This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2021.3131200, IEEE Transactions on Fuzzy Systems

A 0-1 Law in Mathematical Fuzzy Logic

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***Abstract*—This paper continues the theoretical study of weighted structures in mathematical fuzzy logic focusing on the finite model theory of fuzzy logics valued on arbitrary finite MTL-chains. We show that for any first-order (or infinitary with finitely many variables) formula φ, there is a truth-value that φ takes almost surely in every finite many-valued model and such that every other truth-value is almost surely not taken. This generalizes a theorem in the fuzzy setting due to Robert Kosik and Christian G. Fermüller [29].**

***Index Terms*—mathematical fuzzy logic, first-order fuzzy logics, monoidal t-norms, finite weighted structures**

I. Introduction

THE model theory of predicate fuzzy logics has received an increased amount of attention in the last decade [1]–[3], [11]–[17]. In this area, first-order predicate languages are studied in the framework of mathematical fuzzy logic, which deals with graded logics as particular kinds of many-valued inference systems [9].

In this paper, we will be focusing on many-valued predicate logics that arise when we restrict our attention to weighted structures with a finite domain of objects and a finite algebra of linearly ordered truth-values, which provide a generalization of the structures studied by (classical) finite model theory (see e.g. [30]). In particular, we are interested in understanding the relationship between first-order logical languages with many-valued semantics and finite weighted structures. For a concrete example, suppose that we have a domain of three objects $\{d_1, d_2, d_3\}$ and a binary weighted relation "x is preferred to y" (denoted by the symbol $>$) where the weights come from the four-element Gödel algebra G_4, i.e. we work on a four-valued logic. Let the weighted relation $>$ be defined by the following table:

$>$	d_1	d_2	d_3
d_1	0	0	0
d_2	$\frac{1}{3}$	0	0
d_3	1	$\frac{2}{3}$	0

Given this structure, we can learn, for example, that the formula with parameters $(\exists y)(y > d_1 \wedge y > d_2)$ has the truth-value $\frac{2}{3}$, so one could say it is mostly true but not quite. This can be seen as a kind of Valued Constraint Satisfaction Problem [18], [28], where the object that satisfies the constraints to their maximum possible degree (in this case $\frac{2}{3}$) is d_3.

In this setting one can easily define the proportion of models, of a given finite cardinal for their domain, in which a sentence takes a certain value. Indeed, given $n \geqslant 1$ and a signature τ, for any τ-sentence φ and any value $a \in A$, let $l_n^a(\varphi)$ be the cardinality of the (finite) set K_τ^a consisting of all models $\mathfrak{M}$ for the signature τ with domain $M = \{1, 2, \ldots, n\}$ such that φ takes value a in $\mathfrak{M}$. Furthermore, let $l_n(\tau)$ be the cardinality of the (finite) set containing every model for the signature τ with domain $\{1, 2, \ldots, n\}$. Now, the desired proportion of models is given as

$$\mu_n^a(\varphi) = \frac{l_n^a(\varphi)}{l_n(\tau)}.$$

From this, we can define the *asymptotic probability* of φ taking value a as follows:

$$\mu^a(\varphi) = \lim_{n\to\infty} \mu_n^a(\varphi).$$

When restricted to classical models, the only possible choices of a are 1 and 0, and so $\mu^1(\varphi)$ becomes the asymptotic probability of φ being true. In [5] Rudolf Carnap already introduced these notions for classical first-order logic and proved[1] that, whenever the signature is finite and has only unary predicate symbols, for any sentence φ either $\mu(\varphi) = 1$ or $\mu(\varphi) = 0$ (i.e., $\mu(\neg\varphi) = 1$). Thus, he obtained the kind of result that in probability theory is usually called a 0-1 *law*. In the context of systematic development of finite model theory, the result was generalized to finite relational signatures by Ronald Fagin in the foundational paper [23] with a proof that serves as inspiration for the present contribution (a bit earlier, the authors of [24] had obtained the result but using instead a quantifier elimination method). Importantly, not every expressive extension of classical first-order logic satisfies a 0-1 law:

Example 1 (cf. Example 4.1.1 from [19])**.** *Second-order classical logic does not satisfy a 0-1 law. The sentence*

$$\varphi := (\exists X)((\forall x)Xxx \wedge (\forall x, y)(Xxy \to Xyx)$$
$$\wedge(\forall x, y, z)((Xxy \wedge Xyz) \to Xxz)$$
$$\wedge(\forall x)(\exists y)^{=1}(Xxy \wedge y \neq x))$$

is true, among finite structures, in exactly those with an even domain, so the fraction of structures with domain $\{1, \ldots, n\}$ making φ true does not converge to a limit as n tends to ∞. Hence, in first-order logic we cannot axiomatize the class of even structures.

[1]Carnap's motivation seem to have been related to his views on confirmation and the role of probabilistic methods in scientific inquiry (this sort of topic is still actively discussed today in philosophy of science e.g. [32]).

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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2021.3131200, IEEE Transactions on Fuzzy Systems

The goal of the present contribution is to obtain a many-valued extension of Fagin's 0-1 law. More precisely, we want to prove that, for each sentence φ, there is a value $a \in A$ such that $\mu^a(\varphi) = 1$ (and for all other values the asymptotic probability is 0). Our strategy consists in generalizing the classical model-theoretic proof from [23] to many-valued structures now that enough model-theoretic machinery has been developed in the many-valued setting (particularly, the back-and-forth games from [17]). Hence, we will establish a result that will imply the classical 0-1 law as a particular instance.

There is one notable exception to the absence of literature on the finite model theory of predicate fuzzy logics, namely [29]. In that paper, the authors establish (by techniques completely different to ours) a 0-1 law for the particular case of logics based on finite Łukasiewicz linearly ordered algebras (more precisely, those Łukasiewicz algebras where the number of elements is a power of 2). This result now follows as an instance of our more general Theorem 16. One important point to observe is that we have established our theorem for languages with truth-constants for all the elements of the algebra of truth-values, whereas the language of [29] only has the constants $\overline{1}, \overline{0}$. Hence, having shown our result in a more powerful expressive setting and more general algebraic framework, the previous result follows as a particular instance.

Our work will be based in the algebraic setting of MTL-algebras [8], [25]. We cannot allow for completely arbitrary MTL-algebras, though. Instead, we will assume that we have fixed a *finite* MTL-*chain* (i.e. linearly ordered). There are three main reasons for these restrictions:

(1) In the area of fuzzy logic, it is a fundamental requirement that the truth-values of the intended semantics (typically the interval $[0, 1]$ and subsets thereof) are linearly ordered. This algebraic feature allows to model language with vague predicates in such a way that any pair of instances always have *comparable* truth-degrees.
(2) We have a compactness property for first-order languages with semantics given over a fixed finite MTL-chain [15, Theorem 4.4]): every finitely satisfiable set of sentences is satisfiable (in a possibly infinite structure). We use compactness below to obtain a model of the theory T_τ from Definition 4, which is instrumental in the proof of Theorem 16. Compactness is not preserved in general when dealing with infinite MTL-chains: product predicate logic with the standard semantics on the interval $[0, 1]$ is known to be incompact, and Gödel predicate logic in the same interval loses compactness for uncountable languages.
(3) More importantly, the *asymptotic probability* introduced in Definition 5 below would not make sense if the algebra is infinite since there would be infinitely many possible models on a given finite domain. Hence, the question of whether similar work to what we do here can be generalized to infinite algebras remains open.[2]

[2]It might be possible to define some other reasonable probability measure on logics based on particular infinite MTL-chains but we do not know of any nicely motivated measure for this purpose on an arbitrary infinite MTL-chain.

Moreover, this is also the same theoretical framework used recently in [1]–[3], [18].

II. Preliminaries

In this section we introduce the basic notions of graded model theory framework that gives the theoretical context for the kind of finite model theory studied in this paper. Let us start with the syntax and semantics of graded predicate logics, and recall the basic notions we will use in the paper. We (mostly) use the notation and definitions of the Handbook of Mathematical Fuzzy Logic [9].

a) Syntax: The syntactical aspects of our logical setting are (almost) completely classical. We start from a basic propositional language that contains the binary connectives $\wedge$ (lattice conjunction), $\vee$ (lattice disjunction), $\&$ (residuated conjunction), and $\to$ (implication), and two truth-constants: $\overline{0}$ (falsum) and $\overline{1}$ (verum). Two other connectives are defined: $\neg\varphi = \varphi \to \overline{0}$ and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

A *signature* (or *predicate language*) τ is a triple $\langle Pred_\tau, Func_\tau, Ar_\tau \rangle$, where $Pred_\tau$ is a non-empty set of *predicate symbols*, $Func_\tau$ is a set of *function symbols* (disjoint from $Pred_\tau$), and Ar_τ represents the *arity function*, which assigns a natural number to each predicate symbol or function symbol. We call this natural number the *arity* of the symbol. The function symbols with arity zero are named *object constants* (*constants* for short). Object variables, τ-terms, τ-formulas, and the notions of free occurrence of a variable, open formula, substitutability, and sentence are defined as in classical predicate logic. A theory is a set of sentences. When τ is clear from the context, we will refer to τ-terms and τ-formulas simply as *terms* and *formulas*. Also, when no confusion can arise, we will identify τ with the set of its symbols (i.e. $Pred_\tau \cup Func_\tau$) and write expressions such $\tau \subseteq \tau'$, meaning that $Pred_\tau \subseteq Pred_{\tau'}$ and $Func_\tau \subseteq Func_{\tau'}$ and $Ar_{\tau'}$ agrees with Ar_τ in the symbols of τ.

b) Semantics: The non-classicality appears on the semantical side. In graded predicate logics we work with models based on an algebra of (possibly more than two) truth-values. Propositional connectives are semantically interpreted by the notion of an MTL-*algebra* [25], that is, a structure of the form $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ such that

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a bounded lattice,
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ is a commutative monoid,
- for each $a, b, c \in A$, we have:

$$a \&^{\boldsymbol{A}} b \leq c \quad \text{iff} \quad b \leq a \to^{\boldsymbol{A}} c, \qquad \text{(residuation)}$$
$$(a \to^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) = \overline{1}^{\boldsymbol{A}} \qquad \text{(prelinearity)}$$

$\boldsymbol{A}$ is called an MTL-*chain* if its underlying lattice is linearly ordered. Observe that the two-element Boolean algebra, $\boldsymbol{B}_2$, can be seen, in particular, as an MTL-algebra (identifying the operations $\&$ and $\wedge$, and defining the complement as $\neg x = x \to \overline{0}$).

Typical examples of non-Boolean MTL-chains are the algebras $[0, 1]_{\text{G}}$, $[0, 1]_{\text{Ł}}$, and $[0, 1]_{\Pi}$, respectively used in the semantics of Gödel–Dummett, Łukasiewicz, and Product logics (three prominent examples of fuzzy logics; see e.g. [9]). In all cases, $\wedge$, $\vee$, $\overline{0}$, $\overline{1}$ are interpreted respectively as the minimum, the maximum, the number 0, and the number 1, while the interpretations of the other operations differ:

$$\begin{array}{rcl}
a \,\&^{[0,1]_{\mathrm{G}}}\, b & = & \min\{a,b\},\\
a \,\&^{[0,1]_{\text{Ł}}}\, b & = & \max\{a+b-1,0\},\\
a \,\&^{[0,1]_{\Pi}}\, b & = & ab \text{ (standard product of reals)},\\
a \to^{[0,1]_{\mathrm{G}}} b & = & \begin{cases} 1, & \text{if } a \le b,\\ b, & \text{otherwise},\end{cases}\\
a \to^{[0,1]_{\text{Ł}}} b & = & \begin{cases} 1, & \text{if } a \le b,\\ 1-a+b, & \text{otherwise},\end{cases}\\
a \to^{[0,1]_{\Pi}} b & = & \begin{cases} 1, & \text{if } a \le b,\\ b/a, & \text{otherwise}.\end{cases}
\end{array}$$

For the purposes of this paper, it will be very illustrative to introduce as well some examples of finite MTL-chains.

Example 2 (The algebra of Łukasiewicz 3-valued logic [31]). *The algebra* $\text{Ł}_3 = \langle \{0, \frac{1}{2}, 1\}, \wedge^{\text{Ł}_3}, \vee^{\text{Ł}_3}, \&^{\text{Ł}_3}, \to^{\text{Ł}_3}, 0, 1\rangle$ *such that*

- $\wedge^{\text{Ł}_3}(x,y) = min\{x,y\}$
- $\vee^{\text{Ł}_3}(x,y) = max\{x,y\}$
- $\&^{\text{Ł}_3}(x,y) = max\{0, x+y-1\}$
- $\to^{\text{Ł}_3}(x,y) = min\{1, 1-x+y\}$

Example 3 (The algebra of Gödel 4-valued logic [25]). *The algebra* $\boldsymbol{G}_4 = \langle \{0, \frac{1}{3}, \frac{2}{3}, 1\}, \wedge^{\boldsymbol{G}_4}, \vee^{\boldsymbol{G}_4}, \&^{\boldsymbol{G}_4}, \to^{\boldsymbol{G}_4}, 0, 1\rangle$ *such that*

- $\wedge^{\boldsymbol{G}_4}(x,y) = \&^{\boldsymbol{G}_4}(x,y) = min\{x,y\}$
- $\vee^{\boldsymbol{G}_4}(x,y) = max\{x,y\}$
- *and for* $\to^{\boldsymbol{G}_4}$*:*

$$\to^{\boldsymbol{G}_4}(x,y) = \begin{cases} \overline{1}^{\boldsymbol{A}} & \textit{if } x \le y\\ y & \textit{otherwise}.\end{cases}$$

An MTL-chain $\boldsymbol{A}$ may be expanded, for greater expressive power of the logic, with truth-constants (i.e. 0-ary connectives) $\overline{a}$, for each $a \in A$, demanding that they denote their corresponding element (see e.g. [20], [21]). We will do that in the remainder of this paper.

Based on MTL-chains (and their expansions) as algebraic interpretations of the propositional language, now we can give the semantics of first-order predicate formulas. Given a signature $\tau = \langle Pred_\tau, Func_\tau, Ar_\tau\rangle$, we define a *$\tau$-structure* as a pair $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ where $\boldsymbol{A}$ is an MTL-chain and

$$\mathbf{M} = \langle M, (P_{\mathbf{M}})_{P \in Pred_\tau}, (F_{\mathbf{M}})_{F \in Func_\tau}\rangle,$$

where M is a non-empty set (the *domain*), $P_{\mathbf{M}}$ is an n-ary A-valued relation for each n-ary predicate symbol P, i.e., a function from M^n to A, identified with an element of A if $n = 0$; and $F_{\mathbf{M}}$ is a function from M^n to M, identified with an element of M if $n = 0$. We will call $\langle \boldsymbol{A}, \mathbf{M}\rangle$ an *$\boldsymbol{A}$-structure* whenever we need to stress its algebraic part. $\mathfrak{M}$ is said to be *finite* if its part M is.

An $\mathbf{M}$-evaluation of the object variables is a mapping v assigning to each object variable an element of M. If v is an $\mathbf{M}$-evaluation, x is an object variable and $d \in M$, we denote by $v[x \mapsto d]$ the $\mathbf{M}$-evaluation so that $v[x \mapsto d](x) = d$ and $v[x \mapsto d](y) = v(y)$ for y an object variable such that $y \neq x$. We define the *values* of terms and the *truth-values* of formulas in M for an $\mathbf{M}$-evaluation v recursively as follows:

$\|x\|^{\boldsymbol{A}}_{\mathbf{M},v} = v(x)$;
$\|F(t_1,\ldots,t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v},\ldots,\|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for each $F \in Func_\tau$;
$\|P(t_1,\ldots,t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} = P_{\mathbf{M}}(\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v},\ldots,\|t_n\|^{\boldsymbol{A}}_{\mathbf{M},v})$, for each $P \in Pred_\tau$;
$\|\varphi \circ \psi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} \circ^{\boldsymbol{A}} \|\psi_n\|^{\boldsymbol{A}}_{\mathbf{M},v}$, for each binary connective $\circ$;
$\|\overline{a}\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{a}^{\boldsymbol{A}}$;
$\|(\forall x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x\mapsto d]} \mid d \in M\}$;
$\|(\exists x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x\mapsto d]} \mid d \in M\}$.

If the infimum or the supremum do not exist, we take the truth-value of the formula as undefined. A τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is said to be *safe* if the value $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v}$ is defined for each formula φ and each $\mathbf{M}$-evaluation v. Certainly, the semantics can be restricted to models based on completely ordered chains, that is, chains with suprema and infima of all their subsets (for instance, by taking only finite chains as we will do later in this paper) which ensures that all models would be safe. However, in general, this gives rise to serious drawbacks regarding the axiomatizability of the corresponding first-order logics (see [9, Chapter XI]), which justifies the design choice for the general theory of safe models instead. We call the first-order language described with the above semantics $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ [2].

Remark 1. *We can close $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ under infinitary lattice disjunctions and conjunctions, e.g. by allowing formulas $\bigwedge_{i\in I}\varphi_i$ and $\bigvee_{i\in I}\varphi_i$ (where I has any cardinality) with the following semantics:*

$\left\|\bigwedge_{i\in I}\varphi_i\right\|^{\boldsymbol{A}}_{\mathbf{M},v} = \inf\{\|\varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid i \in I\}$;
$\left\|\bigvee_{i\in I}\varphi_i\right\|^{\boldsymbol{A}}_{\mathbf{M},v} = \sup\{\|\varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid i \in I\}$.

We call the resulting language and semantics $\mathscr{L}^{\boldsymbol{A}}_{\infty\omega}$. If, furthermore, we allow only $k \geqslant 1$ many variables in our formulas, we obtain $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$.

For a set of formulas Φ, we write $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$, if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$ for every $\varphi \in \Phi$. We denote by $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ the fact that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$ for all $\mathbf{M}$-evaluations v; analogously for sets Φ. We say that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a *model of a set of formulas* Φ, if $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ (in symbols, $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \Phi$). Observe that in this general presentation we have not required yet the presence of an equality symbol in the language, but it can be added in the form of a binary relational symbol $\approx$ interpreted as (crisp) equality in the models, i.e. $\|t_1 \approx t_2\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$ if $\|t_1\|^{\boldsymbol{A}}_{\mathbf{M},v} = \|t_2\|^{\boldsymbol{A}}_{\mathbf{M},v}$, and $\|t_1 \approx t_2\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{0}^{\boldsymbol{A}}$ otherwise. Hence, we will let $\approx$ stand for crisp equality throughout this paper ($x \not\approx y$ will abbreviate $x \approx y \to \overline{0}$).

We use the notation $\overrightarrow{x}$ for a finite sequence of variables, and $\overrightarrow{d}$ for a finite sequence of elements of a domain M (by a slight abuse of language, we write $\overrightarrow{d} \subseteq M$). Given an

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$\mathbf{M}$-evaluation v, we define $v[\overrightarrow{x} \mapsto \overrightarrow{d}]$ as the $\mathbf{M}$-evaluation such that $v[\overrightarrow{x} \mapsto \overrightarrow{d}](x_i) = d_i$ for each $i \in \{1, \dots, n\}$ and $v[\overrightarrow{x} \mapsto \overrightarrow{d}](y) = v(y)$ for each $y \notin \overrightarrow{x}$. We write $\varphi(\overrightarrow{x})$ to indicate that the free variables of φ are among $\{x_1, \dots, x_n\}$. Given a τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and a formula $\varphi(\overrightarrow{x})$, we say that $\overrightarrow{d} \subseteq M$ *satisfies* $\varphi(\overrightarrow{x})$ (or that $\varphi(\overrightarrow{x})$ is *satisfied* by $\overrightarrow{d}$) if $\|\varphi(\overrightarrow{x})\|^{\boldsymbol{A}}_{\mathbf{M},v[\overrightarrow{x}\mapsto\overrightarrow{d}]} = \overline{1}^{\boldsymbol{A}}$ for any $\mathbf{M}$-evaluation v (also written $\left\|\varphi[\overrightarrow{d}]\right\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ or $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi[\overrightarrow{d}]$). Finally, we say that a set of sentences Φ is *satisfiable* is there is a safe τ-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $\|\Phi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, and we say that it is *finitely satisfiable* if each finite subset of Φ is satisfiable.

Proposition 4. *[15, Theorem 4.4] Given a set of sentences Σ, if every finite subset $\Sigma_0 \subseteq \Sigma$ has a model $\langle \boldsymbol{A}, \mathbf{M}_{\Sigma_0}\rangle$ (finite or infinite), then Σ has a model $\langle \boldsymbol{A}, \mathbf{N}\rangle$.*

Corollary 5 (Finitarity)**.** *Let $\boldsymbol{A}$ be a fixed finite chain. For every set of sentences $\Sigma \cup \{\varphi\}$, if $\Sigma \models_{\boldsymbol{A}} \varphi$, then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models_{\boldsymbol{A}} \varphi$.*

Remark 6. *Observe that when we restrict ourselves to the study of models with finite domains (as those relevant for Definition 5), compactness breaks apart. It is easy to see that the infinite theory*

$$(\forall x_1)(x_1 < x_1 \to \overline{0})$$

$$(\forall x_1, x_2, x_3,)(x_1 < x_2 \wedge x_2 < x_3 \to x_1 < x_3)$$

$$(\exists x_1, \dots, x_n)(\bigwedge_{1 \le i < j \le n} x_i < x_j) \quad \textit{(for all } n \geqslant 1)$$

is finitely satisfiable on finite models (i.e. every finite subset of the theory has a finite model) but not satisfiable as a whole.

c) Equivalent formulas, elementary equivalence, and (partial) isomorphisms: The many-valued semantics brings forth an interesting increase of complexity of basic notions of classical model theory, starting from the very notions of equivalence of formulas and elementary equivalence of structures. Indeed, given two formulas $\varphi(\overrightarrow{x})$ and $\psi(\overrightarrow{x})$, we can define their equivalence in two different ways:

- $\varphi(\overrightarrow{x})$ and $\psi(\overrightarrow{x})$ are *1-equivalent* if for any model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and any sequence of elements $\overrightarrow{d} \in M$, we have: $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi[\overrightarrow{d}]$ iff $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \psi[\overrightarrow{d}]$,
- $\varphi(\overrightarrow{x})$ and $\psi(\overrightarrow{x})$ are *equivalent* if for any model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and any sequence of elements $\overrightarrow{d} \in M$, we have: $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \varphi \leftrightarrow \psi[\overrightarrow{d}]$ (that is, for each v, $\|\varphi(\overrightarrow{x})\|^{\boldsymbol{A}}_{\mathbf{M},v[\overrightarrow{x}\mapsto\overrightarrow{d}]} = \|\psi(\overrightarrow{x})\|^{\boldsymbol{A}}_{\mathbf{M},v[\overrightarrow{x}\mapsto\overrightarrow{d}]}$).

Similarly, equivalence between two structures can be meaningfully defined in two different ways. We say that two safe τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A}, \mathbf{N}\rangle$ are *elementarily equivalent* (in symbols: $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{A}, \mathbf{N}\rangle$) if they are models of the same sentences, i.e. for every τ-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ if and only if $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{N}} = \overline{1}^{\boldsymbol{A}}$.

In case the two structures are based on the same algebra we can define a stronger notion of equivalence by requiring sentences to take the exact same values. More precisely, given safe τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A}, \mathbf{N}\rangle$, we say that they are *strongly elementarily equivalent* (in symbols: $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv^s \langle \boldsymbol{A}, \mathbf{N}\rangle$) if for every τ-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{A}}_{\mathbf{N}}$.

For classical structures, i.e. when $\boldsymbol{A} \cong \boldsymbol{B}_2$, these two definitions give the classical notion of elementary equivalence. But, in general, they differ as shown with counterexamples in [12].

Definition 1 ([12])**.** *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A}, \mathbf{N}\rangle$ be τ-structures, p be a partial mapping from M to N. We say that p is a* partial isomorphism *from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}\rangle$ if*

1) *p is injective,*
2) *for every n-ary functional symbol $F \in Func_\tau$ and every $d_1, \dots, d_n \in M$ such that $d_1, \dots, d_n, F_{\mathbf{M}}(d_1, \dots, d_n) \in \mathrm{dom}(p)$,*
$$p(F_{\mathbf{M}}(d_1, \dots d_n)) = F_{\mathbf{N}}(p(d_1), \dots, p(d_n)),$$
3) *for every n-ary predicate symbol $P \in Pred_\tau$ and $d_1, \dots, d_n \in M$ such that $d_1, \dots, d_n \in \mathrm{dom}(p)$,*
$$P_{\mathbf{M}}(d_1, \dots, d_n) = P_{\mathbf{N}}(p(d_1), \dots, p(d_n)).$$

Definition 2 ([12])**.** *Two τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A}, \mathbf{N}\rangle$ are said to be* finitely isomorphic*, written $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{A}, \mathbf{N}\rangle$, if there is a sequence $\langle I_n \mid n < \omega\rangle$ with the following properties:*

1) *Every I_n is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}\rangle$.*
2) *For each $n < \omega$, $I_{n+1} \subseteq I_n$.*
3) *(Forth-property) For every $p \in I_{n+1}$ and $m \in M$, there is a $p' \in I_n$ such that $p \subseteq p'$ and $m \in \mathrm{dom}(p')$.*
4) *(Back-property) For every $p \in I_{n+1}$ and $n \in N$, there is a $p' \in I_n$ such that $p \subseteq p'$ and $n \in \mathrm{rg}(p')$.*

Definition 3 (k-potentially isomorphic structures)**.** *Given an integer $k \ge 1$, two τ-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{A}, \mathbf{N}\rangle$ are said to be* k-potentially isomorphic*, written $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong^k \langle \boldsymbol{A}, \mathbf{N}\rangle$, if there is a set I of partial isomorphisms with the following properties:*

1) *I is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}\rangle$.*
2) *I is downward-closed: if $p \in I$ and $p' \subseteq p$, then $p' \in I$.*
3) *If $p \in I$ and $|\mathrm{dom}(p)| < k$,*
   a) *(Forth-property) for every $m \in M$, there is a $p' \in I$ such that $p \subseteq p'$ and $m \in \mathrm{dom}(p')$.*
   b) *(Back-property) for every $n \in N$, there is a $p' \in I$ such that $p \subseteq p'$ and $n \in \mathrm{rg}(p')$.*

All these definitions are simplifications of the general case that allows of different algebras in different models, which would require the introduction mappings between the algebras too (see [17]). The key intuitive difference between Definition 2 and Definition 3 is that in the latter the back-and-forth game can only be played for partial isomorphisms defined on less than k-many individuals from the domain of one of the structures. If the propositional language is expanded with truth-constants, all the mentioned notions are extended in the obvious way. In what follows, we fix a finite MTL-chain $\boldsymbol{A}$ for all our models.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2021.3131200, IEEE Transactions on Fuzzy Systems

III. 0-1 LAW: THE CASE OF $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$

We start by recalling a technical result that gives a useful sufficient condition for strong elementary equivalence:

Proposition 7. *[17, Theorem 14 (b)] Consider models $\mathfrak{M}$ and $\mathfrak{N}$ for a finite signature τ. Then, $(i) \implies (ii)$, where*

(i) *$\mathfrak{M} \cong_f \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are finitely isomorphic.*
(ii) *$\mathfrak{M} \equiv^s \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are strongly elementarily equivalent.*

Next, we introduce the definition of a certain theory, which we call T_τ given a particular finite relational signature τ, that will be instrumental in what follows.

Definition 4 (cf. [19])**.** *Consider a finite relational signature τ. For any integer $r \geqslant 0$, let Δ_{r+1} be the (finite) set of all formulas $\varphi(v_1, \ldots, v_r, v_{r+1})$ where φ is an atomic formula $R\overrightarrow{x}$ in the signature τ, v_{r+1} appears in the sequence $\overrightarrow{x}$, and all variables in $\overrightarrow{x}$ are from the list $v_1, \ldots, v_r, v_{r+1}$. Let T_τ be the theory containing, for every A-valued set $\Phi : \Delta_{r+1} \longrightarrow A$, the axiom χ^r_Φ (we will drop the superscript when convenient) defined as:*

$$(\forall v_1, \ldots, v_r)((\neg \bigwedge_{1\leq i<j\leq r} v_i \not\approx v_j)\vee$$

$$(\exists v_{r+1})\begin{pmatrix} \bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1} \\ \wedge \quad \bigwedge_{\varphi\in\Delta_{r+1}} (\varphi \leftrightarrow \overline{\Phi(\varphi)}) \end{pmatrix})$$

We call the above an $r+1$ extension axiom *of T_τ.*

We can show that T_τ has an infinite model by a compactness argument using Corollary 15 below, which provides models for any finite subset of T_τ.

Proposition 8. *Consider a finite relational signature τ. If $\mathfrak{M}, \mathfrak{N}$ are models of T_τ, then $\mathfrak{M} \cong_f \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are finitely isomorphic.*

Proof. For any models $\mathfrak{M}, \mathfrak{N}$ and any finite sequences $\overrightarrow{e}$ of elements of M and $\overrightarrow{d}$ of elements of M, we write $\overrightarrow{e} \equiv_{at} \overrightarrow{d}$ if $\|\varphi[\overrightarrow{e}]\|^{\mathfrak{M}} = \left\|\varphi[\overrightarrow{d}]\right\|^{\mathfrak{N}}$ for every atomic formula φ.

To show that $\mathfrak{M} \cong_f \mathfrak{N}$, we define a system $\langle I_k \mid k < \omega \rangle$ of sets of partial isomorphisms with $I_k = I$ (for each $k < \omega$) where:

$$I = \{p \mid r \geqslant 0,\ p : M \longrightarrow N \text{ is a partial mapping}, p := \overrightarrow{e} \mapsto \overrightarrow{d}, \overrightarrow{e} = e_1, \ldots, e_r, \overrightarrow{d} = d_1, \ldots, d_r, \overrightarrow{e} \equiv_{at} \overrightarrow{d}\}.$$

Note that the I_ks are non-empty since at least they contain $\emptyset$.

Next, we check the forth property. Suppose that $p \in I_{k+1} = I$, $p : M \longrightarrow N$ is a partial mapping $p := \overrightarrow{e} \mapsto \overrightarrow{d}$, $\overrightarrow{e} = e_1, \ldots, e_r$, $\overrightarrow{d} = d_1, \ldots, d_r$, $\overrightarrow{e} \equiv_{at} \overrightarrow{d}$. Then, take $e_{r+1} \in M$ (distinct from any element in the sequence $\overrightarrow{e}$). Consider now the A-valued set Φ with domain Δ_{r+1} defined as follows:

$$\Phi(\varphi(v_1, \ldots, v_r, v_{r+1})) = \|\varphi[\overrightarrow{e}, e_{r+1}]\|^{\mathfrak{M}}.$$

Since $\mathfrak{N} \models T_\tau$, we must have that $\mathfrak{N} \models \chi_\Phi$. Hence,

$$\mathfrak{N} \models (\exists v_{r+1})((\bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1}) \wedge (\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)}))[\overrightarrow{d}],$$

Consequently, for some $d_{r+1} \in N$,

$$\mathfrak{N} \models (\bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1}) \wedge (\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)})[\overrightarrow{d}, d_{r+1}].$$

Now simply consider the partial mapping $p' : M \longrightarrow N$ defined as $p' := \overrightarrow{e}' \mapsto \overrightarrow{d}'$ where $\overrightarrow{e}' = e_1, \ldots, e_r, e_{r+1}$, $\overrightarrow{d} = d_1, \ldots, d_r, d_{r+1}$. This works because $\overrightarrow{e}' \equiv_{at} \overrightarrow{d}'$.

Finally, the back property follows similarly. □

Corollary 9. *For any sentence φ in a finite relational signature τ, there is a unique value $a \in A$ such that $T_\tau \vDash \varphi \leftrightarrow \overline{a}$.*

Proof. If $\mathfrak{M}, \mathfrak{N}$ are models of T_τ, then $\mathfrak{M} \equiv^s \mathfrak{N}$. This follows from Proposition 7 and Proposition 8. Hence, the value of φ is fixed across models of T_τ. The value $\overline{a}$ is unique as if $T_\tau \vDash \varphi \leftrightarrow \overline{a}$ and $T_\tau \vDash \varphi \leftrightarrow \overline{a'}$, then $T_\tau \vDash \overline{a'} \leftrightarrow \overline{a}$, which, since T_τ has a model means that $a = a'$. □

Definition 5 (Asymptotic probabilites; cf. [23])**.** *Take a signature τ. For any formula φ, $a \in A$, and $n \geqslant 1$, let $l^a_n(\varphi)$ be the cardinality of the (finite) set K^a_τ consisting of all models $\mathfrak{M}$ for the signature τ with domain $\{1, 2, \ldots, n\}$ such that $\|\varphi\|^{\mathfrak{M}} = a$. Furthermore, let $l_n(\tau)$ be the cardinality of the (finite) set containing every model $\mathfrak{M}$ for the signature τ with $M = \{1, 2, \ldots, n\}$. Observe that, had our algebra $\boldsymbol{A}$ been infinite, $l_n(\tau)$ would not be an integer (and the fraction below would not be defined). Now, let*

$$\mu^a_n(\varphi) = \frac{l^a_n(\varphi)}{l_n(\tau)}.$$

The asymptotic probability *of φ getting value a is defined as:*

$$\mu^a(\varphi) = \lim_{n\to\infty} \mu^a_n(\varphi).$$

Example 10. *Consider a signature containing a unary relation P and suppose that $3 \leq |A|$. Then,*

$$\mu^{\overline{1}^{\boldsymbol{A}}}_n((\forall x)(Px \vee \neg Px)) = \frac{2^n}{|A|^n}$$

$$\mu^{\overline{1}^{\boldsymbol{A}}}((\forall x)(Px \vee \neg Px)) = \lim_{n\to\infty} \frac{2^n}{|A|^n} = 0.$$

Therefore, almost no structure makes the predicate P crisp.

Example 11. *Consider a signature containing a unary relation P and suppose that $\boldsymbol{A} = \text{Ł}_3$. Then,*

$$\mu^{\frac{1}{2}}_n((\forall x)(Px \vee \neg Px)) = \frac{3^n - 2^n}{3^n}$$

$$\mu^{\frac{1}{2}}((\forall x)(Px \vee \neg Px)) = \lim_{n\to\infty} \frac{3^n - 2^n}{3^n} = 1.$$

Example 12. *Let τ be the empty signature. For any k, let*

$$\varphi^{=k} := (\exists x_1, \ldots, x_k)\begin{pmatrix} \bigwedge_{1\leq i<j\leq k} x_i \not\approx x_j \\ \wedge \quad (\forall x_{k+1})(\bigvee_{1\leq i\leq k} x_{k+1} \approx x_i) \end{pmatrix}.$$

$\mathfrak{M} \models \varphi^{=k}$ iff $|M| = k$. Then, for the infinitary sentence $\bigvee_{k\geqslant 1} \varphi^{=2k+1}$, $\mathfrak{M} \models \bigvee_{k\geqslant 1} \varphi^{=2k+1}$ iff $|M|$ is odd. Then,

$$\mu^{\overline{1}^{\boldsymbol{A}}}_n(\bigvee_{k\geqslant 1} \varphi^{=2k+1}) = \begin{cases} 1 & \textit{if } n \textit{ is odd} \\ 0 & \textit{otherwise.} \end{cases}$$

So, in this case, $\mu^{\overline{1}^{\boldsymbol{A}}}(\bigvee_{k\geqslant 1} \varphi^{=2k+1})$ does not exist.

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Example 13 (cf. Example 4.1.1 from [19]). *Consider a signature containing a unary relation R and an object constant symbol c. Let $a \in A$. Then, for any $n \geqslant 1$,*

$$\mu_n^a(Rc) = \frac{1}{|A|}$$

since in any given model $\mathfrak{M}$ with domain $M = \{1, 2, \ldots, n\}$ where we have fixed the interpretation of c, there are $\frac{1}{|A|}$ chances of interpreting R in such a way that $\|Rc\|^{\mathfrak{M}} = a$. Hence,

$$\mu^a(Rc) = \lim_{n\to\infty} \mu_n^a(Rc) = \frac{1}{|A|}.$$

Lemma 14. *Let τ be a finite relational signature. Fix Δ_{r+1} and some A-valued set $\Phi : \Delta_{r+1} \longrightarrow \boldsymbol{A}$. Then, for the extension axiom χ_Φ, $\mu^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi) = 1$. In other words, χ_Φ takes value $\overline{1}^{\boldsymbol{A}}$ almost surely.*

Proof. Let $a \in A$ be the co-atom of $\boldsymbol{A}$. If we manage to show that $\mu^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi \to \overline{a}) = 0$, then almost surely no model gives $\chi_\Phi \to \overline{a}$ the value $\overline{1}^{\boldsymbol{A}}$, which means that almost every model gives χ_Φ the value $\overline{1}^{\boldsymbol{A}}$, since in every model either χ_Φ or $\chi_\Phi \to \overline{a}$ will take value $\overline{1}^{\boldsymbol{A}}$. Let c be the number of possible A-valued sets with domain Δ_{r+1}. It turns out that for any $n \geqslant 1$,

$$\mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi \to \overline{a}) \leq n^r (\frac{c-1}{c})^{n-r}.$$

As $\lim_{n\to\infty}(\frac{c-1}{c})^{n-r} = 0$ since $\frac{c-1}{c} < 1$, it follows that

$$\lim_{n\to\infty} n^r (\frac{c-1}{c})^{n-r} = 0,$$

so $\mu^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi \to \overline{a}) = \lim_{n\to\infty} \mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi \to \overline{a}) = 0$.

Now let us show where the number $n^r(\frac{c-1}{c})^{n-r}$ comes from. Let $\mathfrak{M}$ be a model for the signature τ with $M = \{1, 2, \ldots, n\}$ such that $\mathfrak{M} \models \chi_\Phi \to \overline{a}$. This means that there is a sequence of elements $\overrightarrow{e} = e_1, \ldots, e_r \in M$ such that the formulas

(1) $(\bigwedge_{1\leq i<j\leq r} v_i \not\approx v_j)$,

(2) $(\exists v_{r+1})\begin{pmatrix} \bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1} \\ \wedge \quad \bigwedge_{\varphi\in\Delta_{r+1}}(\varphi \leftrightarrow \overline{\Phi(\varphi)}) \end{pmatrix} \to \overline{a}$

are both satisfied (take value $\overline{1}^{\boldsymbol{A}}$) by $\overrightarrow{e}$ in $\mathfrak{M}$. (2) being satisfied by $\overrightarrow{e}$ means, furthermore, that for every e_{r+1} at least one of

$$(\bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1})$$

or

$$(\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)})$$

is not satisfied by $\overrightarrow{e}e_{r+1}$. The number $\frac{c-1}{c}$ is the probability that $\overrightarrow{e}e_{r+1}$ will not satisfy $(\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)})$ in a randomly chosen $\mathfrak{M}$ (recall that c is the number of possible A-valued sets with domain Δ_{r+1}) with domain $M = \{1, 2, \ldots, n\}$. Since we have $n - r$ ways of choosing e_{r+1} different from all elements in $\overrightarrow{e}$ once $\overrightarrow{e}$ is fixed, $(\frac{c-1}{c})^{n-r}$ is the probability that we can find an e_{r+1} not satisfying $(\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)})$. Finally, n^r is the number of possible choices of a sequence of r many elements from M. □

Corollary 15. *Let τ be a finite relational signature. For any finite $T'_\tau \subseteq T_\tau$, there is a number k such that for any $n > k$, T'_τ has a model with a universe of objects of size n.*

Proof. Let $a \in A$ be the co-atom of our algebra. If $T'_\tau = \{\chi_{\Phi_1}, \ldots, \chi_{\Phi_r}\}$, we pick k large enough that if $n > k$, $\mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi \to \overline{a}) < \frac{1}{r}$ for each $\chi_\Phi \in T'_\tau$, which is possible as $\mu^{\overline{1}^{\boldsymbol{A}}}(\chi_\Phi \to \overline{a}) = 0$. It then follows that for $n > k$,

$$\mu_n^{\overline{1}^{\boldsymbol{A}}}((\chi_{\Phi_1} \to \overline{a}) \vee \cdots \vee (\chi_{\Phi_r} \to \overline{a})) \leq$$
$$\mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_{\Phi_1} \to \overline{a}) + \cdots + \mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_{\Phi_r} \to \overline{a}) < 1.$$

The second inequality is immediate given the choice of k. Hence, there must be a model $\mathfrak{M}$ of T'_τ with universe $M = \{1, \ldots, n\}$. □

Notice that the proof of Corollary 15 shows how probabilistic techniques can be used in the absence of compactness to indirectly show the existence of models for a (in this case finite) theory, even if we do not necessarily know how to construct them.

Theorem 16 (First 0-1 Law). *If φ is a sentence in the finite relational signature τ, then there is $a \in A$ such that $\mu^a(\varphi) = 1$, and for any other truth-value a', $\mu^{a'}(\varphi) = 0$. In other words, for any φ there is a truth-value that φ takes almost surely in a given model and every other value almost surely does not take.*

Proof. Let $b \in A$ be the co-atom of our algebra. We know from Corollary 9 that for some $a \in A$, $T_\tau \models \varphi \leftrightarrow \overline{a}$. Then, by finitarity (or, equivalently, compactness), for some finite set $\{\chi_{\Phi_1}, \ldots, \chi_{\Phi_r}\} \subseteq T_\tau$, $\bigwedge\{\chi_{\Phi_1}, \ldots, \chi_{\Phi_r}\} \models \varphi \leftrightarrow \overline{a}$, which means that

$$(\varphi \leftrightarrow \overline{a}) \to \overline{b} \models (\bigwedge\{\chi_{\Phi_1}, \ldots, \chi_{\Phi_r}\}) \to \overline{b}, \text{ i.e.}$$
$$(\varphi \leftrightarrow \overline{a}) \to \overline{b} \models \bigvee\{\chi_{\Phi_1} \to \overline{b}, \ldots, \chi_{\Phi_r} \to \overline{b}\}.$$

We can observe that for any n,

$$\mu_n^{\overline{1}^{\boldsymbol{A}}}((\varphi \leftrightarrow \overline{a}) \to \overline{b}) \leq$$
$$\mu_n^{\overline{1}^{\boldsymbol{A}}}((\chi_{\Phi_1} \to \overline{b}) \vee \cdots \vee (\chi_{\Phi_r} \to \overline{b})) \leq$$
$$\mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_{\Phi_1} \to \overline{b}) + \cdots + \mu_n^{\overline{1}^{\boldsymbol{A}}}(\chi_{\Phi_r} \to \overline{b})$$

The latter tends to 0 as n approaches ∞ from Lemma 14 (since for any θ, $\mu_n^{\overline{1}^{\boldsymbol{A}}}(\theta) = 1 - \mu_n^{\overline{1}^{\boldsymbol{A}}}(\theta \to \overline{b})$), hence

$$\mu^{\overline{1}^{\boldsymbol{A}}}((\chi_{\Phi_1} \to \overline{b}) \vee \cdots \vee (\chi_{\Phi_r} \to \overline{b})) = 0, \text{ so}$$
$$\mu^{\overline{1}^{\boldsymbol{A}}}((\varphi \leftrightarrow \overline{a}) \to \overline{b}) = 0.$$

But then $\mu^{\overline{1}^{\boldsymbol{A}}}(\varphi \leftrightarrow \overline{a}) = 1$, which implies that $\mu^a(\varphi) = 1$. □

The classical version of this result (e.g. [27, Thm. 7.4.7]), for $\boldsymbol{B}_2$, simply states that either $\mu^1(\varphi) = 1$ and $\mu^0(\varphi) = 0$, or $\mu^1(\varphi) = 0$ and $\mu^0(\varphi) = 1$. This is an immediate corollary of Theorem 16, and so is the central result from [29].

From our 0-1 law and Example 12, it follows, as in the classical case, that the infinitary sentence $\bigvee_{k\geqslant 1} \varphi^{=2k+1}$ in the empty signature is not 1-equivalent to any first-order

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sentence θ in a relational signature, i.e., there is no θ such that $\mathfrak{M} \models \theta$ iff $\mathfrak{M} \models \bigvee_{k\geqslant 1} \varphi^{=2k+1}$. This is because the formula $\bigvee_{k\geqslant 1} \varphi^{=2k+1}$ is crisp, i.e., it can only take values in the set $\{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$ since equality is assumed to be crisp. And here lies one of the uses of a result of this kind: it allows us to measure the expressive power of a language satisfying a 0-1 law in the way that compactness does in the setting of infinite model theory.

IV. 0-1 LAW: THE CASE OF $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$

In this section, we extend our 0-1 law to the infinitary language $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$, where we admit arbitrarily long infinite lattice conjunctions and disjunctions but only finitely many (in fact, k-many) individual variables. If $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$ is to satisfy a 0-1 law, it must be, of course, expressively weaker than the full $\mathscr{L}^{\boldsymbol{A}}_{\infty\omega}$.

Example 17. *The language $\mathscr{L}^{\omega\boldsymbol{A}}_{\infty\omega} = \bigcup_k \mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$ is, in general, more expressive than its counterpart $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$. For example, in the class of finite orderings, we can write in $\mathscr{L}^{2\boldsymbol{B}_2}_{\infty\omega}$ a sentence in the signature of linear orderings saying that the cardinality of the domain of the ordering is an even number (in particular, this sentence is an infinitary disjunction of sentences χ_n from $\mathscr{L}^{\boldsymbol{B}_2}_{\omega\omega}$ in just two variables saying that the domain is exactly of size n, for n even) [19, Example 3.3.1 (a)]. On the other hand, it is well known [30, Thm. 3.6] that for any formula φ of $\mathscr{L}^{\boldsymbol{B}_2}_{\omega\omega}$ in the signature of linear orderings there are two linear orderings of even and odd size, respectively, satisfying φ.*

Proposition 18. *Consider models $\mathfrak{M}$ and $\mathfrak{N}$ for a finite signature τ. Then, $(i) \implies (ii)$, where*

(i) *$\mathfrak{M} \cong^k \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are k-potentially isomorphic.*
(ii) *$\|\varphi\|^{\mathfrak{M}} = \|\varphi\|^{\mathfrak{N}}$ for any sentence of $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$.*

Proof. This follows by adapting the proof of [17, Theorem 14 (b)]. That proof appeals to the classical two-sorted translation from [16] and classical results for back-and-forth systems. For us it suffices to observe that the aforementioned translation preserves the number of variables allowed in the language from the many-valued point of view in the number of variables allowed in the language in the second sort of the translation. Rather than appealing to the classical Fraïssé theorem as in [16], in our case the reader simply has to appeal to a simple two-sorted variant of the standard finite-variable version of the result [30, Lemma 11.10]. □

Proposition 19. *Consider a finite signature τ. If $\mathfrak{M}, \mathfrak{N}$ are models of χ^r_Φ for all $r \leq k$, then $\mathfrak{M} \cong^k \mathfrak{N}$, i.e., $\mathfrak{M}$ and $\mathfrak{N}$ are k-potentially isomorphic.*

Proof. To show that $\mathfrak{M} \cong^k \mathfrak{N}$, we define the following set I of partial isomorphisms:

$$I = \{p \mid r \leq k,\ p : M \longrightarrow N \text{ is a partial mapping}, p := \overrightarrow{e} \mapsto \overrightarrow{d}, \overrightarrow{e} = e_1, \ldots, e_r, \overrightarrow{d} = d_1, \ldots, d_r, \overrightarrow{e} \equiv_{at} \overrightarrow{d}\}.$$

Note that I is non-empty as it contains $\emptyset$. Furthermore, it clearly is downward-closed on the right: if $p \in I$ and $p' \subseteq p$ (hence p' is also a partial mapping meeting all the requirements) then $p' \in I$.

Next, we check the forth property. Suppose that $p \in I$, $|\mathrm{dom}(p)| < k$, $p : M \longrightarrow N$ is a partial mapping $p := \overrightarrow{e} \mapsto \overrightarrow{d}$, $\overrightarrow{e} = e_1, \ldots, e_r$, $\overrightarrow{d} = d_1, \ldots, d_r$, $\overrightarrow{e} \equiv_{at} \overrightarrow{d}$. Then, take $e_{r+1} \in M$ (distinct from any element in the sequence $\overrightarrow{e}$). Consider now the A-valued set Φ with domain Δ_{r+1} defined as follows:

$$\Phi(\varphi(v_1, \ldots, v_r, v_{r+1})) = \|\varphi[\overrightarrow{e}, e_{r+1}]\|^{\mathfrak{M}}.$$

Since $r < k$, we must have that $\mathfrak{N} \models \chi^r_\Phi$. Hence,

$$\mathfrak{N} \models (\exists v_{r+1})((\bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1}) \wedge (\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)}))[\overrightarrow{d}],$$

Consequently, for some $d_{r+1} \in N$,

$$\mathfrak{N} \models (\bigwedge_{1\leq i\leq r} v_i \not\approx v_{r+1}) \wedge (\bigwedge_{\varphi\in\Delta_{r+1}} \varphi \leftrightarrow \overline{\Phi(\varphi)})[\overrightarrow{d}, d_{r+1}].$$

Now simply consider the partial mapping $p' : M \longrightarrow N$ defined as $p' := \overrightarrow{e}' \mapsto \overrightarrow{d}'$ where $\overrightarrow{e}' = e_1, \ldots, e_r, e_{r+1}$, $\overrightarrow{d} = d_1, \ldots, d_r, d_{r+1}$. This works because $\overrightarrow{e}' \equiv_{at} \overrightarrow{d}'$.

Finally, the back property follows similarly and the proof is complete. □

Theorem 20 (Second 0-1 Law)**.** *If φ is a sentence of $\mathscr{L}^{k\boldsymbol{A}}_{\infty\omega}$ in the finite relational signature τ, then there is $a \in A$ such that $\mu^a(\varphi) = 1$ and for any other truth-value a', $\mu^{a'}(\varphi) = 0$. This immediately give us the result for the language $\mathscr{L}^{\omega\boldsymbol{A}}_{\infty\omega}$ as well.*

Proof. Consider the finitary conjunction $\bigwedge_{r\leq k} \chi^r_\Phi$. Then, from Proposition 18 and Proposition 19, for some $a \in A$, $\bigwedge_{r\leq k} \chi^r_\Phi \models \varphi \leftrightarrow \overline{a}$. The rest of the proof is just as Theorem 16. □

This result serves as a tool for showing that properties of structures whose asymptotic probabilities do not converge to a limit cannot be expressed even in a relatively powerful language like $\mathscr{L}^{\omega\boldsymbol{A}}_{\infty\omega}$.

V. CONCLUDING REMARKS

It is important to observe that our main result (Theorem 16) does not simply follow from applying the classical 0-1 law to the two-sorted language employed in [16] to provide a translation of $\mathscr{L}^{\boldsymbol{A}}_{\omega\omega}$ into classical logic. The problem is that the two-sorted language in question contains several functions and constant symbols, whereas the classical 0-1 law holds, in general, only for languages where those features are absent. Hence, our result cannot be obtained in a lazy manner: one must do the "honest toil", let it be from a classical two-sorted standpoint or from our many-valued one.

Furthermore, there is an interesting dynamics at play when looking at results in many-valued model theory through the different prisms of many-valued and classical logic. For example, from the point of view of classical logic, Theorem 16 is a *prima facie* useless claim about a very specific two-sorted language. In contrast, from the perspective of many-valued logic, Theorem 16 is a generalization of an important classical theorem. This observation provides evidence to believe that the many-valued approach can have certain advantages over the classical approach, at the very minimum methodologically. However, since the 0-1 law is also known to fail in classical logic for languages with function symbols in general [19, Example 4.4.1(b)], we also have that the two-sorted translation we obtain of our result is a classical case where under certain conditions we obtain the 0-1 law for the identities of a language with function symbols.

Regarding future work, we plan to develop a more general and lengthy study of finite model theory of many-valued logics which we conjecture will yield, by different methods to those of the present paper, 0-1 laws for logics with semantics over arbitrary finite lattices.

Acknowledgements

We are grateful to the anonymous referees for their useful remarks. Guillermo Badia was partially supported by the Australian Research Council grant DE220100544. Carles Noguera was supported by the grant project no. 21-23610M of the Czech Science Foundation that is realized at the Institute of Philosophy of the Czech Academy of Sciences. Finally, we acknowledge support by the European Union's Marie Sklodowska–Curie grant no. 101007627 (MOSAIC project).

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