Classes of Conflictness / Non-Conflictness of Belief Functions

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Abstract

Theoretic, descriptive and experimental analysis and description of classes of conflictness, non-conflictness and of conflict hiddeness of belief functions. Theoretic extension of theory of hidden conflicts. Idea of catalogue of belief structures.

1 Introduction

As discussed in [2], the weight of conflict according to the classic Shafer's definition [13] using $m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ is frequently higher than the expected value of conflict even for the partially conflicting belief functions (BFs). On the other hand, a positive value of a conflict (here we have in mind the conflict between BFs based on their non-conflicting parts [3, 4]) was observed even in a situation when $m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ equals zero.

This observation led to the definition of several degrees (up to cardinality of frame of discernment) of hidden conflicts [6, 8], later compared with alternative shades of conflict [12] in [7]. In one-to-one relation to different degrees of conflict hiddenness, there are corresponding classes of non-conflictness [5]. And it is precisely the content of this paper — to explore and analyze different classes of belief functions concerning the hidden conflict.

This study covers a theoretical, descriptive, and a experimental approaches. The first one analyses the definitions and conditions of particular degrees of non-conflictness, resulting in a theoretic characterization of specific classes of non-conflicting BFs in various degrees. The other approaches characterize types of BFs, intending to describe and catalog different structures of BFs concerning hidden conflict or non-conflictness. Due to the complexity of BFs and sets/structures of their focal elements, this approach brings detailed results on small frames of discernment (two and three-element frames Ω_2, Ω_3) and rougher results for larger frames and a general n-element frame Ω_n.

The idea of going through all possible structures of belief functions on different frames of discernment and arranging them in a catalogue was inspired by a similar work [15] on a system of min-balanced systems known from game theory. Indeed, both the system of coalitions and the structure of a belief function is a set of sets. However, unlike game theory, the structure of belief functions is much richer — any restrictive rule does not limit it. For this reason, the resulting catalogue is quite extensive. Therefore it may be appropriate to limit it to some structurally limited subclass of belief functions, such as consonant belief functions in the future. An example of a catalogue for a particular class of min-balanced systems can be found at http://gogo.utia.cas.cz/indecomposable-min-semi-balanced-catalogue. We plan to create a similar catalogue, however, it is not finished at the time of writing this paper.

2 Basic Notions

This section will recall some basic notations needed in this paper.

Assume a finite frame of discernment Ω with elements denoted by lower-case letters from Latin alphabet $a, b, c, \ldots$ and their sets by capital letters. $A = \{a, b\}$. To simplify the notation, we abbreviate $\{a, b\}$ with ab. In the case of $|\Omega| = n$, we will highlight this fact using a subscript as Ω_n. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is a *power-set* of Ω. $\mathcal{P}(\Omega)$ is often denoted also by 2^Ω, e.g., in [12].

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$. The values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed. We sometimes speak about m as of a mass function.

There are other equivalent representations of m: A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. Because there is a unique correspondence between m and corresponding Bel we often speak about m as of a belief function.

Let m be a belief function defined on Ω and $A \subseteq \Omega$. If $m(A) > 0$ we say A is a *focal element* of m. The set of focal elements is denoted by $\mathcal{F}_m$ (or simply $\mathcal{F}$ for short) and we call it a *structure* of m. We say that a focal element $X \in \mathcal{F}$ is proper if $X \neq \Omega$. In the case of $m_{vac}(\Omega) = 1$ we speak about the *vacuous BF* (VBF) and about *a non-vacuous BF* otherwise. We speak about *consistent* BF if all focal elements have a non-empty intersection. If focal elements are nested, we speak about consonant BF.

The (non-normalized) conjunctive rule of combination $\odot$, see e.g. [14], is defined by:

$$(m_1 \odot m_2)(A) = \sum_{X \cap Y = A; X, Y \subseteq \Omega} m_1(X) m_2(Y)$$

for any $A \subseteq \Omega$. $\kappa = \sum_{X \cap Y = \emptyset; X, Y \subseteq \Omega} m_1(X) m_2(Y)$ is usually considered to represent a conflict of respective belief functions when $\kappa > 0$. By normalization of $m_{12} = m_1 \odot m_2$ we obtain Dempster's rule, see [13]. To simplify formulas, we often use $\bigodot_1^3 m = m \odot m \odot m$, and also $\bigodot_1^k (m_1 \odot m_2) = (m_1 \odot m_2) \odot \ldots \odot (m_1 \odot m_2)$, where $(m_1 \odot m_2)$ is repeated k-times.

3 Hidden Conflicts and Internal Hidden Conflicts

After several preliminary studies, two types of hidden conflict were introduced in [8]. We speak either about *internal* hidden conflict of given BF or about a mutual hidden conflict *between* two BFs. Let us recall the hidden conflict definitions and their most important properties here. For introductory examples and more details see [5, 7, 8].

We shall note that hidden conflict and its degrees are just extensions of classic Shafer's definition of conflict. It is not a new alternative definition or approach.

Definition 1 *Assume two BFs m^i and m^{ii} such that for some $k>0$ $(\bigcirc\!\!\!\!\cap_1^k(m^i \bigcirc\!\!\!\!\cap m^{ii}))(\emptyset) = 0$. If there further holds $(\bigcirc\!\!\!\!\cap_1^{k+1}(m^i \bigcirc\!\!\!\!\cap m^{ii}))(\emptyset) > 0$ we say that there is a* conflict of BFs m^i *and* m^{ii} hidden in the k-th degree *(hidden conflict of k-th degree, abbreviated as HC_k). If there is already $(\bigcirc\!\!\!\!\cap_1^{k+1}(m^i \bigcirc\!\!\!\!\cap m^{ii}))(\emptyset) = (m^i \bigcirc\!\!\!\!\cap m^{ii}))(\emptyset) > 0$ for $k = 0$ then there is a conflict of respective BFs which is not hidden or we can say that it is* conflict hidden in degree zero *(HC_0).*

Theorem 1 *Hidden conflict of non-vacuous BFs on $\Omega_n, n > 1$ is always of a degree less or equal to $n-2$; i.e., the condition*

$$(\bigcirc\!\!\!\!\cap_1^{n-1}(m^i \bigcirc\!\!\!\!\cap m^{ii}))(\emptyset) = 0 \tag{1}$$

always means full non-conflictness of respective BFs and there is no hidden conflict.

Definition 2 *Let us assume a BF is given by m such that $(\bigcirc\!\!\!\!\cap_1^2 m)(\emptyset) = 0$ and $(\bigcirc\!\!\!\!\cap_1^s m)(\emptyset) > 0$ for an $s > 2$. Then we say that there is* an internal hidden conflict *in m. More specifically, if $\exists k \geq 0$ such that $(\bigcirc\!\!\!\!\cap_1^{k+1} m)(\emptyset) = 0$ and $(\bigcirc\!\!\!\!\cap_1^{k+2} m)(\emptyset) > 0$, then we say that there is* an internal conflict of BF m hidden in k-th degree[1] *– hidden internal conflict of k-th degree (HIC_k).*

Theorem 2 *Internal hidden conflict of any BF on Ω_n, $n > 1$ is always of a degree less or equal to $n-2$; i.e., the condition*

$$(\bigcirc\!\!\!\!\cap_1^n m)(\emptyset) = 0 \tag{2}$$

always means the full internal non-conflictness of any BF given by any bba m on any Ω_n.

Theorem 3 *(i) Let us assume two BFs m^i, m^{ii} with hidden conflict of k-th degree for $k \geq 2$ and their conjunctive combination $m = m^i \bigcirc\!\!\!\!\cap m^{ii}$. Then there is an internal conflict of m hidden in $k-1$-th degree.*

(ii) Any hidden conflict of any BF m of any degree $k > 1$ can be expressed as a hidden conflict of two BFs of degree $k+1$: $m = m \bigcirc\!\!\!\!\cap m_{vac}$.

Proof. For proofs of both the theorems see [8].

[1]Note that for $k = 0$ there is just $m(\emptyset) = 0$ and $(m \bigcirc\!\!\!\!\cap m)(\emptyset) > 0$; hence m is consistent and the internal conflict is not hidden or we can say *hidden in degree zero*; for $k = 1$ there is just $(m \bigcirc\!\!\!\!\cap m)(\emptyset) = 0$ and $(m \bigcirc\!\!\!\!\cap m \bigcirc\!\!\!\!\cap m)(\emptyset) > 0$ hence the conflict is hidden in the 1^{st} degree.

Note that the definition of k-th degree of hidden conflict differs in powers of $\textcircled{\cap}^k$ (see Definitions 1 and 2). The reason is straightforward: Note that while $(m_1 \textcircled{\cap} m_2)(\emptyset) = (\textcircled{\cap}^1(m_1 \textcircled{\cap} m_2))(\emptyset) > 0$ represents conflict which is not hidden, (i.e., hidden in degree 0), it is already $(\textcircled{\cap}^2(m_1 \textcircled{\cap} m_2))(\emptyset) > 0$ which represents hidden conflict in the degree 1. On the other hand, $(m \textcircled{\cap} m)(\emptyset) = (\textcircled{\cap}^2 m)(\emptyset) > 0$ represents internal conflict (of one BF) (i.e., hidden in degree 0). Thus the first degree of internal hidden conflict appears for $(\textcircled{\cap}^3 m)(\emptyset) > 0$ and k-th degree for $(\textcircled{\cap}^{k+1} m)(\emptyset) = 0$ while $(\textcircled{\cap}^{k+2} m)(\emptyset) > 0$.

Definition 3 *(i) Assume two BFs m^i and m^{ii}. We say that the BFs are non-conflicting in k-th degree if $(\textcircled{\cap}_1^k(m^i \textcircled{\cap} m^{ii})(\emptyset) = 0$.*
(ii) BFs m^i and m^{ii} are fully non-conflicting *if they are non-conflicting in any degree.*

Theorem 4 *Any two BFs on n-element frame of discernment Ω_n non-conflicting in the n-th degree are fully non-conflicting.*

Proof. For the idea of the proof see [5].

In this study, we are not interested in a numeric size of any conflict. What we are interested in are conflictness and non-conflictness. As all the degrees of hidden conflicts are only extensions of classic conjunctive conflict $(m_1 \textcircled{\cap} m_2)(\emptyset)$, all conflictness/non-conflictness depend only on *the sets of focal elements* $\mathcal{F}_1, \mathcal{F}_2$ — on the structures of respective BFs — not on their bbms. More specifically, it depends on the number and cardinalities of focal elements, their intersections, nestedness, etc. We put the masses corresponding to particular focal elements aside in our examples and focus only on the structures of sets of focal elements.

4 Extension and Correction of Hidden Conflict Theory

4.1 Hidden Conflict on Ω_2 and Hidden Conflict of $(n-1)$-th Degree

When preparing this study we have observed hidden conflict also on $\Omega_2 = \{a, b\}$ and analogously we can find a hidden conflict of $(n-1)$-th degree on any n-element frame of discernment. How it is possible? According to the previous section and namely to [6, 8] the maximal degree of hidden conflict is $n-2$ on Ω_n. Since $2-2 = 0$, only a conflict which is not hidden is possible on Ω_2. There is $(m_{vac} \textcircled{\cap} m)(\emptyset) = m(\emptyset) = 0$ for any normalised BF m. Nevertheless, $(\textcircled{\cap}^2(m_{vac} \textcircled{\cap} m))(\emptyset) = (m \textcircled{\cap} m)(\emptyset) > 0$ whenever $m(a) > 0, m(b) > 0$, thus for any general BF m with both the singletons. How this is possible?

The reason is the following. We were looking for a hidden mutual conflict between two BFs in [6, 8]. m_{vac} is considered to be non-conflicting with any BF. Therefore, it is non-conflicting also with any m with both singletons among its focal elements. Hence the hidden conflict of m_{vac} and m is just an internal conflict of m, see also $(m \textcircled{\cap} m)(\emptyset) > 0$ above. I.e., it is an internal conflict of m which is not hidden. In the case of combination, it is hidden by m_{vac}.

The analogous situations appear on any finite frame of discernment; hence we obtain:

Lemma 1 *Let m_1 and m_2 be two belief functions on Ω_n such that they have hidden conflict of $(n-1)$-th degree. Then $\mathcal{F}_1$ contains n subsets of Ω_n of cardinality $n-1$ only, and possibly also entire Ω_n, $\mathcal{F}_2 = \{\Omega_n\}$ or vice versa. Hence one the BFs is vacuous and the corresponding hidden conflict is in fact an internal hidden conflict the other one.*

Proof. Assertion follows Theorem 6 and the above text of this section.

Our current observation and Lemma 1 show the importance of distinguishing internal conflict and of entire/global conflict of two BFs from mutual conflict between them [2] also in a hidden case!

4.2 Belief Structures in Hidden Conflict on Ω_3

Let us present a correction of Lemma 5 from [8] about structures of non-vacuous BFs which have hidden conflict of the $(n-2)$-th degree. The original statement of the Lemma holds for all frames for $n > 3$. There are more belief structures for smaller frames Ω_2 and Ω_3. The corrected version is the following:

Lemma 2 *(i) The only non-vacuous BFs on Ω_n with hidden conflict of degree $(n-2)$ are BFs with focal elements of cardinality $\geq n-1$ for any $n > 3$, such that one has at least $(n-1)$ focal elements of cardinality $(n-1)$ and the other one has just one focal element of cardinality $(n-1)$. Moreover, every $(n-1)$-element subset of Ω_n must be a focal element of either one or both BFs.*
(ii) The only non-vacuous BFs on Ω_n with hidden conflict of degree $(n-2)$ are BFs with focal elements of cardinality $\geq n-1$ for $n = 2, 3$, such that each of them has at least one focal elements of cardinality $(n-1)$ and moreover, every $(n-1)$-element subset of Ω_n must be a focal element of either one or both BFs.

Proof. For proof and explanation see Appendix I.

5 Theoretic Approach

Let suppose a pair of BFs m_1, m_2 with focal elements $\mathcal{F}_1, \mathcal{F}_2$. If $(m_1 \circledcirc m_2)(\emptyset) > 0$, there is a non-hidden conflict, i.e., if there exists $A \in \mathcal{F}_1, B \in \mathcal{F}_2$ with non-empty intersection $A \cap B = \emptyset$. On the other hand the simplest case of non-conflictness of the 1-st degree is characterized by $(m_1 \circledcirc m_2)(\emptyset) = 0$, i.e., by $A \cap B \neq \emptyset$ for any $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$.

What does it mean conflict hidden in degree 1? According to main definition of hidden conflict, Definition 1, a hidden conflict of the first degree arises whenever

$$(m_1 \circledcirc m_2)(\emptyset) = 0 \text{ and } ((m_1 \circledcirc m_2) \circledcirc (m_1 \circledcirc m_2))(\emptyset) > 0.$$

Hence we can characterise the class of pairs of BFs non-conflicting in the second degree by $(\circledcirc^2 (m_1 \circledcirc m_2))(\emptyset) = 0$. Let us turn our attention to the motivation of HC, conflict observation, and the original working definition of hidden conflict: There is the principal assumption that the combination $(m_1 \circledcirc m_2)$ of two mutually non-conflicting BFs m_1, m_2 should be non-conflicting with any of the original m_1 and m_2, hence both

$(m_1 \circledcirc (m_1 \circledcirc m_2))(\emptyset) = 0$ and $((m_1 \circledcirc m_2) \circledcirc m_2)(\emptyset) = 0$. Using associativity and commutativity of conjunctive combination $\circledcirc$ we have $((m_1 \circledcirc m_2) \circledcirc (m_1 \circledcirc m_2))(\emptyset) = ((m_1 \circledcirc m_1) \circledcirc (m_2 \circledcirc m_2))(\emptyset)$ and it is zero whenever any focal element of $(m_1 \circledcirc m_1)$ has non-empty intersection with any focal element of $(m_2 \circledcirc m_2)$ and vice versa.

Thus the class of non-conflictness of the 2-nd degree is specified by $(\bigcirc^2 (m_1 \circledcirc m_2))(\emptyset) = 0$ and alternatively by $(X_i \cap X_j) \cap (Y_r \cap Y_s) \neq \emptyset$ for any focal elements $X_i, X_j \in \mathcal{F}_1$, $Y_r, Y_s \in \mathcal{F}_2$ of m_1, m_2. Note that the first condition corresponds to Yager's pair-wise consistency of $m_{12} = (m_1 \circledcirc m_2)$, which appears if $\sum_{X \cap Y \neq \emptyset, X, Y \in \mathcal{F}_{12}} m_{12}(X) m_{12}(Y) = 1$, see [16].

Analogously, we can continue to hidden conflicts and classes of non-conflictness of higher degrees: hidden conflict of the 2-nd degree and non-conflictness of the 3-rd degree, up to hidden conflict of the $(k-1)$-th degree and related non-conflictness of the k-th degree. Analogously to the classes of the 1-st and 2-nd degrees, we have also two characterizations of the class: (i) one based on the original bbas m_1 and m_2: $\bigcap_1^k X_i \cap \bigcap_1^k Y_j \neq \emptyset$ for any k-tuples of focal elements $X_i \in \mathcal{F}_1$ and $Y_j \in \mathcal{F}_2$, and (ii) the other characterization based on combination $m_{12} = m_1 \circledcirc m_2$: $(\bigcirc^k (m_1 \circledcirc m_2))(\emptyset) = 0$.
Ad (i): If either m_1 or m_2 has less focal elements than k, the focal elements are repeating in the computation of hidden conflict; see, e.g., one of the m_i's in the Introductory and the Little Angel Examples, see [8], thus analogously also in the verification of non-conflictness. Hence intersecting X_i, Y_i need not be different. Hence intersection of any k-tuple of elements of $\mathcal{F}_1$ (possibly with repeating) must be non-empty and must have a non-empty intersection with the intersection of any k-tuple of elements of $\mathcal{F}_2$ (possibly with repeating).
Ad (ii): this correspond to Pichon et al.'s k-consistency of m_{12}, see [12]; for $k = n$ to logical consistency [9].

Theorem 5 *For any pair of BFs Bel_1, Bel_2 given by m_1, m_2 the following is equivalent:*
(i) Bel_1 and Bel_2 are non-conflicting in degree k.
(ii) $\bigcap_1^k X_i \cap \bigcap_1^k Y_j \neq \emptyset$ for any k-tuples of focal elements $X_i \in \mathcal{F}_1$, $Y_j \in \mathcal{F}_2$ of m_1, m_2.
(iii) $(\bigcirc^k (m_1 \circledcirc m_2))(\emptyset) = 0$ for $m_{12} = m_1 \circledcirc m_2$.

6 Descriptive Approach

Let us start with Ω_2. There are $2^3 - 1 = 7$ different belief function structures. We can create a 7×7 table of all pairs of these structures as in Table 1. The black dot represents a singleton, and the black oval is the focal element of cardinality 2, which corresponds to Ω_2 in this case. Table cells correspond to $\circledcirc$ combination of respective structures. Since the $\circledcirc$ operator is commutative, only the right upper part is filled in. White cells correspond to non-conflicting structures, red and cyan to conflicting ones (red represents total conflict). The 4 green cells correspond to hidden conflict, as described in Lemma 1.

This case of Ω_2 has an excellent interpretation. We can easily see that non-conflicting pairs are just the consonant ones (including m_{vac}). Note that either one of them is m_{vac} or both contain the same singleton — the white cells of the table. Conflicting pairs (HC_0)

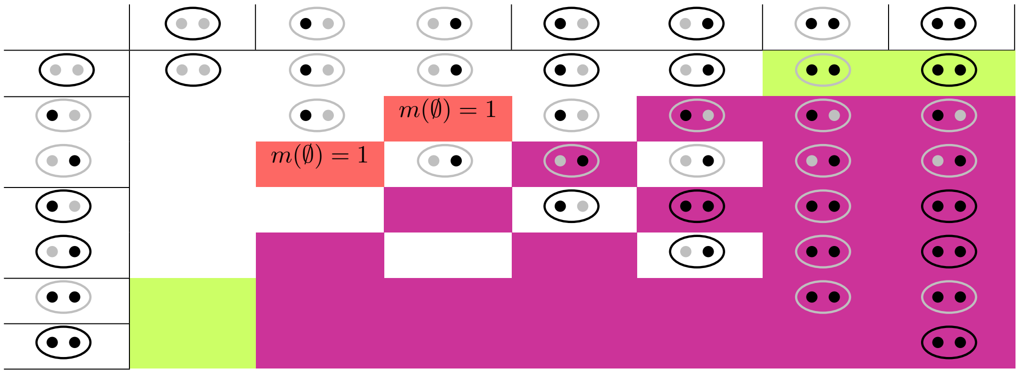


Table 1: All possible combinations of belief function structures on Ω_2

are of two types: (i) one structure contains both singletons and the other is non-vacuous. (ii) both structures have just one singleton, each different. And finally, 4 green HC_1 fields correspond to Lemma 1

Let us continue on Ω_3. This case is significantly more complex. There are $2^7 - 1 = 127$ belief structures here, (note that we have $2^{2^n-1} - 1$ possible structures in general), To give the reader a similar impression as from Table 1, we created a 127×127 bitmap — see Figure 1. Similarly to Ω_2 rows and columns correspond to structures. The structures are ordered by the number of focal elements as the first criterion and their size as the second one. I.e. the ordering is the following: $\{a\}; \{b\}; \{c\}; \{ab\}; \{ac\}; \{bc\}; \{abc\}; \{a, b\}, \{a, c\}, \ldots$. Therefore, e.g. the 7th row and column correspond to vacuous BF. White cells correspond to non-conflict situations, red to HC_0, orange to HC_1, and black to HC_2. Striped cells in Figure 1(a) correspond to pure type of respective conflict as defined later. Note that black cells corresponding to HC_2 appears in row and column corresponding to vacuous BF only.

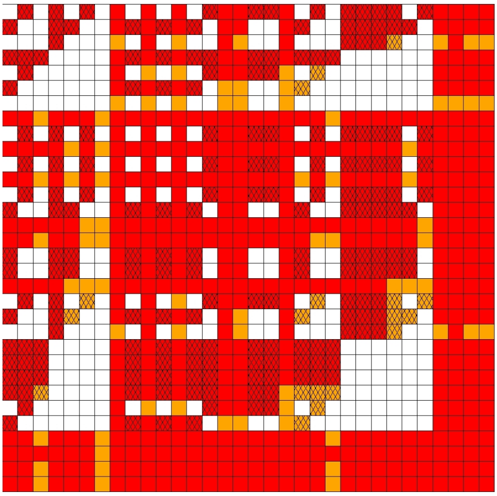
We can easily see white, i.e., non-conflicting area at (23–28)x(23–28) and other areas (23–28)x(60-63), (60-63)x(23-28), and (60-63)x(60-63). Where 22–26 are two couples, 27–28 couple and triple, 60–63 structures with 3 focal elements all 22–28, 60–63 contains c. Thus this is not theoretically very interesting; this area comes from the selected ordering of belief structures. The complete analysis of the bitmap is still under preparation.

Nevertheless, the bitmaps are already part of the experimental results, thus related to the next section.

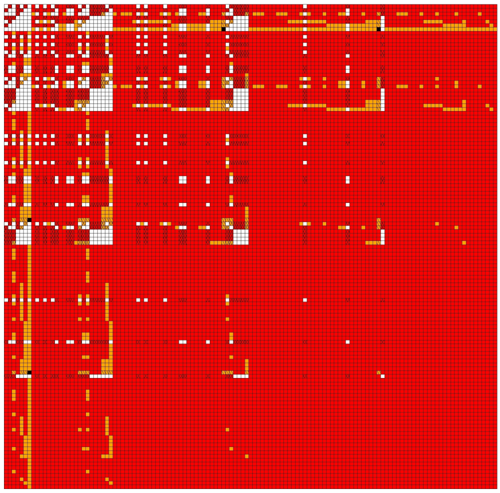
7 Experimental Approach

Conflictness/non-conflictness according to the classical definition of (conjunctive) conflict depends only on the structure of the focal elements given by the bbas. In this section, we will show some results of experiments with these structures.

A conflict based on a mass assigned to an empty set by the conjunctive rule has two



(a) Zoom of left upper part (32×32)



(b) Full 127×127 bitmap

Figure 1: Bitmap of combination of structures on Ω_3

levels. The first relates to the very existence of the conflict, i.e., whether there are two focal elements with an empty intersection. The second level deals with the magnitude of the conflict. The size corresponds to the number of pairs of focal elements with empty intersections and the probability masses they carry.

This paper focuses on the first part of the problem - the theoretical possibility of conflict which is connected with the structure of focal elements only. We are not interested in the probability mass assigned to individual focal elements here.

We know that the number of different structures on a given frame of discernment is super-exponential with respect to frame size. See the first column in Table 2. Suppose we disregard the frame of discernment labelling. Then we can group structures into permutation-equivalent classes and calculate individual properties for only one representative of each class. We can say that we are creating a certain catalogue of structures of belief functions. The number of classes of permutation-equivalent structures for $\Omega_2, \Omega_3, \Omega_4$, and Ω_5 is in Table 2. Note that we were not able to create this catalogue for frame of discernment having more than five elements. Please, be aware that the exact number of classes of permutation equivalent structures for Ω_5 is unknown. By the submission deadline, we were not able to go through all the structures with 15 and 16 focal elements. Therefore, the number from Table 2 is an estimate based on the number of classes for structures with other numbers of focal elements.

How do you recognize that two structures are permutation equivalent? It turns out that the problem corresponds to graph isomorphism – bipartite graphs isomorphism specifically. In this case, focal elements are vertices on one side, and the frame of discernment is represented by vertices on the other side of the graph. To solve graph isomorphism problem, we used the BLISS algorithm by Junttila and Kaski [10, 11] as implemented in

	number of structures	classes of permutation equivalent structures
Ω_2	7	5
Ω_3	127	39
Ω_4	32.767	1.990
Ω_5	2.147.483.648	8.820*

Table 2: Richness of structures

the igraph [1] R package. Note that the algorithm is based on a special heuristic of finding the canonical form of a graph unique for isomorphism.

7.1 Equivalence Classes of Belief Functions Structures

Let us present some interesting statistics on the structures of belief functions and respective equivalence classes. Note that we plan to create an online catalogue of all classes but at the time of writing this paper it is still an ongoing process.
Ω_3: Equivalence classes have three cardinalities (1, 3, and 6). There are 7 classes with only one structure. 24 classes contain 3 structures, and 8 classes contain 6 structures.
cardinality 1: e.g., $\{abc\}$, $\{a, b, c\}$[2] or m_1 and m_2 from Example 1
cardinality 3: e.g., $\{a\}$, $\{a, ab, ac\}$, $\{a, bc, abc\}$
cardinality 6: e.g., $\{a, ab\}$, $\{a, ab, abc\}$, $\{b, ab, ac\}$

Ω_4: In the case of Ω_4, equivalence classes have seven cardinalities (1, 3, 4, 6, 12, 24, and 48). There are 15 classes with only one structure. 16 classes contain 3 structures, and 894 classes contain 24 structures. Interestingly, there is only one class with 48 structures. One of these 48 structures is $\{a, ab, bc, bcd\}$. Note that this structure has internal conflict.

There are more greater classes with increasing n, even for analogous structures, look at the above structures from Ω_3 on greater frame Ω_5:
cardinality 1: e.g., $\{abcde\}$, $\{a, b, c, d, e\}$
cardinality 5: e.g., $\{a\}$,
cardinality 20: e.g., $\{a, ab\}$
cardinality 50: e.g., $\{a, ab, ac\}$
cardinality 60: e.g., $\{a, ab, abc\}$
cardinality 100: e.g., $\{b, ab, ac\}$

7.2 Internal Conflict

A proper survey of all structures aims to provide a detailed insight into the internal structure of individual conflicts and their types. In the case of one belief function, we recognize a hidden internal conflict of the structure of various degrees.

By internal conflict we mean the conflict which is inside a single bba, caused by conflicting masses of the bba, it may appear when we combine the bba with itself or it may remain hidden in some degree, see [8]. For each bba of the permutation-equivalent classes, we calculated whether it is internally non-conflicting or whether it has a hidden internal conflict and of which degree. Recall that the maximum degree of hidden conflict is $n-1$. The degree of conflict hiddeness is k if $\textcircled{\cap}^{k+1} m(\emptyset) = 0$ while $\textcircled{\cap}^{k+2} m(\emptyset) > 0$.

[2]We use abbreviations $\{abc\}$ for $\{\{abc\}\}$, $\{a, b, c\}$ for $\{\{a\}, \{b\}, \{c\}\}$ and analogous, in this section.

In the following table we use the following notation: NC — non-conflicting, IC — internal conflict, HIC_k - hidden internal conflict of k-th degree. The numbers in parentheses refer to all structure. Other numbers to classes of equivalence.

	NC	IC	HIC_1	HIC_2
Ω_2	3(5)	2(2)	–	–
Ω_3	11(37)	26(88)	2(2)	–
Ω_4	79(941)	1.867(31.392)	42(432)	2(2)

Table 3: Number of classes with different internal conflictness/non-conflicness

Example 1 *The only two classes of belief function structures allowing maximal hidden internal conflict for each frame of discernment are represented by the following structures* $\mathcal{F}_1, \mathcal{F}_2$ *for various frames of discernment:*

- $\Omega_2 = \{a, b\}$: $\mathcal{F}_1 = \{a, b\}$; $\mathcal{F}_2 = \{a, b, ab\}$
- $\Omega_3 = \{a, b, c\}$: $\mathcal{F}_1 = \{ab, ac, bc\}$; $\mathcal{F}_2 = \{ab, ac, bc, abc\}$
- $\Omega_4 = \{a, b, c, d\}$: $\mathcal{F}_1 = \{abc, abd, acd, bcd\}$; $\mathcal{F}_2 = \{abc, abd, acd, bcd, abcd\}$

Generally for Ω_n*, the focal elements have to be all subsets of cardinality* $|n-1|$ *with possible focal element covering the whole* Ω_n*.* $\mathcal{F}_1 = \{A \subset \Omega_n : |A| = n-1\}$;$\mathcal{F}_2 = \{A : A \in \mathcal{F}_1 \text{ or } A = \Omega_n\}$*. Then, the respective BFs have internal conflict hidden in* $(n-1)$*-th degree. This corresponds to Theorem 15 in [8].*

7.3 Mutual Conflict

In case of two different bbas, their mutual conflict can be also hidden. Assume m_1 and m_2. The definition of hidden conflict of k degree is that $(\textcircled{\cap}^k(m_1 \textcircled{\cap} m_2))(\emptyset) = 0$ while $(\textcircled{\cap}^{k+1}(m_1 \textcircled{\cap} m_2))(\emptyset) > 0$. In this experiment we tried to distinguish mutual conflict from false mutual conflict caused by possible (hidden) internal conflict of one of the involved bbas. To enumerate all pairs we employ the fact that we already have a catalogue of permutation equivalent structures. Technically, instead of going through all possible pairs of structures, we used only representatives from each permutation equivalent class on the one hand. On the other hand, we had to go through all the structures. This explains why we do not have results for Ω_5. The total number of pairs with a given property is then obtained by multiplying the sizes of a given permutation-equivalent class of structures. The symmetry of the whole operation guarantees the correct result.

Assume $m = m_1 \textcircled{\cap} m_2$. In the following table we use also this notation:

P — pure mutual hidden conflict: $(\textcircled{\cap}^k m)(\emptyset) > 0$ and $(\textcircled{\cap}^n m_1)(\emptyset) = 0$, $(\textcircled{\cap}^n m_2)(\emptyset) = 0$,

C — clear degree of mutual hidden conflict: i.e., $(\textcircled{\cap}^k m)(\emptyset) > 0$ and $(\textcircled{\cap}^k m_1)(\emptyset) = 0$, $(\textcircled{\cap}^k m_2)(\emptyset) = 0$ (degree comes from mutual, not from internal conflict(s)),

F — hidden conflict which may to be caused by internal conflict of either m_1 or m_2. $(\textcircled{\cap}^k m)(\emptyset) > 0$ and simultaneously $(\textcircled{\cap}^k m_1)(\emptyset) > 0$ or $(\textcircled{\cap}^k m_2)(\emptyset) > 0$;

unfortunately we cannon distinguish whether it is false mutual hidden conflict or a mixture of mutual and internal conflicts in general.

	NC	HC_0		HC_1			HC_2			HC_3		
		P=C	F	P	C	F	P	C	F	P	C	F
Ω_2	17	8	20	0	0	4						
Ω_3	649	672	14.048	48	100	656	0	0	4			
Ω_4	258.785	582.016	1.071.094.400	46.696	283.708	1.454.784	32	64	2.528	0	0	4

We have to note that, surprisingly less numbers of HC_2 (both P and C; 32 and 64) on Ω_4 than HC_1 (both P and C; 48 and 100) on Ω_3 (both the cases are conflict hidden on $(n-1)$-th degree on corresponding Ω_n) comes from the situation described in Lemma 1. There are more corresponding structures on Ω_3 than on Ω_4.

Example 2 *(i) Note that the 4 pairs of maximum hidden degree corresponds to both m_1, m_2 from Example 1; they are all combinations of m_1, m_2 with vacuous bba m_{vac}: $m_1 \circledcirc m_{vac}$, $m_2 \circledcirc m_{vac}$, $m_{vac} \circledcirc m_1$, and $m_{vac} \circledcirc m_2$ on any frame. It is generally assumed, that m_{vac} is mutually non-conflicting with any other bba, hence conflicts with the maximum degree of hiddeness $n-2$, are always false, they are always internal hidden conflicts of one of the bbas.*
(ii) Analogously, two numerically identical bbas $m_i \equiv m_j$ with an internal hidden conflict are mutually non-conflicting, hence internal hidden conflict of $m_i \circledcirc m_j$ is also false. Specially, also $m_i \circledcirc m_i$ for bbas from Example 1, nevertheless this time of less degree of hiddeness (degree $\lceil n-2/2 \rceil$).

8 Conclusion

Several theoretic extensions and corrections related to maximal degree of hidden conflict have been presented. Theoretic chracterization of classes of non-conflictness of belief functions has been formulated.

Descriptive and experimental approaches to analysis of combination of belief function structures have been presented. A catalogue of structures of belief functions and of combination of these structures is under preparation.

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Appendix I. Theoretical Corrections — Extensions

Let us present the original version of the theorem on maximal degree of hidden conflict (Theorem 2 in [8]) and a generalisation of the theorem and its corollary.

Theorem 6 (maximum degree of hidden conflict; original version) *For any non-vacuous BFs Bel^i, Bel^{ii} defined by m^i and m^{ii} on Ω_n it holds that*

$$(\circledcap_1^{n-1}(m^i \circledcap m^{ii}))(\emptyset) = 0 \quad \textit{iff} \quad (\circledcap_1^{k}(m^i \circledcap m^{ii}))(\emptyset) = 0$$

for any $k \geq n-1$.

Corollary 1 (original) *A hidden conflict of any non-vacuous BFs on any Ω_n always has has degree less than or equal to $n-2$; i.e., the condition*

$$(\circledcap_1^{n-1}(m^i \circledcap m^{ii}))(\emptyset) = 0 \tag{3}$$

always means the full non-conflictness of any BFs m^i and m^{ii} on any Ω_n. Moreover, there is no hidden conflict on any two-element frame Ω_2

The original version of the theorem is O.K. in [8] because non-vacuous BFs are expectected there. Nevertheless, we can generalized its assertion as it follow:

Theorem 7 (maximum degree of hidden conflict; gneralized) *(i) For any BFs Bel^i, Bel^{ii} defined by m^i and m^{ii} on Ω_n it holds that*

$$(\circledcap_1^{n}(m^i \circledcap m^{ii}))(\emptyset) = 0 \quad \textit{iff} \quad (\circledcap_1^{k}(m^i \circledcap m^{ii}))(\emptyset) = 0$$

for any $k \geq n$.
(ii) For any non-vacouous BFs Bel^i, Bel^{ii} the stronger assertion holds true for any $k \geq n-1$

$$(\circledcap_1^{n-1}(m^i \circledcap m^{ii}))(\emptyset) = 0 \quad \textit{iff} \quad (\circledcap_1^{k}(m^i \circledcap m^{ii}))(\emptyset) = 0.$$

Proof. The assertion follow the proof in [8] and the text in Section 4.1.

The original version of the corollary need a small correction — specfication of mutual conflict between BFs — see the second assertion of the generalised version:

Corollary 2 (generalised) *(i) A hidden conflict of any two BFs on any Ω_n always has a degree less than or equal to $n-1$; i.e., the condition*

$$(\circledcap_1^{n}(m^i \circledcap m^{ii}))(\emptyset) = 0 \tag{4}$$

always means the full non-conflictness of any two BFs m^i and m^{ii} on any Ω_n.
(ii) A hidden conflict of any non-vacuous BFs on any Ω_n always has a degree less than or equal to $n-2$; i.e., the condition

$$(\circledcap_1^{n-1}(m^i \circledcap m^{ii}))(\emptyset) = 0 \tag{5}$$

always means the full mutual non-conflictness between any two BFs m^i and m^{ii} on any Ω_n. Specially, there is neither a hidden mutual conflict nor a hidden internal conflict[3] between any two BFs on two-element frame Ω_2.

[3]There may to be only mutual conflicts of degree $2-2=0$; meaning there are only mutual conflicts $(m^i \circledcap m^{ii})(\emptyset) > 0$ and internal conflicts $(m \circledcap m)(\emptyset) > 0$ which are not hidden.

Let us present a compariosn of the original and updated versions of Lemma 5 in [8] now.

Lemma 3 (original version of Lemma 5) *The only non-vacuous BFs on Ω_n with hidden conflict of degree $(n-2)$ are BFs with focal elements of cardinality $\geq n-1$, such that one has at least $(n-1)$ focal elements of cardinality $(n-1)$ and the other one has just one focal element of cardinality $(n-1)$. Moreover, every $(n-1)$-element subset of Ω_n must be a focal element of either one or both BFs.*

Lemma 4 (updated version of Lemma 5) *(i) The only non-vacuous BFs on Ω_n with hidden conflict of degree $(n-2)$ are BFs with focal elements of cardinality $\geq n-1$ for any $n > 3$, such that one has at least $(n-1)$ focal elements of cardinality $(n-1)$ and the other one has just one focal element of cardinality $(n-1)$. Moreover, every $(n-1)$-element subset of Ω_n must be a focal element of either one or both BFs.*
(ii) The only non-vacuous BFs on Ω_n with hidden conflict of degree $(n-2)$ are BFs with focal elements of cardinality $\geq n-1$ for $n = 2, 3$, such that each of them has at least one focal elements of cardinality $(n-1)$ and moreover, every $(n-1)$-element subset of Ω_n must be a focal element of either one or both BFs.

Proof. The characterisation of the 'other' BF is based on the fact that addition of any other focal element of cardinality $n-1$ decreases focal elements by $\circledcirc$, hence also decreases a degree of conflict hiddeness.

This is true in general. Nevertheless this is not a matter on Ω_3: as $(m_1 \circledcirc m_1) \circledcirc (m_2 \circledcirc m_2)$ cannot to decrease focal element twice; yes, there are three operations $\circledcirc$, each of them theoretically may to decrease the size of focal elements, but $n-1 = 2$ is decreased to zero already by two operations $\circledcirc$, hence other combination cannot further decrease the size of focal elements (decrease of cardinality of empty set). Hence both m_i may contaning two or three focal elements of cardinality 2 and highest degree 1 of hidden conflict is kept. Similarly, $n-1 = 1$, thus size of focal elements and (zero) degree of hidden conflict cannot be decreased twice, even if both singletons are in both bbas m_1 and m_2.