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A methodology for controlling the information quality in interval-valued fusion processes: Theory and application

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ABSTRACT

An important problem faced when dealing with imperfect information in fusion processes the uncertainty regarding values of the membership degrees to be employed in fuzzy modeling. In this scenario, one can apply interval-valued (iv) fuzzy sets, in which the membership degrees are represented by intervals. A recurrent issue is the situation in which the quality of information carried by the intervals, expressed by their widths, suffers degradation during the fusion process. So, the main objective of this paper is to develop a general framework to construct iv-fusion functions whose outputs conserve the information quality of the operated intervals. To achieve that, first, extend important concepts such as width-limiting functions and width-limited iv-functions to the n-dimensional context. Then, we present a characterization for any subclass of increasing fusion function by their set of properties, followed by the interval extension of such characterization to obtain classes of width-limited iv-fusion functions. We show that our methodology is general enough to retrieve several classes of iv-aggregation functions from the literature. Two approaches for constructing width-limited iv-fusion functions are also presented, which enables the application of different subclasses of width-limited iv-fusion functions in fusion/aggregation processes with imperfect information. Finally, we present a case study on a classification problem. Specifically, we use IVTURS, a state-of-the-art iv-fuzzy rule-based classification system, and a particular subclass of width-limited iv-fusion functions (n-dimensional width-limited iv-overlap functions), showing that the control of the information quality through width limitation significantly enhances the accuracy of the classifier.

1. Introduction

Fusion functions are useful operators that combine several numerical values into a single representative one [1]. The most important class of fusion functions is that of aggregation functions [2] (or, more generally, pre-aggregation functions [3]), which are especially suitable to model fuzzy logic operations. For example, t-norms [4] and overlap functions [5] can be applied as fuzzy conjunction operators, while t-conorms [4] and grouping functions [6] can be applied as fuzzy disjunction operators. For that reason, aggregation functions have been widely used in several theoretical and applied fields [2,7]. In particular, we have worked with n-dimensional overlap functions [8], which constitute a subclass of aggregation functions that do not require associativity and have been successfully applied in the reasoning method of fuzzy rule-based classification systems (FRBCSs) [9,10].

When facing problems with imperfect information [11,12], there may be uncertainty regarding the values of the membership degrees or even in the definition of the membership functions to be used in a fuzzy modeling [13,14]. Uncertainty can be modeled by different means for different purposes, such as through extended soft sets [15], fuzzy rough sets [16], Dempster–Shafer evidence theory and its generalizations [17–20], to name a few.

In particular, a viable and popular solution to model uncertainty is the adoption of interval-valued fuzzy sets (IVFSs) [21,22],

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in which the membership degrees are represented by intervals. In this context, the width of the assigned intervals are intrinsically related with the uncertainty/ignorance with respect to the modeling of the fuzzy sets [23,24]. IVFSs have been successfully applied in many different fields, such as image processing [25], game theory [26], multicriteria decision making [27], pest control [28], irrigation systems [29], collaborative clustering [30] and classification [31]. We point out that the modeling of linguistic labels via IVFSs in FRBCSs gave birth to interval-valued rule-based classification systems (IV-FRBCSs) [32–34], in which the aggregation process is a key component for the success of the classifier. Both FRBCSs and IV-FRBCSs have the advantageous aspect of being based on a set of linguistic rules, which makes them highly interpretable classification systems, while still achieving accurate results [32,35].

To accomplish aggregation processes with interval data, different aggregation functions had to be extended to the interval context [36]. Since then, several classes of iv-aggregation were introduced, such as interval-valued t-norms and t-conorms [24], iv-overlap and grouping functions [37,38] and general iv-overlap and grouping functions [39,40]. In most cases, a particular class of iv-aggregation function is defined by extending the definition of a given class of aggregation function based on the concept of best interval representation [23]. That is, the interval output of the iv-aggregation function is defined by the application of the original aggregation function to the endpoints of the input intervals. Besides being intuitive and theoretically sound, extending aggregation functions to the interval context through the best interval representation has other benefits: the computation is generally easy, as one only deals with the endpoints of the input intervals, and correctness is guaranteed, since the output interval contains the exact unknown aggregated value [41].

Nevertheless, there are some drawbacks when applying iv-aggregation functions defined in this manner in practical problems. First, monotonicity is usually evaluated with respect to the product order [42], which also considers only the endpoints of the intervals when comparing them. However, the product order is not a total order, meaning that one may have intervals that are not comparable, a hindrance that has to be avoided in problems such as decision making and classification [43]. To tackle this drawback, Bustince et al. [43] introduced the concept of admissible orders, that is, total orders that refine the product order, and that can be constructed by a pair of aggregation functions. Since then, many works using admissible orders have appeared in the literature [44–46]. In the context of aggregation of interval data, Bustince et al. [47] presented a construction method for iv-aggregation functions that are increasing with respect to a given admissible order. In the same context, Asmus et al. [33] introduced the concept of n-dimensional admissibly ordered iv-overlap functions, which are n-dimensional iv-overlap functions that are increasing with respect to an admissible order, showing good results when applied in IV-FRBCSs.

Another drawback in a practical sense is that, due to some applications constraints concerning the quality of the information [48,49] required for the interval result, the interval output of iv-aggregation functions based on the best interval representation may be larger than a desirable threshold. In this case, the interval result is guaranteed to be correct, however, it may carry no meaningful information about the real value it is approximating.

In an initial study to address this problem, Bustince et al. [47] introduced the concept of width-preserving functions, that is, iv-functions that, under some conditions, can provide outputs with the same width of all the inputs. However, the concept of width preservation only takes into account the very specific case where all the interval inputs have the same width. Then, more recently, Asmus et al. [50] introduced the concept of interval width limitation, where the width of the output of a bivariate iv-function is limited by a function applied to the widths of its inputs. Nevertheless, such theoretical approach for conserving the interval information quality in fusion processes was not considered in any applied problem, which means that there is a challenge yet to be addressed in a practical sense.

Motivated by the discussion above, this paper brings a novel and general methodology to deal with the problem of guaranteeing the information quality by controlling the width of interval outputs that are generated when applying the so called iv-fusion functions (in particular, iv-aggregation functions). Then, the main theoretical objective of this paper is to provide a general framework for n-dimensional width-limited interval-valued (w-iv) fusion functions, which enables the definition and construction methods for different subclasses of w-iv-aggregation functions, capable of retrieving known definitions of iv-aggregation functions from the literature and suitable to be applied on different practical problems in which the information quality has to be controlled. To accomplish this goal, we have the following specific objectives:

1. To extend the concepts of width-limited w-iv-fusion functions and width-limiting fusion functions to the n-dimensional context (Section 3);
2. To present a characterization of any class of increasing fusion function through a set of properties (Section 4);
3. To define classes of w-iv-fusion functions based on an increasing fusion function, the interval extension of its set of properties and a pair of partial orders (Section 4);
4. To present two general approaches to provide construction methods for w-iv-fusion functions, one based on representable interval functions and other on admissibly ordered interval functions, discussing examples (Section 5).

On the application side, we show the beneficial effects of this type of information quality control in classification problems (Section 6). Specifically, we apply the new framework in IVTURS, which is a state-of-the-art IV-FRBCS. For that, we develop a new interval-valued fuzzy reasoning method, in which the information quality is controlled by w-iv-fusion functions, in particular, n-dimensional w-iv-overlap functions. We analyze the effect of the interval width control on the performance of the classifier, since the construction methods allow one to determine the control level by means of a hyper-parameter. Finally, we conduct an experimental study where we compare the results of the original IVTURS classifier versus the best performing configurations of our new approach in order to clearly observe the obtained improvement, regardless of the chosen construction method to obtain w-iv-fusion functions.

Additionally, Section 2 presents some necessary preliminary concepts, and the main conclusions are drawn in Section 7, which completes the organization of the paper.

2. Preliminaries

In this section, we recall some basic concepts on aggregation functions, interval mathematics and iv-aggregation functions.

2.1. Aggregation functions

Denote $\vec{x} = (x_1, \ldots, x_n) \in [0,1]^n$. Any $F : [0,1]^n \to [0,1]$ is called a *fusion function* [1].

Definition 1 ([51]). A function $N : [0,1] \to [0,1]$ is a fuzzy negation if, for all $x, y \in [0,1]$: **(N1)** $N(0) = 1$ and $N(1) = 0$; **(N2)** If $x \leq y$ then $N(y) \leq N(x)$. If (involutive property) **(N3)** $N(N(x)) = x$, then N is a strong fuzzy negation.

Example 1. The Zadeh negation given, for all $x \in [0,1]$, by $N_Z(x) = 1 - x$, is a strong fuzzy negation.

Definition 2. Let H be the set of annihilator elements of a fusion function $F : [0,1]^n \to [0,1]$. F is said to be a strict fusion function is if it is strictly increasing on $([0,1] - H)^n$.

Definition 3 ([51]). Given a strong fuzzy negation $N : [0,1] \to [0,1]$ and a fusion function $F : [0,1]^n \to [0,1]$, then the fusion function $F^N : [0,1]^n \to [0,1]$ defined, for all $\vec{x} \in [0,1]^n$, by $F^N(\vec{x}) = N(F(N(x_1), \ldots, N(x_n)))$, it the N-dual of F.

When it is clear by the context, the N_Z-dual function (dual with respect to the Zadeh negation) of F will be just called dual of F, and will be denoted by F^d.

A particularly important class of fusion function is that of aggregation functions [2], defined as follows.

Definition 4 ([2]). An aggregation function is any fusion function $A : [0,1]^n \to [0,1]$ respecting: **(A1)** A is increasing; **(A2)** $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

An aggregation function A that is strictly increasing in $([0,1] - H)^n$, with H being the set of annihilator elements of F, is said to be a *strict aggregation function*.

Definition 5 ([52]). An aggregation function $A : [0,1]^n \to [0,1]$ is called ultramodular if, for all $\vec{x}, \vec{y}, \vec{\epsilon} \in [0,1]^n$, such that $\vec{y} + \vec{\epsilon} \in [0,1]^n$ and $\vec{x} \leq \vec{y}$: $A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A(\vec{y} + \vec{\epsilon}) - A(\vec{y})$.

Here we extend the concept of (a, b)-ultramodular binary function [50] for the n-dimensional context:

Definition 6. Consider $\vec{a} \in [0,1]^n$. An aggregation function $A : [0,1]^n \to [0,1]$ is called $\vec{a}$-ultramodular if, for all $\vec{x}, \vec{\epsilon} \in [0,1]^n$ and $\vec{x} + \vec{\epsilon}, \vec{a} - \vec{\epsilon} \in [0,1]^n$, it holds that:

$$A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A(\vec{a}) - A(\vec{a} - \vec{\epsilon}). \tag{1}$$

When $\vec{a} = (1, \ldots, 1)$, from Eq. (1) and condition **(A2)** from Definition 4, we have that $A(\vec{x} + \vec{\epsilon}) - A(\vec{x}) \leq A^d(\vec{\epsilon})$, where A^d is the dual of A. In this case, A is said to be $\vec{1}$-ultramodular.

The next result is an extension to the n-dimensional context of Prop. 3.1 from the work of Asmus et al. [50]:

Proposition 1. *Let $A : [0,1]^n \to [0,1]$ be an ultramodular aggregation function. Then, A is an $(\vec{1})$-ultramodular aggregation function, but the converse may not hold.*

There are many classes of aggregation functions defined in the literature. Here we highlight some of them that are going to be of importance on this work.

Definition 7 ([10]). A fusion function $On : [0,1]^n \to [0,1]$ is an n-dimensional overlap function if, for all $\vec{x} \in [0,1]^n$: **(On1)** On is symmetric; **(On2)** $On(\vec{x}) = 0 \Leftrightarrow \prod_{i=1}^{n} x_i = 0$; **(On3)** $On(\vec{x}) = 1 \Leftrightarrow \prod_{i=1}^{n} x_i = 1$; **(On4)** On is increasing; **(On5)** On is continuous.

A 2-dimensional overlap function is just called *overlap function* [5].

Definition 8 ([8]). A fusion function $Gn : [0,1]^n \to [0,1]$ is said to be an n-dimensional grouping function if, for all $\vec{x} \in [0,1]^n$: **(Gn1)** Gn is symmetric; **(Gn2)** $Gn(\vec{x}) = 0 \Leftrightarrow x_i = 0$ for all $i \in \{1, \ldots, n\}$; **(Gn3)** $Gn(\vec{x}) = 1 \Leftrightarrow$ there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$; **(Gn4)** Gn is increasing; **(Gn5)** Gn is continuous.

By duality, one can obtain n-dimensional grouping functions from n-dimensional overlap functions, and vice-versa.

Example 2. **(a)** The arithmetic mean $AM : [0,1]^n \to [0,1]$, defined, for all $\vec{x} \in [0,1]^n$, by $AM(\vec{x}) = \frac{\sum_{i=1}^{n} x_i}{n}$, is an aggregation function that is strict (with $H = \emptyset$) and $\vec{1}$-ultramodular;

(b) The geometric mean: $GM : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by

$$GM(\vec{x}) = \sqrt{\prod_{i=1}^{n} x_i}, \tag{2}$$

is a strict (with $H = \{0\}$) n-dimensional overlap function;

(c) $OnB : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by

$$OnB(\vec{x}) = \sqrt{(\prod_{i=1}^{n} x_i) \cdot (\min\{x_1, \ldots, x_n\})}, \tag{3}$$

is a strict (with $H = \{0\}$) n-dimensional overlap function;

(d) $OnT : [0,1]^n \to [0,1]$, given, for all $\vec{x} \in [0,1]^n$, by

$$OnT(\vec{x}) = \prod_{i=1}^{n} \frac{(2x_i - 1)^3 + 1}{2}, \tag{4}$$

is an n-dimensional overlap function that is also an $\vec{1}$-ultramodular aggregation function, but it is not an ultramodular aggregation function.

(e) The functions $GnB, GnT : [0,1]^n \to [0,1]$ such that $GnB = OnB^d$ and $GnT = OnT^d$, are n-dimensional grouping functions.

2.2. Interval mathematics and admissible orders

Denote by $L([0,1])$ the set of closed subintervals of the unit interval $[0,1]$ and $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$. For any $X = [x_1, x_2] \in L([0,1])$, the left and right projections of X are denoted, respectively, by $\underline{X} = x_1$ and $\overline{X} = x_2$. The width of X is denoted $w(X)$, which is given by $w(X) = \overline{X} - \underline{X}$.

We call by *interval-valued (iv) fusion function* any interval-valued function $IF : L([0,1])^n \to L([0,1])$ that merges n intervals from $L([0,1])$ into a single interval in $L([0,1])$.

Definition 9 ([47]). An iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is called width-preserving (or w-preserving, for simplicity) if, for any $\vec{X} \in L([0,1])^n$ such that $w(X_1) = \cdots = w(X_n)$, it holds that $w(IF(\vec{X})) = w(X_1)$.

An iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is said to be increasing with respect to a partial order $\preceq$ on $L([0,1])$ (or, simply, $\preceq$-increasing) if, for all $\vec{X}, \vec{Y} \in L([0,1])^n$, the following condition holds:

$$X_i \preceq Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IF(\vec{X}) \preceq IF(\vec{Y}).$$

Definition 10 ([50]). Let $IF : L([0,1])^n \to L([0,1])$ be an iv-fusion function and $\preceq_1, \preceq_2$ be two partial order relations on

$L([0,1])$. Then, IF is said to be $(\leq_1, \leq_2)$-increasing if the following condition holds, for all $\vec{X}, \vec{Y} \in L([0,1])^n$:

$X_i \leq_1 Y_i$ for all $i \in \{1,\dots,n\} \Rightarrow IF(\vec{X}) \leq_2 IF(\vec{Y})$.

When an iv-fusion function $IF : L([0,1])^n \to L([0,1])$ is $(\leq,\leq)$-increasing, we denote it simply as $\leq$-increasing, for any partial order relation $\leq$ on $L([0,1])$.

The *product order* [42], denoted by $\leq_{Pr}$, is a partial order relation, defined, for all $X, Y \in L([0,1])$, by:

$X \leq_{Pr} Y \Leftrightarrow \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}$.

Let $f, g : [0,1]^n \to [0,1]$ be two fusion functions such that $f \leq g$. Then, the iv-fusion function $\widehat{f,g} : L([0,1])^n \to L([0,1])$ is given by: $\widehat{f,g}(\vec{X}) = [f(\underline{X_1},\dots,\underline{X_n}), g(\overline{X_1},\dots,\overline{X_n})]$.

Definition 11 (*[24]*). Let $IF : L([0,1])^n \to L([0,1])$ be a $\leq_{Pr}$-increasing iv-fusion function. Then, IF is said to be representable if there exist increasing fusion functions $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$ and $IF = \widehat{f,g}$.

f and g are called the *representatives* of IF. When $IF = \widehat{f,f}$, we denote simply as $\widehat{f}$. In this case, IF is said to be *the best interval representation (BIR)* of f [24].

The next interval operations, defined for all $X, Y \in L([0,1])$, are used in this paper: [42,53]

Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$, with $\overline{X} + \overline{Y} \leq 1$;

Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;

Generalized Hukuhara Division: for $\underline{Y} \neq 0, X \leq_{Pr} Y$:

$$X \div_H Y = [\min\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}, \max\{\underline{X}/\underline{Y}, \overline{X}/\overline{Y}\}]. \quad (5)$$

Here, we recall the concept of admissible orders.

Definition 12 (*[43]*). Let $(L([0,1]), \leq_{AD})$ be a partially ordered set. The order $\leq_{AD}$ is an admissible order if, for all $X, Y \in L([0,1])$: **(i)** $\leq_{AD}$ is a total order on $(L([0,1]), \leq_{AD})$; **(ii)** $X \leq_{Pr} Y \Rightarrow X \leq_{AD} Y$.

Thus, an order $\leq_{AD}$ on $L([0,1])$ is said to be admissible if it is a total order that refines the product order $\leq_{Pr}$ [43]. Since every admissible order $\leq_{AD}$ refines $\leq_{Pr}$, it is immediate that every $\leq_{AD}$-increasing function is also $\leq_{Pr}$-increasing.

Example 3. Here are some examples of admissible orders: **(i)** The lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$, corresponding, respectively, to the first and second coordinates are given by:

$$X \leq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y});$$
$$X \leq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y}).$$

(ii) The order of Xu and Yager $\leq_{XY}$ [54], given by:

$$X \leq_{XY} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}).$$

(iii) The order $\leq_{IQ}$ [33], given by:

$$X \leq_{IQ} Y \Leftrightarrow \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or } (\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{Y} - \underline{Y} \leq \overline{X} - \underline{X}). \quad (6)$$

Observe that the order $\leq_{IQ}$ is based on the order of Xu and Yager, but takes into consideration the information quality [48] when comparing the intervals.

Next, we recall the definition of the admissible order $\leq_{\alpha,\beta}$:

Definition 13 (*[43]*). For $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\leq_{\alpha,\beta}$ is defined, for all $X, Y \in L([0,1])$, by

$$X \leq_{\alpha,\beta} Y \Leftrightarrow K_\alpha(\underline{X}, \overline{X}) < K_\alpha(\underline{Y}, \overline{Y}) \text{ or } (K_\alpha(\underline{X}, \overline{X}) = K_\alpha(\underline{Y}, \overline{Y}) \text{ and } K_\beta(\underline{X}, \overline{X}) \leq K_\beta(\underline{Y}, \overline{Y})),$$

where $K_\alpha, K_\beta : [0,1]^2 \to [0,1]$ are aggregation functions defined, for all $x, y \in [0,1]$, respectively, by

$$K_\alpha(x,y) = x + \alpha \cdot (y - x); \;\; K_\beta(x,y) = x + \beta \cdot (y - x). \quad (7)$$

Remark 1. The order $\leq_{\alpha,\beta}$ can recover other known admissible orders, by an appropriate choice of α and β. For example, **(i)** The lexicographical orders $\leq_{Lex1}$ and $\leq_{Lex2}$ are recovered, respectively, by $\leq_{0,1}$ and $\leq_{1,0}$; **(ii)** The orders $\leq_{XY}$ and $\leq_{IQ}$ are recovered, respectively, by $\leq_{0.5,1}$ and $\leq_{0.5,0}$.

Whenever we apply the mapping K_α on the endpoints of an interval $X \in [0,1]$, we denote $K_\alpha(\underline{X}, \overline{X})$ simply as $K_\alpha(X)$.

Lemma 1 (*[43]*). *For any $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$, it holds that: (i) $\beta > \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,1}$; (ii) $\beta < \alpha \Rightarrow \leq_{\alpha,\beta} = \leq_{\alpha,0}$.*

2.3. Interval-valued fusion functions

Definition 14 (*[55]*). $IN : L([0,1]) \to L([0,1])$ is called an iv-fuzzy negation if it is $\leq_{Pr}$-decreasing, **(IN1)** $IN([1,1]) = [0,0]$ and **(IN2)** $IN([0,0]) = [1,1]$. If $IN(IN(X)) = X$, for all $X \in L([0,1])$, then IN is said to be involutive.

Definition 15 (*[56]*). $IR : L([0,1])^2 \to L([0,1])$ is an iv-restricted equivalence function (IV-REF) with respect to an iv-fuzzy negation IN, if, for all $X, Y, Z \in L([0,1])$: **(IR1)** IR is commutative; **(IR2)** $IR(X,Y) = [1,1] \Leftrightarrow X = Y$; **(IR3)** $IR(X,Y) = [0,0] \Leftrightarrow X = [0,0]$ and $Y = [1,1]$, or $X = [1,1]$ and $Y = [0,0]$; **(IR4)** $IR(X,Y) = IR(IN(X), IN(Y))$; **(IR5)** $X \leq_{Pr} Y \leq_{Pr} Z \Rightarrow IR(X,Y) \geq_{Pr} IR(X,Z)$, $IR(Y,Z) \geq_{Pr} IR(X,Z)$.

Definition 16 (*[39]*). An iv-fusion function $IA : L([0,1])^n \to L([0,1])$ is called an iv-aggregation function if: **(IA1)** IA is $\leq_{Pr}$-increasing; **(IA2)** $IA([0,0],\dots,[0,0]) = [0,0]$ and $IA([1,1],\dots,[1,1]) = [1,1]$.

Definition 17 (*[47]*). Consider $c \in [0,1]$ and $\alpha \in [0,1]$. Then, the maximal possible width of an interval $Z \in L([0,1])$ is denoted by $d_\alpha(c)$, such that $K_\alpha(Z) = c$. Also, define, for any $X \in L([0,1])$,

$$\lambda_\alpha(X) = \frac{w(X)}{d_\alpha(K_\alpha(X))}, \quad (8)$$

where we set $\frac{0}{0} = 1$.

Proposition 2 (*[47]*). *For all $\alpha \in [0,1]$ and $X \in L([0,1])$, one has that*

$$d_\alpha(K_\alpha(X)) = \min\left\{\frac{K_\alpha(X)}{\alpha}, \frac{1 - K_\alpha(X)}{1 - \alpha}\right\}, \quad (9)$$

where we set $\frac{r}{0} = 1$, for all $r \in [0,1]$.

Theorem 1 (*[47]*). *Let $\alpha, \beta \in [0,1]$ be such that $\alpha \neq \beta$. Let $A_1, A_2 : [0,1]^n \to [0,1]$ be two aggregation functions where A_1 is strictly increasing. Then $IF^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by:*

$$IF^\alpha_{A1,A2}(\vec{X}) = R \quad \text{where} \begin{cases} K_\alpha(R) = A_1(K_\alpha(X_1),\dots,K_\alpha(X_n)), \\ \lambda_\alpha(R) = A_2(\lambda_\alpha(X_1),\dots,\lambda_\alpha(X_n)), \end{cases}$$

is an $\leq_{\alpha,\beta}$-increasing iv-aggregation function.

Corollary 1 (*[50]*). *Let $\alpha \in (0,1]$, $\beta \in [0,1]$ be such that $\alpha \neq \beta$. Let $On : [0,1]^n \to [0,1]$ be a strict n-dimensional overlap function and*

A be an aggregation function. Then $IF^{\alpha}_{On,A} : L([0,1])^n \rightarrow L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$, *by:*

$$IF^{\alpha}_{On,A}(\vec{X}) = R$$

$$\text{where } \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = A(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

is an $\leq_{\alpha,\beta}$*-increasing iv-aggregation function.*

Definition 18 (*[39]*)**.** An iv-fusion function $IOn : L([0,1])^n \rightarrow L([0,1])$ is called an n-dimensional iv-overlap function if, for all $\vec{X} \in L([0,1])^n$: **(IOn1)** IOn is symmetric; **(IOn2)** $IOn(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^{n} X_i = [0,0]$; **(IOn3)** $IOn(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^{n} X_i = [1,1]$; **(IOn4)** IOn is $\leq_{Pr}$-increasing; **(IOn5)** IOn is Moore continuous [42].

For $n = 2$, IOn is just called iv-overlap function [37,38].

Definition 19 (*[33]*)**.** An iv-fusion function $AOn : L([0,1])^n \rightarrow L([0,1])$ is called an n-dimensional admissibly ordered iv-overlap function for an admissible order $\leq_{AD}$ ($\leq_{AD}$-overlap function) if it respects the conditions **(IOn1)**, **(IOn2)** and **(IOn3)** of Definition 18, and **(AOn4)** AOn is $\leq_{AD}$-increasing.

Although the construction method presented in Corollary 1 is based on an n-dimensional overlap function, the constructed function is not necessarily an $\leq_{\alpha,\beta}$-overlap function. It is not trivial to obtain such type of function, so we present here a new result, which is an adaptation of Corollary 1 with that purpose, as $\leq_{\alpha,\beta}$-overlap functions are featured throughout our theoretical and practical developments:

Theorem 2. *Consider a strict n-dimensional overlap function* $On : [0,1]^n \rightarrow [0,1]$, *an increasing and symmetric fusion function* $B : [0,1]^n \rightarrow [0,1]$ *and* $\alpha \in (0,1), \beta \in [0,1]$, *such that,* $\alpha \neq \beta$. $AOn^{\alpha}_{B} : L([0,1])^n \rightarrow L([0,1])$ *defined by:*

$$AOn^{\alpha}_{B}(\vec{X}) = R$$

$$\text{where } \begin{cases} K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)), \\ \lambda_\alpha(R) = B(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)), \end{cases}$$

for all $\vec{X} \in L([0,1])^n$, *is an* $\leq_{\alpha,\beta}$*-overlap function.*

Proof. See Appendix A. □

The following result is immediate from Definition 17 and Theorem 2.

Corollary 2. *Let* $\alpha \in (0,1), \beta \in [0,1]$ *be such that,* $\alpha \neq \beta$. *Let* $On : [0,1]^n \rightarrow [0,1]$ *be a strict n-dimensional overlap function,* $B : [0,1]^n \rightarrow [0,1]$ *be an increasing and symmetric fusion function and* $AOn^{\alpha}_{B} : L([0,1])^n \rightarrow L([0,1])$ *be an iv-aggregation function constructed as in Theorem 2. Then, for all* $\vec{X} \in L([0,1])^n$, *we have that*

$$w(AOn^{\alpha}_{B}(\vec{X})) = B(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)) \cdot d_\alpha(K_\alpha(AOn^{\alpha}_{B}(\vec{X}))). \tag{10}$$

3. Width-limited iv-fusion functions

Here, we extend for the n-dimensional context the main results on width-limitation presented by Asmus et al. [50], as these concepts are going to be thoroughly featured on our present theoretical developments and are also required in the practical application presented in Section 6.

Definition 20 (*[50]*)**.** Consider an iv-fusion function $IF : L([0,1])^n \rightarrow L([0,1])$ and $B : [0,1]^n \rightarrow [0,1]$. IF is said to be width-limited by B if $w(IF(\vec{X})) \leq B(w(X_1), \ldots, w(X_n))$, for all $\vec{X} \in L([0,1])^n$. B is called a width-limiting function of IF.

Denote $\mathcal{F}=\{F:[0,1]^n \rightarrow [0,1] | F \text{ is a fusion function}\}$ and $\mathcal{IF}=\{IF:L([0,1])^n \rightarrow L([0,1]) | IF \text{ is an iv-fusion function}\}$. Next theorem shows how to obtain the least width-limiting function for a given iv-fusion function:

Theorem 3 (*[50]*)**.** *The mapping* $\mathfrak{L} : \mathcal{IF} \rightarrow \mathcal{F}$ *defined, for all* $IF \in \mathcal{IF}$ *and* $\vec{\epsilon} \in [0,1]^n$, *by*

$$\mathfrak{L}(IF)(\vec{\epsilon}) = \sup_{\substack{u_1 \in [0, 1-\epsilon_1] \\ \ldots \\ u_n \in [0, 1-\epsilon_n]}} \{w(IF([u_1, u_1+\epsilon_1], \ldots, [u_n, u_n+\epsilon_n]))\}$$

provides the least width-limiting function $\mathfrak{L}(IF) : [0,1]^n \rightarrow [0,1]$ *for IF.*

Denote $\mathcal{A}=\{A:[0,1]^n \rightarrow [0,1] | A \text{ is an aggregation function}\}$,

$\mathcal{IA} = \{IA : L([0,1])^n \rightarrow L([0,1]) \mid IA$ is the best interval representation of an aggregation function $A \in \mathcal{A}\}$.

Then, an approach similar to that one considered in Theorem 3 can be used to obtain the least width-liming aggregation function for a given representable iv-aggregation function.

Theorem 4 (*[50]*)**.** *The mapping* $\mathfrak{L} : \mathcal{IA} \rightarrow \mathcal{F}$ *defined, for all* $IA \in \mathcal{IA}$ *and* $\vec{\epsilon} \in [0,1]^n$, *by*

$$\mathfrak{L}(IA)(\vec{\epsilon}) = \sup_{\substack{u_1 \in [0, 1-\epsilon_1] \\ \ldots \\ u_n \in [0, 1-\epsilon_n]}} \{w(IA([u_1, u_1+\epsilon_1], \ldots, [u_n, u_n+\epsilon_n]))\} \tag{11}$$

provides the least width-limiting function $\mathfrak{L}(IA) : [0,1]^n \rightarrow [0,1]$ *for IA. Moreover,* $\mathfrak{L}(IA)$ *is an aggregation function.*

Now, let us present a characterization for the least width-limiting function of the best interval representation (BIR) of an $\vec{1}$-ultramodular aggregation function, or the BIR of its dual:

Theorem 5 (*[50]*)**.** *Let* $A : [0,1]^n \rightarrow [0,1]$ *be an aggregation function,* $A^d : [0,1]^n \rightarrow [0,1]$ *be the dual of A,* $\mathfrak{L}(\widehat{A}), \mathfrak{L}(\widehat{A^d}) : [0,1]^n \rightarrow [0,1]$ *be the least width-limiting functions for* $\widehat{A}$ *and* $\widehat{A^d}$, *respectively. Then,* $\mathfrak{L}(\widehat{A}) = \mathfrak{L}(\widehat{A^d}) = A^d$ *if and only if A is an* $\vec{1}$*-ultramodular aggregation function.*

In the context of Theorem 5, as $\widehat{A}$ and $\widehat{A^d}$ are representable iv-aggregation functions, then their least width-limiting function A^d is an aggregation function, as stated by Theorem 4. Observe that the function A does not need to be ultramodular.

Example 4. (a) Every iv-fusion function $IF : L([0,1])^n \rightarrow L([0,1])$ is width-limited by the function $B : [0,1]^n \rightarrow [0,1]$ given by $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$.

(b) The n-dimensional iv-overlap function $IOnP$, defined, for all $\vec{X} \in L([0,1])^n$, by $IOnP(\vec{X}) = \prod_{i=1}^{n} X_i$, is width-limited by the probabilistic sum, given by $B(\vec{x}) = 1 - \prod_{i=1}^{n} 1 - x_i$, for all $\vec{x} \in [0,1]^n$. This holds because the IOn_p is the BIR of the product, which is an ultramodular aggregation function, and the probabilistic sum is the dual of the product.

(c) Similar to the last case, the BIR of the n-dimensional overlap function OnT, defined in Eq. (4), denoted by $\widehat{OnT}$, is width-limited by $B = OnT^d$, since OnT is an $\vec{1}$-ultramodular aggregation function.

Remark 2. Observe that a width-limited function (Definition 20) differs from a width-preserving function (Definition 9), since one can only guarantee the uncertainty control in width-preserving functions when all the inputs have the same width. On the other hand, some width-limited functions can guarantee the uncertainty control accordingly to some width-limiting function B and still not be considered width-preserving as by Definition 9. So, width-limitation is a more suitable (and flexible) concept to indicate the potential uncertainty on the outputs of a given interval-valued function than width-preservation.

4. A framework for width-limited iv-fusion functions

The goal of this section is to present a way to obtain different classes of aggregation functions that have their outputs' widths limited by some arbitrary width-limiting function. In the work by Asmus et al. [50], for instance, width-limited interval-valued overlap functions were introduced based on such notion. Here, we recall their definition:

Definition 21 (*[50]*)**.** Let $B : [0,1]^2 \to [0,1]$ be a commutative and increasing function and $\leq_1, \leq_2$ be two partial order relations on $L([0,1])$. Then, the mapping $IOw : L([0,1])^2 \to L([0,1])$ is said to be a width-limited interval-valued overlap function (w-iv-overlap function) with respect to the tuple $(\leq_1, \leq_2, B)$, if the following conditions hold for all $X, Y \in L([0,1])$: **(IOw1)** IOw is commutative; **(IOw2)** $IOw(X,Y) = [0,0] \Leftrightarrow X \cdot Y = [0,0]$; **(IOw3)** $IOw(X,Y) = [1,1] \Leftrightarrow X \cdot Y = [1,1]$; **(IOw4)** IOw is $(\leq_1, \leq_2)$-increasing; **(IOw5)** IOw is width-limited by B.

Notice that Definition 21 is quite similar to Definition 18 for $n = 2$. In fact, conditions **(IOn1)**, **(IOn2)** and **(IOn3)**, for $n = 2$, are the same as **(IOw1)**, **(IOw2)** and **(IOw3)**. That is, both classes of iv-fusion functions share the same properties except for (i) monotonicity, (ii) continuity and (iii) width-limitation. Furthermore, it is easy to observe that those properties that they share, commutativity and boundaries conditions, are interval counterparts of properties **(On1)**, **(On2)** and **(On3)** of n-dimensional overlap functions (Definition 7). In a sense, one can say that n-dimensional overlap functions are the core functions in which both Definitions 18 and 21 derive from. So, one could obtain analogous definitions for width-limited interval-valued aggregation functions based on other core aggregation functions, if the properties of those core functions can be extended to the interval context.

For that, inspired by the approach of directional increasing fusion functions developed by Bustince et al. [57], first we present a characterization of any subclass $\mathcal{F}$ of increasing n-dimensional fusion functions through a set of properties $P_{\mathcal{F}}$ such that: (i) includes boundary conditions for any $F \in \mathcal{F}$ and (ii) possibly includes some other constraints not related to the monotonicity. Such subclass of fusion functions is given by:

$$\mathcal{F} = \{F : [0,1]^n \to [0,1] \mid F \text{ is increasing and satisfies all the properties in } P_{\mathcal{F}}\}. \tag{12}$$

Remark 3. In this paper, we work with the usual definition of monotonicity, however, our characterization is not restricted by this specific definition, meaning that other kinds of monotonicity could be considered, such as weak monotonicity [58], directional monotonicity [59], ordered directionally monotonicity [60] or strengthened ordered directionally monotonicity [61].

Example 5. Based on Eq. (12), the class of aggregation functions $\mathcal{A}$ is given by $\mathcal{A} = \{A : [0,1]^n \to [0,1] \mid A$ is increasing and satisfies all the properties in $P_{\mathcal{A}}\}$, where $P_{\mathcal{A}} = \{A(0,\ldots,0) = 0, A(1,\ldots,1) = 1\}$. Analogously, any subclass of aggregation functions can be denoted in this manner. For example, the class of n-dimensional overlap functions $\mathcal{O}n$ can be defined by: $\mathcal{O}n = \{On : [0,1]^n \to [0,1] \mid On$ is increasing and satisfies all the properties in $P_{\mathcal{O}n}\}$, where $P_{\mathcal{O}n} = \{$(**On1**), (**On2**), (**On3**), (**On5**)$\}$.

Now, given the set $P_{\mathcal{F}}$ of properties of a fusion function $F \in \mathcal{F}$, denote by $IP_{\mathcal{F}}$ the set of interval extensions of the properties in $P_{\mathcal{F}}$. Usually, there are more than one way to extend a given property of a function to the interval context, so $IP_{\mathcal{F}}$ varies accordingly to how one extends such properties.

Finally, consider the function $B \in \mathcal{B}$, where a $\mathcal{B}$ is a subclass of increasing fusion functions (with its corresponding set of properties $P_{\mathcal{B}}$) and let $\leq_1, \leq_2$ be partial orders on $L([0,1])$. Then, denote the class of width-limited interval-valued fusion functions (w-iv-fusion functions) for the tuple $(\leq_1, \leq_2, B)$ by $\mathcal{IFW}^B_{\leq_1,\leq_2}$, which is then given by:

$$\mathcal{IF}w^B_{\leq_1,\leq_2} = \{IF : L([0,1])^n \to L([0,1]) \mid IF \text{ is } (\leq_1, \leq_2)\text{-increasing, width-limited by } B \text{ and satisfies the properties in } IP_{\mathcal{F}}\}. \tag{13}$$

Example 6. Consider the class of aggregation functions $\mathcal{A}$, with its respective set of properties $P_{\mathcal{A}}$ (as shown in Example 5). Also, consider an increasing fusion function $B : [0,1]^n \to [0,1]$ and let $\leq_1, \leq_2$ be partial orders on $L([0,1])$. Then, based on Eq. (13), $\mathcal{IA}w^B_{\leq_1,\leq_2}$ is the class of width-limited interval-valued aggregation functions (w-iv-aggregation functions) for the tuple $(\leq_1, \leq_2, B)$, given by:

$$\mathcal{IA}w^B_{\leq_1,\leq_2} = \{IA : L([0,1])^n \to L([0,1]) \mid IA \text{ is } (\leq_1, \leq_2)\text{-increasing, width-limited by } B \text{ and satisfies the properties in } IP_{\mathcal{A}}\} \tag{14}$$

where $IP_{\mathcal{A}}$ is an interval extension of $P_{\mathcal{A}}$, given by $IP_{\mathcal{A}} = \{IA([0,0],\ldots,[0,0]) = [0,0], IA([1,1],\ldots,[1,1]) = [1,1]\}$. Observe that if $\leq_1 = \leq_2 = \leq_{Pr}$ and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$, then $\mathcal{IA}w^B_{\leq_1,\leq_2}$ is the class of iv-aggregation functions defined in Definition 16. Similarly, other subclasses of iv-aggregation functions can be retrieved by Eq. (13), depending on the set of properties $IP_{\mathcal{F}}$. For example, take $IP_{\mathcal{F}} = IP_{\mathcal{O}n}$, where $IP_{\mathcal{O}n}$ is the interval extension of the set $P_{\mathcal{O}n}$ (see Example 5): $IP_{\mathcal{O}n} = \{$(**IOn1**), (**IOn2**), (**IOn3**), (**IOn5**)$\}$, and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$. Then, $\mathcal{IO}nw^B_{\leq_{Pr}}$, given by

$$\mathcal{IO}nw^B_{\leq_{Pr}} = \{IOn : L([0,1])^n \to L([0,1]) \mid IOn \text{ is } \leq_{Pr}\text{-increasing, width-limited by } B \text{ and satisfies the properties in } IP_{On}\}, \tag{15}$$

is the class of n-dimensional iv-overlap functions (Definition 18).

Remark 4. The representation of w-iv-fusion functions by Eq. (13) is general enough so that different iv-aggregation functions defined in the literature may be retrieved, such as interval-valued t-norms and t-conorms [24], general interval-valued overlap functions [39], general interval-valued grouping functions [40], among others, by restricting to the case where $\leq_1 = \leq_2 = \leq_{Pr}$ and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$. However, those functions clearly have no limitation regarding their output widths and may not be applicable in problems where admissible orders must be considered.

Example 7. Consider a function $B \in \mathcal{B}$, where $\mathcal{B}$ is the subclass of increasing fusion functions, such that $P_{\mathcal{B}} = \{\text{simmetry}\}$, and two partial orders $\leq_1, \leq_2$ on $L([0,1])$. Then, $\mathcal{IO}nw^B_{\leq_1,\leq_2}$ is the class of width-limited n-dimensional interval-valued overlap functions (w-iv-overlap functions) for the tuple $(\leq_1, \leq_2, B)$, given by:

$$\mathcal{IO}nw^B_{\leq_1,\leq_2} = \{IOnw : L([0,1])^n \to L([0,1]) \mid IOnw \text{ is } (\leq_1, \leq_2)\text{-increasing, width-limited by } B \text{ and satisfies the properties in } IP_{\mathcal{O}n'}\}, \tag{16}$$

where $IP_{\mathcal{O}n'} = \{$(**IOn1**), (**IOn2**), (**IOn3**)$\}$.

Observe that when $n = 2$, $\mathcal{IO}nw^B_{\leq_1,\leq_2}$ is the class of w-iv-overlap functions as shown in Definition 21. In other case, when $\leq_1=\leq_2=\leq_{\alpha,\beta}$, with $\alpha,\beta \in [0,1]$, such that $\alpha \neq \beta$, and $B(\vec{x}) = 1$, for all $\vec{x} \in [0,1]^n$, then, also by Eq. (16), $\mathcal{IO}nw^B_{\leq_{\alpha,\beta}}$ is the class of n-dimensional admissibly ordered interval-valued overlap function as presented in Definition 19.

Then, one can see that classes of iv-fusion functions that may not be $\leq_{Pr}$-increasing can also be retrieved by Eq. (13).

Remark 5. It is noteworthy that the continuity (**On5**) was not extended to the interval context, so was not considered in $IP'_{\mathcal{O}n}$. Neither Definition 19 nor Definition 21 has continuity as a condition, as its interval extension it is not fully developed in the context of admissible orders. This is not a drawback, since the continuity requirement was added to the definition of overlap functions just because it was introduced firstly to be applied in image processing [5]. Actually, it is well known that the continuity can be disregarded in several applications, which is the case, for example, when overlap functions are applied in classification problems [33,39,62,63]. Nevertheless, if one is defining a class of width-limited $\leq_{Pr}$-increasing fusion functions, than the continuity can be extended to the interval-context by the Moore-continuity [42], as in Example 6, when defining the class of n-dimensional iv-overlap functions.

Remark 6. The associativity property was proved to be difficult to maintain in iv-fusion functions that have controlled width-limitation (see the construction methods in Section 5), so we do not include it when defining a set $IP_{\mathcal{F}}$. We point out that this is not a drawback of our approach, in the sense that we explain below. Observe that, for several applications (e.g., classification), it is necessary to consider n-dimensional inputs for the aggregation process. That is why the associativity property has been considered an important requirement for extending bi-variate aggregation functions to the n-dimensional context in a very direct way, which is the case, for example, of t-norms and t-conorms [51]. However, it is well known that possibly non-associative bi-variate functions (e.g., overlap/grouping functions) can be extended to the n-dimensional context in many ways (e.g., n-dimensional overlap/grouping functions and general overlap/grouping functions). Also, one can find in the literature several possibly non-associative aggregation functions which can be used as alternatives to t-norms/t-conorms, such as t-seminorms or semi-copulas [64], weak t-norms [65], pseudo-t-norms [66], semi-uninorms [67], MICA operators [68], and micanorms [69].

Remark 7. In general, there are no restrictions regarding the width-limiting function B, or its set of properties $P_{\mathcal{B}}$, when defining a class $\mathcal{IFW}^B_{\leq_1,\leq_2}$ by Eq. (13). But, to construct some examples of w-iv-fusion functions of the class $\mathcal{IFW}^B_{\leq_1,\leq_2}$ respecting the properties of $IP_{\mathcal{F}}$, it may be necessary that $P_{\mathcal{B}}$ shares some properties with $P_{\mathcal{F}}$. That is the reason for which we required that B to be symmetric in Example 7, a shared property with On. This relation between the core function F and the width-limiting function B becomes clear in the construction methods presented in Section 5.

5. Construction methods

With the concepts of width-limited functions and least width-limiting functions, by Theorem 3, one can expect the maximum amount of uncertainty on the outputs of a given iv-fusion function. However, in order to control such uncertainty to an arbitrary degree (given by a chosen width-limiting function B), one can apply the developed theory to obtain some construction methods for width-limited iv-fusion functions. This is the main goal of this section.

In the following, we present a key concept to be applied in the construction methods for w-iv-fusion functions:

Definition 22. Consider a fusion function $B : [0,1]^n \to [0,1]$ and let $IF : L([0,1])^n \to L([0,1])$ be an iv-fusion function. Then, the function $m_{IF,B} : L([0,1])^n \to [0,1]$, defined for all $\vec{X} \in L([0,1])^n$ by:

$$m_{IF,B}(\vec{X}) = \min\{w(IF(\vec{X})), B(w(X_1),\ldots,w(X_n))\},$$

is called the maximal width threshold for the pair (IF, B).

5.1. Construction method based on representable iv-fusion functions (CMR)

The main idea of the Construction Method based on Representable iv-fusion functions (CMR) is to reduce the output's width of a representable iv-fusion function when it surpasses the limit imposed by a width-limiting fusion function B. The outputs of the constructed function are given by the maximal threshold for the pair $(\widehat{F}, B)$, where $\widehat{F}$ is the best interval representation (BIR) of a strict fusion function F. The reduction of the output occurs in the direction of a point of the interval, accordingly to a chosen value of $\alpha \in [0,1]$. For example, if $\alpha = 0.5$, then the output is "narrowed" towards the medium point of the interval obtained through the BIR.

The formalization of this concept is presented in the following three theorems, each one with some specificity regarding the chosen strict fusion function F and its respective restriction on the choice of admissible order that is suitable for the construction method.

Theorem 6. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1]$ and $\beta \in [0,\alpha)$. Then, the iv-fusion function $IFw^\alpha_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}), \quad K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{F},B}(\vec{X})], \tag{17}$$

is a width-limited fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix B. □

Theorem 7. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 1$ as its annihilator element $\alpha \in [0,1)$ and $\beta \in (\alpha,1]$. Then, the iv-fusion function $IFw^\alpha_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}), \quad K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{F},B}(\vec{X})],$$

is a width-limited fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 6. □

The following theorem follows from Theorems 6 and 7.

Theorem 8. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strictly increasing fusion function $F : [0,1]^n \to [0,1]$ and $\alpha,\beta \in [0,1]$ with $\alpha \neq \beta$. Then, the iv-fusion function $IFw^\alpha_B : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}),$$

$K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{F},B}(\vec{X})]$,

is a width-limited fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

One can impose certain conditions on the functions B and F, reflected on the sets $P_{\mathcal{B}}$ and $P_{\mathcal{F}}$, respectively, in order to obtain specific subclasses of fusion functions. In the following, consider the class of w-iv-aggregation functions $\mathcal{IA}w^B_{\leq_{Pr},\leq_{\alpha,\beta}}$ given by Eq. (14), in Example 6.

Theorem 9. *Consider an increasing function* $B:[0,1]^n \to [0,1]$, *a strict aggregation function* $A:[0,1]^n \to [0,1]$ *with* $h=0$ *as its annihilator element,* $\alpha \in (0,1]$ *and* $\beta \in [0,\alpha)$. *Then, the iv-fusion function* $IAw^\alpha_B : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$, *by*

$$IAw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{A}(\vec{X})) - \alpha\cdot m_{\widehat{A},B}(\vec{X}),\ K_\alpha(\widehat{A}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{A},B}(\vec{X})],$$

is a width-limited aggregation function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. From the proof of Theorem 6, IAw^α_B is well defined, $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing and width-limited by B. We prove that: **(i)** $IAw^\alpha_B([0,0],\ldots,[0,0])=[0,0]$, **(ii)** $IAw^\alpha_B([1,1],\ldots,[1,1])=[1,1]$.

(i) From **(A2)**, one has that $\widehat{A}([0,0],\ldots,[0,0]) = [0,0]$. Then: $w(\widehat{A}([0,0],\ldots,[0,0])) = 0 = m_{\widehat{A},B}([0,0],\ldots,[0,0])$.
So, $IAw^\alpha_B([0,0],\ldots,[0,0]) = \widehat{A}([0,0],\ldots,[0,0]) = [0,0]$.

(ii) Also, from **(A2)**, one has that $\widehat{A}([1,1],\ldots,[1,1]) = [1,1]$. Then: $w(\widehat{A}([1,1],\ldots,[1,1])) = 0 = m_{\widehat{A},B}([1,1],\ldots,[1,1])$. So, $IAw^\alpha_B([1,1],\ldots,[1,1])=\widehat{A}([1,1],\ldots,[1,1])=[1,1]$. □

Theorem 10. *Consider an increasing function* $B:[0,1]^n \to [0,1]$, *a strict aggregation function* $A:[0,1]^n \to [0,1]$ *with* $h=1$ *as its annihilator element,* $\alpha \in [0,1)$ *and* $\beta \in (\alpha,1]$. *Then, the iv-fusion function* $IAw^\alpha_B : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$, *by*

$$IAw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{A}(\vec{X})) - \alpha\cdot m_{\widehat{A},B}(\vec{X}),\ K_\alpha(\widehat{A}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{A},B}(\vec{X})],$$

is a width-limited aggregation function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 9. □

The next theorem follows from Theorems 9 and 10:

Theorem 11. *Consider an increasing function* $B:[0,1]^n \to [0,1]$, *an aggregation function* $A:[0,1]^n \to [0,1]$ *that is strictly increasing on* $[0,1]^n$, *and* $\alpha,\beta \in [0,1]$ *with* $\alpha \neq \beta$. *Then, the iv-fusion function* $IAw^\alpha_B : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$, *by*

$$IAw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{A}(\vec{X})) - \alpha\cdot m_{\widehat{A},B}(\vec{X}),\ K_\alpha(\widehat{A}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{A},B}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Example 8. Consider $B=\min$, $A=AM$ (arithmetic mean), $\alpha=0.5$ and $\beta=0$ (admissible order $\leq_{IQ}$). Then, the iv-fusion function $IAMw^\alpha_{\min} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by

$$IAMw^\alpha_{\min}(\vec{X}) = [K_\alpha(\widehat{AM}(\vec{X})) - \alpha\cdot m_{\widehat{AM},\min}(\vec{X}),\ K_\alpha(\widehat{AM}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{AM},\min}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{Pr}, \leq_{IQ}, \min)$.

Remark 8. It is noteworthy that, by Theorem 5, $\widehat{AM}$ (BIR of the arithmetic mean) is width-limited by AM, as AM is an 1-ultramodular aggregation function and $AM = AM^d$. However, if one must control the output's width by a width-limiting function B such that $B \leq AM$ (which is the case for $B=\min$), then the construction method shown in Theorem 11 may be employed, with the result presented in Example 8.

Analogously, one can construct w-iv-fusion functions that are interval extensions of specific types of aggregation functions. For instance, consider the class of w-iv-overlap functions $\mathcal{IO}nw^B_{\leq_{Pr},\leq_{\alpha,\beta}}$ given by Eq. (15), in Example 7.

Theorem 12. *Consider a symmetric and increasing function* $B: [0,1]^n \to [0,1]$, *a strict* n*-dimensional overlap function* $On: [0,1]^n \to [0,1]$, $\alpha \in (0,1)$ *and* $\beta \in [0,\alpha)$. *Then, the iv-fusion function* $IOnw^\alpha_B : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$, *by*

$$IOnw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{On}(\vec{X})) - \alpha\cdot m_{\widehat{On},B}(\vec{X}),\ K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha)\cdot m_{\widehat{On},B}(\vec{X})], \tag{18}$$

is a width-limited overlap function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Proof. From the proof of Theorem 6, it is immediate that $IOnw^\alpha_B$ is well defined, $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing and width-limited by B. It remains to be proven that $IOnw^\alpha_B$ has the properties of the set $IP_{\mathcal{O}n'} = \{(\mathbf{IOn1}), (\mathbf{IOn2})$ *and* $(\mathbf{IOn3})\}$:

(IOn1) It is immediate, since On and B are both symmetric.

(IOn2) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$, such that $IOnw^\alpha_B(\vec{X}) = [0,0]$. Then, we have the following cases:

(1) $m_{\widehat{On},B}(\vec{X}) = w(\widehat{On}(\vec{X}))$:
From Eqs. (7) and (18), it follows that:

$$\begin{aligned}
&[K_\alpha(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),\\
&\quad K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha) w(\widehat{On}(\vec{X}))] = [0,0]\\
&\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}) + \alpha w(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),\\
&\quad On(\underline{X_1},\ldots,\underline{X_n}) + \alpha w(\widehat{On}(\vec{X})) + w(\widehat{On}(\vec{X})) -\\
&\quad \alpha w(\widehat{On}(\vec{X}))] = [0,0]\\
&\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}),\\
&\quad On(\underline{X_1},\ldots,\underline{X_n}) + w(\widehat{On}(\vec{X}))] = [0,0]\\
&\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}), On(\overline{X_1},\ldots,\overline{X_n})] = [0,0]\\
&\Rightarrow \widehat{On}(\vec{X}) = [0,0] \Leftrightarrow \prod_{i=1}^n X_i = [0,0], \text{ by } (\mathbf{On2}).
\end{aligned}$$

(2) $m_{\widehat{On},B}(\vec{X}) = B(w(X_1),\ldots,w(X_n))$:
From Eqs. (7) and (18), it holds that:

$$\begin{aligned}
&[K_\alpha(\widehat{On}(\vec{X})) - \alpha B(w(X_1),\ldots,w(X_n)),\\
&\quad K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha) B(w(X_1),\ldots,w(X_n))]=[0,0]
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\Rightarrow -\alpha B(w(X_1),\ldots,w(X_n)) =\\
&\quad (1-\alpha) B(w(X_1),\ldots,w(X_n))\\
&\Rightarrow B(w(X_1),\ldots,w(X_n)) = 0\\
&\Rightarrow [K_\alpha(\widehat{On}(\vec{X})), K_\alpha(\widehat{On}(\vec{X}))] = [0,0], \text{ by Eq. (19)}\\
&\Rightarrow K_\alpha(\widehat{On}(\vec{X})) = 0\\
&\Rightarrow \widehat{On}(\vec{X}) = [0,0], \text{ since } \alpha \neq 0\\
&\Leftrightarrow \prod_{i=1}^n X_i = [0,0], \text{ by } (\mathbf{On2}).
\end{aligned}$$

($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$, such that $\prod_{i=1}^n X_i = [0,0]$. Then, it is immediate that $\widehat{On}(\vec{X}) = [0,0]$ and $m_{\widehat{On},B}(\vec{X}) = 0$. Furthermore, from Eq. (18):

$$IOnw_B^\alpha(\vec{X}) = [K_\alpha([0,0]) - \alpha\cdot 0, K_\alpha([0,0]) + (1-\alpha)\cdot 0] = [0,0].$$

(IOn3) ($\Rightarrow$) Take $\vec{X} \in L([0,1])^n$, such that $IOnw_B^\alpha(X,Y) = [1,1]$. We have the following cases:

(1) $m_{\widehat{On},B}(\vec{X}) = w(\widehat{On}(\vec{X}))$
From Eqs. (7) and (18), it follows that:

$$\begin{aligned}
&[K_\alpha(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),\\
&\quad K_\alpha(\widehat{On}(\vec{X})) + (1-\alpha)tw(\widehat{On}(\vec{X}))] = [1,1]\\
&\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}) + \alpha w(\widehat{On}(\vec{X})) - \alpha w(\widehat{On}(\vec{X})),\\
&\quad O(\underline{X},\underline{Y}) + \alpha w(\widehat{O}(X,Y)) + w(\widehat{O}(X,Y))\\
&\quad -\alpha w(\widehat{On}(\vec{X}))] = [1,1]\\
&\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}), On(\underline{X_1},\ldots,\underline{X_n}) + w(\widehat{On}(\vec{X}))] = [1,1]\\
&\Rightarrow [On(\underline{X_1},\ldots,\underline{X_n}), On(\overline{X_1},\ldots,\overline{X_n})] = [1,1]\\
&\Rightarrow \widehat{On}(\vec{X}) = [1,1] \Leftrightarrow \prod_{i=1}^n X_i = [1,1], \text{ by } (\mathbf{On3}).
\end{aligned}$$

(2) $m_{\widehat{On},B}(\vec{X}) = B(w(X_1),\ldots,w(X_n))$
From Eqs. (7) and (18), it holds that:

$$\begin{aligned}
&[K_\alpha(\widehat{On}(\vec{X})) - \alpha B(w(X_1),\ldots,w(X_n)), K_\alpha(\widehat{On}(\vec{X}))\\
&\quad + (1-\alpha)B(w(X_1),\ldots,w(X_n))] = [1,1] \qquad (20)\\
&\Rightarrow -\alpha B(w(X_1),\ldots,w(X_n)) =\\
&\quad (1-\alpha)B(w(X_1),\ldots,w(X_n))\\
&\Rightarrow B(w(X_1),\ldots,w(X_n)) = 0 \qquad (21)\\
&\Rightarrow [K_\alpha(\widehat{On}(\vec{X})), K_\alpha(\widehat{On}(\vec{X}))] = [1,1], \text{ by Eq. (20)}\\
&\Rightarrow K_\alpha(\widehat{On}(\vec{X})) = 1\\
&\Rightarrow \widehat{On}(\vec{X}) = [1,1], \text{ since } \alpha \neq 1\\
&\Leftrightarrow \prod_{i=1}^n X_i = [1,1], \text{ by } (\mathbf{On3}).
\end{aligned}$$

($\Leftarrow$) Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. Then, it is immediate that $\widehat{On}(\vec{X}) = [1,1]$ and $m_{\widehat{On},B}(\vec{X}) = 0$. Furthermore, from Eq. (18): $IOnw_B^\alpha(\vec{X}) = [K_\alpha([1,1]) - \alpha\cdot 0, K_\alpha([1,1])(1-\alpha)\cdot 0] = [1,1]$. □

Remark 9. In Theorem 12, we have that $\alpha \neq 1$, which is not necessary in Theorems 6 and 9. This is to ensure that $IOnw_B^\alpha$ respects condition **(IOn3)**. Furthermore, B must be symmetric for $IOnw_B^\alpha$ to respect condition **(IOn1)**. These restrictions on B and α may vary accordingly to the class of aggregation function on which the construction method is based.

Here, we present an example of applying CMR based on the n-dimensional overlap function OnB, given in Eq. (3), which is a function that produces good results when applied in classification problems, as shown in Section 6.

Example 9. Consider $B = \max$, $On = OnB$, given in Eq. (3), $\alpha = 0.5$ and $\beta = 0$ (admissible order $\leq_{IQ}$). Then, by Theorem 12, the iv-fusion function $IOnw_{OnB,\max}^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by

$$\begin{aligned}
&IOnw_{OnB,\max}^{\alpha=0.5}(\vec{X}) = \qquad (22)\\
&\quad [K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) - 0.5\cdot m_{\widehat{OnB},\max}(\vec{X}),\\
&\quad K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) + 0.5\cdot m_{\widehat{OnB},\max}(\vec{X})],
\end{aligned}$$

is a w-iv-overlap function for the tuple ($\leq_{Pr}, \leq_{IQ}, \max$).

(a) Let $n=2$, $X_1 = [0.2, 0.8]$ and $X_2 = [0.5, 1]$. So, we have that $w(X_1) = 0.6$, $w(X_2) = 0.5$ and $\max\{w(X_1), w(X_2)\} = 0.6$. Also, it holds that:

$$\widehat{OnB}(\vec{X}) = [OnB(0.2, 0.5), OnB(0.8, 1)] = [0.1414, 0.8].$$

Observe that

$$w(\widehat{OnB}(\vec{X})) = 0.6586 > 0.6 = \max\{w(X_1), w(X_2)\},$$

meaning that $\widehat{OnB}$ is not width-limited by max.
However, from Eq. (22), one has that:

$$\begin{aligned}
&IOnw_{OnB,\max}^{\alpha=0.5}(\vec{X})\\
&= [K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) - 0.5\cdot m_{\widehat{OnB},\max}(\vec{X}),\\
&\quad K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) + 0.5\cdot m_{\widehat{OnB},\max}(\vec{X})]\\
&= [K_{\alpha=0.5}([0.1414, 0.8]) - 0.5\cdot \min\{w(\widehat{OnB}(\vec{X})),\\
&\quad \max\{w(X_1), w(X_2)\}\}, K_{\alpha=0.5}([0.1414, 0.8])\\
&\quad +0.5\cdot\min\{w(\widehat{OnB}(\vec{X})), \max\{w(X_1), w(X_2)\}\}]\\
&= [0.4707 - 0.5\cdot\min\{0.6586, 0.6\}, 0.4707\\
&\quad +0.5\cdot\min\{0.6586, 0.6\}] = [0.1707, 0.7707].
\end{aligned}$$

So, $w(IOnw_{OnB,\max}^{\alpha=0.5}(\vec{X})) = 0.6 \leq \max\{w(X_1), w(X_2)\}$, which is expected from a function that is width-limited by max.

One can visualize the way that the method works by taking the interval $[0.1414, 0.8]$ (output of the BIR) and "narrowing" it in the direction of its K_α point. In this case, as $\alpha = 0.5$, its the midpoint (0.4707). This can be verified, since:

$$\begin{aligned}
&K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) = K_{\alpha=0.5}([0.1414, 0.8]) = 0.4707\\
&\quad = K_{\alpha=0.5}([0.1707, 0.7707]) = K_{\alpha=0.5}(IOnw_{OnB,\max}^{\alpha=0.5}(\vec{X})).
\end{aligned}$$

If we consider $\alpha = 0.99$, then the narrowing of the interval $[0.1414, 0.8]$ would occur towards the value 0.7934, near its right endpoint. In this case, $IOnw_{OnB,\max}^{\alpha=0.99}(\vec{X}) = [0.1994, 0.7994]$.

(b) Take $X_1 = [0.6, 0.9]$ and $X_2 = [0.8, 0.8]$, then, $w(X_1) = 0.3$, $w(X_2) = 0$ and $\max\{w(X_1), w(X_2)\} = 0.3$. In this case,

$$\widehat{OnB}(\vec{X}) = [OnB(0.6, 0.8), OnB(0.9, 0.8)] = [0.5367, 0.759].$$

So, $w(\widehat{OnB}(\vec{X})) = 0.2223 < 0.3 = \max\{w(X_1), w(X_2)\}$. Also, from Eq. (22):

$$\begin{aligned}
&IOnw_{OnB,\max}^{\alpha=0.5}(\vec{X})\\
&= [K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) - 0.5\cdot m_{\widehat{OnB},\max}(\vec{X}),\\
&\quad K_{\alpha=0.5}(\widehat{OnB}(\vec{X})) + 0.5\cdot m_{\widehat{OnB},\max}(\vec{X})]\\
&= [K_{\alpha=0.5}([0.5367, 0.759]) - 0.5\cdot\min\{w(\widehat{OnB}),\\
&\quad \max\{w(X_1), w(X_2)\}\}, K_{\alpha=0.5}([0.5367, 0.759])\\
&\quad +0.5\cdot\min\{w(\widehat{OnB}), \max\{w(X_1), w(X_2)\}\}]\\
&= [0.6479 - 0.5\cdot 0.2223, 0.6479 + 0.5\cdot 0.2223]\\
&= [0.5367, 0.759].
\end{aligned}$$

Observe that, although $\widehat{OnB}$ is not width-limited by max, in this case, the width of the output does not exceed the limit imposed by the chosen width-limiting function (max). That is why, by applying the construction method as by Eq. (22), it follows that $IOnw_{OnB,\max}^{\alpha=0.5}(\vec{X}) = [0.5367, 0.759] = \widehat{OnB}(\vec{X})$.

In Table 1, we show the results obtained for $IOnw_{OnB,B}^{\alpha}(\vec{X})$, given in Example 9, by different choices of α and width-limiting function B. In this table, it is possible to compare the results obtained by CMR with the ones given by the BIR. Also, one can

Table 1
Ex. of CMR, comparing with the BIR.

	CMR	Best interval representation
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\max}(\vec{X}) = [0.1707, 0.7707]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.99$	$IOnwp^{0.99}_{OnB,\max}(\vec{X}) = [0.1994, 0.7994]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\min}(\vec{X}) = [0.2207, 0.7207]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 0.99$	$IOnwp^{0.99}_{OnB,\min}(\vec{X}) = [0.2984, 0.7984]$	$\widehat{OnB}(\vec{X}) = [0.1414, 0.8]$

observe that, in every case shown in Table 1, $IOnw^{\alpha}_{OnB,B}(\vec{X}) \subseteq \widehat{OnB}(\vec{X})$.

By extending n-dimensional grouping functions to the interval context in a similar manner as done in Example 7 with n-dimensional overlap functions, one can obtain the class of n-dimensional w-iv-grouping functions, denoted by $\mathcal{IG}nw^{B}_{\leq_{Pr},\leq_{\alpha,\beta}}$. As n-dimensional grouping functions are the dual of n-dimensional overlap functions, the following result is immediate from Theorems 10 and 12.

Theorem 13. *Consider a symmetric and increasing function $B : [0,1]^n \to [0,1]$, a strict n-dimensional grouping function $Gn : [0,1]^n \to [0,1]$, $\alpha \in (0,1)$ and $\beta \in (\alpha, 1]$. Then, the iv-fusion function $IOnw^{\alpha}_{B} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IOnw^{\alpha}_{B}(\vec{X}) = [K_{\alpha}(\widehat{On}(\vec{X})) - \alpha \cdot m_{\widehat{On},B}(\vec{X}), \\ K_{\alpha}(\widehat{On}(\vec{X})) + (1-\alpha) \cdot m_{\widehat{On},B}(\vec{X})], \tag{23}$$

is n-dimensional w-iv-grouping function for $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$.

Example 10. Consider $B = AM$ (arithmetic mean), $Gn = Gn_p$ (probabilistic sum), $\alpha = 0.5$ and $\beta = 1$ (admissible order $\leq_{XY}$). Then, the iv-fusion function $IGPw^{\alpha}_{\min} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by

$$IGPw^{\alpha}_{AM}(\vec{X}) = [K_{\alpha}(\widehat{Gn_p}(\vec{X})) - \alpha m_{\widehat{Gn_p},AM}(\vec{X}), \\ K_{\alpha}(\widehat{Gn_p}(\vec{X})) + (1-\alpha) m_{\widehat{Gn_p},AM}(\vec{X})], \tag{24}$$

is a w-iv-grouping function for the tuple $(\leq_{Pr}, \leq_{XY}, AM)$.

Regardless of the core fusion function F employed on CMR, the following result holds:

Proposition 3. *Let $IFw^{\alpha}_{B} : L([0,1])^n \to L([0,1])$ be a w-iv-fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$ obtained through Theorem 6, Theorem 7 or Theorem 8, with the respective choices of $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$. Then, for any $\vec{X} \in L([0,1])^n$ one has that $IFw^{\alpha}_{B}(\vec{X}) \subseteq \widehat{F}(\vec{X})$.*

Remark 10. Proposition 3 ensures that any w-iv-fusion function obtained through the CMR never generates outputs outside of the interval output of the BIR of the base fusion function F.

5.2. A construction method based on admissibly ordered iv-fusion functions (CMA)

This method follows a similar approach as CMR, by applying the maximal threshold to limit the outputs' widths of the constructed function around a K_{α} point. The main difference is that, instead of being based on representable iv-fusion functions, it is based on $\leq_{\alpha,\beta}$-increasing fusion functions.

Theorem 14. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1]$, $\beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-fusion function $IF^{\alpha} : L([0,1])^n \to L([0,1])$, such that $K_{\alpha}(IF^{\alpha})(\vec{X}) = F(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv- fusion function $IFw^{\alpha}_{B} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^{\alpha}_{B}(\vec{X}) = [K_{\alpha}(IF^{\alpha}(\vec{X})) - \alpha \cdot m_{IF^{\alpha},B}(\vec{X}), \\ K_{\alpha}(IF^{\alpha}(\vec{X})) + (1-\alpha) \cdot m_{IF^{\alpha},B}(\vec{X})],$$

is a width-limited fusion function for $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. See Appendix C. □

Theorem 15. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 1$ as its annihilator element, $\alpha \in [0,1)$, $\beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-fusion function $IF^{\alpha} : L([0,1])^n \to L([0,1])$, such that $K_{\alpha}(IF^{\alpha})(\vec{X}) = F(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv- fusion function $IFw^{\alpha}_{B} : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by*

$$IFw^{\alpha}_{B}(\vec{X}) = [K_{\alpha}(IF^{\alpha}(\vec{X})) - \alpha \cdot m_{IF^{\alpha},B}(\vec{X}), \\ K_{\alpha}(IF^{\alpha}(\vec{X})) + (1-\alpha) \cdot m_{IF^{\alpha},B}(\vec{X})],$$

is a width-limited fusion function for $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$.

Proof. Analogous to the proof of Theorem 14. □

The next theorem follows from Theorems 14 and 15.

Theorem 16. *Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strictly increasing fusion function $F : [0,1]^n \to [0,1]$, $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$-increasing iv-fusion function $IF^{\alpha} : L([0,1])^n \to L([0,1])$, such that $K_{\alpha}(IF^{\alpha})(\vec{X}) = F(K_{\alpha}(X_1), \ldots, K_{\alpha}(X_n))$, for all $\vec{X} \in L([0,1])^n$. Then, the iv- fusion* *function* $IFw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IFw_B^\alpha(\vec{X}) = [K_\alpha(IF^\alpha(\vec{X})) - \alpha \cdot m_{IF^\alpha,B}(\vec{X}),\ K_\alpha(IF^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IF^\alpha,B}(\vec{X})],$$

is a width-limited fusion function for $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$*.*

Similarly as done with CMR, one can obtain w-iv-aggregation functions as follows:

Theorem 17. *Consider an increasing fusion function* $B : [0,1]^n \to [0,1]$*, a strict aggregation function* $A : [0,1]^n \to [0,1]$ *with* $h = 0$ *as its annihilator element,* $\alpha \in (0,1]$*,* $\beta \in [0,1]$ *such that* $\alpha \neq \beta$*, and an* $\leq_{\alpha,\beta}$*-increasing iv-aggregation function* $IA^\alpha : L([0,1])^n \to L([0,1])$*, such that* $K_\alpha(IA)(\vec{X}) = A(K_\alpha(X_1), \ldots, K_\alpha(X_n))$*, for all* $\vec{X} \in L([0,1])^n$*. Then, the iv-fusion function* $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IAw_B^\alpha(\vec{X}) = [K_\alpha(IA^\alpha(\vec{X})) - \alpha \cdot m_{IA^\alpha,B}(\vec{X}),\ K_\alpha(IA^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IA,B}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$*.*

Proof. From the proof of Theorem 16, it is immediate that IAw_B^α is well defined, $\leq_{\alpha,\beta}$-increasing and width-limited by B. It remains to be proven that: **(i)** $IAw_B^\alpha([0,0],\ldots,[0,0]) = [0,0]$ and **(ii)** $IAw_B^\alpha([1,1],\ldots,[1,1]) = [1,1]$.

(i) By **(IA2)**, one has that $IA([0,0],\ldots,[0,0]) = [0,0]$. Then $w(IA([0,0],\ldots,[0,0])) = 0 = m_{IA^\alpha,B}([0,0],\ldots,[0,0])$. So, $IAw_B^\alpha([0,0],\ldots,[0,0]) = IA^\alpha([0,0],\ldots,[0,0]) = [0,0]$.

(ii) By **(IA2)**, $IA^\alpha([1,1],\ldots,[1,1]) = [1,1]$. So, we have that $w(IA^\alpha([1,1],\ldots,[1,1])) = 0 = m_{IA^\alpha,B}([1,1],\ldots,[1,1])$, and $IAw_B^\alpha([1,1],\ldots,[1,1]) = IA^\alpha([1,1],\ldots,[1,1]) = [1,1]$. □

Theorem 18. *Consider an increasing fusion function* $B : [0,1]^n \to [0,1]$*, a strict aggregation function* $A : [0,1]^n \to [0,1]$ *with* $h = 1$ *as its annihilator element,* $\alpha \in [0,1)$*,* $\beta \in [0,1]$ *such that* $\alpha \neq \beta$*, and an* $\leq_{\alpha,\beta}$*-increasing iv-aggregation function* $IA^\alpha : L([0,1])^n \to L([0,1])$*, such that*

$$K_\alpha(IA^\alpha)(\vec{X}) = A(K_\alpha(X_1), \ldots, K_\alpha(X_n)),$$

for all $\vec{X} \in L([0,1])^n$*. Then, the iv-fusion function* $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IAw_B^\alpha(\vec{X}) = [K_\alpha(IA^\alpha(\vec{X})) - \alpha \cdot m_{IA^\alpha,B}(\vec{X}),\ K_\alpha(IA^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IA^\alpha,B}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$*.*

Proof. Analogous to the proof of Theorem 17. □

The next theorem follows from Theorems 17 and 18:

Theorem 19. *Consider an increasing fusion function* $B : [0,1]^n \to [0,1]$*, a strictly increasing aggregation function* $A : [0,1]^n \to [0,1]$*,* $\alpha, \beta \in [0,1]$ *such that* $\alpha \neq \beta$*, and an* $\leq_{\alpha,\beta}$*-increasing iv-aggregation function* $IA^\alpha : L([0,1])^n \to L([0,1])$*, such that* $K_\alpha(IA^\alpha)(\vec{X}) = A(K_\alpha(X_1), \ldots, K_\alpha(X_n))$*, for all* $\vec{X} \in L([0,1])^n$*. Then, the iv-fusion function* $IAw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IAw_B^\alpha(\vec{X}) = [K_\alpha(IA^\alpha(\vec{X})) - \alpha \cdot m_{IA^\alpha,B}(\vec{X}),\ K_\alpha(IA^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IA^\alpha,B}(\vec{X})],$$

is a w-iv-aggregation function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$*.*

Width-limited functions based on a specific class of aggregation functions can also be constructed. In the following, we present the construction method for w-iv-overlap functions:

Theorem 20. *Consider an increasing and symmetric fusion function* $B : [0,1]^n \to [0,1]$*, a strict n-dimensional overlap function* $On : [0,1]^n \to [0,1]$*,* $\alpha \in (0,1)$*,* $\beta \in [0,1]$ *with* $\alpha \neq \beta$*, and an* $\leq_{\alpha,\beta}$*-overlap function* $IOn^\alpha : L([0,1])^n \to L([0,1])$*, such that* $K_\alpha(IOn)(\vec{X}) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n))$*, for all* $\vec{X} \in L([0,1])^n$*. Then, the iv-fusion function* $IOnw_B^\alpha : L([0,1])^n \to L([0,1])$ *defined, for all* $\vec{X} \in L([0,1])^n$*, by*

$$IOnw_B^\alpha(\vec{X}) = [K_\alpha(IOn^\alpha(\vec{X})) - \alpha \cdot m_{IOn^\alpha,B}(\vec{X}),\ K_\alpha(IOn^\alpha(\vec{X})) + (1-\alpha) \cdot m_{IOn^\alpha,B}(\vec{X})],$$

is a w-iv-overlap function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$*.*

Proof. From the proof of Theorem 19, it is immediate that $IOnw_B^\alpha$ is well defined, $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing and width-limited by B. It remains to be proven that $IOnw_B^\alpha$ has the properties of the set $IP_{\mathcal{O}n'} = \{(\textbf{IOn1}), (\textbf{IOn2}), (\textbf{IOn3})\}$:

(IOn1) It is immediate, since On and B are both symmetric.

(IOn2) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$ and suppose that $IOnw_B^\alpha(\vec{X}) = [0,0]$. Then, we have that

$$K_\alpha(IOnw_B^\alpha(\vec{X})) = K_\alpha([0,0]) = 0 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)),$$

since $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_\alpha(X_i) = 0$ for some $i \in \{0, \ldots, n\}$, and, therefore, $\prod_{i=1}^n X_i = [0,0]$;

$(\Leftarrow)$ Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [0,0]$. So, $K_\alpha(X_i) \cdot \ldots \cdot K_\alpha(X_n) = 0$, since $\alpha \in (0,1)$. Then, by **(On2)**, one has that

$$K_\alpha(IOnw_B^\alpha(\vec{X})) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 0,$$

meaning that $IOnw_B^\alpha(\vec{X}) = [0,0]$;

(IOn3) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$ such that $IOnw_B^\alpha(\vec{X}) = [1,1]$. Then, one has that

$$K_\alpha(IOnw_B^\alpha(\vec{X})) = K_\alpha([1,1]) = 1 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)).$$

By **(On3)**, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$, meaning that $\prod_{i=1}^n X_i = [1,1]$;

$(\Leftarrow)$ Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$. Then, by **(On3)**, one has that

$$K_\alpha(IOnw_B^\alpha(\vec{X})) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 1,$$

meaning that $IOnw_B^\alpha(\vec{X}) = [1,1]$. □

Example 11. Consider the $\leq_{IQ}$-overlap function $AOn_{OnB,\max}^\alpha$ given by Theorem 2 for $B = \max$, $On = OnB$ given by Eq. (3), $\alpha = 0.5$ and $\beta = 0$ (admissible order $\leq_{IQ}$). Then, the iv-fusion function $IOnw_{OnB,\max}^\alpha : L([0,1])^n \to L([0,1])$ defined, for all $\vec{X} \in L([0,1])^n$, by

$$IOnw_{OnB,\max}^\alpha(\vec{X}) = [K_\alpha(AOn_{OnB,\max}^\alpha(\vec{X})) - \alpha m_{AOn_{OnB,\max}^\alpha,\max}(\vec{X}),\ K_\alpha(AOn_{OnB,\max}^\alpha(\vec{X})) + (1-\alpha) m_{AOn_{OnB,\max}^\alpha,\max}(\vec{X})], \tag{25}$$

is a w-iv-overlap function for the tuple $(\leq_{IQ}, \leq_{IQ}, \max)$.

Let us consider the same cases as in Example 9, but now applying CMA. So, take $n = 2$, $X_1 = [0.2, 0.8]$ and $X_2 = [0.5, 1]$, meaning that $\max\{w(X_1), w(X_2)\} = 0.6$. Also, from Theorem 2, it holds that:

$$AOn^{\alpha}_{OnB,\max}(\vec{X}) = R \quad \text{where} \begin{cases} K_\alpha(R) = OnB(K_\alpha(X_1), K_\alpha(X_2)), \\ \lambda_\alpha(R) = \max\{\lambda_\alpha(X_1), \lambda_\alpha(X_2)\}, \end{cases} \tag{26}$$

with $\lambda_\alpha(X_1)$ and $\lambda_\alpha(X_2)$ given by Definition 17. So, we have that:

$$\begin{aligned}
&w(AOn^{\alpha}_{OnB,\max}(\vec{X})) \\
&= \lambda_\alpha(AOn^{\alpha}_{OnB,\max}) \cdot \min\left\{\frac{K_\alpha(AOn^{\alpha}_{OnB,\max})}{\alpha}, \frac{1-K_\alpha(AOn^{\alpha}_{OnB,\max})}{1-\alpha}\right\} \text{ by Eq. (10)} \\
&= \max\{\lambda_\alpha([0.2,0.8]), \lambda_\alpha([0.5,1])\} \cdot \\
&\quad \min\left\{\frac{OnB(K_\alpha([0.2,0.8]), K_\alpha([0.5,1]))}{0.5}, \frac{1-OnB(K_\alpha([0.2,0.8]), K_\alpha([0.5,1]))}{1-0.5}\right\} \text{ by Eq. (26)} \\
&= \max\left\{\frac{w([0.2,0.8])}{d_\alpha(K_\alpha([0.2,0.8]))}, \frac{w([0.5,1])}{d_\alpha(K_\alpha([0.5,1]))}\right\} \cdot \\
&\quad \min\left\{\frac{OnB(0.5,0.75)}{0.5}, \frac{1-OnB(0.5,0.75)}{0.5}\right\} \text{ by Eq. (8)} \\
&= \max\left\{\frac{0.6}{\min\left\{\frac{0.5}{0.5},\frac{0.5}{0.5}\right\}}, \frac{0.5}{\min\left\{\frac{0.75}{0.5},\frac{0.25}{0.5}\right\}}\right\} \cdot \\
&\quad \min\left\{\frac{0.433}{0.5}, \frac{1-0.433}{0.5}\right\} \text{ by Eq. (9)} \\
&= \max\left\{\frac{0.6}{1}, \frac{0.5}{0.5}\right\} \cdot \min\{0.866, 1.134\} = 0.866.
\end{aligned}$$

So, it follows that:

$$w(AOn^{\alpha}_{OnB,\max}(\vec{X})) = 0.866 > 0.6 = \max\{w(X_1), w(X_2)\},$$

meaning that $AOn^{\alpha}_{OnB,\max}$ is not width-limited by max. By applying CMA, from Eq. (25), one has that:

$$\begin{aligned}
&IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X}) \\
&= [K_{\alpha=0.5}(AOn^{\alpha}_{OnB,\max}(\vec{X})) - 0.5 \cdot m_{AOn^{\alpha}_{OnB,\max},\max}(\vec{X}), \\
&\quad K_{\alpha=0.5}(AOn^{\alpha}_{OnB,\max}(\vec{X})) + 0.5 \cdot m_{AOn^{\alpha}_{OnB,\max},\max}(\vec{X})] \\
&= [OnB(0.5,0.75) - 0.5 \cdot \min\{0.866, 0.6\}, \\
&\quad OnB(0.5,0.75) + 0.5 \cdot \min\{0.866, 0.6\}] \\
&= [0.433 - 0.5 \cdot 0.6, 0.433 + 0.5 \cdot 0.6] \\
&= [0.133, 0.733].
\end{aligned}$$

So, $w(IOnw^{\alpha=0.5}_{OnB,\max}(\vec{X})) = 0.6 \leq \max\{w(X_1), w(X_2)\}$, which is expected from a function that is width-limited by max.

For a similar reason as observed in Example 9, whenever one has that $w(AOn^{\alpha}_{OnB,\max}(\vec{X})) \leq \max\{w(X_1), w(X_2)\}$, for some $\vec{X} \in L([0,1])^n$ and $\alpha \in (0,1]$, then the output already respects the width control dictated by max, and, in this case, $IOnw^{\alpha}_{OnB,\max}(\vec{X}) = AOn^{\alpha}_{OnB,\max}(\vec{X})$.

Some values obtained by both $IOnw^{\alpha}_{OnB,B}(\vec{X})$ and $AOn^{\alpha}_{OnB,B}(\vec{X})$ (from Example 11) can be seen on Table 2, by varying the value of α and the chosen width-limiting function B. One can observe that, in Table 2, every result obtained by CMA ($IOnw^{\alpha}_{OnB,B}(\vec{X})$) is contained on the interval given by the corresponding $\leq_{\alpha,\beta}$-overlap function without width-limitation ($AOn^{\alpha}_{OnB,B}(\vec{X})$). Also, different from the BIR, $AOn^{\alpha}_{OnB,B}$ varies accordingly to the chosen α. Finally, it is noteworthy that, in both Example 11 and Table 2, we applied the same function B for both $IOnw^{\alpha}_{OnB,B}(\vec{X})$ and $AOn^{\alpha}_{OnB,B}(\vec{X})$, but this is not a requirement. We decided to keep both iv-functions based on the same B for simplicity.

At this point, it is clear that other classes of w-iv-aggregation functions can be obtained through CMA, by the appropriate choices of a strict aggregation function A, a width-limiting function B, α, β and an $\leq_{\alpha,\beta}$-increasing iv-aggregation function IA^{α}.

5.3. Applications to practical problems

When applying either of the construction methods of width-limited iv-fusion functions in practical problems, the domain expert has to make some key choices, as explained bellow:

1. *The choice of fusion function F*: usually, when extending a fusion/aggregation process modeled by a fusion function F to the interval context, one can maintain the same fusion function F as the core of the construction method to be employed. For example, it was shown by Asmus et al. [33] that the n-dimensional overlap functions GM and OnB, defined in Eqs. (2) and (3), respectively, are well fitted to be employed in classification problems. Thus, it is natural that those functions are chosen to be the core of some w-iv-fusion functions to be applied in IV-FRBCSs with width-limitation (see Section 6);

2. *The choice of α*: It is entirely determined by the admissible order $\leq_{\alpha,\beta}$ that is indicated for the application. The choice of the interval order depends on how the intervals are obtained or interpreted [43,45,47]. To keep the same example, Asmus et al. [33] showed that the admissible order $\leq_{IQ}$ ($\alpha = 0.5, \beta = 0$), defined in Eq. (6), is a suitable choice for IV-FRBCSs.

3. *The choice of the width-limiting function B*: it depends on how much the length of the output interval's width has to be controlled to conserve the information quality of the interval inputs, since the larger the width of the interval output, the lesser is the information quality carried by it [48]. This level of control to be determined may not be obvious, but there are ways to test/compare different configurations of the same construction method by taking into account different width-limiting functions, as we show in the application in a classification problem, presented in Section 6.

6. Application to classification problems

To showcase the applicability of our developments in practical problems, in this section we apply interval operators of specific subclasses of w-iv-fusion functions in the IVTURS IV-FRBCS. In the work by Asmus et al. [33], it was shown that n-dimensional overlap functions (and interval-valued functions based on them) are recommended to be applied in this type of problem. Furthermore, the best performing methods on that paper are based on n-dimensional $\leq_{\alpha,\beta}$-increasing iv-overlap functions that are width-limited by the minimum, that is, the outputs' widths are lesser or equal than the widths of the inputs. So, the class of n-dimensional w-iv-overlap functions as defined in Example 5 seems a natural choice to provide functions for this application.

In the following, first we recall some key concepts regarding IV-FRBCSs. After that, we present the experimental framework, followed by the analysis of the results.

6.1. IV-FRBCSs

A classification problem is composed by P training examples $\vec{x_p} = (x_{p1}, \ldots, x_{pn})$, $p \in \{1, \ldots, P\}$ where x_{pi} is the value of the ith variable of the pth example. Each example belongs to one of M classes in $C=\{C_1, \ldots, C_M\}$. The goal of the learned classifier is to identify the class of new testing examples.

Table 2
CMA, comparing with $\leq_{\alpha,\beta}$-overlap functions.

	CMA	$\leq_{\alpha,\beta}$-overlap function
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\max}(\vec{X}) = [0.13, 0.73]$	$AOn^{0.5}_{OnB,\max}(\vec{X}) = [0, 0.87]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \max$ $\alpha = 0.99$	$IOnw^{0.99}_{OnB,\max}(\vec{X}) = [0.19, 0.79]$	$AOn^{0.99}_{OnB,\max}(\vec{X}) = [0, 0.80]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 0.5$	$IOnw^{0.5}_{OnB,\min}(\vec{X}) = [0.18, 0.68]$	$AOn^{0.5}_{OnB,\min}(\vec{X}) = [0.17, 0.69]$
$X_1 = [0.2, 0.8]$ $X_2 = [0.5, 1]$ $B = \min$ $\alpha = 1$	$IOnwp^{0.99}_{OnB,\min}(\vec{X}) = [0.29, 0.79]$	$AOn^{0.99}_{OnB,\min}(\vec{X}) = [0.19, 0.80]$

One of the most frequently applied techniques to deal with classification problems are the FRBCSs. They can achieve accurate results by using highly interpretable models, since the fuzzy rules are expressed by linguistic labels [35]. The structure of the fuzzy rules is given by:

$$\text{Rule } R_j : \text{If } x_1 \text{ is } A_{j1} \text{ and } \dots \text{ and } x_n \text{ is } A_{jn} \quad \text{then } Class = C_j' \text{ with } RW_j, \tag{27}$$

where R_j is the label of the jth rule, $x = (x_1, \dots, x_n)$ is an n-dimensional example vector, A_{ji} is the fuzzy set modeling the linguistic term of the jth rule in the ith antecedent, $C_j' \in C$ is a class label, and $RW_j \in [0, 1]$ is the rule weight [70]. In particular, the rule weight is computed by the fuzzy confidence value (or certainty factor), as follows:

$$RW_j = \frac{\sum_{x_p \in C_j'} A_j(x_p)}{\sum_{p=1}^{P} A_j(x_p)}, \tag{28}$$

where $A_j(x_p)$ is the matching degree of the pattern x_p with the antecedent part of the fuzzy rule R_j, given by

$$A_j(x_p) = \mathfrak{c}(A_{j1}(x_{p1}), \dots, A_{jn}(x_{pn})), \tag{29}$$

where $\mathfrak{c}$ is an n-dimensional conjunction operator and $j \in \{1, \dots, L\}$.

A FRBCS becomes an IV-FRBCS when some of the linguistic labels (or all of them) are modeled using IVFSs. This means that the fuzzy reasoning method must work with intervals instead of numbers, being called an iv-fuzzy reasoning method (IV-FRM), to take into account the interval widths (uncertainty) throughout the whole inference process [71] (see Section 6.2). As a novelty for this kind of classifier, we apply width-limited functions to control the information quality of the interval operations that occur on the IV-FRM, and analyze if such width control improves the performance of the system.

6.2. New iv-fuzzy reasoning method

For the following experimentation, we apply our new theoretical concepts in the IVTURS algorithm, which is a state of the art IV-FRBCS (an in-depth look at each step of the IVTURS algorithm can be seen in the work by Sanz et al. [72]). Here, we recall the steps of its learning process:

(1) *The building of an IV-FRBCS*, by the following procedures:

(a) To generate an initial FRBCS by applying the two first stages of FARC-HD [73], a technique that is based on the Apriori algorithm [74] to build fuzzy rules (Eq. (28)) in its first learning stage. In those fuzzy rules, the product t-norm is usually applied as the conjunction operator $\mathfrak{c}$ in Eq. (29). However, as shown by Asmus et at. [33], one can benefit from replacing the product t-norm by other n-dimensional overlap functions On, such as the Geometric Mean and the n-dimensional OnB-overlap, given by Eqs. (2) and (3), respectively. This allows for the chosen n-dimensional overlap functions to be considered as the core functions for the construction of the n-dimensional w-iv-overlap functions to be used in the IV-FRM (described in the sequence). Thus, the function On impacts the learning process of the fuzzy rules, which means that if On is not the product, then the obtained fuzzy rules would not necessarily be the same than those obtained by the original IVTURS.

(b) To define IVFSs to model the linguistic labels of the learned FRBCS;

(c) To generate initial IV-REFs for each variable of the problem.

(2) *The application of an optimization approach*, so that:

(a) It learns the best values of the IV-REFs' parameters;

(b) It applies a rule selection process to decrease the system's complexity.

After creating the interval-valued fuzzy rules that compose the system, we define the new mechanism for classifying examples, as follows.

Let $\vec{x}_{\vec{p}} = (x_{p1}, \dots, x_{pn})$ be a new example to be classified, L be the number of rules in the rule base and M be the number of classes of the problem. The new IV-FRM is defined by the following steps:

(1) *Interval matching degree*: It expresses activation strength the rules' antecedents for each x_p. The similarity between the interval membership degrees (of each variable of the pattern to the corresponding IVFS) and the ideal membership degree $[1, 1]$ is computed by an IV-REF IR and, then, we use an interval-valued fusion function $F_O : L([0, 1])^n \to L([0, 1])$, for $j \in \{1, \dots, L\}$ to combine their results as follows:

$$\mathcal{A}_j(x_p) = F_O\left(IR\left(\mathcal{A}_{j1}(x_{p1}), [1, 1]\right), \dots, IR\left(\mathcal{A}_{jn}(x_{pn}), [1, 1]\right)\right),$$

with F_O being an n-dimensional w-iv-overlap function based on the n-dimensional overlap function On (applied as the conjunction operator when generating the initial FRBCS) and an increasing and symmetric width-limiting function B, which will determine how much information quality control we desire on the IV-FRM.

(2) *Interval association degree*: For the respective class of each rule, the interval matching degree is weighted with the corresponding iv-rule weight $IRW_j^k \in L([0,1])$, through an iv-fusion function $F_P : L([0,1])^n \to L([0,1])$, as follows:

$$b_j^k = F_P\left(\mathcal{A}_j(x_p), IRW_j^k\right), \tag{30}$$

with $k = 1, \ldots, M$, $j = 1, \ldots, L$ and F_P being an interval-valued product operation, applied with the same criteria for width-limitation as the one for F_O.

The rule weight is defined by the interval-valued confidence value [75], given by:

$$IRW_j = \sum_{x_p \in C_j'} \mathcal{A}_j(x_p) \div_H \sum_{p=1}^{P} \mathcal{A}_j(x_p),$$

with $\div_H$ being defined as in Eq. (5).

(3) *Interval pattern classification soundness degree for all classes*: All interval association degrees of each class (obtained in Step (2)) with upper bounds that are greater than 0 are aggregated by an interval-valued aggregation function IA:

$$Y_k = IA\left(b_j^k, j = 1, \ldots, L \text{ and } \overline{b_j^k} > 0\right),$$

with $k = 1, \ldots, M$.

(4) *Classification*: A decision function F is applied over the interval pattern classification soundness degrees for all classes, as follows:

$$F(Y_1, \ldots, Y_M) = \arg\max_{k=1,\ldots,M} (Y_k).$$

In this final step of the IV-FRM, the system selects the greatest interval soundness degree, which is done by comparing the resulting intervals through an admissible order (in order to avoid a stalemate). As discussed by Asmus et al. [33], the order $\leq_{IQ}$ (order of Xu-Yager based on the quality of information, or $\leq_{\alpha,\beta}$ with $\alpha = 0.5$ and $\beta = 0$) is a suitable option for this type of classification problem, so we opted for this admissible order in our experiment.

6.3. Experimental framework

The general goal of our experiment is to analyze the classification performance of our new methodology, by testing and comparing different configurations of the proposed IV-FRM, some of which allow controlling the information quality. To that end, we apply different n-dimensional w-iv-overlap functions obtained by either the construction method based on representable fusion functions (CMR) or the construction method based on $\leq_{\alpha,\beta}$-increasing fusion functions (CMA).

To conduct our experiment, we have selected 31 real-world data-sets from the KEEL repository [76], which are publicly available on the webpage (http://www.keel.es/dataset.php). Table 3 summarizes the properties of the considered data-sets, presenting, for each data-set, the numbers of attributes (Atts.), examples (Ex.), and classes (Class.). To improve the learning process efficiency, the *magic, page-blocks, penbased, ring, satimage, shuttle*, and *twonorm* data-sets have been stratified sampled at 10%. Also, missing values from *bands*, *cleaveland* and *wisconsin* data-sets have been removed before our experiments.

We apply a *5-fold cross-validation model*, by dividing each data-set in 5 random partitions. Four of them (80%) are combined to train the system and the remaining one (20%) is reserved for testing. This process is executed 5 times, changing the testing partition in each iteration. The accuracy rate (percentage of well classified examples) is used to measure the system's performance.

We follow the recommendation provided by Sanz et al. [72] for the set-up of the IVTURS classifier, with the modifications explained in Section 6.2. Then, we analyze the classification performance by comparing different configurations based on the function F_O applied on Step (1) of the IV-FRM, which is determined by both the corresponding On used on the learning process of the fuzzy rules and the weighting operation (F_P) used in the Step (2) of the IV-FRM, as shown in Table 4. The selected n-dimensional overlap functions (On) to be used as the core of F_O were based on the best performing operations for this kind of classifier [33], namely, GM and OnB (Eqs. (2) and (3), respectively), as well as the product (OnP), since it is the operation used on the original IVTURS.

From Table 4, it can be seen that there are nine methods to be tested and compared, belonging to three groups:

(i) REP: F_O is obtained by the BIR of On ($\widehat{On}$);

(ii) CONR: F_O is obtained by CMR, via Theorem 12 ($IOnw^{\alpha}_{On,B}$);

(iii) CONA: F_O is obtained by CMA, via Theorem 20 ($AOnw^{\alpha}_{On,B}$).

In each group, we have three methods, and each one of them is based on one of the three n-dimensional overlap functions that were selected for this study (OnP, OnB and GM). For instance, in group REP, we have REP-Prod, REP-GM and REP-OnB. In summary, from the nine considered methods, the six from groups CONR and CONA are based on our new developed concepts that allow for an arbitrary level of width limitation, while the three from group REP are instances of the new IV-FRM that coincide with methods that were developed on a previous work [33] and have no width control.

When F_O is given by an n-dimensional w-iv-overlap function (all the approaches derived for the CONR and CONA groups), we check the effect of the width-limitation by comparing the results of the classifier when varying the width-limiting function B, given by:

$$\begin{aligned} B^{\rho}(w(x_1), \ldots, w(x_n)) = {} & \min\{w(x_1), \ldots, w(x_n)\} + \\ & \rho(\max\{w(x_1), \ldots, w(x_n)\} - \min\{w(x_1), \ldots, w(x_n)\}), \end{aligned} \tag{31}$$

with $\rho \in [0,1]$. Specifically, we test each configuration with five possible values for ρ: $\rho = 0$ ($B = \min$); $\rho = 0.25$; $\rho = 0.5$; $\rho = 0.75$; $\rho = 1$ ($B = \max$). In this manner, the parameter ρ indicates the amount of width control that we are imposing on the system. When $\rho = 0$, the output's width is limited by the minimum of the inputs' widths, representing the most strict width limitation. Conversely, when $\rho = 1$, the output's width is limited by the maximum of the inputs' widths, representing the less width control.

To detect if there are statistical differences in performance among the methods in a selected group, first, we use the aligned Friedman ranks test [77], reporting the obtained ranks of each method (the less the rank, the better). The best ranking method of such group is compared with the others through a Holm's post-hoc test [78]. When the goal is to provide a pairwise comparison, we apply a Wilcoxon Signed-Ranks test [79]. García et al. [80] showed that this selection of statistical tests is highly recommended to be used in machine learning.

6.4. Discussion of the results

In Table 5 we present results in testing (average of the accuracy rate obtained in the 31 datasets) for all the possible configurations of the new method, one in each row (the same ones as shown in Table 4). All approaches based on the construction methods CONR and CONA allow the control of the interval widths by means of the hyper-parameter ρ, whose results are shown in columns. On the other hand, approaches belonging to the REP group do not allow such control and, therefore, we present their results in a single column as they are not affected by the hyper-parameter ρ. For each ρ, we highlight in bold face

Table 3
Summary of the employed datasets.
Source: KEEL, 2011.

id	Data-set	Atts.	Ex.	Class.
app	appendicitis	7	106	2
bal	balance	4	625	3
ban	banana	2	5300	2
bds	bands	19	365	2
bup	bupa	6	345	2
clv	cleveland	13	297	5
con	contraceptive	9	1473	3
eco	ecoli	7	336	8
gla	glass	9	214	7
hab	haberman	3	306	2
hay	hayes-hoth	4	160	3
ion	ionosphere	33	351	2
iri	iris	4	150	3
led	led7digit	7	500	10
mag	magic	10	19020	2
new	newthyroid	5	215	3
pag	pageblocks	10	5472	5
pen	penbased	16	10992	10
pho	phoneme	5	5404	2
pim	pima	8	768	2
rin	ring	20	7400	2
sah	saheart	9	462	2
sat	satimage	36	6435	7
shu	shuttle	9	58000	7
spe	spectfheart	44	267	2
tit	titanic	3	2201	2
two	twonorm	20	7400	2
veh	vehicle	18	846	4
win	wine	13	178	3
wis	wisconsin	9	683	2
yea	yeast	8	1484	10

Table 4
Configuration schemes for the used classifiers.

Classifier identifier	On	F_O	F_P
REP-Prod	OnP	$IOn_P = \widehat{OnP}$	$IOn_P = \widehat{OnP}$
REP-OnB	OnB	$IOnB = \widehat{OnB}$	$IOn_P = \widehat{OnP}$
REP-Gm	GM	$IGM = \widehat{GM}$	$IOn_P = \widehat{OnP}$
CONR-Prod	OnP	$IOnw^{\alpha}_{OnP,B}$	$IOnw^{\alpha}_{OnP,B}$
CONR-OnB	OnB	$IOnw^{\alpha}_{OnB,B}$	$IOnw^{\alpha}_{OnP,B}$
CONR-Gm	GM	$IOnw^{\alpha}_{GM,B}$	$IOnw^{\alpha}_{OnP,B}$
CONA-Prod	OnP	$AOnw^{\alpha}_{OnP,B}$	$AOnw^{\alpha}_{OnP,B}$
CONA-OnB	OnB	$AOnw^{\alpha}_{OnB,B}$	$AOnw^{\alpha}_{OnP,B}$
CONA-Gm	GM	$AOnw^{\alpha}_{GM,B}$	$AOnw^{\alpha}_{OnP,B}$

Table 5
Results in testing for the different methods.

Method					
REP-Prod			79.56		
REP-OnB			79.83		
REP-Gm			79.75		
	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
CONR-Prod	79.82	79.62	79.58	79.61	79.20
CONR-OnB	**80.11**	79.76	79.79	79.87	79.76
CONR-Gm	79.65	79.90	79.71	79.71	79.95
CONA-Prod	79.54	79.48	79.46	79.43	79.60
CONA-OnB	80.06	**80.00**	**80.02**	**80.09**	79.87
CONA-Gm	79.91	79.94	79.94	79.91	**80.04**

the best result, that is, the configuration of the classifier that produced the greatest global mean. The detailed results for all the datasets (in all the partitions), with every possible combination, can be queried on https://github.com/tiagoasmus/testingResults-w-iv-overlaps.

By observing Table 5, we see that the methods from group REP are not able to obtain better averaged behaviors than those highlighted in the second part of the table, that is, the best configurations of the methods that allow one to control de output's interval width. Moreover, most of the highlighted results come from the group CONA, with the exception of one method that is from group CONR (CONR-OnB with $\rho = 0$), which is also the method with the best global mean. Therefore, we can affirm that methods from groups CONR and CONA produce good results,

Table 6
Average rankings of the algorithms (Aligned Friedman) – Comparing levels of width control ρ.

Method	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
CONR-Prod	**60.61** (−)	77.87 (0.390)	72 (0.390)	77.76 (0.390)	101.76 (0.001)*
CONR-OnB	**61.31** (−)	84.81 (0.118)	81.40 (0.156)	75.34 (0.218)	87.15 (0.094)*
CONR-GM	86.58 (0.383)	70.47 (0.800)	80.18 (0.539)	85.19 (0.383)	**67.58** (−)
CONA-Prod	73.02 (0.583)	80.19 (0.477)	83.97 (0.394)	86.07 (0.362)	**66.76** (−)
CONA-OnB	80.16 (1.000)	78.42 (1.000)	76.79 (1.000)	**72.24** (−)	82.39 (1.000)
CONA-GM	81.65 (0.529)	78.36 (0.529)	79.69 (0.529)	84.08 (0.469)	**66.23** (−)

Table 7
Average rankings of the algorithms (Aligned Friedman) – Comparison by groups.

Method	Rank (APV)	Method	Rank (APV)	Method	Rank (APV)
REP-Prod	50.61 (0.500)	CONR-Prod^{0}	49.44 (0.628)	CONA-Prod^{1}	57.31 (0.029*)
REP-OnB	**42.73** (−)	CONR-OnB^{0}	**42.53** (−)	CONA-OnB$^{0.75}$	**40.55** (−)
REP-GM	47.66 (0.500)	CONR-GM^{1}	49.03 (0.628)	CONA-GM^{1}	43.15 (0.705)

possibly due to the control of the interval widths in the interval operations. In particular, the method CONA-OnB achieves a very robust performance for every considered level of width limitation.

Next, we study the effect of the level of width control performed in each method (the different values of ρ). To do it, we compare the five possible values of ρ for each method, by applying the aligned Friedman's test. The obtained ranks, as well as the Adjusted P-Values (APVs, presented in brackets) obtained by the Holm's post hoc test are shown in Table 6, where we have highlighted in **bold-face** the best rank (the less one) and we have stressed with an asterisk (*) those cases where there are statistical differences (using $\alpha = 0.1$) in favor to the ρ associated to the less rank. Looking at Table 6, one can observe that the benefit from a more rigid width control depends on the applied interval-valued function:

(a) For the group CONR, two algorithms with a more strict width control produced better results (CONR-Prod and CONR-OnB, both with $\rho = 0$ as the control method). In both cases, there are significant differences from the control method ($\rho = 0$, strong width-limitation), and the algorithm with $\rho = 1$ (least considered width-limitation);

(b) Considering the group CONA, the method CONA-Prod appear to perform better with a less aggressive width limitation (control method has $\rho = 1$). Confirming the previous observation, CONA-OnB achieved good results for every considered level of width control, as indicated by the same APV = 1 obtained for each of its configurations when compared to the control method ($\rho = 0.75$);

(c) Independently of the employed Construction method, the algorithms that are based on the geometric mean also seem more accurate with less strict width control, as the control methods have $\rho = 1$ for both CONR-GM and CONA-GM.

Next, we apply three Aligned Friedman and Holm's tests, one for each group (REP, CONR and CONA), to compare the best performing algorithms from each group. In the case of the group REP, we test the three considered methods as they do not depend on the values of ρ. For the groups CONR and CONA, we compare the control methods we obtained from Table 6. We indicate the value of ρ of each method as a superindex, when necessary. For example, the method CONR-Prod with $\rho = 0$ is denoted by CON-PROD^{0}. The results of these tests are shown in Table 7, with the best ranking method in each group highlighted in **bold-face** and methods that present significant difference from the control method are marked with an asterisk (*).

Observing Table 7, it is clear that the methods based on the n-dimensional overlap function *OnB* have good performance, as they are the control methods in each of the groups (REP, CONR and CONA). Next, we statistically compare those three best performing methods in another aligned test, whose obtained results are presented on Table 8. From Table 8, we see that the method

Table 8
Average rankings of the algorithms (Aligned Friedman) – Comparing construction methods.

Method	Rank (APV)
REP-OnB	55.87 (0.069*)
CONR-OnB^{0}	43.77 (0.724)
CONA-OnB$^{0.75}$	**41.36** (−)

Table 9
Pairwise comparisons via Wilcoxon test.

Comparison	R^{+}	R^{-}	p-value
IVTURS vs REP-OnB	181.5	314.5	0.214
IVTURS vs CONR-OnB^{0}	131.5	364.5	0.024*
IVTURS vs CONA-OnB$^{0.75}$	125.5	370.5	0.018*

CONA-OnB$^{0.75}$ is the best option, although CONR-OnB^{0} also performs well. The method REP-OnB, does not achieve the same level of performance of the other two compared methods, being significantly less accurate than the control method. As those three methods are all based on the same core n-dimensional overlap function (*OnB*), which is used throughout all the components of those algorithms, the main difference between them lies on the construction of the interval-valued operations that take place in the IV-FRM, which may or may not control the widths of the outputs of such operations. Thus, we can conclude that controlling the width of the intervals, which implies having intervals with better information quality, is beneficial for the system's performance.

Finally, to further analyze the benefits of the new proposed methods, we carry out three pairwise comparisons between the best performing method from each group with the original configuration of the IVTURS algorithm (REP-Prod), through the Wilcoxon test. These results can be seen on Table 9, with the results with significant differences marked with an asterisk (*). Analyzing Table 9, it is clear that the configurations of both CONR-OnB^{0} and CONA-OnB$^{0.75}$ improve significantly the performance of the IVTURS algorithm, whereas the method REP-OnB does not improve the accuracy of IVTURS in the same manner.

To illustrate the superiority in performance of CONR-OnB^{0} and CONA-OnB$^{0.75}$ when compared to IVTURS, firstly, Fig. 1 shows the difference in accuracy between CONR-OnB^{0} and IVTURS on each dataset (as denoted on Table 3). The positive values (orange bars) indicate that CONR-OnB^{0} was more accurate on a dataset, while the negative values (blue bars) indicate that IVTURS was more accurate on a dataset. Analogously, Fig. 2 illustrate the pairwise comparison between IVTURS and CONA-OnB$^{0.75}$. Thus, the superior performance of both CONR-OnB^{0} and CONA-OnB$^{0.75}$ over IVTURS can be visualized by the predominance of orange bars on Figs. 1 and 2.

Fig. 1. Difference in accuracies between IVTURS and CONR-OnB^{0}.

Fig. 2. Difference in accuracies between IVTURS and CONA-OnB$^{0.75}$.

Therefore, we can conclude that the exchange from the product to the n-dimensional overlap *OnB* is not the sole reason for the better performance of CONR-OnB^{0} and CONA-OnB$^{0.75}$, indicating that these new methods benefit from a certain amount of width limitation.

7. Conclusion

When aggregating interval data through iv-aggregation functions, usually by means of the BIR, one may face the problem of dealing with interval outputs with large widths and, thus, low information quality. With the motivation to tackle this sort of challenge, in this paper, we presented a general framework to define and construct different subclasses of w-iv-fusion functions, allowing for the control of the interval output widths provided by interval aggregation operations that occur in practical problems, such as in IV-FRBCSs. From that, we have the following contributions:

1. The development of the concept of width-limitation, with the extension to the n-dimensional context of width-limited iv-fusion functions and width-limiting fusion functions;

2. The characterization of increasing fusion functions through a set of properties, a form of representation that facilitates the definition of interval-valued counterparts of such functions;

3. The definition of classes of w-iv-fusion functions based on an increasing fusion function, the interval extension of its set of properties and a pair of partial orders. This general methodology is capable of retrieving known definitions of iv-aggregation functions from the literature while also providing a flexible way to obtain iv-fusion functions with a desirable amount of width-limitation;

4. Two approaches to provide construction methods for w-iv-fusion functions, one based on the representable interval functions (CMR) and another based on admissibly ordered interval functions (CMA), where the interval outputs are "narrowed" in the direction of a K_α point and whose widths to not surpass a given threshold.

One of the key aspects of the developed framework and the presented construction methods is their flexibility, derived from the different possible choices of fusion functions, interval orders and width-limiting functions. This flexibility translates into potential applicability, as one can define a particular class of w-iv-fusion function accordingly to the constraints/requirements of a given practical problem. This aspect was highlighted in our case study, where we developed and applied a new IV-FRM for IV-FRBCSs, in which the information quality is controlled by n-dimensional w-iv-overlap functions, whose class is defined through our general framework. From our experimentation and subsequent statistical analysis, we can draw the following conclusions:

1. Configurations of the classifier based on n-dimensional w-iv-overlap functions constructed via either CMR or CMA have good classification accuracy, in general;

2. Although the amount of width control that benefits the performance of the system varies for each choice of w-iv-fusion function, this information can be retrieved by defining the width-limiting function through a parameter (ρ). In this manner, we can compare different values of ρ for each algorithm and possibly determine how much the widths of the outputs have to be constrained;

3. Configurations of the classifier based on the n-dimensional overlap function *OnB* produce the best results. Among those configurations, the one based on the CMA method have a significantly higher classification accuracy than the one based on the BIR of *OnB*. Particularly, CMA-OnB produces good results for every considered level of width limitation, presenting itself as a very stable method;

4. The two best performing methods, one based on CMR and other based on CMA, significantly enhances the classification accuracy of the state-of-the-art IVTURS algorithm, showing that both construction method approaches are suitable to provide w-iv-fusion functions to be applied in classification problems, which can benefit from the width control provided by such methods.

From these conclusions, we can state that the new IV-FRM, which is based on w-iv-fusion functions constructed via the proposed framework, is recommended to be considered as a key part of IV-FRBCSs where the information quality carried by the operated intervals is taken into account and kept under control throughout the inference process.

In future works we intend to apply our general framework to define and construct w-iv-fusion functions to be employed in aggregation processes with uncertainty (e.g., sensor data fusion), as in techniques for multicriteria decision making [27] and image processing [25].

CRediT authorship contribution statement

Tiago da Cruz Asmus: Writing – original draft, Conceptualization, Methodology, Investigation, Software, Writing – review & editing. **José Antonio Sanz:** Investigation, Software, Writing – review & editing. **Graçaliz Pereira Dimuro:** Conceptualization, Methodology, Writing – review & editing. **Javier Fernandez:** Conceptualization, Methodology, Writing – review & editing. **Radko Mesiar:** Conceptualization, Methodology, Writing – review & editing. **Humberto Bustince:** Conceptualization, Writing – review & editing, Supervision, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 2

Proof. Consider a symmetric aggregation function $B : [0,1]^n \to [0,1]$, a strict n-dimensional overlap function $On : [0,1]^n \to [0,1]$ and let $\alpha \in (0,1)$ and $\beta \in [0,1]$ such that $\alpha \neq \beta$. Observe that AOn^α_B is well defined. In fact, considering that $AOn^\alpha_B(\vec{X}) = R$, one has that $w(R) = On(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_n)) \cdot d_\alpha(K_\alpha(R))$, $\underline{R} = K_\alpha(R) - \alpha \cdot w(R)$ and $\overline{R} = K_\alpha(R) + (1-\alpha) \cdot w(R)$. Now, let us verify if $IAOn^\alpha_B$ respects all conditions from Definition 19.

(IOn1) Immediate, since On and B are both symmetric;

(IOn2) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$ and suppose that $AOn^\alpha_B(\vec{X}) = R = [0,0]$. Then, we have that

$$K_\alpha(R) = K_\alpha([0,0]) = 0 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)),$$

since $\alpha \in (0,1)$. Thus, by condition **(On2)**, $K_\alpha(X_i) = 0$ for some $i \in \{1,\ldots,n\}$, and, therefore, $\prod_{i=1}^n X_i = [0,0]$;
$(\Leftarrow)$ Consider $\vec{X_i} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [0,0]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 0$, since $\alpha \in (0,1)$. Then, by **(O2)**, one has that $K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 0$, meaning that $AOn^\alpha_B(\vec{X}) = R = [0,0]$;

(IOn3) $(\Rightarrow)$ Take $\vec{X} \in L([0,1])^n$ such that $AOn^\alpha_B(\vec{X}) = R = [1,1]$. Then, one has that

$$K_\alpha(R) = K_\alpha([1,1]) = 1 = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)).$$

By **(On3)**, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$, meaning that $\prod_{i=1}^n X_i = [1,1]$;
$(\Leftarrow)$ Consider $\vec{X} \in L([0,1])^n$ such that $\prod_{i=1}^n X_i = [1,1]$. So, $K_\alpha(X_1) \cdot \ldots \cdot K_\alpha(X_n) = 1$, since $\alpha \in (0,1)$. Then, by **(i)** and **(O3)**, one has that $K_\alpha(R) = On(K_\alpha(X_1), \ldots, K_\alpha(X_n)) = 1$, meaning that $AOn^\alpha_B(\vec{X}) = R = [1,1]$;

(AOn4) Consider $Z \in L([0,1])$, $\vec{X}, \vec{Y} \in L([0,1])^n$, such that there exist $k \in \{1,\ldots,n\}$ for which $X_k \leq_{\alpha,\beta} Y_k$ and $X_i = Y_i = Z$ for all $i \in \{1,\ldots,n\} - \{k\}$. So, it holds that $X_i \leq_{\alpha,\beta} Y_i$ for all $i \in \{1,\ldots,n\}$. By Lemma 1, one can consider $\beta = 0$ or $\beta = 1$. Here, we present the proof for $\beta = 0$.

If $K_\alpha(Z) = 0$, then we have that $K_\alpha(AOn^\alpha_B(\vec{X})) = 0 = K_\alpha(AOn^\alpha_B(\vec{Y}))$, which means that $AOn^\alpha_B(\vec{X}) = [0,0] = AOn^\alpha_B(\vec{Y})$, since $\alpha \neq 0$. Then, $AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y})$.

If $K_\alpha(Z) \neq 0$, then we have the following cases:

(a) $K_\alpha(X_k) < K_\alpha(Y_k)$: Since On is strict, one has that $K_\alpha(AOn^\alpha_B(\vec{X})) < K_\alpha(AOn^\alpha_B(\vec{Y}))$. Therefore,

$$K_\alpha(AOn^\alpha_B(\vec{X})) < K_\alpha(AOn^\alpha_B(\vec{Y}))$$
$$\Rightarrow AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y}).$$

(b) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) < K_\beta(Y_k)$: Then, $\underline{X_k} < \underline{Y_k} \leq \overline{Y_k} < \overline{X_k}$, meaning that $w(X_k) > w(Y_k)$ and, therefore, by Definition 17, $\lambda_\alpha(X_k) > \lambda_\alpha(Y_k)$. So,

$$\begin{aligned} K_\alpha(AOn^\alpha_B(\vec{X})) &= On(K_\alpha(Z), \ldots, K_\alpha(X_k), \ldots, K_\alpha(Z)) \\ &= On(K_\alpha(Z), \ldots, K_\alpha(Y_k), \ldots, K_\alpha(Z)) \\ &= K_\alpha(AOn^\alpha_B(\vec{Y})), \text{ and} \end{aligned}$$

$$\begin{aligned} &K_{\beta=0}(AOn^\alpha_B(\vec{X})) \\ &= K_\alpha(AOn^\alpha_B(\vec{X})) - \alpha w(AOn^\alpha_B(\vec{X})) \text{ by Definition 13} \\ &= K_\alpha(AOn^\alpha_B(\vec{X})) - \\ &\quad \alpha B(\lambda_\alpha(Z), \ldots, \lambda_\alpha(X_k), \ldots, \lambda_\alpha(Z)) d_\alpha(K_\alpha(AOn^\alpha_B(\vec{X}))) \\ &\leq K_\alpha(AOn^\alpha_B(\vec{Y})) - \\ &\quad \alpha B(\lambda_\alpha(Z), \ldots, \lambda_\alpha(Y_k), \ldots, \lambda_\alpha(Z)) d_\alpha(K_\alpha(AOn^\alpha_B(\vec{Y}))) \\ &= K_{\beta=0}(AOn^\alpha_B(\vec{Y})), \end{aligned}$$

since B is increasing. Therefore, $AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y})$.

(c) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) = K_\beta(Y_k)$: In this case, $\vec{X} = \vec{Y}$, so, it is immediate that $AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y})$. So, for every scenario when $\beta = 0$, it holds that if $X_i \leq_{\alpha,\beta} Y_i$, for all $i \in \{1,\ldots,n\}$, then $AOn^\alpha_B(\vec{X}) \leq_{\alpha,\beta} AOn^\alpha_B(\vec{Y})$. The proof for $\beta = 1$ is obtained analogously. □

Appendix B. Proof of Theorem 6

Proof. Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element and take $\alpha \in (0,1]$, $\beta \in [0,\alpha)$. Observe that, for all $\vec{X} \in L([0,1])^n$:

(i) $K_\alpha(IFw^\alpha_B(\vec{X})) = K_\alpha(\widehat{F}(\vec{X}))$;

(ii) $K_\beta(IFw^\alpha_B(\vec{X})) = K_\alpha(\widehat{F}(\vec{X})) - \alpha m_{\widehat{F},B}(\vec{X}) + \beta m_{\widehat{F},B}(\vec{X})$;

(iii) $w(IFw^\alpha_B(\vec{X})) = m_{\widehat{F},B}(\vec{X}) = \min\{w(\widehat{F}(\vec{X})), B(w(X_1), \ldots, w(X_n))\}$.

So, it is immediate that IFw^α_B is well defined and, by **(iii)**, that is width-limited by B. Now, consider $Z \in L([0,1])$, $\vec{X}, \vec{Y} \in L([0,1])^n$, such that there exists $k \in \{1,\ldots,n\}$ for which $X_k \leq_{Pr} Y_k$ and $X_i = Y_i = Z$ for all $i \in \{1,\ldots,n\} - \{k\}$. So, it holds that $X_i \leq_{Pr} Y_i$ for all $i \in \{1,\ldots,n\}$. As $\beta < \alpha$, by Lemma 1, one can consider $\beta = 0$. Thus:

$$K_{\beta=0}(IFw^\alpha_B(\vec{X})) = K_\alpha(\widehat{F}(\vec{X})) - \alpha \cdot m_{\widehat{F},B}(\vec{X}) \tag{B.1}$$
$$K_{\beta=0}(IFw^\alpha_B(\vec{Y})) = K_\alpha(\widehat{F}(\vec{Y})) - \alpha \cdot m_{\widehat{F},B}(\vec{Y}). \tag{B.2}$$

Now, we have the following possibilities regarding $m_{\widehat{F},B}(\vec{X})$ and $m_{\widehat{F},B}(\vec{Y})$ that affects the values of $IFw^\alpha_B(\vec{X})$ and $IFw^\alpha_B(\vec{Y})$, respectively:

(1) $m_{\widehat{F},B}(\vec{X}) = w(\widehat{F}(\vec{X}))$ and $m_{\widehat{F},B}(\vec{Y}) = w(\widehat{F}(\vec{Y}))$: In this case, we have $IFw^\alpha_B(\vec{X}) = \widehat{F}(\vec{X}) \leq_{Pr} \widehat{F}(\vec{Y}) = IFw^\alpha_B(\vec{Y})$, meaning that $IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y})$.

(2) $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{\widehat{F},B}(\vec{Y}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$: It follows that

$$IFw^\alpha_B(\vec{X}) = [K_\alpha(\widehat{F}(\vec{X})) - \alpha B(w(Z), \ldots, w(X_k), \ldots, w(Z)), \\ K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha) B(w(Z), \ldots, w(X_k), \ldots, w(Z))], \text{ and}$$

$$IFw^\alpha_B(\vec{Y}) = [K_\alpha(\widehat{F}(\vec{Y})) - \alpha B(w(Z), \ldots, w(Y_k), \ldots, w(Z)), \\ K_\alpha(\widehat{F}(\vec{Y})) + (1-\alpha) B(w(Z), \ldots, w(Y_k), \ldots, w(Z))].$$

Now, let us verify all the cases in which $X_k \leq_{Pr} Y_k$ holds:

(a) $\underline{X_k} = \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$: We have that $X_k = Y_k$, meaning that $IFw^\alpha_B(\vec{X}) = IFw^\alpha_B(\vec{Y}) \Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y})$.

(b) $\underline{X_k} = \underline{Y_k}$ and $\overline{X_k} < \overline{Y_k}$: When $\underline{Z} \neq h = 0$, it holds that $K_\alpha(\widehat{F}(\vec{X})) < K_\alpha(\widehat{F}(\vec{Y}))$, since F is strictly increasing on $(0,1]^n$ and $\alpha \in (0,1]$. So, it follows that

$$K_\alpha(IFw^\alpha_B(\vec{X})) < K_\alpha(IFw^\alpha_B(\vec{Y})) \Rightarrow IFw^\alpha_B(\vec{X}) \leq_{\alpha,\beta} IFw^\alpha_B(\vec{Y}).$$

If $\underline{Z} = h = 0$ and $\overline{Z} \neq h = 0$, one has that

$$\widehat{F}(\vec{X}) = [0, F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z})],$$
$$\widehat{F}(\vec{Y}) = [0, F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z})].$$

Since $\overline{X_k} < \overline{Y_k}$ and F is strict, then

$$\begin{aligned} K_\alpha(IFw_B^\alpha(\vec{X})) &= K_\alpha(\widehat{F}(\vec{X})) \\ &< K_\alpha(\widehat{F}(\vec{Y})) \\ &= K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

If $\underline{Z} = h = 0$ and $\overline{Z} = h = 0$, then

$$\widehat{F}(\vec{X}) = IFw_B^\alpha(\vec{X}) = [0,0] = IFw_B^\alpha(\vec{Y}) = \widehat{F}(\vec{Y}).$$

So, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(Y, Z)$, for all $X_k, Y_k, Z \in L([0,1])$, such that $\underline{X_k} = \underline{Y_k}$ and $\overline{X_k} < \overline{Y_k}$.

(c) $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$: When $\underline{Z} \neq h = 0$ and $\alpha \neq 1$, we have that $K_\alpha(\widehat{F}(\vec{X})) < K_\alpha(\widehat{F}(\vec{Y}))$. So, it holds that

$$K_\alpha(IFw_B^\alpha(\vec{X})) < K_\alpha(IFw_B^\alpha(\vec{Y})) \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

When taking $\underline{Z} \neq h = 0$ and $\alpha = 1$, we have that $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) = K$. Moreover, from Eqs. (C.1) and (C.2):

$$K_\beta(IFw_B^\alpha(\vec{X})) = K - B(w(Z), \ldots, w(X_k), \ldots, w(Z))$$
$$K_\beta(IFw_B^\alpha(\vec{Y})) = K - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)).$$

As $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$, we have that $w(Y_k) < w(X_k)$, and, since B is increasing,

$$B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \leq B(w(Z), \ldots, w(X_k), \ldots, w(Z)).$$

So:

$$\begin{aligned} K_\beta(IFw_B^\alpha(\vec{X})) &= K - B(w(Z), \ldots, w(X_k), \ldots, w(Z)) \\ &\leq K - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})). \text{ Then:} \end{aligned}$$

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(Ifw_B^\alpha(\vec{Y})) \\ &\quad \text{and } K_\beta(IFw_B^\alpha(\vec{X})) \leq K_\beta(IFw_B^\alpha(\vec{Y})) \\ &\quad \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

If $\underline{Z} = h = 0$, one has that

$$\widehat{F}(\vec{X}) = [0, F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z})],$$
$$\widehat{F}(\vec{Y}) = [0, F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z})].$$

Since $\overline{X_k} = \overline{Y_k}$, then $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y}))$ and, analogously to the previous case, when $\underline{Z} \neq h = 0$ and $\alpha = 1$, we have that

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\quad \text{and } K_\beta(IFw_B^\alpha(\vec{X})) \leq K_\beta(IFw_B^\alpha(\vec{Y})) \\ &\quad \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

So, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$, for all $X_k, Y_k, Z \in L([0,1])$, such that $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$.

(d) $\underline{X} < \underline{Y}$ and $\overline{X} < \overline{Y}$: When $\underline{Z} \neq h = 0$, it holds that $K_\alpha(\widehat{F}(\vec{X})) < K_\alpha(\widehat{F}(\vec{Y}))$. So, we have that

$$K_\alpha(IFw_B^\alpha(\vec{X})) < K_\alpha(IFw_B^\alpha(\vec{Y})) \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{X}).$$

If $\underline{Z} = h = 0$ and $\overline{Z} \neq h = 0$, one has that

$$\widehat{F}(\vec{X}) = [0, F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z})],$$
$$\widehat{F}(\vec{X}) = [0, F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z})].$$

Since $\overline{X_k} < \overline{Y_k}$ and F is strict, then

$$\begin{aligned} K_\alpha(IFw_B^\alpha(\vec{X})) &= K_\alpha(\widehat{F}(\vec{X})) \\ &< K_\alpha(\widehat{F}(\vec{X})) \\ &= K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}). \end{aligned}$$

If $\underline{Z} = \overline{Z} = h = 0$, then

$$\widehat{F}(\vec{X}) = IFw_B^\alpha(\vec{X}) = [0,0] = IFw_B^\alpha(\vec{Y}) = \widehat{F}(\vec{Y}).$$

So, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$, for all $X_k, Y_k, Z \in L([0,1])$, such that $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} < \overline{Y_k}$. Thus, we conclude that, for all $X_k, Y_k, Z \in L([0,1])$, when

$$m_{\widehat{F},B}(X, Z) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$$
$$m_{\widehat{F},B}(Y, Z) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z)), \text{ then}$$

$$X_i \leq_{Pr} Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IOw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IOw_B^\alpha(\vec{Y}).$$

(3) $m_{\widehat{F},B}(\vec{X}) = w(\widehat{F}(\vec{X}))$ and $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$: It follows that

$$IFw_B^\alpha(\vec{X}) = \widehat{F}(\vec{X}) \text{ and}$$

$$IFw_B^\alpha(\vec{Y}) = [K_\alpha(\widehat{F}(\vec{Y})) - \alpha B(w(Z), \ldots, w(Y_k), \ldots, w(Z)),$$
$$K_\alpha(\widehat{F}(\vec{Y})) + (1-\alpha) B(w(Z), \ldots, w(Y_k), \ldots, w(Z))].$$

Again, we analyze all the possibilities in which $X_k \leq_{Pr} Y_k$ holds. The results are exactly the same as the ones presented on the proof when $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$, with the exception of the particular case when $\underline{X_k} < \underline{Y_k}$, $\overline{X_k} = \overline{Y_k}$, $\underline{Z} \neq h = 0$ and $\alpha = 1$. In this case, we have that $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) = K$. Moreover, from Eqs. (C.1) and (C.2): $K_\beta(IFw_B^\alpha(\vec{X})) = K - w(\widehat{F}(\vec{X}))$ and $K_\beta(IFw_B^\alpha(\vec{Y})) = K - B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$.

As $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$, we have that:

$$\begin{aligned} &B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &\leq w(\widehat{F}(\vec{Y})) \\ &= F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - F(\underline{Z}, \ldots, \underline{Y_k}, \ldots, \underline{Z}) \\ &\leq F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z}) - F(\underline{Z}, \ldots, \underline{X_k}, \ldots, \underline{Z}) \\ &= w(\widehat{F}(\vec{X})), \end{aligned}$$

since F is strictly increasing. So,

$$\begin{aligned} &K_\beta(IFw_B^\alpha(\vec{X})) \\ &= F(\overline{Z}, \ldots, \overline{X}, \ldots, \overline{Z}) - w(\widehat{F}(\vec{X})) \\ &\leq F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})). \text{ Then:} \end{aligned}$$

$$\begin{aligned} &K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y})) \\ &\text{and }, K_\beta(IFw_B^\alpha(\vec{X})) \leq K_\beta(IFw_B^\alpha(\vec{Y})) \\ &\Rightarrow \; IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IOw_B^\alpha(Y, Z). \end{aligned}$$

Thus, one can conclude that, for all $X_k, Y_k, Z \in L([0,1])$, when $m_{\widehat{F},B}(\vec{X}) = w(\widehat{F}(\vec{X}))$ and $m_{\widehat{F},B}(\vec{Y}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$, then

$$X_i \leq_{Pr} Y_i \text{ for all } i \in \{1, \ldots, n\} \Rightarrow IOw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IOw_B^\alpha(\vec{Y}).$$

(4) $m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{\widehat{F},B}(\vec{Y}) = w(\widehat{F}(\vec{Y}))$: It follows that

$$IFw_B^\alpha(\vec{X}) = [K_\alpha(\widehat{F}(\vec{X})) - \alpha B(w(Z), \ldots, w(X_k), \ldots, w(Z)),$$
$$K_\alpha(\widehat{F}(\vec{X})) + (1-\alpha) B(w(Z), \ldots, w(X_k), \ldots, w(Z))] \text{ and}$$

$$IFw_B^\alpha(\vec{Y}) = \widehat{F}(\vec{Y}).$$

Once more, by analyzing every possibility in which $X_k \leq_{Pr} Y_k$ holds, one may observe that the results are exactly the same as the ones presented previously, with the exception when $\underline{X_k} < \underline{Y_k}$, $\overline{X_k} = \overline{Y_k}$, $\underline{Z} \neq h = 0$ and $\alpha = 1$. In this case, we have that $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y}))$. Moreover, from Eqs. (C.1) and (C.2):

$$K_\beta(IFw_B^\alpha(\vec{X}))$$
$$= F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z}) - B(w(Z), \ldots, w(X_k), \ldots, w(Z)) \text{ and}$$

$$K_\beta(IFw_B^\alpha(\vec{Y})) = F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - w(\widehat{F}(\vec{Y})).$$

As $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$, we have that $w(Y_k) < w(X_k)$, and, since B is increasing:

$$w(\widehat{F}(\vec{Y})) \leq B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$$
$$\leq B(w(Z), \ldots, w(X_k), \ldots, w(Z)), \text{ So:}$$

$$K_\beta(IFw_B^\alpha(\vec{X}))$$
$$= F(\overline{Z}, \ldots, \overline{X_k}, \ldots, \overline{Z}) - w(\widehat{F}(\vec{X}))$$
$$\leq F(\overline{Z}, \ldots, \overline{Y_k}, \ldots, \overline{Z}) - B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$$
$$= K_\beta(IFw_B^\alpha(\vec{Y})). \text{ Then:}$$

$$K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IFw_B^\alpha(\vec{Y}))$$
$$\text{and } K_\beta(IFw_B^\alpha(\vec{X})) \leq K_\beta(IFw_B^\alpha(\vec{Y}))$$
$$\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

Then, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$, for all $X_k, Y_k, Z \in L([0,1])$, such that $\underline{X_k} < \underline{Y_k}$ and $\overline{X_k} = \overline{Y_k}$.

Thus, one conclude that, for all $X_k, Y_k, Z \in L([0,1])$, when

$$m_{\widehat{F},B}(\vec{X}) = B(w(Z), \ldots, w(X), \ldots, w(Z))$$
$$m_{\widehat{F},B}(\vec{Y}) = w(\widehat{F}(\vec{Y})), \text{ then}$$
$$\forall i \in \{1, \ldots, n\}: \quad X_i \leq_{Pr} Y_i \Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

As verified for all possible scenarios, it holds that IFw_B^α is $(\leq_{Pr}, \leq_{\alpha,\beta})$-increasing, for all $\alpha \in (0,1]$ and $\beta \in [0,\alpha)$, which completes the proof that IFw_B^α is a w-iv-fusion function for the tuple $(\leq_{Pr}, \leq_{\alpha,\beta}, B)$. □

Appendix C. Proof of Theorem 14

Proof. Consider an increasing fusion function $B : [0,1]^n \to [0,1]$, a strict fusion function $F : [0,1]^n \to [0,1]$ with $h = 0$ as its annihilator element, $\alpha \in (0,1]$, $\beta \in [0,1]$ such that $\alpha \neq \beta$, and an $\leq_{\alpha,\beta}$ increasing fusion function $IF^\alpha : L([0,1])^n \to L([0,1])$ such that $K_\alpha(IF)(\vec{X}) = F(K_\alpha(X_1), \ldots, K_\alpha(X_n))$, for all $\vec{X} \in L([0,1])^n$. Observe that, for all $\vec{X} \in L([0,1])^n$: **(i)** $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X}))$; **(ii)** $K_\beta(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) - \alpha \cdot m_{IF^\alpha,B}(\vec{X}) + \beta \cdot m_{IF^\alpha,B}(\vec{X})$; **(iii)** $w(IFw_B^\alpha(\vec{X})) = m_{IF^\alpha,B}(\vec{X}) = \min\{w(IF^\alpha(\vec{X})), B(w(X_1), \ldots, w(X_n))\}$. So, it is immediate that IFw_B^α is well defined and, by **(iii)**, that is width-limited by B.

Now, consider $Z \in L([0,1])$, $\vec{X}, \vec{Y} \in L([0,1])^n$, such that there exist $k \in \{1, \ldots, n\}$ for which $X_k \leq_{\alpha,\beta} Y_k$ and $X_i = Y_i = Z$ for all $i \in \{1, \ldots, n\} - \{k\}$. So, it holds that $X_i \leq_{\alpha,\beta} Y_i$ for all $i \in \{1, \ldots, n\}$. By Lemma 1, one can consider $\beta = 0$ or $\beta = 1$. First, we present the proof for $\beta = 0$. Thus:

$$K_{\beta=0}(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) - \alpha m_{IF^\alpha,B}(\vec{X}) \tag{C.1}$$
$$K_{\beta=0}(IFw_B^\alpha(\vec{Y})) = K_\alpha(IF^\alpha(\vec{Y})) - \alpha m_{IF^\alpha,B}(\vec{Y}). \tag{C.2}$$

Next, if $K_\alpha(Z) = 0$, then $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) = 0 = K_\alpha(IF^\alpha(\vec{Y})) = K_\alpha(IFw_B^\alpha(\vec{Y}))$, which means that $IFw_B^\alpha(\vec{X}) = IF^\alpha(\vec{X}) = [0,0] = IF^\alpha(\vec{Y}) = IFw_B^\alpha(\vec{Y})$, since $\alpha \neq 0$. Then, $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$.

If $K_\alpha(Z) \neq 0$, then we have the following cases:

(a) $K_\alpha(X_k) < K_\alpha(Y_k)$: Since F is strict, one has that:

$$K_\alpha(IFw_B^\alpha(\vec{X}))$$
$$= K_\alpha(IF^\alpha(\vec{X})) < K_\alpha(IF^\alpha(\vec{Y})) = K_\alpha(IFw_B^\alpha(\vec{Y})). \text{ Thus:}$$

$$K_\alpha(IFw_B^\alpha(\vec{X})) < K_\alpha(IFw_B^\alpha(\vec{Y}))$$
$$\Rightarrow IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

(b) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) < K_\beta(Y_k)$: Since $K_\alpha(X_k) = K_\alpha(Y_k)$, then $K_\alpha(IFw_B^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{X})) = K_\alpha(IF^\alpha(\vec{Y})) = K_\alpha(IFw_B^\alpha(\vec{Y})) = K$ and, since IF^α is $\leq_{\alpha,\beta}$-increasing, we have that $K_\beta(IF^\alpha(\vec{X})) \leq K_\beta(IF^\alpha(\vec{Y}))$. As $\beta = 0$, then $w(X_k) > w(Y_k)$ and $w(IF^\alpha(\vec{X})) \geq w(IF^\alpha(\vec{Y}))$. From Eqs (C.1) and (C.2), the values of $K_\beta(IFw_B^\alpha(\vec{X}))$ and $K_\beta(IFw_B^\alpha(\vec{Y}))$ depend on the values of $m_{IF^\alpha,B}(\vec{X})$ and $m_{IF^\alpha,B}(\vec{Y})$, respectively. So, let us analyze each possibility regarding those maximal thresholds.

(1) $m_{IF^\alpha,B}(\vec{X}) = w(IF^\alpha(\vec{X}))$ and $m_{IF^\alpha,B}(\vec{Y}) = w(IF^\alpha(\vec{Y}))$: In this case, $IFw_B^\alpha(\vec{X}) = IF^\alpha(\vec{X}) = IF^\alpha(\vec{Y}) = IFw_B^\alpha(\vec{Y})$.

(2) $m_{IF^\alpha,B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{IF^\alpha,B}(\vec{Y}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$: From Eqs. (C.1) and (C.2):

$$\begin{aligned} K_\beta(IFw_B^\alpha(\vec{X})) &= K - B(w(Z), \ldots, w(X_k), \ldots, w(Z)) \\ &\leq K - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})), \end{aligned}$$

since B is increasing and $w(X_k) > w(Y_k)$. Thus,

$$IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

(3) $m_{IF^\alpha,B}(\vec{X}) = w(IF^\alpha(\vec{X}))$ and $m_{IF^\alpha,B}(\vec{Y}) = B(w(Z), \ldots, w(Y_k), \ldots, w(Z))$: From Eqs. (C.1) and (C.2):

$$\begin{aligned} K_\beta(IFw_B^\alpha(\vec{X})) &= K - w(IF^\alpha(\vec{X})) \\ &\leq K - B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &= K_\beta(IFw_B^\alpha(\vec{Y})), \end{aligned}$$

since $B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \leq w(IF^\alpha(\vec{Y})) \leq w(IF^\alpha(\vec{X}))$. Thus,

$$IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y}).$$

(4) $m_{IF^\alpha,B}(\vec{X}) = B(w(Z), \ldots, w(X_k), \ldots, w(Z))$ and $m_{IF^\alpha,B}(\vec{Y}) = w(IF^\alpha(\vec{Y}))$: Since $w(Y_k) \leq w(X_k)$, then

$$\begin{aligned} w(IF^\alpha(\vec{Y})) &\leq B(w(Z), \ldots, w(Y_k), \ldots, w(Z)) \\ &\leq B(w(Z), \ldots, w(X_k), \ldots, w(Z)). \end{aligned}$$

From Eqs. (C.1) and (C.2):

$$K_\beta(IFw_B^\alpha(\vec{X}))$$
$$= K - B(w(Z), \ldots, w(X_k), \ldots, w(Z)) \leq K - w(IF^\alpha(\vec{Y})),$$

meaning that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$. So, when $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) < K_\beta(Y_k)$, we have that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$.

(c) $K_\alpha(X_k) = K_\alpha(Y_k)$ and $K_\beta(X_k) = K_\beta(Y_k)$: In this case, $\vec{X} = \vec{Y}$, so, it is immediate that $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$. So, for every scenario when $\beta = 0$, it holds that, if $X_i \leq_{\alpha,\beta} Y_i$ for all $i \in \{1, \ldots, n\}$, then $IFw_B^\alpha(\vec{X}) \leq_{\alpha,\beta} IFw_B^\alpha(\vec{Y})$.

The proof for $\beta = 1$ is obtained analogously.

Thus, as verified for all possible scenarios, it holds that IFw_B^α is $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta})$-increasing, for all $\alpha \in (0,1]$ and $\beta \in [0,1]$, which completes the proof that IFw_B^α is a w-iv-fusion function for the tuple $(\leq_{\alpha,\beta}, \leq_{\alpha,\beta}, B)$. □

References

[1] R. Mesiar, A. Kolesárová, H. Bustince, G. Dimuro, B. Bedregal, Fusion functions based discrete Choquet-like integrals, European J. Oper. Res. 252 (2) (2016) 601–609, http://dx.doi.org/10.1016/j.ejor.2016.01.027.

[2] G. Beliakov, H. Bustince, T. Calvo, A Practical Guide to Averaging Functions, Springer, Berlin, New York, 2016.

[3] G. Lucca, J. Sanz, G. Pereira Dimuro, B. Bedregal, R. Mesiar, A. Kolesárová, H. Bustince Sola, Pre-aggregation functions: construction and an application, IEEE Trans. Fuzzy Syst. 24 (2) (2016) 260–272, http://dx.doi.org/10.1109/TFUZZ.2015.2453020.

[4] V. Kreinovich, M. Mukaidono, Interval (pairs of fuzzy values), triples, etc.: Can we thus get an arbitrary ordering? in: Proceedings of 9th IEEE International Conference on Fuzzy Systems, San Antonio, TX, USA, 2000, vol. 1, IEEE, Los Alamitos, 2000, pp. 234–238.

[5] H. Bustince, J. Fernandez, R. Mesiar, J. Montero, R. Orduna, Overlap functions, Nonlinear Anal. TMA 72 (3–4) (2010) 1488–1499.

[6] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, Grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons, IEEE Trans. Fuzzy Syst. 20 (3) (2012) 405–415, http://dx.doi.org/10.1109/TFUZZ.2011.2173581.

[7] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Berlin, 2007.

[8] D. Gómez, J.T. Rodríguez, J. Montero, H. Bustince, E. Barrenechea, n-dimensional overlap functions, Fuzzy Sets and Systems 287 (2016) 57–75, http://dx.doi.org/10.1016/j.fss.2014.11.023.

[9] M. Elkano, M. Galar, J. Sanz, H. Bustince, Fuzzy rule-based classification systems for multi-class problems using binary decomposition strategies: On the influence of n-dimensional overlap functions in the fuzzy reasoning method, Inform. Sci. 332 (2016) 94–114.

[10] M. Elkano, M. Galar, J. Sanz, A. Fernández, E. Barrenechea, F. Herrera, H. Bustince, Enhancing multi-class classification in FARC-HD fuzzy classifier: On the synergy between n-dimensional overlap functions and decomposition strategies, IEEE Trans. Fuzzy Syst. 23 (5) (2015) 1562–1580.

[11] L.A. Zadeh, Is there a need for fuzzy logic? Inform. Sci. 178 (13) (2008) 2751–2779.

[12] L.A. Zadeh, Toward a generalized theory of uncertainty (GTU) – an outline, Inform. Sci. 172 (1–2) (2005) 1–40.

[13] J.M. Mendel, Computing with words and its relationships with fuzzistics, Inform. Sci. 177 (4) (2007) 988–1006, http://dx.doi.org/10.1016/j.ins.2006.06.008.

[14] H.T. Nguyen, V. Kreinovich, Q. Zuo, Interval-valued degrees of belief: applications of interval computations to expert systems and intelligent control, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 5 (3) (1997) 317–358.

[15] F. Feng, J. Cho, W. Pedrycz, H. Fujita, T. Herawan, Soft set based association rule mining, Knowl.-Based Syst. 111 (2016) 268–282, http://dx.doi.org/10.1016/j.knosys.2016.08.020.

[16] J. Ye, J. Zhan, W. Ding, H. Fujita, A novel fuzzy rough set model with fuzzy neighborhood operators, Inform. Sci. 544 (2021) 266–297, http://dx.doi.org/10.1016/j.ins.2020.07.030.

[17] C. Zhu, F. Xiao, Z. Cao, A generalized rényi divergence for multi-source information fusion with its application in eeg data analysis, Inform. Sci. 605 (2022) 225–243, http://dx.doi.org/10.1016/j.ins.2022.05.012.

[18] L. Zhang, F. Xiao, A novel belief χ^2 divergence for multisource information fusion and its application in pattern classification, Int. J. Intell. Syst. 37 (2022) 7968–7991, http://dx.doi.org/10.1002/int.22912.

[19] F. Xiao, J. Wen, W. Pedrycz, Generalized divergence-based decision making method with an application to pattern classification, IEEE Trans. Knowl. Data Eng. (2022) 1, http://dx.doi.org/10.1109/TKDE.2022.3177896, in press, early access.

[20] F. Xiao, W. Pedrycz, Negation of the quantum mass function for multisource quantum information fusion with its application to pattern classification, IEEE Trans. Pattern Anal. Mach. Intell. (2022) 1, http://dx.doi.org/10.1109/TPAMI.2022.3167045, in press, early access.

[21] H. Bustince, E. Barrenechea, M. Pagola, J. Fernandez, Z. Xu, B. Bedregal, J. Montero, H. Hagras, F. Herrera, B. De Baets, A historical account of types of fuzzy sets and their relationships, IEEE Trans. Fuzzy Syst. 24 (1) (2016) 179–194, http://dx.doi.org/10.1109/TFUZZ.2015.2451692.

[22] I. Grattan-Guiness, Fuzzy membership mapped onto interval and many-valued quantities, Z. Math. Log. Grundl. Math. 22 (1) (1976) 149–160.

[23] B.C. Bedregal, G.P. Dimuro, R.H.N. Santiago, R.H.S. Reiser, On interval fuzzy S-implications, Inform. Sci. 180 (8) (2010) 1373–1389, http://dx.doi.org/10.1016/j.ins.2009.11.035.

[24] G.P. Dimuro, B.C. Bedregal, R.H.N. Santiago, R.H.S. Reiser, Interval additive generators of interval t-norms and interval t-conorms, Inform. Sci. 181 (18) (2011) 3898–3916, http://dx.doi.org/10.1016/j.ins.2011.05.003.

[25] M. Galar, J. Fernandez, G. Beliakov, H. Bustince, Interval-valued fuzzy sets applied to stereo matching of color images, IEEE Trans. Image Process. 20 (7) (2011) 1949–1961.

[26] T.C. Asmus, G.P. Dimuro, B. Bedregal, On two-player interval-valued fuzzy Bayesian games, Int. J. Intell. Syst. 32 (6) (2017) 557–596, http://dx.doi.org/10.1002/int.21857.

[27] F. Kutlu Gündogdu, C. Kahraman, A novel fuzzy TOPSIS method using emerging interval-valued spherical fuzzy sets, Eng. Appl. Artif. Intell. 85 (2019) 307–323, http://dx.doi.org/10.1016/j.engappai.2019.06.003.

[28] L.M. Rodrigues, G.P. Dimuro, D.T. Franco, J.C. Fachinello, A system based on interval fuzzy approach to predict the appearance of pests in agriculture, in: Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting(IFSA/NAFIPS), 2003, IEEE, Los Alamitos, 2013, pp. 1262–1267, http://dx.doi.org/10.1109/IFSA-NAFIPS.2013.6608583.

[29] K. Hu, Q. Tan, T. Zhang, S. Wang, Assessing technology portfolios of clean energy-driven desalination-irrigation systems with interval-valued intuitionistic fuzzy sets, Renew. Sustain. Energy Rev. 132 (2020) 109950, http://dx.doi.org/10.1016/j.rser.2020.109950.

[30] L.T. Ngo, T.H. Dang, W. Pedrycz, Towards interval-valued fuzzy set-based collaborative fuzzy clustering algorithms, Pattern Recognit. 81 (2018) 404–416, http://dx.doi.org/10.1016/j.patcog.2018.04.006.

[31] U. Bentkowska, Interval-Valued Methods in Classifications and Decisions, Springer, Cham, 2020.

[32] J.A. Sanz, A. Fernández, H. Bustince, F. Herrera, Improving the performance of fuzzy rule-based classification systems with interval-valued fuzzy sets and genetic amplitude tuning, Inform. Sci. 180 (19) (2010) 3674–3685.

[33] T.C. Asmus, J.A. Sanz, G.P. Dimuro, B. Bedregal, J. Fernandez, H. Bustince, N-dimensional admissibly ordered interval-valued overlap functions and its influence in interval-valued fuzzy rule-based classification systems, IEEE Trans. Fuzzy Syst. 30 (4) (2022) 1060–1072, http://dx.doi.org/10.1109/TFUZZ.2021.3052342.

[34] A wrapper methodology to learn interval-valued fuzzy rule-based classification systems, Appl. Soft Comput. 104 (2021) 107249, http://dx.doi.org/10.1016/j.asoc.2021.107249.

[35] H. Ishibuchi, T. Nakashima, M. Nii, Classification and Modeling with Linguistic Information Granules: Advanced Approaches to Linguistic Data Mining, Springer-Verlag, Berlin, 2004.

[36] M. Komorníková, R. Mesiar, Aggregation functions on bounded partially ordered sets and their classification, Fuzzy Sets and Systems 175 (1) (2011) 48–56, http://dx.doi.org/10.1016/j.fss.2011.01.015, theme: Aggregation Functions, Generalised Measure Theory.

[37] J. Qiao, B.Q. Hu, On interval additive generators of interval overlap functions and interval grouping functions, Fuzzy Sets and Systems 323 (2017) 19–55, http://dx.doi.org/10.1016/j.fss.2017.03.007.

[38] B. Bedregal, H. Bustince, E. Palmeira, G. Dimuro, J. Fernandez, Generalized interval-valued OWA operators with interval weights derived from interval-valued overlap functions, Internat. J. Approx. Reason. 90 (2017) 1–16, http://dx.doi.org/10.1016/j.ijar.2017.07.001.

[39] T.C. Asmus, G.P. Dimuro, B. Bedregal, J.A. Sanz, S.P. Jr., H. Bustince, General interval-valued overlap functions and interval-valued overlap indices, Inform. Sci. 527 (2020) 27–50, http://dx.doi.org/10.1016/j.ins.2020.03.091.

[40] T. da Cruz Asmus, G. Pereira Dimuro, H. Bustince, B. Bedregal, H. Santos, J.A. Sanz, General interval-valued grouping functions, in: 2020 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2020, pp. 1–8, http://dx.doi.org/10.1109/FUZZ48607.2020.9177558.

[41] R.H.N. Santiago, B.R.C. Bedregal, B.M. Acióly, Formal aspects of correctness and optimality of interval computations, Form. Asp. Comput. 18 (2006) 231–243, http://dx.doi.org/10.1007/s00165-006-0089-x.

[42] R.E. Moore, R.B. Kearfott, M.J. Cloud, Introduction To Interval Analysis, SIAM, Philadelphia, 2009.

[43] H. Bustince, J. Fernandez, A. Kolesárová, R. Mesiar, Generation of linear orders for intervals by means of aggregation functions, Fuzzy Sets and Systems 220 (2013) 69–77, http://dx.doi.org/10.1016/j.fss.2012.07.015.

[44] H. Zapata, H. Bustince, S. Montes, B. Bedregal, G. Dimuro, Z. Takáč, M. Baczyński, J. Fernandez, Interval-valued implications and interval-valued strong equality index with admissible orders, Internat. J. Approx. Reason. 88 (2017) 91–109, http://dx.doi.org/10.1016/j.ijar.2017.05.009.

[45] U. Bentkowska, H. Bustince, A. Jurio, M. Pagola, B. Pekala, Decision making with an interval-valued fuzzy preference relation and admissible orders, Appl. Soft Comput. 35 (2015) 792–801, http://dx.doi.org/10.1016/j.asoc.2015.03.012.

[46] Z. Takáč, J. Fernandez, J. Fumanal, C. Marco-Detchart, I. Couso, G. Dimuro, H. Santos, H. Bustince, Distances between interval-valued fuzzy sets taking into account the width of the intervals, in: 2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2019, pp. 1–6, http://dx.doi.org/10.1109/FUZZ-IEEE.2019.8858908.

[47] H. Bustince, C. Marco-Detchart, J. Fernandez, C. Wagner, J. Garibaldi, Z. Takáč, Similarity between interval-valued fuzzy sets taking into account the width of the intervals and admissible orders, Fuzzy Sets and Systems 390 (2020) 23–47, http://dx.doi.org/10.1016/j.fss.2019.04.002, Similarity, Orders, Metrics.

[48] G.P. Dimuro, A.C.R. Costa, D.M. Claudio, A coherence space of rational intervals for a construction of IR, Reliab. Comput. 6 (2) (2000) 139–178, http://dx.doi.org/10.1023/A:1009913122021.

[49] S. Abramsky, A. Jung, Domain theory, in: T. Maimbaum S. Abramsky (Ed.), Handbook of Logic in Computer Science, vol. 3, Oxford Svience Publications, 1994, pp. 1–168.
[50] T.C. Asmus, G.P. Dimuro, B. Bedregal, J.A. Sanz, R. Mesiar, H. Bustince, Towards interval uncertainty propagation control in bivariate aggregation processes and the introduction of width-limited interval-valued overlap functions, Fuzzy Sets and Systems 441 (5) (2022) 130–168, http://dx.doi.org/10.1016/j.fss.2021.09.005.
[51] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publisher, Dordrecht, 2000.
[52] E.P. Klement, M. Manzi, R. Mesiar, Ultramodular aggregation functions, Inform. Sci. 181 (19) (2011) 4101–4111, http://dx.doi.org/10.1016/j.ins.2011.05.021.
[53] L. Stefanini, A generalization of hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Systems 161 (11) (2010) 1564–1584, http://dx.doi.org/10.1016/j.fss.2009.06.009.
[54] Z. Xu, R.R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. Gen. Syst. 35 (4) (2006) 417–433, http://dx.doi.org/10.1080/03081070600574353.
[55] B.C. Bedregal, On interval fuzzy negations, Fuzzy Sets and Systems 161 (17) (2010) 2290–2313, http://dx.doi.org/10.1016/j.fss.2010.04.018.
[56] A. Jurio, M. Pagola, D. Paternain, C. Lopez-Molina, P. Melo-Pinto, Interval-valued restricted equivalence functions applied on clustering techniques, in: Proceedings of the Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference, Lisbon, 2009, pp. 831–836.
[57] H. Bustince, R. Mesiar, A. Kolesárová, G. Dimuro, J. Fernandez, I. Diaz, S. Montes, On some classes of directionally monotone functions, Fuzzy Sets and Systems 386 (2020) 161–178, http://dx.doi.org/10.1016/j.fss.2019.01.024, Aggregation Operations.
[58] T. Wilkin, G. Beliakov, Weakly monotonic averaging functions, Int. J. Intell. Syst. 30 (2) (2015) 144–169.
[59] H. Bustince, J. Fernandez, A. Kolesárová, R. Mesiar, Directional monotonicity of fusion functions, European J. Oper. Res. 244 (1) (2015) 300–308.
[60] H. Bustince, E. Barrenechea, M. Sesma-Sara, J. Lafuente, G.P. Dimuro, R. Mesiar, A. Kolesárová, Ordered directionally monotone functions, justification and application, IEEE Trans. Fuzzy Syst. 26 (4) (2017) 2237–2250, http://dx.doi.org/10.1109/TFUZZ.2017.2769486.
[61] M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, H. Bustince, Strengthened ordered directionally monotone functions, links between the different notions of monotonicity, Fuzzy Sets and Systems 357 (2019) 151–172, http://dx.doi.org/10.1016/j.fss.2018.07.007.
[62] G.P. Dimuro, J. Fernández, B. Bedregal, R. Mesiar, J.A. Sanz, G. Lucca, H. Bustince, The state-of-art of the generalizations of the choquet integral: From aggregation and pre-aggregation to ordered directionally monotone functions, Inf. Fusion 57 (2020) 27–43, http://dx.doi.org/10.1016/j.inffus.2019.10.005.
[63] G. Lucca, J.A. Sanz, G.P. Dimuro, B. Bedregal, M.J. Asiain, M. Elkano, H. Bustince, CC-integrals: Choquet-like copula-based aggregation functions and its application in fuzzy rule-based classification systems, Knowl.-Based Syst. 119 (2017) 32–43, http://dx.doi.org/10.1016/j.knosys.2016.12.004.
[64] F. Spizzichino, Semi-copulas and interpretations of coincidences between stochastic dependence and ageing, in: P. Jaworski, F. Durante, W.K. Härdle, T. Rychlik (Eds.), Copula Theory and Its Applications, Springer, Berlin, Heidelberg, 2010, pp. 237–254.
[65] J.C. Fodor, Strict preference relations based on weak t-norms, Fuzzy Sets and Systems 43 (3) (1991) 327–336, http://dx.doi.org/10.1016/0165-0114(91)90259-S.
[66] Z. Wang, Y. Yu, Pseudo-t-norms and implication operators on a complete brouwerian lattice, Fuzzy Sets and Systems 132 (1) (2002) 113–124, http://dx.doi.org/10.1016/S0165-0114(01)00210-X.
[67] H. Zhan, Y.-M. Wang, H.-W. Liu, The modularity condition for semi-t-operators and semi-uninorms, Fuzzy Sets and Systems 334 (2018) 36–59, http://dx.doi.org/10.1016/j.fss.2017.05.025.
[68] R.R. Yager, Aggregation operators and fuzzy systems modeling, Fuzzy Sets and Systems 67 (2) (1994) 129–145, http://dx.doi.org/10.1016/0165-0114(94)90082-5.
[69] E. Yang, Weakening-free, Non-associative fuzzy logics: Micanorm-based logics, Fuzzy Sets and Systems 276 (2015) 43–58, http://dx.doi.org/10.1016/j.fss.2014.11.020.
[70] H. Ishibuchi, T. Nakashima, Effect of rule weights in fuzzy rule-based classification systems, IEEE Trans. Fuzzy Syst. 9 (4) (2001) 506–515, http://dx.doi.org/10.1109/91.940964.
[71] J. Sanz, A. Fernandez, H. Bustince, F. Herrera, IIVFDT: Ignorance functions based interval-valued fuzzy decision tree with genetic tuning, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 20 (2012) 1–30.
[72] J. Sanz, A. Fernández, H. Bustince, F. Herrera, IVTURS: A linguistic fuzzy rule-based classification system based on a new interval-valued fuzzy reasoning method with tuning and rule selection, IEEE Trans. Fuzzy Syst. 21 (3) (2013) 399–411, http://dx.doi.org/10.1109/TFUZZ.2013.2243153.
[73] J. Alcala-Fdez, R. Alcala, F. Herrera, A fuzzy association rule-based classification model for high-dimensional problems with genetic rule selection and lateral tuning, IEEE Trans. Fuzzy Syst. 19 (5) (2011) 857–872.
[74] R. Agrawal, R. Srikant, Fast algorithms for mining association rules, in: Proceedings of 20th International Conference Very Large Data Bases (VLDB), Santiago Do Chile, Morgan Kaufmann, San Francisco, 1994, pp. 487–499.
[75] H. Ishibuchi, T. Yamamoto, Rule weight specification in fuzzy rule-based classification systems, IEEE Trans. Fuzzy Syst. 13 (4) (2005) 428–435, http://dx.doi.org/10.1109/TFUZZ.2004.841738.
[76] J. Alcalá-Fdez, L. Sánchez, S. García, M. Jesus, S. Ventura, J. Garrell, J. Otero, C. Romero, J. Bacardit, V. Rivas, J. Fernández, F. Herrera, Keel: a software tool to assess evolutionary algorithms for data mining problems, Soft Comput. 13 (3) (2009) 307–318.
[77] J.L. Hodges, E.L. Lehmann, Ranks methods for combination of independent experiments in analysis of variance, Ann. Math. Stat. 33 (1962) 482–497.
[78] S. Holm, A simple sequentially rejective multiple test procedure, Scand. J. Stat. 6 (1979) 65–70.
[79] F. Wilcoxon, Individual comparisons by ranking methods, Biometrics 1 (1945) 80–83.
[80] S. García, A. Fernández, J. Luengo, F. Herrera, A study of statistical techniques and performance measures for genetics–based machine learning: Accuracy and interpretability, Soft Comput. 13 (10) (2009) 959–977.