# Problem Books in Mathematics

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# Mathematical Logic

## Exercises and Solutions

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# PREFACE

Delivering a Mathematical Logic course is always a challenge. Students enroll the course expecting to learn what "logical thinking" is and what the infallible rules of mathematical and rational thinking are. Instead, they get boring stuff like Zorn's lemma, torsion-free divisible Abelian groups, or dense linear ordering. They probably have heard of the infamous incompleteness theorem, and are eager to know how it destroys mathematics and rational thinking in general. Instead, they get the technicalities of Gödel's theorems and a page-long list of conditions under which they apply. The Mathematical Logic course covers a large and diverse segment of mathematics, and a significant part comes in the form of *problem-solving*. The weekly assignments play another important role: a crucial part of learning mathematics is gaining intuition about how the various concepts operate and interact, which, much like learning to drive a car, cannot be done without hands-on experience, and trial and error.

Problems in this volume have been collected over more than 30 years of teaching undergraduate students Mathematical Logic at Eötvös Loránd University, Budapest. The problems come in great variety: routine applications of a newly introduced technique, checking whether the conditions of a particular theorem are really necessary, extending or finding the limitations of various methods, to amusing puzzles and interesting applications of established results. They range from easy questions and riddles to proving hard theorems when all the necessary ingredients are—hopefully—available.

Several of the problems are part of the "mathematical folklore": well-known ones often used in teaching that everyone changes or twists slightly to fit their taste and the problem to illustrate. They are like good jokes or anecdotes that one keeps telling to new guests at a dinner party, although no one is sure exactly where they have come from. Others are extensions, details, or crucial points of the often hard and ingenious proofs of major theorems. Still others are based on solutions submitted by our students where an unexpected, clever method was used, or where the proposed solution had an interesting flaw or omission. And, some of the problems originated from intriguing questions from our students.

Chapters 1–4 contain problems from supporting fields: set-theoretical constructions, interesting applications in (perfect information) games, basic (and not so basic) results in formal languages, and recursion theory. Mathematical Logic proper is covered in Chapters 5–11 with problems in propositional and multi-valued logic, compactness and derivation; basic properties of first-order logic, derivation, compactness and completeness, elementary equivalence, and the ultraproduct technique. Chapter 10 covers arithmetics and incompleteness, and Chapter 11 touches on two advanced topics: the insufficiency of the Ehrenfeucht–Fraïssé game, and the zero-one law of random (universal) graphs.

The book concludes with solutions to all of the problems. We strongly encourage the reader to try to solve the problems before reading the solution included. Someone pointed out that there is a difference between doing push-ups and watching someone doing them (however fortunate it would be if it was otherwise!), which also seems to apply to mental exercise. The problems are organized such that ideas and techniques from previous ones can often be used to solve the next problem, so looking at the problems and their solutions leading up to the current one can be a good way of getting some inspiration.

One of the first homework problems in the Mathematical Logic course is a famous *sorosites* created by Lewis Carroll, so let us start our collection with it.

### THE PIGS AND BALLOONS PUZZLE

The following facts are known:

(1) All, who neither dance on tight ropes nor eat penny-buns, are old.
(2) Pigs, that are liable to giddiness, are treated with respect.
(3) A wise balloonist takes an umbrella with him.
(4) No one ought to lunch in public who looks ridiculous and eats penny-buns.
(5) Young creatures, who go up in balloons, are liable to giddiness.
(6) Fat creatures, who look ridiculous, may lunch in public, provided that they do not dance on tight ropes.
(7) No wise creatures dance on tight ropes, if liable to giddiness.
(8) A pig looks ridiculous, carrying an umbrella.
(9) All who do not dance on tight ropes, and who are treated with respect are fat.

Show that no wise young pigs go up in balloons.

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