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Multi-criteria decision analysis without consistency in pairwise comparisons

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ABSTRACT

Life itself is colorful and brings situations where making the right decision is a matter of compromise given the various criteria, often conflicting with each other. To handle such situations, a plethora of mathematical methods supporting decision-making has been developed. A little attention has been paid to cases where either criteria or expert preferences are not transitive by nature. Usually, standard decision-making methods handle such a case as an input error (input inconsistency). Being designed for consistent cases, standard methods may conclude in wrong results. We present a novel framework aimed at dealing with inconsistent preferences, without forcing experts to reconsider their initial judgments thus distorting their spontaneous assessments. A simulation analysis has been led to check the methodological validity of our proposal. Specifically, by setting different consistency ranges, thousands of experiments on simulated matrices confirm that our framework represents a valid alternative to the traditional practice. The applicability of the proposed approach has been eventually demonstrated through a real-world case study focused on supply chain management of a relevant industrial problem.

1. Introduction and State of Art

Multi-criteria decision-making methods enable decision-makers to establish which solution (or which set of alternatives) represents the best trade-off according to differently weighted evaluation criteria referring to such practical aspects as, for instance, safety & security, cost, productivity, and so on. Among the plethora of existing methods, literature agrees on considering the *analytic hierarchy process* (AHP) as one of the most popular. See, for example, (Vaidya & Kumar, 2006; de FSM Russo & Camanho, 2015; Petruni et al., 2019). The AHP, initially developed by Saaty (1980), has been widely demonstrated to be suitable to narrow the gap between theory and practice, by combining the objectivity of the traditional scientific method with the real behavior of humans and complex systems in decision-making problems (Benítez, Carpitella, Certa, Izquierdo, & La Fata, 2017).

The AHP is based on the concept of pairwise comparisons between pairs of elements expressed by a decision-making team (Dong & Cooper, 2016), or maybe, by a single expert in the field of interest, in the form of linguistic variables (Franek & Kresta, 2014). Judgements of pairwise comparisons have to be collected and aggregated into input matrices, called *pairwise comparison matrices* (PCMs) - see Grzybowski and Starczewski, 2020. PCMs will be mathematically manipulated to obtain the vector of weights of the involved elements, the last ones being eventually ranked based on the calculated weights Liu, Zhang, Zhang, and Pedrycz, 2020. The final ranking well represents evaluations of the expert(s) given an additional assumption of consistency is met, that may be easily verified mathematically. The key point of the AHP is the consistency of pairwise comparisons attributed by experts, which directly influence the quality of final decisions Hsieh et al., 2018.

For various reasons, it is however impossible to achieve a complete degree of consistency when expressing judgments. A certain degree of inconsistency is expected (and allowed) due to the limits of human reasoning Benítez, Carpitella, Certa, and Izquierdo, 2020. When a given threshold is not met, that is in the case of inconsistency, either the tool of the AHP may not be used, or the evaluations of expert(s) have to be manipulated in such a way that matrices are consistent (at least to some degree). As shown by Benítez, Delgado-Galván, Izquierdo, and Pérez-García (2011), increasing consistency by manipulating PCMs necessarily yields decisions that no more reflect the initial opinions expressed by experts. This evidence leads to establishing a feedback-based

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relationship with the decision making group Benítez, Carpitella, Certa, Ilaya-Ayza, and Izquierdo, 2018 to find a good trade-off between the twofold objective: reflecting the reality on the one hand and keeping PCMs within the allowed threshold of consistency on the other hand.

Such a normative position of the AHP may be limiting. Indeed, one may easily design decision-making problems leading to inconsistent PCMs. This corresponds well to the current discussion in the domain of the *expected utility theory* (EUT) introduced by Von and Morgenstern (1953), a canonical theory of individual decision-making. Many observations of systematic violations of the EUT axioms, see e.g. Tversky, 1969, motivated various authors to develop alternative decision-making theories, such as Fishburn (1988),tarmer (2000), or Machina (2004). In particular, the axiom of *transitivity* of preferences is not always supported by empirical evidence Bar-Hillel and Margalit, 1988; Butler, Pogrebna, and Blavatskyy, 2016. A concise mathematical model of non-transitive decision-making has been proposed in Kreweras, 1961; Fishburn, 1982, representing preferences with a skew-symmetric bi-linear (SSB) functional. Such theory may deal with inconsistent PCMs seamlessly, therefore it will be used to transform the above-elicited PCMs into weight vectors. Thus one obtains a method converging to a shared choice among various decision-makers that may express their preferences with no additional limitations on their judgments.

The present paper is organised as follows. The next Section 2 states the motivation of this research and describes the existent methodologies that will be used to elaborate on a new approach to handle inconsistency. In particular, this new methodological approach is presented in Section 3. Section 4 applies the new method first to numerical examples, then to simulated data, and finally to a real-world case study of industrial reality. In particular, a real logistic problem will be sorted out to test the validity of our method and its applicability for practical problems.

2. Motivation and Existent Methodologies

First we elaborate the motivation to deal with inconsistent preferences, see SubSection 2.1. In SubSection 2.2 basic notation and definitions are summarized together with a concise introduction of the AHP method. An aggregated preference matrix is defined in SubSection 2.3, and the theory of SSB representation is introduced in SubSection 2.4.

2.1. Motivation

In case of a consistent input, the AHP offers an effective tool to find a solution of the decision problem described using pairwise comparison matrices. One may use the AHP for inconsistent input as well, but there is no guarantee that the solution of the process leads to the best alternative. From the AHP perspective, an inconsistent input is wrong by principle and the decision maker internally ambivalent and irrational. However, real-world problems are often inconsistent in the above sense (e.g. non-transitive evaluation of alternatives given by an expert). In such a case the AHP necessarily loses a considerable amount of information in the process of representing an inconsistent PCM by a weight vector. To avoid such a bottleneck, we propose an alternative to the AHP employing the theory of SSB representation of (potentially intransitive) preferences, see e.g. Fishburn, 1988; Pištěk, 2018; Pištěk, 2019. First we aggregate all the data into a PCM yielding the aggregated preferences. On such a basis we find a maximal preferred element in the sense of the SSB representation, that is a probability vector yielding a more preferred outcome more likely than any other probability vector. Based on such vector we finally rank the options in the same way the global score vector is used in the AHP.

2.2. Notation, Basic Definitions and Analytic Hierarchy Process

Let k be a positive integer. By $\mathscr{P}(k)$ we denote the set of all probability distributions having a finite support of cardinality k, i.e. $\mathscr{P}(k) = \left\{p \in \mathbb{R}^k : p \geqslant 0, \sum_{i=1}^k p_i = 1\right\}$. Depending on the context, elements of $\mathscr{P}(k)$ may be called convex combinations of k elements, or lotteries over k elements.

Let X be a square matrix, we denote the transpose of X by $X^\intercal$. Matrix X is *skew-symmetric* if $X^\intercal = -X$. A square matrix with positive entries obtained from comparisons between certain attributes following a predefined scale is a *pairwise comparison matrix* (PCM). A PCM matrix X of order k is *reciprocal*, $X \in \mathscr{R}(k)$, if $x_{ji} = 1/x_{ij}$ for all $i, j = 1, \dots, k$, and *homogeneous* if $x_{ii} = 1$ for all $i = 1, \dots, k$. Note that homogeneity derives from reciprocity, since for $i = j, x_{ij}x_{ji} = 1$ and $x_{ij} > 0$ gives $x_{ii} = 1$. However, for the sake of clarity it is customary to present these properties separately. Finally, we say that a reciprocal matrix $X \in \mathscr{R}(k)$ is *consistent* with a weight vector $w \in \mathscr{P}(k)$ if w reflects the priorities expressed by elements of X in such a way that $x_{ij} = w_i/w_j$ for all $i,j = 1, \dots, k$.

For a (binary) relation $\succ$ defined on a set S, an element $s \in S$ is a *maximal element* of S with respect to $\succ$ if set $\{q \in S : q \succ s\}$ is empty. We say that $\succ$ is *asymmetric* if $p \succ q$ implies $q \neg \succ p$ for all $p, q \in S$, and $\succ$ is *transitive* if $p \succ q$ and $q \succ r$ implies $p \succ r$ for all $p, q, r \in S$. Further, we assume that S is $\mathscr{P}(k)$. If there is a skew-symmetric matrix X that represents relation $\succ$ on set $\mathscr{P}(k)$ as follows:

$$p \succ q \quad \Leftrightarrow \quad p^\intercal X q > 0 \quad \text{for all} \quad p, q \in \mathscr{P}(k),$$

we say that matrix X is *skew-symmetric bi-linear* (SSB) *representation* of $\succ$, see e.g. Fishburn, 1988; Pištěk, 2018; Pištěk, 2019. A relation admitting a SSB representation will further be called *SSB preference relation*. Note that such a (preference) relation $\succ$ on $\mathscr{P}(k)$ is asymmetric, but not necessarily transitive. Finally, the *indifference* relation $\sim$ and *preference-or-indifference* relation $\gtrsim$ are

$$p \sim q \quad \Leftrightarrow \text{neither } p \succ q \text{ nor } q \succ p \Leftrightarrow \quad p^\intercal X q = 0,$$
$$p \gtrsim q \Leftrightarrow p \succ q \text{ or } p \sim q \Leftrightarrow \quad p^\intercal X q \geqslant 0.$$

Next we introduce a quick overview on the AHP that, as already underlined, is a useful and flexible decision-support tool based on the use of the above defined PCMs. The AHP application allows converging to a shared decision among different stakeholders who express their judgments of preference about pairs of various elements of analysis (criteria, sub-criteria, and alternatives).

The AHP decomposes the decision-making problem into sets of elements to be organised through different levels of a hierarchical structure. The first step to apply the AHP technique consists in breaking the problem down and representing it in a hierarchical way Saaty and Vargas, 1994. Then the elements of each level are pairwise compared to each element belonging to the immediate upper level. Such pairwise comparisons are attributed through numerical values associated with evaluations from one of the various scales available in the literature; the most used is the nine-point scale by Saaty (1977).

Performing such a comparison for a finite set of k elements (criteria or alternatives) yields a $k \times k$ PCM $X = (x_{ij})$. Then, the problem is to produce a set of numerical values $w_1, \dots, w_k$ reflecting the priorities of the compared elements according to the elicited judgments x_{ij}. As previously explained,if all judgments have been expressed in a completely consistent way, the relations between weights w_i and judgments x_{ij} are simply given by $w_i/w_j = x_{ij}$ $(i,j = 1,2,\dots,k)$, and matrix X is then said to be (fully) consistent.

However, as some degree of inconsistency is always expected during the process of judgments attribution because of the natural lack of consistency of human thinking, the reciprocal PCM X is, in general, not consistent. As shown by Saaty (2003), the eigenvector is necessary for obtaining priorities. The hypothesis is that the estimates of these values are small perturbations of the "right" values guarantees a small perturbation of the eigenvalues, see, e.g., Stewart, 2001. For non-consistent matrices, one has to solve the eigenvalue problem, $X\mathbf{w} = \lambda\mathbf{w}$, to find λ_{max} the unique largest eigenvalue of X that gives the Perron eigenvector as an estimate of the priority vector. Note finally that also the geometric mean method may be used to estimate the priority vector, see Saaty and Vargas, 1984. However, in this paper, we consider only the above-described eigenvector method.

The AHP theory developed by Saaty provides a measure of consistency in each set of judgments. It is determined through the so-called *consistency ratio CR*:

$$CR = \frac{CI}{RI}, \tag{1}$$

where *CI* is the *consistency index*, and *RI* is the *random index*. For matrices of order k, *CI* is defined as:

$$CI = \frac{\lambda_{max} - k}{k - 1}, \tag{2}$$

Furthermore, Saaty (2000) provided average consistencies (*RI* values) of randomly generated matrices. In general, when $CR \leqslant 0.1$ it implies acceptable consistency. Otherwise, pairwise comparison judgments may not be reliable and should be reconsidered. Judgment modifications can be performed either by employing tools for improving consistency or by asking for a new elicitation from decision-makers.

Let us consider a decision-making problem with m criteria $\{c_1, c_2, \ldots, c_m\}$ and n options $\{o_1, o_2, \ldots, o_n\}$. The input data for the AHP application can be collected by using several pairwise comparison matrices $\boldsymbol{A}, \boldsymbol{B}^{(k)}$, $k = 1, \ldots, m$, where $\boldsymbol{A} \in \mathscr{R}(m)$ with $a_{ij} > 0$ is a PCM representing the importance of different criteria. $\boldsymbol{B}^{(k)} \in \mathscr{R}(n)$ with $b_{ij}^{(k)} > 0$ is expressing the degree of preference of option o_i over option o_j with respect to k-th criterion, $k \in \{1, \ldots, m\}$.

The standard AHP approach is based on the assumption that given the set of k elements (criteria or alternatives) to be pairwise compared, all the input matrices faithfully estimate their respective priority vectors. As mentioned above, the assumption of consistency can be easily violated. Hereinafter, we propose to take a more general position by assuming matrix $\boldsymbol{A} \in \mathscr{R}(m)$ to be acceptably consistent and contemplating the possibility that matrices $\boldsymbol{B}^{(k)} \in \mathscr{R}(n)$ may be even highly inconsistent. We will denote as $\boldsymbol{w} \in \mathscr{P}(m)$ the vector of *criteria weights* derived from matrix $\boldsymbol{A}$; as $\boldsymbol{v}^{(k)} \in \mathscr{P}(n)$ the *local priorities* that are the vectors of options weights from matrices $\boldsymbol{B}^{(k)}$ for all $k = 1, \ldots, m$; and as $z \in \mathscr{P}(n)$ the vector of *final (options) weights* calculated on the basis of the criteria weights and options' local priorities. The exact mathematical procedure required to calculate vectors $\boldsymbol{w}$ and $\boldsymbol{v}^{(k)}$ involves the computation of eigenvalues and eigenvectors, through the Perron vector Perron, 1907; Frobenius, 1912 which can be easily calculated, for example, using the power method Peretti, 2014. Once obtained criteria and options priorities, vector z of final weights is obtained as multiplication of matrix $\boldsymbol{V}$ whose columns rows are vectors of local priorities $\boldsymbol{v}^{(k)}$ and vector $\boldsymbol{w}$ of criteria weights:

$$z = \boldsymbol{V}\boldsymbol{w}. \tag{3}$$

Example. (*Leader example*) To illustrate the standard AHP approach let us consider a decision-making problem with $n = 3$ alternatives and $m = 4$ evaluation criteria. We will use the "Tom, Dick, and Harry" example by Wikipedia contributors (2020). This example introduces the real situation in choosing a leader for a company whose founder is about to retire. There are three competing candidates (Tom, Dick, and Harry) and four different criteria (Age, Charisma, Education, Experience) for choosing the most suitable candidate. Criteria are pairwise compared and the related judgements of preference a_{ij} are collected in the following input matrix $\boldsymbol{A}$:

Similarly, we can evaluate the preference of each candidate to a given criterion. We will denote respective preference matrices as $\boldsymbol{B}^{(l)}$ where $l \in \{1, 2, 3, 4\}$. Consistency can be now easily checked by calculating the consistency ratio according to (1). Since for $\boldsymbol{A}$ and all $\boldsymbol{B}^{(l)}, l = 1, \ldots, 4$, this ratio is smaller than 0.1, see the last columns in Tables 1 and 2, matrices $\boldsymbol{A}$ and $\boldsymbol{B}^{(l)}$ are acceptably consistent.

As it has been already mentioned, to calculate priorities, it is enough to get the so-called Perron vector, which is the eigenvector corresponding to the dominant eigenvalue of the respective matrix. The leading eigenvalue is 4.1184 in this case, with corresponding eigenvector $[0.0893, 0.4312, 0.2022, 0.8747]^{\mathrm{T}}$. After normalizing it (to sum up to one), we get the vector of criteria weights for matrix $\boldsymbol{A}$:

$$\boldsymbol{w} = [0.0559, 0.2699, 0.1266, 0.5476]^{\mathrm{T}}.$$

The matrix $\boldsymbol{V}$, composed of vectors $\boldsymbol{v}^{(k)}$ derived from matrices in Table 2, is in Table 3.

The vector of final weights given by (3) is in Table 4. Based on this, one can see that Dick is the best candidate for the position according to the AHP.

2.3. Aggregated Preference Matrix

We assume that the pairwise comparison of criteria represented by matrix $\boldsymbol{A}$ is consistent (to a high-enough degree), thus we may compute the vector of evaluation criteria weights, $\mathbf{w} \in \mathscr{P}(m)$, in the standard AHP-way. However, in our problem setting, this is not assumed for matrices $\boldsymbol{B}^{(l)}, l = 1, \ldots, k$, representing the evaluation of options by individual criteria. Trying to represent $\boldsymbol{B}^{(l)} \in \mathscr{P}(n)$ by a vector of weights may thus lead to high information loss. To avoid this issue, we combine matrices $\boldsymbol{B}^{(l)}$ into a PCM matrix called *aggregated preference matrix* $\boldsymbol{P} \in \mathscr{R}(n)$ employing also the weight vector of evaluation criteria $\boldsymbol{w}$.

As underlined by Blagojevic, Srdjevic, Srdjevic, and Zoranovic (2016), there are various possible procedures for aggregating judgments of pairwise comparisons and obtaining matrix $\boldsymbol{P}$. The most common is the aggregation of individual judgments and the aggregation of individual priorities Abel, Mikhailov, and Keane, 2015; Ramanathan and Ganesh, 1994, but also models based on consensus convergence Lehrer and Wagner, 2012 and 'soft' consensus computations Wu and Xu, 2012 have been applied. We will employ an element-wise (weighted) geometric mean of matrices $\boldsymbol{B}^{(l)}$:

$$\boldsymbol{P} = \prod_l \left(\boldsymbol{B}^{(l)}\right)^{w_l}, \tag{4}$$

where $w_l \geqslant 0, l = 1, 2, \ldots, m$ such that $\sum_l w_l = 1$ are criteria weights obtained from matrix A. Let us highlight that both product $\prod$ and power $\boldsymbol{B}^{w}$ in (4) are performed element wise. Note that aggregated preference matrix $\boldsymbol{P}$ is reciprocal; moreover, $\boldsymbol{P}$ is also consistent provided all $\boldsymbol{B}^{(l)}$ are consistent.

Example. (*Continuation of the Leader example*) Note that in case of matrices $\boldsymbol{A}, \boldsymbol{B}^{(l)}$ from the Leader Example, as defined in Tables 1 and 2, respectively, matrix $\boldsymbol{P}$ can be found in Table 5. For the sake of readability, we have rounded the numbers to three decimal places.

2.4. Skew-symmetric Bi-linear Representation of Preferences

Let k be a positive integer and $\succ$ be a (preference) relation on $\mathscr{P}(k)$. From the perspective of the EUT, preference relation $\succ$ is rational if and only if it may be represented by a linear functional on $\mathscr{P}(k)$, i.e. there

Table 1
$\boldsymbol{A}$: criteria pairwise comparison.

	Age	Charisma	Education	Experience	CR
Age	1	1/5	1/3	1/7	
Charisma	5	1	3	1/3	0.04435
Education	3	1/3	1	1/4	
Experience	7	3	4	1	

Table 2
$B^{(k)}$: alternative pairwise comparison in each criterion.

	Tom	Dick	Harry	CR
		(a) $B^{(1)}$: Age		
Tom	1	1/3	5	
Dick	3	1	9	0.02795
Harry	1/5	1/9	1	
		(b) $B^{(2)}$: Charisma		
Tom	1	5	9	
Dick	1/5	1	4	0.06852
Harry	1/9	1/4	1	
		(c) $B^{(3)}$: Education		
Tom	1	3	1/5	
Dick	1/3	1	1/7	0.06239
Harry	5	7	1	
		(d) $B^{(4)}$: Experience		
Tom	1	1/4	4	
Dick	4	1	9	0.03548
Harry	1/4	1/9	1	

Table 3
Matrix V of local priorities $v^{(k)}$.

	Age	Charisma	Education	Experience
Tom	0.267	0.193	0.220	0.735
Dick	0.669	0.083	0.713	0.199
Harry	0.064	0.724	0.067	0.065

Table 4
Vector z of final weights.

	final weights
Tom	0.304
Dick	0.454
Harry	0.242

Table 5
Aggregated preference matrix P for the Leader example.

	Tom	Dick	Harry
Tom	1	0.781	3.451
Dick	1.280	1	4.280
Harry	0.290	0.234	1

has to exist $x \in \mathscr{P}(k)$ such that

$$p \succ q \quad \Leftrightarrow \; x^{\intercal}p > x^{\intercal}q \text{ for all } p, q \in \mathscr{P}(k).$$

Representing a preference scale for any element of $\mathscr{P}(k)$ by a real number, the EUT-based model of rationality may not account for possible intransitives of individual preferences. To this end a more general[1] theory of the SSB representation of preferences has been proposed Fishburn, 1982. Omitting the ongoing discussion about intransitivity of preferences from the normative perspective, see e.g., Anand, Pattanaik, and Puppe (2009, Chapter 6), we simply note that in the domain of the AHP a need for a tool that may deal with possibly intransitive experts' evaluations is evident.

Assume next that an asymmetric matrix X is a SSB representation of $\succ$, that is

$$p \succ q \quad \Leftrightarrow \quad p^{\intercal}Xq > 0 \quad \text{for all} \quad p, q \in \mathscr{P}(k).$$

[1] Let $\succ$ be a rational preference in the sense of the EUT, then there exist vectors $u, w \in \mathscr{P}(k)$ such that matrix X given by $x_{ij} = u_i w_j - w_i u_j$ is a SSB representation of $\succ$, see, e.g. Fishburn (1988, page 77).

For such a general X one may easily find examples of $p, q, r \in \mathscr{P}(k)$ such that $p \succ q, q \succ r$, and $r \succ p$. This seems to be an insurmountable obstacle for decision-making using such preferences. However, the well-known Minimax Theorem, see, e.g., Von and Morgenstern, 1953, implies that there is a maximal element $s \in \mathscr{P}(k)$ such that $s \succ q, s \succ p$, and $s \succ r$.

Theorem 2.1. Let $\succ$ be a preference relation on $\mathscr{P}(k)$ that has a SSB representation, then there exists a maximal element of $\mathscr{P}(k)$ w.r.t. $\succ$.

More general existence theorems have been proposed in, e.g., Fishburn, 1982; Pištěk, 2018; Pištěk, 2019.

Recall that elements x_{ij} of X are proportional to the scale of preference of alternative i over j. Thus, for any $p, q \in \mathscr{P}(k)$, one may evaluate the probability vector of p yielding a more preferred outcome than q by $p^{\intercal}Xq$. This gives a clear interpretation to the maximal element $s \in \mathscr{P}(k)$: satisfying $s^{\intercal}Xq \geqslant 0$ for all $q \in \mathscr{P}(k)$, element s yields a more preferred outcome more (or equally) likely than any other probability vector in $\mathscr{P}(k)$. This condition can be equivalently[2] stated as $s^{\intercal}X \geqslant 0$, and finally transformed to

$$Xs \leqslant 0 \tag{5}$$

using skew-symmetry of X, i.e., $X = -X^{\intercal}$. Solution of (5) may be found directly by using the methods of polyhedral geometry. We conclude this section by stating that for a typical skew-symmetric matrix X, there exists a unique maximal element.

Remark 2.2. For almost all[3] SSB preference relations on $\mathscr{P}(k)$ there is a unique maximal element in $\mathscr{P}(k)$.

The above remark is precisely mathematically formulated in Brandl (2017, Propositon 4).

3. New Approach to Cope with Inconsistency

By applying the tools introduced in Section 2, we will obtain a new method that may well handle the possible inconsistency of experts' judgements (note, however, that from the perspective of the SSB representation, the AHP-inconsistency is, actually, not an inconsistency), see SubSection 3.1. In SubSection 3.2 a relationship of this method to the AHP in the consistent case is discussed; its resistance to the so-called order reversal is elaborated in Section 3.3. These observations are illustrated by basic examples in SubSection 4.1.

3.1. New Method

Let $P \in \mathscr{R}(n)$ be the aggregated preference matrix P given by (4). To apply the theory of the SSB representation, one needs a skew-symmetric matrix X such that x_{ij}, if positive, represents the scale of preference of i over j. One may come with many ways how to transform aggregated preference matrix P into a SSB matrix; we propose to use an element-wise logarithm

$$X = \log P \tag{6}$$

to this end. Indeed, such a matrix is skew-symmetric using reciprocity of P, the sign of x_{ij} indicates if i is preferred to j or vice versa, and the absolute value of x_{ij} corresponds to the scale of such a preference.

[2] To show that $s^{\intercal}Xq \geqslant 0$ for all $q \in \mathscr{P}(k)$ implies $s^{\intercal}X \geqslant 0$, consider all $q \in \mathscr{P}(k)$ such that $q_i = 1$ for a particular $i = 1, \dots, k$.

[3] Almost all in the sense of the measure theory. Indeed, in Brandl, 2017 necessary conditions for distribution of SSB preference relations are examined such that a randomly chosen preference relation almost surely admits a unique maximal element. This result is presented in the realm of the game theory; however, there is a one-to-one correspondence between SSB maximal elements and optimal strategies of symmetric two-players zero-sum games (identifying the pay-off matrix of a game with the matrix of SSB preference relation).

The maximal preferred element with respect to such matrix will be called *the final distribution of preference*, and denoted by $\zeta \in \mathscr{P}(n)$. By using (5), vector ζ satisfies $(\log \boldsymbol{P})\zeta \leqslant 0$. In other words, all final distributions of preference form a non-empty polyhedron determined by

$$\left(\log \boldsymbol{P}\right)\zeta \leqslant 0, \zeta \geqslant 0, \sum_{i=1}^{n} \zeta_i = 1, \tag{7}$$

which is, typically, degenerated into just one point, cf. Remark 2.2. Let us recall that such $\zeta \in \mathscr{P}(n)$ leads to a more preferred outcome more (or equally) likely than any other probability distribution in $\mathscr{P}(n)$.

Example. (*Continuation of the Leader example*) To illustrate the fact that $log(\boldsymbol{P})$ is an SSB matrix, let us apply the element-wise log transformation to matrix $\boldsymbol{P}$ of Table 5. We obtain a matrix $\log \boldsymbol{P}$ shown in Table 6. The final distribution of preference ζ calculated by solving the problem from polyhedral geometry (7) is shown in Table 7; note that for the given matrix $\boldsymbol{P}$ it is unique.

Contrary to the final weights vector z of the AHP, see (3), the final distribution of preference $\zeta \in \mathscr{P}(n)$ determined by (7) often has many zero elements, see, e.g. the solution of the Leader example in Table 7. Thus, ζ indicates well which element of $\mathscr{P}(n)$ leads to the best choice more (or equally) likely than any other, but it may not be reasonably used to rank all alternatives. To this end one may evaluate the *pair-wise degree of preference* of ζ, defined by

$$\boldsymbol{\pi} = \boldsymbol{P}^{\intercal}\zeta. \tag{8}$$

Such π_i corresponds to the (expected) degree of preference of ζ over any alternative $i = 1, \ldots, k$ that is measured on the Saaty's scale[4]. Observe that by skew-symmetry of $\log P$ together with (7) we have $(\log \boldsymbol{P})^{\intercal}\zeta \geqslant 0$ and so

$$\pi = \boldsymbol{P}^{\intercal}\zeta \geqslant \exp\{(\log \boldsymbol{P}^{\intercal})\zeta\} \geqslant 1$$

using Jensen inequality. Thus $\pi_i \geqslant 0$ for all $i = 1, \ldots, k$ well indicating that ζ is optimal.

On the other hand, when $\zeta \in \mathscr{P}(n)$ determined by (7) has a unique positive value at the i-th component, to obtain the second best option, we may apply (7) again with a reduced aggregated preference matrix $\boldsymbol{P}(\bar{i})$ obtained by erasing the i-th row and column from $\boldsymbol{P}$. Then, according to $\zeta' \in \mathscr{P}(n-1)$ determined by (7) with substitution $\boldsymbol{P}(\bar{i})$ for $\boldsymbol{P}$, the second best option is found. Moreover, if $\zeta' \in \mathscr{P}(n-1)$ a unique positive value, we can find the third best option by repeating the same procedure.

Example. (*Continuation of the Leader example*) Having matrix $\boldsymbol{P}$, the pair-wise degree of preference $\boldsymbol{\pi}$ is shown in Table 8.

Note that for the case of final distribution ζ being concentrated on one optimal alternative, like in the Leader example, see Table 7, the pair-wise degree of preference $\boldsymbol{\pi}$ corresponds directly to the respective row of $\boldsymbol{P}$, cf. Tables 5 and 8. The values 1.196 and 2.654 in Table 8 mean that Dick is preferred 1.196 times more than Tom and 2.654 times than Harry in the holistic evaluation.

Table 6
$\log \boldsymbol{P}$ - an SSB matrix.

	Tom	Dick	Harry
Tom	0	−0.247	1.239
Dick	0.247	0	1.454
Harry	−1.239	−1.454	0

[4] This interpretation is actually the main motivation for setting $\boldsymbol{X} = \log \boldsymbol{P}$ in (6). One may use other skew-symmetric matrix being somehow proportional to preference relation $\boldsymbol{P}$, e.g. $\widetilde{X} = \boldsymbol{P} - \boldsymbol{P}^{\intercal}$. Then, however, incompatibility of such $\widetilde{X}$ with both multiplicative reciprocity assumed for $B^{(l)}$ as well as geometrical mean used to aggregated preferences into P would not allow such an interpretation of $\boldsymbol{\pi}$.

Table 7
The final distribution of preference ζ.

Tom	Dick	Harry
0	1	0

Table 8
The pair-wise degree of preference $\boldsymbol{\pi}$.

Tom	Dick	Harry
1.280	1	4.280

3.2. Consistent Case: Relation to Analytic Hierarchy Process

In this subsection, we will show the similarity and differences between the AHP and the proposed SSB-based method. To this end, we consider a case when all given comparison matrices are consistent.

Let $\boldsymbol{A} \in \mathscr{R}(m)$ and $\boldsymbol{B}^{(l)} \in \mathscr{R}(n)$ for all $l = 1, .., m$ be such that, given positive vectors $\boldsymbol{w} \in \mathscr{P}(m)$ and $\boldsymbol{v}^{(l)} \in \mathscr{P}(n)$ for all $l = 1, .., m$, it holds

$$a_{kl} = \frac{w_k}{w_l}, \quad \text{and} \quad b_{ij}^{(l)} = \frac{v_i^{(l)}}{v_j^{(l)}} \tag{9}$$

for all $i, j = 1, .., n$ and $k, l = 1, ..m$. It is well known that, in this fully consistent case, the weights obtained in the conventional AHP procedure are given by $w \in \mathscr{P}(m)$ and $v^{(l)} \in \mathscr{P}(n)$ for all $l = 1, .., m$. Therefore, the best alternative is determined as the j^*-th alternative satisfying

$$\sum_l w_l v_{j^*}^{(l)} = \max_{j=1,\ldots,n} \sum_l w_l v_j^{(l)}. \tag{10}$$

Theorem 3.1. Consider matrices $\boldsymbol{A}, \boldsymbol{B}^{(l)}$ defined by (9) with $\boldsymbol{w} \in \mathscr{P}(m)$ and $v^{(l)} \in \mathscr{P}(n)$ for all $l = 1, .., m$. Let $j^* \in \{1, .., n\}$ be the index of element which maximizes $\prod_l {v_j^{(l)}}^{\boldsymbol{w}_l}$, namely, j^* satisfies

$$\prod_l {v_{j^*}^{(l)}}^{w_l} = \max_{j=1,\ldots,n} \prod_l {v_j^{(l)}}^{w_l}, \text{or equivalently}, \sum_l w_l \log\left(v_{j^*}^{(l)}\right) = \max_{j=1,\ldots,n} \sum_l w_l \log\left(v_j^{(l)}\right). \tag{11}$$

Then, the vector of final distribution of preferences ζ satisfies

$$\zeta_i = \begin{cases} 1, & \text{if } i = j^*, \\ 0, & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, n. \tag{12}$$

Accordingly, $\boldsymbol{\pi}$ given by (8) satisfies

$$\pi_i = \prod_l {v_{j^*}^{(l)}}^{w_l} \Big/ \prod_l {v_i^{(l)}}^{w_l}, \quad i = 1, 2, .., n. \tag{13}$$

Proof. Because matrix $\boldsymbol{A}$ is consistent, the positive vector $\boldsymbol{w} = [w_1, w_2, \ldots, w_m] \in \mathscr{P}(m)$ is estimated by the estimation method of the standard AHP. The (i,j)-component p_{ij} of (P) is obtained as

$$p_{ij} = \prod_l \left(b_{ij}^{(l)}\right)^{w_l} = \frac{\prod_l \left(v_i^{(l)}\right)^{w_l}}{\prod_l \left(v_j^{(l)}\right)^{w_l}}, \quad i, j = 1, \ldots, n.$$

From the definition of j^*, we have

$$\prod_l \left(v_i^{(l)}\right)^{w_l} \leqslant \prod_l \left(v_{j^*}^{(l)}\right)^{w_l}, \ i = 1, \ldots, n.$$

This implies

$$p_{ij^*} = \frac{\prod_l \left(v_i^{(l)}\right)^{w_l}}{\prod_l \left(v_{j^*}^{(l)}\right)^{w_l}} \leqslant 1, \ i = 1, \ldots, n.$$

We obtain

$$\log p_{ij^*} \leqslant 0, \ i = 1, \ldots, n.$$

Hence, ζ defined by (8) is a solution satisfying (6). Corresponding to this solution, we obtain $\boldsymbol{\pi}$ defined by (9). □ From Theorem 3.1, we observe that the holistic evaluation is made by the weighted geometric mean of local priorities (or equivalently, by the weighted arithmetic mean of logarithms of local priorities) whereas in the AHP it is made by the weighted arithmetic mean of local priorities. We note that $p_{ij} = \prod_l (v_i^{(l)})^{w_l} / \prod_l (v_j^{(l)})^{w_l} = \prod_l (v_i^{(l)}/v_j^{(l)})^{w_l}$ in the proof of Theorem 3.1 reminds us the weighted product model (WPM) method (Triantaphyllou, 2000). In the WPM method, the j^*-th alternative with $p_{j^*i} \geqslant 1$ for all $i \in \{1, 2, \ldots, n\}$ is considered as the best option. As $\log(p_{ij^*}) \leqslant 0$ corresponds to $p_{ij^*} \leqslant 1$ equivalent to $p_{j^*i} \geqslant 1$, we may see the proposed method corresponds to WPM method when evaluations by criteria and the weights of criteria are not exactly given. On the other hand, it is known that the AHP corresponds to the weighted sum model (WSM) method (Triantaphyllou, 2000) when evaluations by criteria and the weights of criteria are not exactly given. From these correspondences, we can see that the proposed SSB-based method provides a dimensionless analysis (Triantaphyllou, 2000) because p_{ij} eliminates any units of measure which is an advantage of the proposed method.

3.3. Considerations about Order Reversal

Belton and Gear (1983) demonstrated that a ranking inconsistency can occur in the AHP analysis when an identical alternative is added to the original set of alternatives. As shown in the following theorem, such order reversal never occurs in our method even when the added alternative is not equivalent to any of original alternatives as far as the scores under criteria are given, see also Example 4.3 below.

Theorem 3.2. Let $sc_i^{(l)}, i = 1, \ldots, n$ be scores of n alternatives under the l-th criterion ($l = 1, \ldots, m$). Consider the $(n+1)$-th alternative having a score $sc_{n+1}^{(l)}$ under the l-th criterion $l = 1, \ldots, m$. Then, in the proposed method, the order among the first n alternatives is not changed by adding the $(n+1)$-th alternative regardless a pairwise comparison matrix A among criteria.

Proof. Let $\mathbf{w} = (w_1, \ldots, w_m)^{\mathrm{T}}$ be the weight vector obtained from A. Before adding the $(n+1)$-th alternative, the normalized scores $v_i^{(l)}, i = 1, \ldots, n$ under the l-th criterion are obtained by

$$v_i^{(l)} = \frac{sc_i^{(l)}}{\sum_{k=1}^{n} sc_k^{(l)}}, \ l = 1, \ldots, m. \tag{14}$$

Then the overall score Sc_i of the i-th alternative is obtained by

$$Sc_i = \prod_{l=1}^{m} v_i^{(l)w_l} = \frac{\prod_{l=1}^{m} sc_i^{(l)}}{\prod_{l=1}^{m} \left(\sum_{k=1}^{n} sc_k^{(l)}\right)}, \ i = 1, \ldots, n. \tag{15}$$

In the same way, after adding the $(n+1)$-th alternative, the overall score $\overline{Sc}_j$ of the j-th alternative is obtained by

$$\overline{Sc}_i = \frac{\prod_{l=1}^{m} sc_j^{(l)}}{\prod_{l=1}^{m} \left(\sum_{k=1}^{n+1} sc_k^{(l)}\right)}, \ j = 1, \ldots, n+1. \tag{16}$$

The denominators of (15) are same among n alternatives and the denominators of (16) are same among $(n+1)$ alternatives. The numerators of Sc_i in (15) and $\overline{Sc}_i$ in (16) are same for $i = 1, \ldots, n$. Therefore, we have

$$Sc_i \geqslant Sc_j \text{ if and only if } \overline{Sc}_i \geqslant \overline{Sc}_j, \ i, j \in \{1, \ldots, n\}. \tag{17}$$

Therefore, the order among the original n alternatives does not change by adding the $(n+1)$-th alternative having any scores. □ Theorem 3.2 implies that the order reversal never occurs by adding/deleting any alternatives in the proposed method as far as the scores under criteria are given.

4. Applications

Now we apply the above introduced method to various decision making problems. First, we will solve several illustrative examples. Then, a simulation analysis is provided to compare the AHP and the proposed method statistically. Finally, we show a real-world study to demonstrate our method in detail.

4.1. Illustrative Examples

The following example demonstrates the difference between the AHP and the proposed method.

Example 4.1. To see the difference between the AHP and the proposed method, let us consider the case where $n = 5$ and $m = 2$, namely, we compare five options $o_1, o_2, \ldots, o_5$ by two criteria. Both criteria have the same importance, i.e. $\boldsymbol{A}$ is a matrix of ones and corresponding $w_1 = w_2 = 0.5$. The pairwise comparison matrices $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are given as in Table 9.

Applying the conventional AHP approach (eigenvalue method), we obtain local priorities as

$$\boldsymbol{v}^{(1)} = [0.4058, 0.4101, 0.1008, 0.04841, 0.03488]^{\mathrm{T}},$$
$$\boldsymbol{v}^{(2)} = [0.3822, 0.3842, 0.1149, 0.07596, 0.04272]^{\mathrm{T}}.$$

Then we obtain final weight vector z as

$$z = \begin{bmatrix} 0.4058 & 0.3822 \\ 0.4101 & 0.3842 \\ 0.1008 & 0.1149 \\ 0.04841 & 0.07596 \\ 0.03488 & 0.04272 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.3940 \\ 0.3971 \\ 0.1079 \\ 0.06219 \\ 0.03880 \end{bmatrix}.$$

Therefore, we obtain $o_2 \succ o_1 \succ o_3 \succ o_4 \succ o_5$.

Table 9
Matrices $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$.

	o_1	o_2	o_3	o_4	o_5	CR
			(a) $\boldsymbol{B}^{(1)}$: Criterion 1			
o_1	1	2	4	6	8	
o_2	1/2	1	9	8	9	
o_3	1/4	1/9	1	4	3	0.09321
o_4	1/6	1/8	1/4	1	2	
o_5	1/8	1/9	1/3	1/2	1	
			(b) $\boldsymbol{B}^{(2)}$: Criterion 2			
o_1	1	2	2	5	6	
o_2	1/2	1	4	8	9	
o_3	1/2	1/4	1	1	3	0.06160
o_4	1/5	1/8	1	1	2	
o_5	1/6	1/9	1/3	1/2	1	

However, the decision maker evaluates that option o_1 is qualitatively better than any other options in both criteria because we have $b_{1j}^{(1)} > 1$, $b_{1j}^{(2)} > 1, j = 2, \ldots, 5$ and $b_{ij}^{(l)} > 1$ implies the i-th option is more important than j-th option in the l-th criterion. From this point of view, from Table 9, we see that decision maker orders the options qualitatively as $o_1 \succ o_2 \succ o_3 \succ o_4 \succ o_5$ in both criteria through pairwise comparisons. Hence, we have the same order in the holistic evaluation.

When we apply the proposed approach, we obtain

$$\boldsymbol{P} = \begin{bmatrix} 1 & 2 & 2.8284 & 5.4772 & 6.9282 \\ 0.5 & 1 & 6 & 8 & 9 \\ 0.3536 & 0.16667 & 1 & 2 & 3 \\ 0.1826 & 0.125 & 0.5 & 1 & 2 \\ 0.1443 & 0.1111 & 0.3333 & 0.5 & 1 \end{bmatrix}.$$

and thus,

$$\log\boldsymbol{P} = \begin{bmatrix} 0 & 0.6931 & 1.0397 & 1.70060 & 1.93560 \\ -0.6931 & 0 & 1.7918 & 2.0794 & 2.1972 \\ -1.0397 & -1.79176 & 0 & 0.6931 & 1.0986 \\ -1.7006 & -2.0794 & -0.6931 & 0 & 0.6931 \\ -1.9356 & -2.1972 & -1.0986 & -0.6931 & 0 \end{bmatrix}.$$

Then we obtain $\boldsymbol{\zeta} = (1, 0, 0, 0, 0)^{\mathrm{T}}$ as the final distribution of preference, which implies that o_1 is surely the most preferred option. Repeating the application of (7) to the reduced aggregated preference matrix of $\boldsymbol{P}$ by erasing the option corresponding to the unique positive value of $\boldsymbol{\zeta}$, we can rank the options. In this example, we obtain $o_1 \succ o_2 \succ o_3 \succ o_4 \succ o_5$ which is the same as the decision maker's preference obtained through pairwise comparisons. Example 4.1 demonstrates that the proposed approach respects the qualitative meanings of pairwise comparisons rather than the degrees of intensity of pairwise comparisons which AHP respects.

The following example demonstrates the difference between AHP and the proposed approach when the pairwise comparison matrices are intransitive.

Example 4.2. Consider a virtual singer selection problem. A music company would like to employ a singer for selling her/his music records. There are five candidates of singers, o_1, o_2, o_3, o_4 and o_5. The singers are evaluated by their singing prowess and marketability by the respective domain experts. Those two criteria are equally important, i.e., $w_1 = w_2 = 0.5$. The pairwise comparison matrices $\boldsymbol{B}^{(1)}$ (singing prowess) and $\boldsymbol{B}^{(2)}$ (marketability) are given as in Table 10.

As the evaluations of the singing prowess and the marketability are complex, we found the non-transitivity among o_1, o_2 and o_3 in both evaluations in the singing prowess and the marketability. For example, in the pairwise comparison matrix $\boldsymbol{B}^{(1)}, o_1$ is preferred to o_2 and o_2 is preferred to o_3 because of $b_{12}^{(1)} > 1$ and $b_{23}^{(1)} > 1$. However, o_3 is preferred to o_1 because of $b_{31}^{(1)} > 1$.

Table 10
Matrices $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$.

	o_1	o_2	o_3	o_4	o_5	CR
			(a) $\boldsymbol{B}^{(1)}$: Criterion 1			
o_1	1	4	1/3	6	8	
o_2	1/4	1	4	8	7	
o_3	3	1/4	1	7	4	0.313741
o_4	1/6	1/4	1/7	1	2	
o_5	1/8	1/7	1/4	1/2	1	
			(b) $\boldsymbol{B}^{(2)}$: Criterion 2			
o_1	1	1/3	2	5	6	
o_2	3	1	1/2	8	9	
o_3	1/2	2	1	6	5	0.107289
o_4	1/5	1/8	1/6	1	1	
o_5	1/6	1/9	1/5	1	1	

Applying the AHP, we obtain local priorities as

$$\boldsymbol{v}^{(1)} = [0.3397, 0.3159, 0.2750, 0.03702, 0.03235]^{\mathrm{T}},$$
$$\boldsymbol{v}^{(2)} = [0.2598, 0.3639, 0.2933, 0.04184, 0.04112,]^{\mathrm{T}}.$$

As CR values of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are obtained as shown in Table 10. We found $\boldsymbol{B}^{(1)}$ is too inconsistent to utilize $\boldsymbol{v}^{(1)}$. Then we need to ask the domain experts for singing prowess to revise the pairwise comparison matrix. However, we guess that the revision might be difficult because some part of the singing prowess evaluation would not be explained mathematically.

If we enforce to use $\boldsymbol{v}^{(1)}$, we obtain the final weights as

$$z = [0.2997, 0.3399, 0.2842, 0.03943, 0.03673]^{\mathrm{T}}.$$

This implies that o_1, o_2 and o_3 are better than o_4 and o_5 and that o_2 is the best and o_1 is the second best.

Now, we apply the proposed method, we obtain

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1.1547 & 0.8165 & 5.4772 & 6.9282 \\ 0.8660 & 1 & 1.4142 & 8 & 7.9373 \\ 1.2247 & 0.7071 & 1 & 6.4807 & 4.4721 \\ 0.1826 & 0.125 & 0.1543 & 1 & 1.4142 \\ 0.1443 & 0.1260 & 0.2236 & 0.7071 & 1 \end{bmatrix}.$$

and thus,

$$\log\boldsymbol{P} = \begin{bmatrix} 0 & 0.1438 & -0.2027 & 1.7006 & 1.9356 \\ -0.1438 & 0 & 0.3466 & 2.0794 & 2.0716 \\ 0.2027 & -0.3466 & 0 & 1.8688 & 1.4979 \\ -1.7006 & -2.0794 & -1.8688 & 0 & 0.3466 \\ -1.9356 & -2.0716 & -1.4979 & -0.3466 & 0 \end{bmatrix}.$$

From (7), we obtain

$$\boldsymbol{\zeta} = [0.5, 0.2925, 0.2075, 0, 0]^{\mathrm{T}}.$$

$\zeta_4 = \zeta_5 = 0$ implies that candidates o_4 and o_5 cannot be selected. On the other hand, the first three candidates o_1, o_2 and o_3 can be selected because $\zeta_i, i = 1, 2, 3$ are positive. This corresponds to the fact that the evaluations of those three candidates are intransitive and the other two are inferior in pairwise comparison matrices $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$. The fact that $\boldsymbol{\zeta}$ has multiple positive elements implies the non-conclusive. However, the values of ζ_i's show the tendencies (probabilities) to be selected as the final solution.

In the current selection problem, candidate o_1 has the highest tendency to be selected. Therefore, selecting o_1 would be convincing. This result is different from the result by enforced application of the AHP.

As demonstrated in Example 4.2, the proposed approach solves the multi-criteria decision problem with inconsistent pairwise comparison matrices by giving a probability distribution where each probability shows the tendency to be selected as the final solution.

Finally, using the example given by Belton and Gear (1983) where the weights of criteria and scores of alternatives under each criterion are given exactly, we demonstrate that the order reversal discussed in Subection 3.3 does not occur in the proposed SSB-based model.

Example 4.3. Consider a multi-criteria decision problem with three alternatives A_1, A_2, A_3 and three criteria C_1, C_2, C_3 shown in Table 11a.

As the weights of criteria and scores of alternatives are given, this corresponds to a consistent case with $w_l = 1/3, l = 1, 2, 3$ and $v_i^{(l)}$, i = 1,2,3, l = 1,2,3 are given by

Table 11
Original and extended multi-criteria decision problem.

	C_1	C_2	C_3
	(a) The original problem		
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1
weight	1/3	1/3	1/3
	(b) The extended problem		
A_1	1	9	8
A_2	9	1	9
A_3	1	1	1
A_4	9	1	9
weight	1/3	1/3	1/3

$$\begin{aligned}
&v_1^{(1)} = \frac{1}{1+9+1} = \frac{1}{11},\ v_2^{(1)} = \frac{9}{1+9+1} = \frac{9}{11},\ v_3^{(1)} = \frac{1}{1+9+1} = \frac{1}{11},\\
&v_1^{(2)} = \frac{9}{9+1+1} = \frac{9}{11},\ v_2^{(2)} = \frac{1}{9+1+1} = \frac{1}{11},\ v_3^{(2)} = \frac{1}{9+1+1} = \frac{1}{11},\\
&v_1^{(3)} = \frac{8}{8+9+1} = \frac{4}{9},\ v_2^{(3)} = \frac{9}{8+9+1} = \frac{1}{2},\ v_3^{(3)} = \frac{1}{8+9+1} = \frac{1}{18}.
\end{aligned} \tag{18}$$

Based on the AHP procedure, we obtain the holistic evaluation scores of alternatives A_1, A_2 and A_3 as $134/297, 31/66$ and $47/594$, respectively. Namely the holistic scores of alternatives A_1, A_2 and A_3 are approximately, $0.45, 0.47$ and 0.08, respectively. Therefore, the three alternatives are ranked as $A_2 \succ A_1 \succ A_3$. However, if a new alternative A_4 is evaluated same as alternative A_2, i.e., the multi-criteria problem shown in Table 11b.

In the case of Table 11b, normalized scores $\overline{v}_i^{(l)}$, i = 1,2,3,4, l = 1,2,3 are given by

$$\begin{aligned}
&\overline{v}_1^{(1)} = \frac{1}{1+9+1+9} = \frac{1}{20},\ \overline{v}_2^{(1)} = \frac{9}{1+9+1+9} = \frac{9}{20},\ \overline{v}_3^{(1)} = \frac{1}{1+9+1+9} = \frac{1}{20},\ \overline{v}_4^{(1)} = \frac{9}{1+9+1+9} = \frac{9}{20}\\
&\overline{v}_1^{(2)} = \frac{9}{9+1+1+1} = \frac{9}{12},\ \overline{v}_2^{(2)} = \frac{1}{9+1+1+1} = \frac{1}{12},\ \overline{v}_3^{(2)} = \frac{1}{9+1+1+1} = \frac{1}{12},\ \overline{v}_4^{(2)} = \frac{1}{9+1+1+1} = \frac{1}{12}\\
&\overline{v}_1^{(3)} = \frac{8}{8+9+1+9} = \frac{8}{27},\ \overline{v}_2^{(3)} = \frac{9}{8+9+1+9} = \frac{1}{3},\ \overline{v}_3^{(3)} = \frac{1}{8+9+1+9} = \frac{1}{27},\ \overline{v}_4^{(3)} = \frac{9}{8+9+1+9} = \frac{1}{3}.
\end{aligned} \tag{19}$$

Then, by the AHP procedure, the holistic evaluation scores of alternatives A_1, A_2, A_3 and A_4 are obtained as $148/405, 13/45, 23/405$ and $13/45$ which are approximately $0.37, 0.29, 0.06$ and 0.29, respectively. This implies $A_1 \succ A_2 \sim A_4 \succ A_3$. The order between A_1 and A_2 are reversed in the rankings without A_4 and with A_4.

On the other hand, we have

$$\begin{aligned}
&\prod_{l=1,2,3} v_1^{(l)^{w_l}} = \sqrt[3]{\frac{1\cdot 9\cdot 4}{11\cdot 11\cdot 9}} = \sqrt[3]{\frac{4}{121}} \approx 0.32,\\
&\prod_{l=1,2,3} v_2^{(l)^{w_l}} = \sqrt[3]{\frac{9\cdot 1\cdot 1}{11\cdot 11\cdot 2}} = \sqrt[3]{\frac{9}{242}} \approx 0.33,\\
&\prod_{l=1,2,3} v_3^{(l)^{w_l}} = \sqrt[3]{\frac{1\cdot 1\cdot 1}{11\cdot 11\cdot 18}} = \sqrt[3]{\frac{1}{2178}} \approx 0.08.
\end{aligned} \tag{20}$$

Therefore, we obtain $j^* = 2$ (the index j of the alternative maximizing $\prod_l v_j^{(l)^{w_l}}$) in the original multi-criteria decision problem. Moreover, we have

$$\begin{aligned}
&\prod_{l=1,2,3} \overline{v}_1^{(l)^{w_l}} = \sqrt[3]{\frac{1\cdot 9\cdot 8}{20\cdot 12\cdot 27}} = \sqrt[3]{\frac{1}{90}} \approx 0.22,\\
&\prod_{l=1,2,3} \overline{v}_2^{(l)^{w_l}} = \sqrt[3]{\frac{9\cdot 1\cdot 1}{20\cdot 12\cdot 3}} = \sqrt[3]{\frac{1}{80}} \approx 0.23,\\
&\prod_{l=1,2,3} \overline{v}_3^{(l)^{w_l}} = \sqrt[3]{\frac{1\cdot 1\cdot 1}{20\cdot 12\cdot 27}} = \sqrt[3]{\frac{1}{6480}} \approx 0.05,\\
&\prod_{l=1,2,3} \overline{v}_3^{(l)^{w_l}} = \sqrt[3]{\frac{9\cdot 1\cdot 1}{20\cdot 12\cdot 3}} = \sqrt[3]{\frac{1}{80}} \approx 0.23.
\end{aligned} \tag{21}$$

Then, we obtain $k^* = 2$ or 4 (the index k of the alternative maximizing $\prod_l \overline{v}_k^{(l)^{w_l}}$) in the extended multi-criteria decision problem by adding alternative A_4. As we obtain $A_2 \sim A_4 \succ A_1 \succ A_3$, the order among A_1, A_2 and A_3 does not change between without A_4 and with A_4.

4.2. Simulation Analysis

To show the effectiveness of the proposed method, we performed 15.000 of simulated experiments. Since the preferential matrix $\boldsymbol{A}$ is treated in the same way as in the AHP, we kept $\boldsymbol{A} \in \mathscr{R}(5)$ fixed. Yet we did two rounds of simulations to eliminate the influence of such a choice. First we used matrix $\boldsymbol{A}$ from the Industrial application below, see Table 15, and then we repeated simulations with $\boldsymbol{A}$ being the matrix of ones. As there were no qualitative differences in the obtained results, we shall present them together. Further, given the dimension of matrix $\boldsymbol{A}$, it was necessary to generate preferential matrices $\boldsymbol{B}^{(k)}, k = 1, \dots, 5$ for each experiment. Each matrix $\boldsymbol{B}^{(k)}$ was repeatedly randomly chosen from $\mathscr{R}(4)$ until the consistency ratio $CR(B^{(k)})$ was in the preset threshold interval. Next, the best options were identified using both the AHP as well as the newly designed procedure. By progressively setting different consistency thresholds CR we thus obtained results reported in Table 12.

Typically, both methods found the same best option. The number of different outcomes increases when limits for CR threshold increase; i.e. when pairwise comparisons are less consistent. Indeed, for acceptably consistent matrices with $CR \leqslant 0.1$, we have observed divergence of the best options determined by the two methods in 18.5% of the simulated cases. This percentage increases to 33.1% given the threshold for CR is

Table 12
Overview of the results from the simulation analysis.

CR limits	divergent results (%)	mean of differences
$0 \leqslant CR \leqslant 0.1$	18.5%	0.0354
$0.1 < CR \leqslant 0.2$	21.6%	0.0396
$0.2 < CR \leqslant 0.3$	22.3%	0.0375
$0.3 < CR \leqslant 0.4$	22.9%	0.0421
$0.4 < CR \leqslant 0.5$	25.0%	0.0422
$0.5 < CR \leqslant 0.6$	28.0%	0.0458
$0.6 < CR \leqslant 0.7$	28.1%	0.0444
$0.7 < CR \leqslant 0.8$	27.1%	0.0483
$0.8 < CR \leqslant 0.9$	31.3%	0.0490
$0.9 < CR \leqslant 1$	33.1%	0.0480

between 0.9 and 1. For the divergent cases, we measured the difference of the two options by the difference of the corresponding elements of (the AHP based) weight vector z; the means of such differences are reported in the table for each considered range of CR. Thus obtained differences confirm to be quantitatively almost insignificant in the vast majority of the experiments, especially for lower values of CR, and so any of these two best options is equally efficient with respect to the considered criteria. From the above presented results we can conclude that the methodological approach based on the SSB is a valid alternative to the AHP for consistent judgments. Moreover, it may be used also in the presence of inconsistent judgments when the traditional AHP loses its scientific validity.

4.3. Industrial Application

The proposed case study presents a real-world decision-making problem whose main goal consists in optimising the number of stocked materials in industrial warehouses. In particular, we aim to establish which plan of storage management should be implemented for an Italian manufacturing enterprise operating in the alimentary sector. The decision has to be made among four ($n = 4$) different options, each one expressing a different planning possibility in terms of numerical quantities of stock materials. Items to be stored are essentially packaging materials (PMs) according to the production of finite products (FPs) to be led based on the received orders. In detail, the planning involves three types of FP produced by the company (FP_1, FP_2, FP_3) and six types of related packaging materials stored in the industrial warehouse (PM_1, PM_2, PM_3, PM_4, PM_5, PM_6). Four stock quantity plans (o_1, o_2, o_3, o_4), synthesised in Table 13, have been elaborated by the management of the company and represent the options of the decision-making problem currently under evaluation for establishing short time logistic strategies. As we can observe, data of Table 13 refer to the expected quantities of FPs to be produced in the first trimester of the next year and contemplate various quantities of PMs to face possible demand changes. In particular, quantities of FPs are the same for all the four plans over a given month, and quantities of PMs (highlighted in bold) vary based on the FPs to be packed, from a minimum to a maximum stockpile level. These levels have been determined by taking into account the storage capacity and the average performance indices from historical data.

We search for a solution, among the set of considered options, that represents the best trade-off according to various aspects (or criteria) to support the decision-making process of the company and to have a positive impact on its whole level of performance.

To such an aim, the chosen solution will have to simultaneously optimise the following five aspects ($m = 5$), herein considered as evaluation criteria: c_1 supplies management, c_2 relations with clients, c_3 order risk, c_4 packaging time, c_5 storage management. Let us briefly explain the meaning of the criteria: the first criterion c_1 refers to cost efficiency as well as relationships with suppliers above all in terms of communication quality and flexibility in managing possible sudden changes during the process of orders management. The second and the third criteria, namely c_2 and c_3, respectively, refer to the flexibility of customers for product delivery times and the presence of penalties and/or extra costs due to possible delays in delivery. Lastly, the fourth and fifth criteria refer to processing times for packaging and the costs related to the process of stock management as well as security aspects, respectively.

A decision-making group made of four equally weight stakeholders has been involved to get the vector of criteria weights by means of the AHP technique. The four decision-makers (DMs) have been involved on the basis of their experience and complementary points of view, to achieve as accurate knowledge about the problem under analysis as possible. Experts' roles are the following: DM_1 general manager, DM_2

Table 13
Stock quantity plans.

Month	Type	Plan o_1	Plan o_2	Plan o_3	Plan o_4
Jan. 2021	FP_1	1200	1200	1200	1200
	FP_2	900	900	900	900
	FP_3	800	800	800	800
	PM_1	**1400**	**1700**	**2000**	**2400**
	PM_2	**2100**	**2400**	**2700**	**3100**
	PM_3	**1100**	**1400**	**1700**	**2100**
	PM_4	**1200**	**1500**	**1800**	**2200**
	PM_5	**1000**	**1300**	**1600**	**2000**
	PM_6	**850**	**1150**	**1450**	**1850**
Feb. 2021	FP_1	1000	1000	1000	1000
	FP_2	1000	1000	1000	1000
	FP_3	900	900	900	900
	PM_1	**1200**	**1500**	**1800**	**2200**
	PM_2	**1600**	**1900**	**2200**	**2600**
	PM_3	**1200**	**1500**	**1800**	**2200**
	PM_4	**1600**	**1900**	**2200**	**2600**
	PM_5	**1100**	**1400**	**1700**	**2100**
	PM_6	**1200**	**1500**	**1800**	**2200**
Mar. 2021	FP_1	1100	1100	1100	1100
	FP_2	900	900	900	900
	FP_3	800	800	800	800
	PM_1	**1300**	**1600**	**1900**	**2300**
	PM_2	**1900**	**2200**	**2500**	**2900**
	PM_3	**1100**	**1400**	**1700**	**2100**
	PM_4	**1200**	**1500**	**1800**	**2200**
	PM_5	**1000**	**1300**	**1600**	**2000**
	PM_6	**850**	**1150**	**1450**	**1850**

Table 14
Criteria evaluations issued by the decision makers.

(a) DM_1: general manager

DM_1	c_1	c_2	c_3	c_4	c_5	CR
c_1	1	1	$\frac{1}{3}$	2	$\frac{1}{3}$	
c_2	1	1	$\frac{1}{3}$	2	$\frac{1}{3}$	
c_3	3	3	1	2	1	0.0445
c_4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	
c_5	3	3	1	2	1	

(b) DM_2: logistic manager

DM_2	c_1	c_2	c_3	c_4	c_5	CR
c_1	1	2	4	2	$\frac{1}{3}$	
c_2	$\frac{1}{2}$	1	1	$\frac{1}{3}$	$\frac{1}{5}$	
c_3	$\frac{1}{4}$	1	1	$\frac{1}{3}$	$\frac{1}{7}$	0.0291
c_4	$\frac{1}{2}$	3	3	1	$\frac{1}{3}$	
c_5	3	5	7	3	1	

(c) DM_3: company consultant

DM_3	c_1	c_2	c_3	c_4	c_5	CR
c_1	1	1	$\frac{1}{3}$	1	$\frac{1}{2}$	
c_2	1	1	$\frac{1}{3}$	1	$\frac{1}{2}$	
c_3	3	3	1	2	$\frac{1}{2}$	0.0299
c_4	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	
c_5	2	2	2	2	1	

(d) DM_4: storage responsible (operator)

DM_4	c_1	c_2	c_3	c_4	c_5	CR
c_1	1	1	4	$\frac{1}{3}$	$\frac{1}{2}$	
c_2	1	1	3	$\frac{1}{3}$	$\frac{1}{2}$	
c_3	$\frac{1}{4}$	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{4}$	0.0594
c_4	3	3	3	1	$\frac{1}{2}$	
c_5	2	2	4	2	1	

Table 15
Aggregated PCM A and vector of criteria weights $\mathbf{w}$.

A	c_1	c_2	c_3	c_4	c_5	$\mathbf{w}$
c_1	1	1.189	1.155	1.075	0.408	16.59%
c_2	0.841	1	0.760	0.687	0.359	12.71%
c_3	0.866	1.316	1	0.816	0.366	14.79%
c_4	0.931	1.456	1.225	1	0.452	17.32%
c_5	2.449	2.783	2.736	2.213	1	38.59%

logistic manager, DM_3 company consultant, DM_4 storage responsible (operator). Each DM was asked to fill in a PCM by pairwise comparing evaluation criteria with respect to the main decision-making goal. The four PCMs collecting input evaluations are reported in Table 14. The last columns of matrices report the related consistency ratio and, in all the cases, we have $CR \leqslant 0.1$, which confirms the consistency of judgments. Decision-makers' opinions have to be successively aggregated to produce the final consensus priority vector.

The aggregation of individual judgments is herein applied as an aggregation procedure for obtaining a group priority vector supporting the decision-making process. The individual comparison matrices are merged into a single PCM, reported in Table 15 so that the group can be treated as a new individual. The aggregated PCM has been obtained employing the weighted geometrical mean, see Eq. (4), by assuming a weight equal to 0.25 associated with each stakeholder since, as already mentioned, decision-makers have been considered as having the same mutual importance. The Perron vector $\mathbf{w}$ expressing the mutual importance of evaluation criteria, obtained via the power method Peretti, 2014, is also given in Table 15.

Table 16 presents the options' evaluations related to the five considered criteria. The last columns give the values of consistency ratios CR. In particular, we can observe that judgments' consistency is verified just in the first and in the second cases since all the remaining CR values surpass the threshold of 0.1. The list of vectors $\mathbf{v}^{(k)}$ derived from matrices in Table 16 is in Table 17a.

Having matrix $\boldsymbol{P}$, the pair-wise degree of preference is

$$\boldsymbol{\pi} = [1.092, 1, 2.898, 3.046]^{\mathrm{T}}.$$

As emerged from $\boldsymbol{\zeta} = [0, 1, 0, 0]^{\mathrm{T}}$, plan o_2 seems to be the most suitable candidate to sort out the decision-making problem discussed in the proposed case study. The selected solution is expected to be more expensive with respect to the condition of minimum cost represented by plan o_1, the last one characterised by the minimum amount of stocked materials. However, one has to observe that the implementation of plan o_2 is far less expensive with respect to the remaining plans o_3 and o_4. Moreover, plan o_2 guarantees acceptable values of fire load as well as acceptable order risk evaluations. These aspects can be translated into the possibility of reaching a good degree of flexibility for relationships with both clients and suppliers. We can conclude that plan o_2 (now scheduled as a part of the company strategy for the warehouse management) represents a good trade-off by satisfactorily matching all the criteria considered as relevant by the company. This result fully corresponds with a standard AHP solution $z = [0.248, 0.387, 0.178, 0.186]$, suggesting also $o_2 \succ o_1 \succ o_4 \succ o_3$. Our method, however, represents an alternative to traditional decision-making approaches that is particularly valuable for real-life situations where transitivity of judgements cannot be granted. On the one hand, the AHP claims that inconsistent solutions are not reliable according to the mathematical point of view. On the other hand, our proposal bypasses this perspective by simultaneously guaranteeing mathematical accuracy. By encouraging the application of a valid methodology to practical engineering problems we pursue the integration between the domains of scientific research and industry. This can be fundamental for improving the performance of core industrial processes, as the analysed logistic problem of storage and orders management.

Table 16
Evaluation of options with respect to criteria and CR values.

	o_1	o_2	o_3	o_4	CR
		(a) $B^{(1)}$			
o_1	1	1	2	3	
o_2	1	1	3	3	0.0077
o_3	$\frac{1}{2}$	$\frac{1}{3}$	1	1	
o_4	$\frac{1}{3}$	$\frac{1}{3}$	1	1	
		(b) $B^{(2)}$			
o_1	1	$\frac{1}{2}$	1	$\frac{1}{3}$	
o_2	2	1	1	1	0.0442
o_3	1		1	1	
o_4	3	1	1	1	
		(c) $B^{(3)}$			
o_1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	
o_2	1	1	1	1	0.1159
o_3	3	1	1	$\frac{1}{3}$	
o_4	3	1	3	1	
		(d) $B^{(4)}$			
o_1	1	1	1	1	
o_2	1	1	3	4	0.1100
o_3	1	$\frac{1}{3}$	1	$\frac{1}{2}$	
o_4	1	$\frac{1}{4}$	2	1	
		(e) $B^{(5)}$			
o_1	1	1	2	2	
o_2	1	1	6	6	0.1515
o_3	$\frac{1}{2}$	$\frac{1}{6}$	1	4	
o_4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{4}$	1	

Table 17
Case study.

(a) Matrix $\boldsymbol{V}$ of local priorities $\mathbf{v}^{(k)}$

	c_1	c_2	c_3	c_4	c_5
o_1	0.347	0.159	0.141	0.233	0.283
o_2	0.383	0.280	0.234	0.420	0.468
o_3	0.142	0.245	0.234	0.150	0.163
o_4	0.128	0.316	0.391	0.197	0.086

(b) $\boldsymbol{P}$

	o_1	o_2	o_3	o_4
o_1	1	0.916	1.246	1.159
o_2	1.092	1	2.898	3.046
o_3	0.803	0.345	1	1.287
o_4	0.863	0.328	0.777	1

5. Conclusion

A novel approach for solving complex real-world decision-making problems has been proposed, that is based on the preliminary collection of judgements of pairwise comparisons from selected stakeholders. Being elicited by human decision-makers, such judgments are often inconsistent. If minimal consistency requirements are not met in the AHP, experts are requested to revise - and possibly distort - their original judgments. We solve this issue by using the SSB representation of preferences. In this way a mathematically well founded method is obtained with no consistency assumptions whatsoever. We established basic properties of this method: its relation to the AHP in the consistent case as well as the fact that contrary to the AHP, this method does not exhibit the so-called order reversal when adding/deleting alternatives. In the simulation analysis we have shown that for consistent data our method typically yields the same best option as the AHP, see Table 12. These results validate that our approach provides a rigorous calculation independent on consistency limits. Thus analysts can easily contemplate real-life situations in which preferences expressed from expert(s) may be naturally inconsistent. Eventually, the practical applicability of our method has been proved by means of a case study related to the complex industrial reality. In particular, the field of supply chain management in the alimentary industry has been explored and a problem of storage management has been solved in a structured way for an Italian manufacturing company based on (inconsistent) opinions provided by a decision-making group. As demonstrated, the presented research can be helpful by conjugating the reliability of a scientific decision-making framework with the subjective experience of decision-makers reflecting practical reality. Indeed, integrating the SSB within the AHP framework has the advantage to propose a final decision on the basis of real experience without forcing experts to reconsider their opinions for potential AHP reiterations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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