(x) = 1$ if and only if $x \in A$.

Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{A}$ and all the considered monotone measures are supposed to be finite, i.e., for any $\mu \in \mathcal{M}$, $\mu(X) < \infty$.

From Even and Lehrer [5] (see also Lehrer [9]), and Mesiar and Stupňanová [16], we recall some results related to decomposition integrals. Their construction copies the idea of lower integral sums and it is based on a system $\mathcal{H}$ of finite set systems from $\mathcal{A} \setminus \{\varnothing\}$ (called collections in [5]).

For a fixed measurable space $(X, \mathcal{A})$, the set of all systems $\mathcal{H}$ of finite set systems from $\mathcal{A} \setminus \{\varnothing\}$ will be denoted by $\mathbb{X}$.

Let $\mathcal{H} \in \mathbb{X}$ be fixed. The mapping $I_{\mathcal{H}} : \mathcal{M} \times \mathcal{F}^+ \to [0, +\infty]$ given by

$$I_{\mathcal{H}}(\mu, f) = \sup\left\{\sum_{i\in J} a_i \mu(A_i) : \{A_i\}_{i\in J} \in \mathcal{H}, \sum_{i\in J} a_i \chi_{A_i} \le f\right\}, \tag{2.1}$$

where all constants $a_i \ge 0$, is called a *decomposition integral*.

The $\sum_{i\in J} a_i \chi_{A_i}$ in formula (2.1) is called a $\mathcal{H}$-sub-decomposition of f (with respect to collection $\{A_i\}_{i\in J} \in \mathcal{H}$).

Depending on $\mathcal{H}$, several well-known nonlinear integrals can be constructed ([5,7,16]). We will adopt some notations used in [7,13,16].

- Let $\mathcal{H}_{Sh} = \big\{\{A\} : A \in \mathcal{A} \setminus \{\varnothing\}\big\}$. Then $I_{\mathcal{H}_{Sh}}(\mu, f)$ is the Shilkret integral ([7,22]), i.e., for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$,

$$I_{\mathcal{H}_{Sh}}(\mu, f) = \sup\big\{t \cdot \mu(\{f \ge t\}) : t \in [0, \infty]\big\}.$$

- Let $\mathcal{H}_{pan}$ be the set of all finite measurable partitions of X (also denoted by $\hat{\mathcal{P}}$, i.e., $\mathcal{H}_{pan} = \hat{\mathcal{P}}$). Then $I_{\mathcal{H}_{pan}}(\mu, f)$ is the $(+, \cdot)$-based pan-integral (see also [24]).
- Let $\mathcal{H}_{Ch} = \big\{\mathcal{C} : \mathcal{C}$ is a finite chain in $\mathcal{A} \setminus \{\varnothing\}\big\}$. Then $I_{\mathcal{H}_{Ch}}(\mu, f)$ is the Choquet integral [1,2,21], i.e.,

$$I_{\mathcal{H}_{Ch}}(\mu, f) = \int_0^\infty \mu(\{x : f(x) \ge t\})\, dt.$$

- Let $\mathcal{H}_{cav} = \{\mathcal{D} : \mathcal{D}$ is a finite subset in $\mathcal{A} \setminus \{\varnothing\}\}$. Then $I_{\mathcal{H}_{cav}}(\mu, f)$ is the concave integral introduced by Lehrer [8] (see also [10]). Observe that there are also several other decomposition systems yielding the concave integral. For example, when $\mathcal{A}$ is finite, this is the case of $\mathcal{H} = \mathcal{A} \setminus \{\varnothing\}$.

The Choquet integral is based on finite chains of sets, the pan-integral is related to finite partitions of X and the concave integral to any finite set systems of measurable subsets of X.

For the convenience of discussion, the following symbols from [13] will be adopted: for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$, denote $\mathbf{Sh}_\mu(f) = I_{\mathcal{H}_{Sh}}(\mu, f)$, $\mathbf{Ch}_\mu(f) = I_{\mathcal{H}_{Ch}}(\mu, f)$, $\mathbf{Pan}_\mu(f) = I_{\mathcal{H}_{pan}}(\mu, f)$ and $\mathbf{Cav}_\mu(f) = I_{\mathcal{H}_{cav}}(\mu, f)$.

The basic properties of these types of integrals can be found in [2,5–8,10,14–16,24].

For any $\mathcal{H}_s, \mathcal{H}_t \in \mathbb{X}$, it is easy to see that if $\mathcal{H}_s \subseteq \mathcal{H}_t$, then

$$I_{\mathcal{H}_s}(\mu, f) \le I_{\mathcal{H}_t}(\mu, f)$$

for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$.

We have further results. Let $\mathcal{H}_s, \mathcal{H}_t \in \mathbb{X}$. $\mathcal{H}_t$ is said to be *a refinement* of $\mathcal{H}_s$ ([16], see also [11]), denoted by $\mathcal{H}_s \preceq \mathcal{H}_t$, if for any $\mathcal{D}_1 \in \mathcal{H}_s$ there is $\mathcal{D}_2 \in \mathcal{H}_t$ such that $\mathcal{D}_1 \subseteq \mathcal{D}_2$ Obviously, if $\mathcal{H}_s \subseteq \mathcal{H}_t$ then $\mathcal{H}_s \preceq \mathcal{H}_t$. The converse is not true. Let $\mathcal{H}_s, \mathcal{H}_t \in \mathbb{X}$, then

$$\mathcal{H}_s \preceq \mathcal{H}_t \text{ implies } I_{\mathcal{H}_s}(\mu, f) \le I_{\mathcal{H}_t}(\mu, f)$$

for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$ ([16]).

Note that

$$\mathcal{H}_{Sh} \subsetneq \mathcal{H}_{pan} \subsetneq \mathcal{H}_{cav} \quad \text{and} \quad \mathcal{H}_{Sh} \subsetneq \mathcal{H}_{Ch} \subsetneq \mathcal{H}_{cav},$$

then, for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$,

$$\mathbf{Sh}_\mu(f) \le \mathbf{Pan}_\mu(f) \wedge \mathbf{Ch}_\mu(f)$$

and

$$\mathbf{Pan}_\mu(f) \vee \mathbf{Ch}_\mu(f) \le \mathbf{Cav}_\mu(f)$$

and, $\mathbf{Pan}_\mu(f)$ and $\mathbf{Ch}_\mu(f)$ are incomparable.

For any $(m, f) \in \mathcal{M}_a \times \mathcal{F}^+$,

$$\mathbf{Pan}_m(f) = \mathbf{Ch}_m(f) = \mathbf{Cav}_m(f) = \mathbf{Leb}_m(f) \tag{2.2}$$

where $\mathbf{Leb}_m(f)$ denotes the Lebesgue integral of f on X with respect to σ-additive measure m. The Shilkret integral does not possess this property.

The following Hasse diagram represents the relationships among four types of decomposition integrals (Fig. 1).

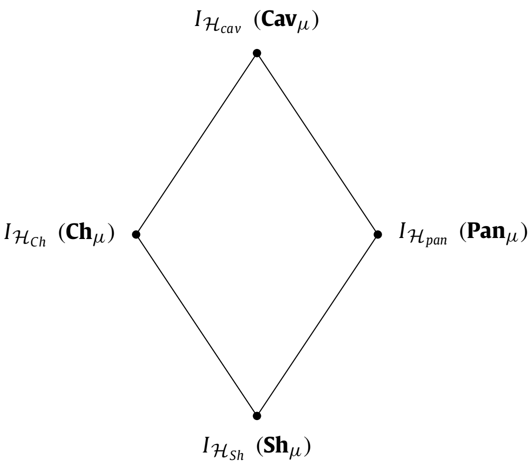

Fig. 1. Hasse diagram of four types of decomposition integrals.

3. Lebesgue decomposition systems

In the rest of sections, we shall confine our discussion to finite spaces. If not stating explicitly, the set X is fixed to $\{1, 2, \ldots, n\}$ and the algebra $\mathcal{A} = 2^X$. In this case, $\mathbb{X} = 2^{2^{2^X \setminus \{\emptyset\}} \setminus \{\emptyset\}} \setminus \{\emptyset\}$.

Consider the system $\mathbb{L} \subset \mathbb{X}$ given by

$$\mathbb{L} \triangleq \left\{ \mathcal{H} \in \mathbb{X} : \ \forall\, m \in \mathcal{M}_a, I_{\mathcal{H}}(m, \cdot) \text{ is a Lebesgue integral on } X \right\},$$

where $I_{\mathcal{H}}(\cdot, \cdot)$ is decomposition integral as defined in the formula (2.1).

A system $\mathcal{H}$ will be called a *Lebesgue decomposition system* if $\mathcal{H} \in \mathbb{L}$.

For any additive measure $m : 2^X \to [0, +\infty[$,

$$\mathbf{Leb}_m(f) = \sum_{i \in X} f(i) \cdot m(\{i\}) = \sum_{j \in J} a_j \cdot m(E_j)$$

for any $a_j \geq 0$, E_j, $j \in J$ such that $\sum_{j \in J} a_j \chi_{E_j} = f$.

In this paper, only Lebesgue integral acting on non-negative measurable functions is considered, as for real-valued functions, it is then enough to consider their positive parts and the negative parts, respectively. This approach allows us to define decomposition integrals on real functions, as it was considered, e.g., in [3,4].

From the equation (2.2), it can be seen that $\mathcal{H}_{pan}, \mathcal{H}_{Ch}, \mathcal{H}_{cav} \in \mathbb{L}$. However, $\mathcal{H}_{Sh} \notin \mathbb{L}$. Some more results on the extension of the Lebesgue integral were presented in [11].

Example 3.1. (i) Let $\mathcal{H} = \Big\{ \big\{ \{1\}, \{2\}, \ldots, \{n\} \big\} \Big\} \in \mathbb{X}$. Then for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$,

$$I_{\mathcal{H}}(\mu, f) = \sum_{i \in X} f(i) \cdot \mu(\{i\})$$

and hence if m is an additive measure on 2^X, then for any $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}}(m, f) = \sum_{i \in X} f(i) \cdot m(\{i\}) = \mathbf{Leb}_m(f).$$

Thus, $\mathcal{H} = \Big\{ \big\{ \{1\}, \{2\}, \ldots, \{n\} \big\} \Big\} \in \mathbb{L}$.

(ii) Let $\mathcal{H} = \big\{ \{A_1, A_2, \ldots, A_k\} \big\}$ with $\{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$. Then for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$,

$$I_{\mathcal{H}}(\mu, f) = \sum_{i \in X} \big(\inf_{x \in A_i} f(x) \big) \cdot \mu(\{A_i\}).$$

For $k < n$, i.e., $\{A_j\}_{j=1}^k \neq \{\{i\}\}_{i=1}^n$, $\mathcal{H} = \big\{ \{A_1, A_2, \ldots, A_k\} \big\} \notin \mathbb{L}$. If $k = n$, then $\{A_j\}_{j=1}^k = \{\{i\}\}_{i=1}^n$, it goes back to the case (i), $\mathcal{H} = \Big\{ \big\{ \{1\}, \{2\}, \ldots, \{n\} \big\} \Big\} \in \mathbb{L}$.

(iii) Let $\mathcal{H} = \Big\{ \big\{ \{1\} \big\}, \big\{ \{2\} \big\}, \ldots, \big\{ \{n\} \big\} \Big\} \in \mathbb{X}$ $(n \geq 2)$. Then for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$,

$$I_{\mathcal{H}}(\mu, f) = \max_{i \in X} \big\{ f(i) \cdot \mu(\{i\}) \big\}.$$

For such a system $\mathcal{H} \in \mathbb{X}$, $I_{\mathcal{H}}(m, f) = \mathbf{Leb}_m(f)$ only if m is a positive multiple of some Dirac measure on X. For other $m \in \mathcal{M}_a$, $I_{\mathcal{H}}(m, f) \neq \mathbf{Leb}_m(f)$, therefore $\mathcal{H} = \Big\{ \big\{ \{1\} \big\}, \big\{ \{2\} \big\}, \ldots, \big\{ \{n\} \big\} \Big\} \notin \mathbb{L}$.

Now we discuss the maximal element of $(\mathbb{L}, \subseteq)$.

The following result has been shown in [11], we provide here an alternative proof.

Proposition 3.2. *Let $\mathcal{H} \in \mathbb{X}$. If there exists $\mathcal{H}_s \in \mathbb{L}$ such that $\mathcal{H}_s \preceq \mathcal{H}$, then $\mathcal{H} \in \mathbb{L}$, and hence if $\mathcal{H}_s \in \mathbb{L}$ and $\mathcal{H}_t \in \mathbb{X}$, then $\mathcal{H}_s \cup \mathcal{H}_t \in \mathbb{L}$.*

Proof. Let $f \in \mathcal{F}^+$. For any collection $\mathcal{D} = \{E_j\}_{j \in J} \in \mathcal{H}$, if $m \in \mathcal{M}_a$, i.e., m is additive measure on 2^X and $\sum_{j \in J} a_j \chi_{E_j} \leq f$, then $\sum_{j \in J} a_j m(E_j) \leq \mathbf{Leb}_m(f)$. Therefore for $f \in \mathcal{F}^+$, $I_{\mathcal{H}}(m, f) \leq \mathbf{Leb}_m(f)$. On the other hand, $\mathcal{H}_s \preceq \mathcal{H}$ and $\mathcal{H}_s \in \mathbb{L}$, therefore

$$I_{\mathcal{H}}(m, f) \geq I_{\mathcal{H}_s}(m, f) = \mathbf{Leb}_m(f).$$

Consequently, for any $f \in \mathcal{F}^+$, $I_{\mathcal{H}}(m, f) = \mathbf{Leb}_m(f)$, i.e., $\mathcal{H} \in \mathbb{L}$. $\square$

Example 3.3. Consider $\mathcal{H}_{pan}$, i.e., the set of all finite measurable partitions of X. Let $\mathcal{H} = \{\{\{1\}, \{2\}, \dots, \{n\}\}\}$, then $\mathcal{H} \in \mathbb{L}$ and $\mathcal{H}_{pan} \supseteq \mathcal{H}$. From Proposition 3.2, we have $\mathcal{H}_{pan} \in \mathbb{L}$. Then for additive measure m, it holds $\mathbf{Pan}_m(f) = I_{\mathcal{H}_{pan}}(m, f) = \mathbf{Leb}_m(f)$.

Since $\mathcal{H} \subseteq \mathcal{H}_{cav}$ holds for any $\mathcal{H} \in \mathbb{X}$, the following result is a consequence of Proposition 3.2.

Proposition 3.4. *The* $(\mathbb{L}, \subseteq)$ *is an upper semilattice with maximal element*

$$\mathcal{H}^* = \left\{\mathcal{D} : \mathcal{D} \text{ is a finite subset in } 2^X \setminus \{\emptyset\}\right\},$$

that is, $\mathcal{H}^* = \mathcal{H}_{cav}$.

Consequently, the concave integral $I_{\mathcal{H}_{cav}}(\mu, f)$ introduced by Lehrer [8] is the greatest decomposition integral defined on $\mathcal{M} \times \mathcal{F}^+$.

Now we discuss the minimal Lebesgue decomposition systems in $(\mathbb{L}, \subseteq)$.

Let $\wp = \{A_1, A_2, \dots, A_k\} \in \hat{\mathcal{P}}$ be given, where $\hat{\mathcal{P}}$ is the class of all (finite measurable) partitions of X. Define

$$\mathcal{H}^c_{(\wp)} = \left\{\bigcup_{i=1}^{k} \mathcal{C}_i : \mathcal{C}_i \text{ is a maximal chain in } 2^{A_i} \setminus \{\emptyset\}, i = 1, 2, \dots, k\right\}. \tag{3.1}$$

Note: In the formula (3.1), each $\mathcal{C}_i$ is required to be a maximal chain in $2^{A_i} \setminus \{\emptyset\}$ just for the convenience of discussion. In fact, if we consider arbitrary chains in each A_i, then the results are the same.

The following presents some properties of the decomposition systems $\mathcal{H}^c_{(\wp)}$ $(\wp \in \hat{\mathcal{P}})$ and the corresponding decomposition integrals $I_{\mathcal{H}^c_{(\wp)}}(\mu, \cdot)$.

- $|\mathcal{H}^c_{(\wp)}| = \prod_{i=1}^{k}(|A_i|!)$, where $|\cdot|$ stands for the cardinality of a set.
- Let $\wp_n = \{\{1\}, \{2\}, \dots, \{n\}\}$, then $\mathcal{H}^c_{(\wp_n)} = \{\{\{1\}, \{2\}, \dots, \{n\}\}\}$, and hence

$$I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f) = \sum_{i=1}^{n} f(i) \cdot \mu(\{i\}). \tag{3.2}$$

  As shown in Example 3.1, if μ is an additive measure on 2^X, then $I_{\mathcal{H}^c_{(\wp_n)}}(\mu, \cdot)$ is a Lebesgue integral on X. Therefore $\mathcal{H}^c_{(\wp_n)} \in \mathbb{L}$.

  Note that for such $\mathcal{H}^c_{(\wp_n)}$ and any given $\mu \in \mathcal{M}$ (not necessarily additive), the decomposition integral $I_{\mathcal{H}^c_{(\wp_n)}}(\mu, \cdot)$ determines a positive homogeneous and linear functional on $\mathcal{F}^+$, i.e., for any $f, g \in \mathcal{F}^+$, $\alpha \geq 0$,

$$I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f + g) = I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f) + I_{\mathcal{H}^c_{(\wp_n)}}(\mu, g),$$

  and

$$I_{\mathcal{H}^c_{(\wp_n)}}(\mu, \alpha f) = \alpha I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f).$$

- If $\wp = \{X\}$, then $\mathcal{H}^c_{(\wp)}$ is the family of all maximal chains in $2^X \setminus \{\emptyset\}$. It is easy to see that $|\mathcal{H}^c_{(\{X\})}| = n!$ and $\mathcal{H}^c_{(\{X\})} \subseteq \mathcal{H}_{Ch}$. But,

$$I_{\mathcal{H}^c_{(\{X\})}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f) = \mathbf{Ch}_\mu(f),$$

  and hence $\mathcal{H}^c_{(\{X\})} \in \mathbb{L}$.

Given $\wp = \{A_1, A_2, \dots, A_k\} \in \hat{\mathcal{P}}$, then it is easy to see that for any $(m, f) \in \mathcal{M}_a \times \mathcal{F}^+$,

$$I_{\mathcal{H}^c_{(\wp)}}(m, f) = \sum_{i=1}^{n} f(i) \cdot m(\{i\}) = \mathbf{Leb}_m(f).$$

Proposition 3.5. *For any* $\wp \in \hat{\mathcal{P}}$*,* $\mathcal{H}^c_{(\wp)}$ *is a Lebesgue decomposition system, i.e.,* $\mathcal{H}^c_{(\wp)} \in \mathbb{L}$.

The following is a direct result from Propositions 3.2 and 3.5.

Proposition 3.6. *Let $\mathcal{H} \in \mathbb{X}$. If there exists some $\wp \in \hat{\mathcal{P}}$ such that $\mathcal{H} \supseteq \mathcal{H}^c_{(\wp)}$, then $\mathcal{H} \in \mathbb{L}$.*

The converse of Proposition 3.6 is not true, i.e., for a Lebesgue decomposition system $\mathcal{H} \in \mathbb{L}$, it may not contain any $\mathcal{H}^c_{(\wp)}$ ($\wp \in \hat{\mathcal{P}}$). This is illustrated by the following example.

Example 3.7. Let $X = \{1, 2, 3\}$ and

$$\mathcal{H} = \Big\{\big\{\{1\}, \{2\}, \{1, 2, 3\}\big\}, \big\{\{1\}, \{3\}, \{1, 2, 3\}\big\}, \big\{\{2\}, \{3\}, \{1, 2, 3\}\big\}\Big\}.$$

Then $\mathcal{H} \in \mathbb{L}$. In fact, for any $(m, f) \in \mathcal{M}_a \times \mathcal{F}^+$, if $f(1) = \min_{1\le i\le 3}\{f(i)\}$ then

$$f(1)\chi_{\{1,2,3\}} + (f(2) - f(1))\chi_{\{2\}} + (f(3) - f(1))\chi_{\{3\}} = f$$

and

$$f(1) \cdot m(\{1, 2, 3\}) + (f(2) - f(1)) \cdot m(\{2\}) + (f(3) - f(1)) \cdot m(\{3\}) = \sum_{i=1}^{3} f(i) \cdot m(\{i\}) = \mathbf{Leb}(m, f),$$

i.e., $I_{\mathcal{H}}(m, f) = \mathbf{Leb}(m, f)$. Other cases can be treated similarly. Therefore, $\mathcal{H} \in \mathbb{L}$. However, there is no $\wp \in \hat{\mathcal{P}}$ such that $\mathcal{H} \supseteq \mathcal{H}^c_{(\wp)}$.

Let $\mathcal{H} \in \mathbb{L}$. It is said to be a *minimal Lebesgue decomposition system* (in $\mathbb{L}$), if there is no Lebesgue decomposition system $\mathcal{H}'$ such that $\mathcal{H}' \subsetneq \mathcal{H}$, i.e., $\mathcal{H} \in \mathbb{L}$ and for any $\mathcal{D} \in \mathcal{H}$, $\mathcal{H} \setminus \{\mathcal{D}\} \notin \mathbb{L}$.

From the above discussions on $\mathcal{H}^c_{(\wp_n)}$, we can see that

$$\mathcal{H}^c_{(\wp_n)} = \Big\{\big\{\{1\}, \{2\}, \ldots, \{n\}\big\}\Big\}$$

is a minimal Lebesgue decomposition system. In general the following result holds.

Proposition 3.8. *For every $\wp \in \hat{\mathcal{P}}$, $\mathcal{H}^c_{(\wp)}$ is a minimal Lebesgue decomposition system in $\mathbb{L}$.*

Proof. Let $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$. It suffices to show that if we take an arbitrary element $\cup_{i=1}^k \mathcal{C}_i$ from $\mathcal{H}_{(\wp)}$, where for each $1 \le i \le k$, $\mathcal{C}_i$ is a maximal chain in $2^{A_i} \setminus \{\emptyset\}$, then there will be the case that

$$I_{\mathcal{H}^c_{(\wp)} \setminus \{\cup_{i=1}^k \mathcal{C}_i\}}(m, f) \ne \mathbf{Leb}_m(f)$$

for some $(m, f) \in \mathcal{M}_a \times \mathcal{F}^+$.

For simplicity, suppose that $A_1 = \{1, 2, \ldots, l\}$ and

$$\mathcal{C}_1 = \Big\{\{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, l\}\Big\}.$$

$\mathcal{C}_1$ is a maximal chain in $2^{A_1} \setminus \{\emptyset\}$.

Define $f \in \mathcal{F}^+$ as

$$f(i) = \begin{cases} l - i + 1 & \text{if } 1 \le i \le l, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $m \in \mathcal{M}_a$ with $m(\{i\}) > 0$ for all $1 \le i \le l$, we have $I_{\mathcal{H}^c_{(\wp)} \setminus \{\cup_{i=1}^k \mathcal{C}_i\}}(m, f) < \mathbf{Leb}_m(f)$. In fact, for any maximal chain

$$\mathcal{C}_1' = \{E_1', E_2', \ldots, E_l'\}$$

in $2^{A_1} \setminus \{\emptyset\}$ which differs from $\mathcal{C}_1$ and any nonnegative number $\lambda_1, \lambda_2, \ldots, \lambda_l$ such that $\sum_{i=1}^l \lambda_i \chi_{E_i'} \le f$ it must be that $\sum_{i=1}^l \lambda_i \chi_{E_i'} < f$. So, the finiteness of X implies that $I_{\mathcal{H}^c_{(\wp)} \setminus \{\cup_{i=1}^k \mathcal{C}_i\}}(m, f) < \mathbf{Leb}_m(f)$. □

In particular, by Propositions 3.5 and 3.6, for $\wp_n = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$ and $\wp = \big\{X\big\}$, the related systems $\mathcal{H}^c_{(\wp_n)}$ and $\mathcal{H}^c_{(\{X\})}$ are all minimal Lebesgue decomposition system in $\mathbb{L}$.

Note 3.9. The system $\mathcal{H}$ defined as in Example 3.7 is also a minimal Lebesgue decomposition system in $\mathbb{L}$. But, for any $\wp \in \hat{\mathcal{P}}$, $\mathcal{H} \neq \mathcal{H}^c_{(\wp)}$.

The following theorem provides a way to calculate decomposition integral $I_{\mathcal{H}^c_{(\wp)}}(\mu, \cdot)$.

Theorem 3.10. *For any monotone measure $\mu : 2^X \to [0, +\infty[$ and any $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$, we have*

$$I_{\mathcal{H}^c_{(\wp)}}(\mu, f) = \sum_{i=1}^{k} \mathbf{Ch}_{\mu|_{A_i}}\big(f\,|_{A_i}\big) \tag{3.3}$$

for all $f \in \mathcal{F}^+$, where $\mu\,|_{A_i}$ is the restriction of μ to A_i and $f\,|_{A_i}$ is the restriction of f to A_i.

Proof. For any $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$,

$$\mathcal{H}^c_{(\wp)} = \left\{ \bigcup_{i=1}^{k} \mathcal{C}_i : \mathcal{C}_i \text{ is a maximal chain in } 2^{A_i} \setminus \{\emptyset\} \right\}.$$

Now let f be given and $\sum_{j\in J} a_j \chi_{E_j}$ be an arbitrary sub-decomposition of f, i.e., $\sum_{j\in J} a_j \chi_{E_j} \le f$ and $\{E_j\}_{j\in J} \in \mathcal{H}^c_{(\wp)}$. Then $\sum_{i=1}^{k} \big(\sum_{E_j \subseteq A_i} a_j \chi_{E_j}\big) = \sum_{j\in J} a_j \chi_{E_j} \le f$. On the other hand,

$$\sum_{j\in J} a_j \mu(E_j) = \sum_{i=1}^{k} \Big(\sum_{E_j \subseteq A_i} a_j \mu(E_j) \Big)$$

and for every $i = 1, 2, \ldots, k$,

$$\mathbf{Ch}_{\mu|_{A_i}}\big(f\,|_{A_i}\big) = \sup \left\{ \sum_{E_j \subseteq A_i} a_j \mu(E_j) : \{E_j : E_j \subseteq A_i\} \text{ is a maximal chain in } A_i,\ \sum_{E_j \subset A_i} a_j \chi_{E_j} \le f \cdot \chi_{A_i} \right\}.$$

Noting that $\sum_{j\in J} a_j \chi_{E_j} \le f$ is equivalent to $\sum_{E_j \subset A_i} a_j \chi_{E_j} \le f \cdot \chi_{A_i}$ for every $i = 1, 2, \ldots, k$. Therefore

$$\begin{aligned} I_{\mathcal{H}^c_{(\wp)}}(\mu, f) &= \sup \left\{ \sum_{J\in J} a_j \mu(E_j) : \{E_j\}_{j\in J} \in \mathcal{H}^c_{(\wp)},\ \sum_{j\in J} a_j \chi_{E_j} \le f \right\} \\ &= \sum_{i=1}^{k} \left[\sup \left\{ \sum_{E_j \subseteq A_i} a_j \mu(E_j) : \{E_j : E_j \subseteq A_i\} \text{ is a maximal chain in } A_i,\ \sum_{E_j \subset A_i} a_j \chi_{E_j} \le f \cdot \chi_{A_i} \right\} \right] \\ &= \sum_{i=1}^{k} \mathbf{Ch}_{\mu|_{A_i}}\big(f\,|_{A_i}\big). \end{aligned}$$

The proof is completed. □

A general version of Theorem 3.10 has been shown in [11]. For the sake of self-containedness, the above proof of Theorem 3.10 is not omitted.

Observe that for any given $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$, the decomposition system $\mathcal{H}^c_{(\wp)}$ is related to maximal chains in $2^{A_i} \setminus \{\emptyset\}, i = 1, 2, \ldots, k$. Similar to the decomposition system $\mathcal{H}^c_{(\wp)}$, consider a family of decomposition systems which are related to partitions in $A_i, i = 1, 2, \ldots, k$.

Let $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$ be given. We define

$$\mathcal{H}^p_{(\wp)} = \left\{ \bigcup_{i=1}^{k} \mathcal{D}_i : \mathcal{D}_i \text{ is a partition of } A_i, i = 1, 2, \ldots, k \right\}. \tag{3.4}$$

We present some properties of the decomposition systems $\mathcal{H}^p_{(\wp)}$ ($\wp \in \hat{\mathcal{P}}$) and the integrals $I_{\mathcal{H}^p_{(\wp)}}(\mu, \cdot)$.

- For $\wp_n = \Big\{\{1\}, \{2\}, \ldots, \{n\}\Big\}$, then $\mathcal{H}^p_{(\wp_n)} = \Big\{\big\{\{1\}, \{2\}, \ldots, \{n\}\big\}\Big\}$, and we have

$$I_{\mathcal{H}^p_{(\wp_n)}}(\mu, f) = I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f) = \sum_{i=1}^{n} f(i) \cdot \mu\big(\{i\}\big), \tag{3.5}$$

  and $|\mathcal{H}^p_{(\wp_n)}| = 1$. Obviously, if μ is an additive measure on 2^X, then $I_{\mathcal{H}^p_{(\wp_n)}}(\mu, \cdot)$ is a Lebesgue integral on X, and hence $\mathcal{H}^p_{(\wp_n)} \in \mathbb{L}$.
- Consider $\wp = \big\{X\big\}$, then $\mathcal{H}^p_{\{X\}} = \mathcal{H}_{pan}$. Thus,

$$I_{\mathcal{H}^p_{\{X\}}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f) = \mathbf{Pan}_\mu(f),$$

  and hence $\mathcal{H}^p_{(\{X\})} \in \mathbb{L}$.

For any $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$, evidently

$$\mathcal{H}^p_{(\wp_n)} \subseteq \mathcal{H}^p_{(\wp)} \subseteq \mathcal{H}^p_{(\{X\})} = \mathcal{H}_{pan},$$

i.e., $\mathcal{H}^p_{(\wp_n)}$ and $\mathcal{H}^p_{(\{X\})}$ are respectively minimal and maximal elements in family $\{\mathcal{H}^p_{(\wp)}\}_{\wp\in\hat{\mathcal{P}}}$ in the sense of standard set inclusion. However, for the family $\{\mathcal{H}^c_{(\wp)}\}_{\wp\in\hat{\mathcal{P}}}$, there is no similar property. In fact, $\mathcal{H}^c_{(\wp_n)} \nsubseteq \mathcal{H}^p_{(\{X\})}$ (recall that $I_{\mathcal{H}^c_{\{X\}}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f) = \mathbf{Ch}_\mu(f)$).

Similar to Proposition 3.5 and 3.6, we have the following results.

Proposition 3.11. *(i) For any $\wp \in \hat{\mathcal{P}}$, $\mathcal{H}^p_{(\wp)}$ is a Lebesgue decomposition system, i.e., $\mathcal{H}^p_{(\wp)} \in \mathbb{L}$.*
(ii) If for some $\wp \in \hat{\mathcal{P}}$ such that $\mathcal{H} \supseteq \mathcal{H}^p_{(\wp)}$, then $\mathcal{H} \in \mathbb{L}$.

The following is a special case of Theorem 2 in [11].

Theorem 3.12. *For any monotone measure $\mu : 2^X \to [0, +\infty[$ and any $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$, we have*

$$I_{\mathcal{H}^p_{(\wp)}}(\mu, f) = \sum_{i=1}^{k} \mathbf{Pan}_{\mu|_{A_i}}\big(f|_{A_i}\big) \tag{3.6}$$

for all $f \in \mathcal{F}^+$.

4. A new type of decomposition integral

4.1. PC-decomposition integrals

In the following we recall a decomposition system $\mathcal{H}_{(pc)}$ which was introduced by Stupňanová [23], defined as

$$\mathcal{H}_{(pc)} \triangleq \Big\{\mathcal{D} \in \mathbb{X} : \forall\, E_s, E_t \in \mathcal{D}, \text{such that } E_s \cap E_t \in \big\{\emptyset, E_s, E_t\big\}\Big\}.$$

The system $\mathcal{H}_{(pc)}$ is called a *PC-collection* and the decomposition integral $I_{\mathcal{H}_{(pc)}}(\mu, f)$ with respect to $\mathcal{H}_{(pc)}$ is called *PC-integral*.

The following relations are obviously true for any $\wp \in \hat{\mathcal{P}}$:

$$\mathcal{H}^p_{(\wp)} \subseteq \mathcal{H}_{(pc)} \subseteq \mathcal{H}_{cav} \quad\text{and}\quad \mathcal{H}^c_{(\wp)} \subseteq \mathcal{H}_{(pc)} \subseteq \mathcal{H}_{cav}.$$

In particular,

$$\mathcal{H}^p_{(\{X\})} = \mathcal{H}_{pan} \subseteq \mathcal{H}_{(pc)} \quad\text{and}\quad \mathcal{H}^c_{(\{X\})} \subseteq \mathcal{H}_{Ch} \subseteq \mathcal{H}_{(pc)},$$

and from Proposition 3.6 (or Proposition 3.11), it is evident that $\mathcal{H}_{(pc)} \in \mathbb{L}$. Therefore, the PC-integral extends the Lebesgue integral.

From the above inclusion relations, for any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$, it holds that

$$\mathbf{Pan}_\mu(f) \vee \mathbf{Ch}_\mu(f) \le I_{\mathcal{H}_{(pc)}}(\mu, f) \le \mathbf{Cav}_\mu(f).$$

In Example 1 in [23] it was shown that there is some $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$ such that

$$\mathbf{Pan}_\mu(f) \vee \mathbf{Ch}_\mu(f) < I_{\mathcal{H}_{(pc)}}(\mu, f) < \mathbf{Cav}_\mu(f).$$

For the lattice structure of these integrals, see Fig. 2.

4.2. The new integral

The following problem has attracted our interest:

Can we introduce an integral such that it is greater than both the Choquet integral and the pan integral, but smaller than PC-integral?

To answer this question, let us begin by constructing a decomposition system $\mathcal{H}^c_{\hat{\mathcal{P}}}$ via the family $\{\mathcal{H}^c_{(\wp)}\}_{\wp\in\hat{\mathcal{P}}}$ of decomposition systems.

Definition 4.1. Let X be a finite set and $\hat{\mathcal{P}}$ be the class of all partitions of X. The decomposition system $\mathcal{H}^c_{\hat{\mathcal{P}}}$ is defined as the union of the family $\{\mathcal{H}^c_{(\wp)}\}_{\wp\in\hat{\mathcal{P}}}$, i.e.,

$$\begin{aligned}\mathcal{H}^c_{\hat{\mathcal{P}}} &\triangleq \bigcup_{\wp\in\hat{\mathcal{P}}}\mathcal{H}^c_{(\wp)}\\ &= \Big\{\mathcal{C}\in\mathbb{X} : \{A_i\}_{i=1}^n\in\hat{\mathcal{P}},\ \{C\cap A_i \mid C\in\mathcal{C}, C\cap A_i\neq\emptyset\} \text{ is a maximal chain in } A_i, i=1,2,\dots,k\Big\}. \qquad (4.1)\end{aligned}$$

The following is a concrete example of $\mathcal{H}^c_{\hat{\mathcal{P}}}$

Example 4.2. Let $X=\{1,2,3\}$. Then

$$\begin{aligned}\mathcal{H}^c_{\hat{\mathcal{P}}} = \Big\{&\{\{1\},\{2\},\{3\}\},\{\{1\},\{1,2\},\{3\}\},\{\{2\},\{1,2\},\{3\}\},\{\{1\},\{1,3\},\{2\}\},\\ &\{\{3\},\{1,3\},\{2\}\},\{\{2\},\{2,3\},\{1\}\},\{\{3\},\{2,3\},\{1\}\},\\ &\{\{1\},\{1,2\},\{1,2,3\}\},\{\{1\},\{1,3\},\{1,2,3\}\},\{\{2\},\{1,2\},\{1,2,3\}\},\\ &\{\{2\},\{2,3\},\{1,2,3\}\},\{\{3\},\{1,3\},\{1,2,3\}\},\{\{3\},\{2,3\},\{1,2,3\}\}\Big\}.\end{aligned}$$

Definition 4.3. The decomposition system $\mathcal{H}^c_{\hat{\mathcal{P}}}$ induces a new type of decomposition integral $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\cdot,\cdot)$, which is given by

$$I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu,f)=\sup\left\{\sum_{i\in J}a_i\mu(A_i) : \{A_i\}_{i\in J}\in\mathcal{H}^c_{\hat{\mathcal{P}}}, \sum_{i\in J}a_i\chi_{A_i}\le f\right\},$$

where all constants $a_i\ge 0$.

Since $\mathcal{H}_{pan}\preceq\mathcal{H}^c_{\hat{\mathcal{P}}}$ and $\mathcal{H}^c_{(\{X\})}\subset\mathcal{H}^c_{\hat{\mathcal{P}}}\subset\mathcal{H}_{(pc)}$, the decomposition integral $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\cdot,\cdot)$ has the following desired property

$$\mathbf{Pan}_\mu(f)\vee\mathbf{Ch}_\mu(f)\le I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu,f)\le I_{\mathcal{H}_{(pc)}}(\mu,f).$$

Since for any $\wp\in\hat{\mathcal{P}}$, $\mathcal{H}^c_{(\wp)}\subseteq\mathcal{H}^c_{\hat{\mathcal{P}}}$, and $\mathcal{H}^c_{(\wp)}\in\mathbb{L}$, it follows from Proposition 3.6 that $\mathcal{H}^c_{\hat{\mathcal{P}}}\in\mathbb{L}$, i.e., $\mathcal{H}^c_{\hat{\mathcal{P}}}$ is a Lebesgue decomposition system, that is, the decomposition integral $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\cdot,\cdot)$ extends the Lebesgue integral.

The following theorem provides a way to calculate $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\cdot,\cdot)$.

Theorem 4.4. *For any $(\mu,f)\in\mathcal{M}\times\mathcal{F}^+$, we have*

$$I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu,f)=\bigvee_{\wp\in\hat{\mathcal{P}}} I_{\mathcal{H}^c_{(\wp)}}(\mu,f) \qquad (4.2)$$

$$=\max\left\{\sum_{i=1}^k\mathbf{Ch}_{\mu|_{A_i}}\left(f\,|_{A_i}\right) : \wp=\{A_i\}_{i=1}^k\in\hat{\mathcal{P}}\right\}. \qquad (4.3)$$

Proof. Bearing in mind that for any $\wp\in\hat{\mathcal{P}}$, $\mathcal{H}^c_{(\wp)}\subseteq\mathcal{H}^c_{\hat{\mathcal{P}}}$. As a consequence, for any $(\mu,f)\in\mathcal{M}\times\mathcal{F}^+$, it holds

$$I_{\mathcal{H}^c_{(\wp)}}(\mu,f)\le I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu,f),$$

and hence $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu,f)\ge\bigvee_{\wp\in\hat{\mathcal{P}}} I_{\mathcal{H}^c_{(\wp)}}(\mu,f)$.

On the other hand, let $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$ be given. For any $\mathcal{H}^c_{\hat{\mathcal{P}}}$-sub-decomposition of f, $\sum_{j=1}^s a_j\chi_{E_j}$ (i.e., $\sum_{j=1}^s a_j\chi_{E_j} \le f, a_j \ge 0, \{E_j\}_{j=1}^s \in \mathcal{H}^c_{\hat{\mathcal{P}}}$), there is some $\wp_0 \in \hat{\mathcal{P}}$ such that $\{E_j\}_{j=1}^s \in \mathcal{H}^c_{(\wp_0)}$, thus $\sum_{j=1}^s a_j\chi_{E_j}$ is also a $\mathcal{H}^c_{(\wp_0)}$-sub-decomposition of f. So

$$\left\{\sum_{j=1}^s a_j\mu(E_j) : \sum_{j=1}^s a_j\chi_{E_j} \le f, \{E_j\}_{j=1}^s \in \mathcal{H}^c_{\hat{\mathcal{P}}}, a_j \ge 0\right\}$$
$$\subseteq \bigcup_{\wp\in\hat{\mathcal{P}}} \left\{\sum_{k=1}^t b_k\mu(F_k) : \sum_{k=1}^t b_k\chi_{F_k} \le f, (F_k)_{k=1}^t \in \mathcal{H}^c_{(\wp)}, b_k \ge 0\right\}.$$

Therefore,

$$\begin{aligned} I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) &= \max\left\{\sum_{j=1}^s a_j\mu(E_j) : \sum_{j=1}^s a_j\chi_{E_j} \le f, \{E_j\}_{j=1}^s \in \mathcal{H}^c_{\hat{\mathcal{P}}}, a_j \ge 0\right\} \\ &\le \max_{\wp\in\hat{\mathcal{P}}} I_{\mathcal{H}^c_{(\wp)}}(\mu, f) \\ &= \max\left\{\sum_{i=1}^k \mathbf{Ch}_{\mu|_{A_i}}\left(f\,|_{A_i}\right) : \wp = \{A_i\}_{i=1}^k \in \hat{\mathcal{P}}\right\}. \end{aligned}$$

The proof is completed. □

For the fixed partition $\wp_n = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$, we have

$$I_{\mathcal{H}^p_{(\wp_n)}}(\mu, f) = I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f) = \sum_{i=1}^n f(i)\cdot\mu\big(\{i\}\big)$$

(see Eq. (3.5)), and hence

$$I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) \ge I_{\mathcal{H}^c_{(\wp_n)}}(\mu, f) = \sum_{i=1}^n f(i)\cdot\mu\big(\{i\}\big).$$

Being similar to Eq. (4.1), one may want to consider the following decomposition system

$$\mathcal{H}^p_{\hat{\mathcal{P}}} \triangleq \bigcup_{\wp\in\hat{\mathcal{P}}} \mathcal{H}^p_{(\wp)}.$$

However, this is not a new decomposition system. In fact, for any $\wp \in \hat{\mathcal{P}}$, $\mathcal{H}^p_{(\wp)} \subseteq \mathcal{H}^p_{\{X\}}$, then

$$\mathcal{H}^p_{\hat{\mathcal{P}}} = \mathcal{H}^p_{(\{X\})} = \mathcal{H}_{pan} \quad \text{and} \quad I_{\mathcal{H}^p_{\hat{\mathcal{P}}}}(\mu, f) = \mathbf{Pan}_\mu(f).$$

Thus, for any $\wp \in \hat{\mathcal{P}}$, it holds

$$I_{\mathcal{H}^p_{(\wp)}}(\mu, f) \le \mathbf{Pan}_\mu(f) \le I_{\mathcal{H}_{(pc)}}(\mu, f) \le \mathbf{Cav}_\mu(f)$$

(see Fig. 2).

The following result indicates that the pan-integral $\mathbf{Pan}_\mu(\cdot)$ is a lower bound of $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, \cdot)$, see Fig. 2.

Proposition 4.5. *Let X be finite space and $\mu \in \mathcal{M}$ be fixed. Then for all $f \in \mathcal{F}^+$, we have*

$$\mathbf{Pan}_\mu(f) \le I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f).$$

Proof. Given $f \in \mathcal{F}^+$. Then there is a partition $\wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}}$ and $a_j \ge 0, 1 \le j \le k$ with $\sum_{j=1}^k a_j\chi_{A_j} \le f$ such that

$$\mathbf{Pan}_\mu(f) = \sum_{j=1}^k a_j\mu(A_j).$$

By the monotonicity of Choquet integral, we have

$$\sum_{j=1}^{k} a_j \mu(A_j) = \sum_{j=1}^{k} \mathbf{Ch}_{\mu|_{A_j}}(a_j) \le \sum_{j=1}^{k} \mathbf{Ch}_{\mu|_{A_j}}(f|_{A_j})$$
$$\le \max\left\{ \sum_{j=1}^{k} \mathbf{Ch}_{\mu|_{A_j}}(f|_{A_j}) : \wp = \{A_1, A_2, \ldots, A_k\} \in \hat{\mathcal{P}} \right\}$$
$$= I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f).$$

The proof is completed. □

The following example shows that there is some $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$ such that

$$\mathbf{Pan}_\mu(f) < I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) < I_{\mathcal{H}_{(pc)}}(\mu, f)$$

and

$$\mathbf{Ch}_\mu(f) < I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) < I_{\mathcal{H}_{(pc)}}(\mu, f).$$

Example 4.6. Let $X = \{1, 2, 3\}$, $\mathcal{A} = 2^X$. Define $f(1) = 2$, $f(2) = 3$, $f(3) = 4$.

(i) Define a monotone measure μ as follows:

$$\mu(A) = \begin{cases} 3 & \text{if } |A| \ge 2, \\ 1 & \text{if } |A| = 1, \\ 0 & A = \emptyset. \end{cases}$$

Then

$$I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) = 2\mu(\{1\}) + 3\mu(\{2,3\}) + \mu(\{3\}) = 12,$$
$$\mathbf{Ch}_\mu(f) = 2\mu(\{1,2,3\}) + \mu(\{2,3\}) + \mu(\{3\}) = 10,$$
$$\mathbf{Pan}_\mu(f) = 2\mu(\{1\}) + 3\mu(\{2,3\}) = 11.$$

This shows that $\mathbf{Ch}_\mu(f) < I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f)$ and $\mathbf{Pan}_\nu(f) < I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\nu, f)$.

(ii) Define a monotone measure ν as follows:

$$\nu(A) = \begin{cases} 4 & \text{if } A = X, \\ 0 & \text{if } A = \emptyset, \\ 1 & \text{else}. \end{cases}$$

Then

$$I_{\mathcal{H}_{(pc)}}(\nu, f) = 2\nu(\{1,2,3\}) + \nu(\{2\}) + 2\nu(\{3\}) = 11,$$
$$I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\nu, f) = 2\nu(\{1,2,3\}) + \nu(\{2,3\}) + \nu(\{3\}) = 10,$$
$$\mathbf{Pan}_\nu(f) = I_{\mathcal{H}^p_{\hat{\mathcal{P}}}}(\nu, f) = 2\nu(\{1\}) + 3\nu(\{2\}) + 4\nu(\{3\}) = 9.$$

Therefore,

$$\mathbf{Pan}_\nu(f) < I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) < I_{\mathcal{H}_{(pc)}}(\mu, f).$$

We illustrate the relationships among the above several types of decomposition integrals by the following Hasse diagram (they form a join semilattice, compare with Fig. 1).

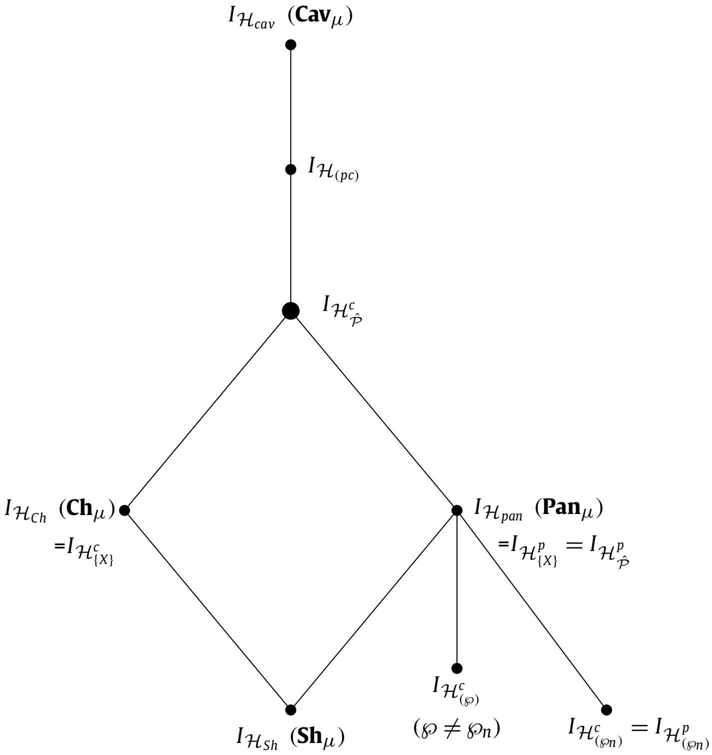

Fig. 2. Hasse diagram of decomposition integrals.

5. The coincidences among several kinds of integrals

In this section, we discuss the coincidences among several types of decomposition integrals discussed in the above sections.

Lehrer and Teper [10] showed that the Choquet integral coincides with the concave integral if and only if the monotone measure μ is supermodular. In [19] we showed that subadditivity of monotone measures is a sufficient condition that the concave integrals coincide with the pan-integral. In [20] it is showed that the (M)-property of monotone measures (which was proposed by Mesiar [13], as follows: for any $P, Q \in \mathcal{A}$ with $P \subseteq Q$, there exists $R \in \mathcal{A}$ such that $R \subseteq P$, $\mu(R) = \mu(P)$ and $\mu(Q) = \mu(R) + \mu(Q \setminus R)$) is sufficient to the equivalence of the pan-integral and the Choquet integral.

From Proposition 2 in [10], Theorem 4.6 in [18] (see also Theorem 4.1 in [20]), Theorem 4.1 in [17], and Propositions 3.10 and 3.12, the following results are immediate.

Proposition 5.1. *Let X be finite space and $\mu \in \mathcal{M}$ be fixed. Then*

(i) μ is supermodular, i.e., for any $P, Q \in \mathcal{A}$, $\mu(P \cup Q) + \mu(P \cap Q) \geq \mu(P) + \mu(Q)$, if and only if for all $f \in \mathcal{F}^+$,

$$\mathbf{Ch}_\mu(f) = I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) = I_{\mathcal{H}_{(pc)}}(\mu, f) = \mathbf{Cav}_\mu(f);$$

(ii) μ has (M)-property if and only if for all $f \in \mathcal{F}^+$,

$$\mathbf{Ch}_\mu(f) = I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) = \mathbf{Pan}_\mu(f);$$

(iii) for all $f \in \mathcal{F}^+$,

$$\mathbf{Pan}_\mu(f) = I_{\mathcal{H}^p_{\hat{\mathcal{P}}}}(\mu, f) = I_{\mathcal{H}_{(pc)}}(\mu, f) = \mathbf{Cav}_\mu(f)$$

if and only if the conditions (1) and (2) in Theorem 4.1 in [17] are satisfied.

Note: In [17] we introduced the concept of minimal atom of a monotone measure and used its characteristics to present a necessary and sufficient condition that the concave integral coincides with the pan-integral (see Theorem 4.1 in [17]).

From Corollary 4.7 in [17] (see also Theorem 9 in [19], Theorem 2 in [23]), if μ is subadditive, i.e., for any $A, B \in \mathcal{A}$, $\mu(A \cup B) \le \mu(A) + \mu(B)$, then for all $f \in \mathcal{F}^+$, it holds

$$\mathbf{Pan}_\mu(f) = I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) = I_{\mathcal{H}_{(pc)}}(\mu, f) = \mathbf{Cav}_\mu(f).$$

Given a monotone measure space $(X, \mathcal{A}, \mu)$. A set $A \in \mathcal{A}$ is called a *minimal atom* of monotone measure μ if $\mu(A) > 0$ and for every $B \in \mathcal{A}$ and $B \subseteq A$ holds either (1) $\mu(B) = 0$, or (2) $A = B$ (see [17]). When X is a finite space, for given $\mu \in \mathcal{M}$, X can be expressed as

$$X = E_0 \cup E_1 \cup E_2 \cup \cdots \cup E_k,$$

where $\{E_i\}_{i=1}^k$ is a family of pairwise disjoint minimal atoms of μ contained in X and $\mu(E_0) = 0$. Then $\wp_a = \{E_0, E_1, E_2, \ldots, E_k\} \in \hat{\mathcal{P}}$. Combining Proposition 4.6 in [12], Propositions 3.10 and 3.12, for all $f \in \mathcal{F}^+$, the following result (which is a general version of Eq. (3.5)) holds:

$$I_{\mathcal{H}^c_{(\wp_a)}}(\mu, f) = I_{\mathcal{H}^p_{(\wp_a)}}(\mu, f) = \sum_{i=1}^{k} \inf_{x \in E_i} f(x) \cdot \mu(E_i).$$

This implies $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) \ge \sum_{i=1}^{k} \inf_{x \in E_i} f(x) \cdot \mu(E_i)$.

From Theorem 5.3 in [18] and Proposition 5.1, we have the following result: for all $f \in \mathcal{F}^+$,

$$\mathbf{Pan}_\mu(f) = \mathbf{Ch}_\mu(f) = I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, f) = I_{\mathcal{H}_{(pc)}}(\mu, f) = \mathbf{Cav}_\mu(f)$$

if and only if the following two conditions hold:

(1) μ has (M)-property;
(2) μ possesses the *minimal atoms disjointness property*, i.e., for every pair of minimal atoms T_1 and T_2 of μ, $T_1 \ne T_2$ implies $T_1 \cap T_2 = \emptyset$.

6. Conclusion

We have constructed a new type of decomposition integral, $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, \cdot)$, by using a family of decomposition integrals $\{I_{\mathcal{H}^c_{(\wp)}}(\mu, \cdot)\}_{\wp \in \hat{\mathcal{P}}}$ (Theorem 4.4). As we have seen, this integral is based on the decomposition systems $\mathcal{H}^c_{\hat{\mathcal{P}}}$ related to partitions and maximal chains of sets, and it extends the Lebesgue integral. Each of the family of integrals also extends the Lebesgue integral, not only that, every $\mathcal{H}^c_{(\wp)} (\wp \in \hat{\mathcal{P}})$ is a minimal Lebesgue decomposition system in $(\mathbb{L}, \subseteq)$ (Proposition 3.8). In the structure of a lattice on the class of decomposition integrals, the introduced integral $I_{\mathcal{H}^c_{\hat{\mathcal{P}}}}(\mu, \cdot)$ is lower bound of the PC-integral and is between the concave integral and the Choquet integral, and also between the concave integral and the pan-integral (see Fig. 2).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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