Sticky polymatroids on at most five elements

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Dedicated to the memory of Frantisek Matúš

***Abstract*—The sticky polymatroid conjecture states that any two extensions of the polymatroid have an amalgam if and only if the polymatroid has no non-modular pairs of flats. We show that the conjecture holds for polymatroids on five or less elements.**

***Index Terms*—Polymatroid; sticky polymatroid conjecture; modular cut.**

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I. Introduction

A polymatroid is *sticky* if any two of its extensions have an amalgam. This is a direct generalization of the same property of matroids. If every pair of flats is modular then the polymatroid is sticky; the proof in [8] generalizes to polymatroids. The *sticky polymatroid conjecture* states that the converse also holds: in a sticky polymatroid, each pair of flats is a modular pair. The corresponding conjecture for matroids has been stated by Poljak and Turzik [10], and received considerable attention. Poljak and Turzik showed that the sticky matroid conjecture holds for rank-3 matroids. Bachem and Kern [1] showed that the same conjecture holds in general if it is true for every rank-4 matroid. Generalizing a result of Bonin in [2] which states that a matroid of rank at least three with two disjoint hyperplanes is not sticky, Hochstättler and Wilhelmi showed that matroids having a non-principal modular cut are not sticky [5]. The same statement for polymatroids was proved in [3] using a convolution-type construction. Thus the sticky polymatroid conjecture follows from the following stronger

Conjecture 1. $(*)$ *If a polymatroid has a non-modular pair of flats, then it also has a non-principal modular cut.*

There does not seem to be a direct connection between the two notions of stickiness. While the sticky polymatroid conjecture seems to be stronger than the same conjecture for matroids (as the former requires more extensions to have an amalgam), it may not be the case. The reason is that two matroids could have a polymatroid amalgam but no matroid amalgam (alas no such an example is known). Such a polymatroid amalgam exists iff the system of linear inequalities describing the properties of the unknown ranks has *any* solution, while there is a matroid amalgam iff the same system has an *integer* solution. In contrast, $(*)$ in Conjecture 1 is easily seen to be equivalent to the same statement stated for matroids. Consequently, if true, it also implies the sticky matroid conjecture.

In this note we show that Conjecture 1 holds for polymatroids on ground set with at most five elements. Thus a smallest counterexample to the sticky polymatroid conjecture, if exists at all, must have at least six atoms.

All sets in this paper are finite. Following the usual practice ground sets and their subsets are denoted by capital letters, their elements by lower case letters. The union sign $\cup$ and the curly brackets around singletons are omitted, thus abA denotes the set $\{a,b\}\cup A$. The *modular defect* of subsets A and B is defined as

$$\delta_f(A,B) = f(A)+f(B)-f(A\cup B)-f(A\cap B).$$

It is non-negative, and if zero, then (A,B) is a *modular pair*.

The paper is organized as follows. Section II contains the definition and basic properties of polymatroids, one-point extension, and the excess function. Section III introduces the notion of *linear polymatroid*. This is an intrinsic property shared by all linearly representable polymatroids. We hope that this notion has further applications. Two lemmas in Section IV describe some properties of a minimal counterexample to Conjecture 1. Using these lemmas and a property of linear polymatroids, Section V shows that no polymatroid on five or less elements violates Conjecture 1.

II. Definitions

A *polymatroid* $\mathcal{M}=(f,M)$ is a real-valued, non-negative, monotone and submodular function f defined on the set of subsets of the finite set M such that $f(\emptyset)=0$. Here M is the *ground set*, and f is the *rank function*. The polymatroid is a *matroid* if all ranks are integers and $f(A)\le|A|$ for all $A\subseteq M$. For details see [6], [8]. The rank function can be identified with a $(2^{|M|}-1)$-dimensional real vector, where the indices are the non-empty subsets of M. Vectors corresponding to polymatroids on the ground set M form the pointed polyhedral cone Γ_M [13]. Its facets are the hyperplanes determined by the basic submodular inequalities $\delta_f(iK,jK)\ge 0$ with distinct $i,j\in M-K$ and $K\subseteq M$ (K can be empty), and the monotonicity requirements $f(M)\ge f(M-i)$; see [7, Theorem 2]. Much less is known about the extremal rays of this cone. They have been computed for ground sets up to five elements [12] without indicating any structural property. Fixing a polymatroid on each extremal ray, every polymatroid in Γ_M is a non-negative linear combination (also called *conic combination*) of these extremal polymatroids.

A. Flats, modular cuts and filters

Let $\mathcal{M}=(f,M)$ be a fixed polymatroid. A subset $F\subseteq M$ is a *flat* if every proper superset of F has strictly larger rank. The *closure of A*, denoted by $\mathsf{cl}(A)$, is the smallest flat containing A. A collection $\mathcal{F}$ of flats is a *modular cut* if it has properties (i)–(iii) below:

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(i) closed upwards: if $F \in \mathcal{F}$ and the flat F' is a superset of F, then $F' \in \mathcal{F}$;

(ii) closed for modular intersection: if $F_1, F_2 \in \mathcal{F}$ and (F_1, F_2) is a modular pair (that is, $\delta_f(F_1, F_2) = 0$), then $F_1 \cap F_2 \in \mathcal{F}$ (note that the intersection of flats is a flat);

(iii) not empty, which is equivalent to $M \in \mathcal{F}$.

In standard textbooks the empty collection is also considered to be a modular cut, see, e.g., [8]. It has been excluded here to emphasize the similarity to modular filters defined below.

The modular cut *generated by the flats* $F_1, \dots, F_k$ is the smallest modular cut containing all of these sets. This modular cut is denoted by $\mathcal{F}(F_1, \dots, F_k)$.

A modular cut $\mathcal{F}$ is *principal* if it is generated by a single flat; or, equivalently, if the intersection of all elements of $\mathcal{F}$ is also an element of $\mathcal{F}$. When $\mathcal{F}$ is not principal, there are two flats $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap F_2 \notin \mathcal{F}$. In this case $F_1 \cap F_2 \notin \mathcal{F}(F_1, F_2)$ as the modular cut generated by F_1 and F_2 is a subcollection of $\mathcal{F}$.

The collection $\mathcal{G}$ of subsets of M is a *modular filter* if it satisfies the following properties:

(i) closed upwards: if $A \in \mathcal{G}$, $A \subseteq B$, then $B \in \mathcal{G}$;

(ii) closed for modular intersection: if $A, B \in \mathcal{G}$ and (A, B) is a modular pair, then $A \cap B \in \mathcal{G}$;

(iii) non-trivial: if $f(X) = f(M)$, then $X \in \mathcal{G}$.

Modular filters generated by certain subsets as well as principal and non-principal modular filters can be defined similarly to modular cuts. In particular, $\mathcal{G}(A, B)$ is the smallest modular filter containing A and B.

The following Proposition shows how to get a modular filter from a modular cut.

Proposition 2. *Suppose $\mathcal{F}$ is a modular cut. The collection $\mathsf{cl}^{-1}(\mathcal{F}) = \{A \subseteq M : \mathsf{cl}(A) \in \mathcal{F}\}$ is a modular filter.*

Proof. Properties (i) and (iii) of the definition of a modular filter clearly hold, thus suppose (A, B) is a modular pair and both $F_A = \mathsf{cl}(A)$ and $F_B = \mathsf{cl}(B)$ are in $\mathcal{F}$. Then $\mathsf{cl}(F_A F_B) = \mathsf{cl}(AB)$ and $F_A \cap F_B \supseteq A \cap B$, thus

$$\delta_f(F_A, F_B) = \delta_f(A, B) + f(A \cap B) - f(F_A \cap F_B) \le \\ \le \delta_f(A, B) = 0.$$

Consequently (F_A, F_B) is a modular pair and $f(A \cap B) = f(F_A \cap F_B)$, thus $\mathsf{cl}(A \cap B) = F_A \cap F_B \in \mathcal{F}$, and $A \cap B \in \mathsf{cl}^{-1}(\mathcal{F})$. □

B. Extensions

The polymatroid $\mathcal{M}' = (f', M')$ is an *extension* of $\mathcal{M} = (f, M)$ if $M' \supset M$, and $f(X) = f'(X)$ for all $X \subseteq M$. This is a *one-point extension* if $M' - M$ has a single element.

Given a polymatroid (f, M) and extensions (f_1, M_1) and (f_2, M_2) of (f, M) with $M_1 - M$ and $M_2 - M$ disjoint, an *amalgam* is a polymatroid on $M_1 \cup M_2$ that extends both (f_1, M_1) and (f_2, M_2). The polymatroid $\mathcal{M}$ is *sticky* if any two of its extensions have an amalgam.

The function e defined on the set of subsets of M is an *excess function of* $\mathcal{M} = (f, M)$ if there is a one-point extension $\mathcal{M}' = (f', M \cup \{x\})$ of $\mathcal{M}$ such that $e(A) = f'(xA) - f'(A)$ for all $A \subseteq M$. If the polymatroid $\mathcal{M}$ is clear from the context, e is called simply an *excess function*.

Proposition 3. *The function e is an excess function if and only if the following conditions hold.*

(i) *e is non-negative and decreasing: $e(A) \ge e(B) \ge 0$ for $A \subseteq B \subseteq M$,*

(ii) *$\big(e(M) - e(M{-}i)\big) + \big(f(M) - f(M{-}i)\big) \ge 0$ for all $i \in M$,*

(iii) *$\delta_e(aA, bA) + \delta_f(aA, bA) \ge 0$ for all $abA \subseteq M$, $a, b \notin A$ (including $A = \emptyset$).*

Proof. It is clear that the conditions are necessary. For sufficiency, it is enough the check that f' defined on subsets of Mx as $f'(Ax) = f(A) + e(A)$ and $f'(A) = f(A)$ is the rank function of a polymatroid. According to [7, Theorem 2] it is enough to check $f'(M) \ge f'(M{-}i)$ for $i \in Mx$ and $\delta_{f'}(aA, bA) \ge 0$ for $a, b \in Mx{-}A$. These inequalities follow easily from the listed conditions. □

The identically zero function clearly satisfies these assumptions, thus it is an excess function. Actually, it adds a *loop* to the polymatroid. The inequality $\delta_e(A, B) + \delta_f(A, B) \ge 0$ holds for arbitrary subsets $A, B \subseteq M$ as this is just the modular defect of the pair (Ax, Bx) in the extension.

The following statements connect one-point extensions, modular cuts and modular filters.

Proposition 4. *The collection $\mathcal{G}$ of subsets of M is a modular filter if and only if there is an excess function e such that $e(M) = 0$ and $\mathcal{G} = \{A \subseteq M : e(A) = 0\}$.*

Proof. If e is an excess function with $e(M) = 0$, then Proposition 3 and $\delta_e(A, B) + \delta_f(A, B) \ge 0$ trivially imply that $\mathcal{G}$ is a modular filter. To show the converse, let $\mathcal{G}$ be a modular filter, and choose the function e as $e(A) = 0$ for $A \in \mathcal{G}$, and $e(A) = \varepsilon$ otherwise where ε is a sufficiently small positive value. We claim that conditions (i)–(iii) of Proposition 3 hold. This is clear for (i). For (ii) observe that $e(M) - e(M{-}i)$ is either 0 or $-\varepsilon$, and the latter holds when $M{-}i \notin \mathcal{G}$, but then $f(M{-}i) \ne f(M)$. Thus choosing ε smaller than all positive $f(M) - f(M{-}i)$ ensures condition (ii).

Finally, $\delta_e(aA, bA)$ is either non-negative or equals $-\varepsilon$. This latter happens when both aA and bA are in $\mathcal{G}$ but $A \notin \mathcal{G}$. In this case (aA, bA) is not a modular pair. Choosing ε smaller than all possible positive modular defects in the polymatroid gives condition (iii). □

Claim 5. *The following statements are equivalent:*

(i) *The polymatroid $\mathcal{M}$ has a non-principal modular cut.*

(ii) *The polymatroid has flats F_1 and F_2 such that $F_1 \cap F_2 \notin \mathcal{F}(F_1, F_2)$.*

(iii) *The polymatroid has flats F_1 and F_2 and an excess function e such that $e(F_1) = e(F_2) = 0$ and $e(F_1 \cap F_2) > 0$.*

Proof. (i) $\to$ (ii) If $\mathcal{F}$ is not a principal cut, then there are $F_1, F_2 \in \mathcal{F}$ such that $S = F_1 \cap F_2 \notin \mathcal{F}$. As $\mathcal{F}(F_1, F_2)$ is a subcollection of $\mathcal{F}$, $S \notin \mathcal{F}(F_1, F_2)$.

(ii) $\to$ (iii) Let $\mathcal{F} = \mathcal{F}(F_1, F_2)$. By Proposition 2, $\mathcal{G} = \mathsf{cl}^{-1}(\mathcal{F})$ is a modular filter, and by Proposition 4 there is an excess function e such that $e(A) = 0$ for $A \in \mathcal{G}$, and $e(A) > 0$ otherwise. As $F_1, F_2 \in \mathcal{G}$, the first required property holds, and the second also holds if we show that $S = F_1 \cap F_2 \notin \mathcal{G}$. But S is a flat, $S \notin \mathcal{F}$, thus $S \notin \mathsf{cl}^{-1}(\mathcal{F}) = \mathcal{G}$.

(iii) $\to$ (i) Let e be the excess function, and consider the collection of flats $\mathcal{F} = \{F : e(F) = 0\}$. Clearly, $F_1, F_2 \in \mathcal{F}$ and $F_1 \cap F_2 \notin \mathcal{F}$. We claim that $\mathcal{F}$ is a modular cut. The fact that it is non-principal is clear. Properties (i) and (iii) clearly hold. For (ii) observe that $\delta_f(A, B) + \delta_e(A, B) \geq 0$, thus if $A, B \in \mathcal{F}$, then $e(A) = e(B) = e(A \cup B) = 0$ (as e is decreasing and non-negative), and if (A, B) is a modular pair, that is, $\delta_f(A, B) = 0$, then

$$0 \leq \delta_f(A, B) + \delta_e(A, B) = -e(A \cap B),$$

meaning that $e(A \cap B) = 0$, thus the flat $A \cap B$ is in $\mathcal{F}$. ☐

III. Linear polymatroids

The polymatroid $\mathcal{M} = (f, M)$ is *linearly representable* if there is a (finite dimensional) vector space V over some finite field, and a linear subspace V_i of V for each $i \in M$ such that for all $A \subseteq M$, the rank of A is the dimension of the subspace spanned by $V_A = \bigcup\{V_i : i \in A\}$; see [8].

A linearly representable polymatroid is clearly integer, and there are linearly representable polymatroids whose sum is not linearly representable. Frequently when linearly representable polymatroids have some interesting (or desired) property, so do polymatroids in their conic hull. The definition of *linear polymatroids* below illustrates such a case. As it captures one of the most important aspects of linear representability, we hope that this notion has other applications.

Definition 6. *The polymatroid (f, M) is* linear *if for every pair X, Y of subsets of the ground set M, either (X, Y) is a modular pair, or there is an excess function e with $e(X) = e(Y) = 0$ and $e(X \cap Y) > 0$.*

By Proposition 4 this is equivalent to require $X \cap Y \notin \mathcal{G}(X, Y)$ for non-modular pairs (X, Y), where $\mathcal{G}(X, Y)$ is the modular filter generated by X and Y.

Claim 7. *Linearly representable polymatroids are linear.*

Proof. Suppose $\mathcal{M} = (f, M)$ is linearly representable; let $V_i \subseteq V$ be the linear subspace assigned to $i \in M$. For $A \subseteq M$, its rank is the dimension of the subspace spanned by V_A. If the subsets X and Y are not modular, adjoin a new element to $\mathcal{M}$ represented by the intersection of the linear span of V_X and the linear span of V_Y. Let e be the excess function of this one-point extension. Then $e(X) = e(Y) = 0$, and $e(X \cap Y)$ equals the modular defect of X and Y, which is non-zero. ☐

Claim 8. *Conic combination of linear polymatroids is linear.*

Proof. The definition of linear polymatroids is clearly invariant under multiplication by a constant. So suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined on the same ground set and both are linear. If (X, Y) is modular in $\mathcal{M}_i$, then let e_i be identically zero, otherwise let it be the excess function guaranteed by linearity. If (X, Y) is not modular in $\mathcal{M}_1 + \mathcal{M}_2$ then $e_1 + e_2$ is the excess function showing the required extension. ☐

Claim 9. *Linear polymatroids satisfy $(*)$ of Conjecture 1.*

Proof. Suppose (F_1, F_2) is a non-modular pair of flats, we need to find a non-principal modular cut in the polymatroid. As the polymatroid is linear, there is an excess function e with $e(F_1) = e(F_2) = 0$ and $e(F_1 \cap F_2) > 0$, and then the existence of non-principal modular cut follows from Claim 5. ☐

IV. Main lemmas

A non-negative linear (conic) combination of polymatroids on the same set M is again a polymatroid on M. If F is a flat in any constituent with positive coefficient, then F is a flat in the sum; however the sum can have a flat which is not a flat in any of the constituents. The next lemmas establish properties of the constituents when their conic combination violates Conjecture 1.

Lemma 10. *Let $\mathcal{M}$ and $\mathcal{N}$ be two polymatroids on the same set. Suppose $\mathcal{M}$ has two flats F_1, F_2 such that $F_1 \cap F_2 \notin \mathcal{F}_{\mathcal{M}}(F_1, F_2)$. Then for any $\lambda > 0$, $\mathcal{N} + \lambda\mathcal{M}$ satisfies Conjecture 1.*

Proof. This holds since there is a non-principal modular cut in $\mathcal{N} + \lambda\mathcal{M}$. This follows from Claim 5 once we show that a) F_1 and F_2 are flats in $\mathcal{N} + \lambda\mathcal{M}$ (this is trivial from the discussion above), and b) there exists an appropriate excess function e for $\mathcal{N} + \lambda\mathcal{M}$.

Let $e_{\mathcal{M}}$ be the excess function for $\mathcal{M}$ with $e_{\mathcal{M}}(F_1) = e_{\mathcal{M}}(F_2) = 0$, and $e_{\mathcal{M}}(F_1 \cap F_2) > 0$, guaranteed by the condition and Claim 5, and define $e = \lambda e_{\mathcal{M}}$. Conditions in Proposition 3 trivially hold (as they are linear), thus e is the required excess function for $\mathcal{N} + \lambda\mathcal{M}$. ☐

Lemma 11. *Suppose $\lambda > 0$ and $\mathcal{N} + \lambda\mathcal{M}$ is a minimal counterexample to Conjecture 1. In $\mathcal{M}$ every intersecting pair of flats is modular.*

Proof. As $\mathcal{M}^* = \mathcal{N} + \lambda\mathcal{M}$ is a counterexample, it has a non-modular pair of flats but no non-principal modular cut. If (F_1, F_2) is a non-modular pair of flats in $\mathcal{M}^*$ and $S = F_1 \cap F_2$ is not empty, then the contraction M^*/S is a smaller counterexample to Conjecture 1. Consequently $\mathcal{M}^*$ has no intersecting non-modular flat pairs.

To finish the proof one has to notice that if F_1 and F_2 are intersecting non-modular flats in $\mathcal{M}$, then they remain the same in $\mathcal{M}^*$ as well. ☐

From here the strategy for checking Conjecture 1 should be clear. Every polymatroid on a given ground set is a conic combination of finitely many extremal polymatroids which can be listed explicitly when the polymatroid has five or less elements [12]. Suppose $\mathcal{M}$ violates Conjecture 1 and no counterexample exists on a smaller ground set. This $\mathcal{M}$ is a conic combination of the extremal polymatroids. The combining coefficient is zero if the corresponding extremal polymatroid a) contains two intersecting non-modular flats (Lemma 11), or b) contains disjoint flats F_1, F_2 such that the modular cut $\mathcal{F}(F_1, F_2)$ is not principal (Lemma 10). This hopefully leaves only a few extremal polymatroids which can be checked individually.

V. Sticky polymatroids on five or fewer elements

A. Polymatroids on two elements

Let $M = \{a, b\}$. Polymatroids on M are conic combinations of the three extremal ones listed in Table I. Each of them is linearly representable, thus linear. By Claim 8 their conic combination remains linear. Thus every polymatroid on two elements is linear, and by Claim 9 they satisfy Conjecture 1.

	a	b	ab
$\mathcal{M}_a$	1	0	1
$\mathcal{M}_b$	0	1	1
$\mathcal{M}_{ab}$	1	1	1

TABLE I
Extremal polymatroids on $\{a, b\}$

B. Polymatroids on three elements

There are eight extremal polymatroids on $M = \{a, b, c\}$. Up to isomorphism there are four different ones listed in Table II; the others can be obtained by permuting the elements of M. As

	a	b	c	ab	ac	bc	abc
$\mathcal{M}_a$	1	0	0	1	1	0	1
$\mathcal{M}_{ab}$	1	1	0	1	1	1	1
$\mathcal{M}_{abc}$	1	1	1	1	1	1	1
$\mathcal{M}_*$	1	1	1	2	2	2	2

TABLE II
Extremal polymatroids on $\{a, b, c\}$

in the two-element case, all of them are linearly representable, thus every polymatroid on $\{a, b, c\}$ is linear. By Claim 9, Conjecture 1 holds for these polymatroids. In $\mathcal{M}_*$, every pair of singletons is independent (modular), but any two determine the third one.

C. Polymatroids on four elements

Extremal polymatroids on four and five elements have been reported in [12]. The software package Polco [11] can generate the extremal rays from the collection of the defining inequalities. The polymatroid cone Γ_{abcd} has 41 extremal rays. Up to isomorphism there are only 11 different ones among the corresponding polymatroids. Table III lists one element from each isomorphism class; the ranks are shown as follows: first one-element subsets, then two-element subsets, etc., each group in alphabetical order. Polymatroids $\mathcal{M}_1$–$\mathcal{M}_{10}$ are linearly representable. For $\mathcal{M}_{10}$ take three linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$; they span the one-dimensional subspaces assigned to a, b, c, respectively, while d gets the 2-dimensional subspace spanned by $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$. The polymatroid $\mathcal{M}_{11}$ is not linearly representable, but the modular cut generated by the flats ac and bd is $\{ac, bd, abcd\}$. Thus by Lemma 10 it cannot contribute to a polymatroid violating Conjecture 1. Consequently each polymatroid on four elements satisfies Conjecture 1.

$\mathcal{M}_1$	0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1
$\mathcal{M}_2$	0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
$\mathcal{M}_3$	0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
$\mathcal{M}_4$	0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2
$\mathcal{M}_5$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
$\mathcal{M}_6$	1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
$\mathcal{M}_7$	1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
$\mathcal{M}_8$	1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3
$\mathcal{M}_9$	1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
$\mathcal{M}_{10}$	1, 1, 1, 2, 2, 2, 3, 2, 3, 3, 3, 3, 3, 3, 3
$\mathcal{M}_{11}$	2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4

TABLE III
Extremal polymatroids on $\{a, b, c, d\}$

D. Polymatroids on five elements

According to [12] there are 117983 extremal polymatroids on a five element set, and 1320 different ones up to isomorphism. By Lemmas 10 and 11 we can eliminate those extremal polymatroids that a) contain two intersecting non-modular flats, or b) contain disjoint flats F_1, F_2 such that $\mathcal{F}(F_1, F_2)$ is not principal (i.e., the generated cut does not contain the empty set). After this thinning we get a quite meager set of

0,0,0,0,1,0,0,0,1,0,0,1,0,1,1,0,0,1,0,1,1,0,1,1,1,0,1,1,1,1,1
0,0,0,1,1,0,0,1,1,0,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
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1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2
1,1,1,2,2,2,2,2,2,2,3,3,3,3,3,2,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3

TABLE IV
Remaining extremal polymatroids on $\{a, b, c, d, e\}$

17 isomorphism classes; representatives are listed in Table IV. The ranks are shown by cardinality of the subset, and within that alphabetically. By inspection, all of them are linearly representable, thus linear. By Claim 8, any conic combination of linear polymatroids is linear, and Claim 9 says that these linear polymatroids satisfy $(*)$. Consequently all polymatroids on 5 elements satisfy Conjecture 1.

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