On the equivalence of the three-link to the almost linear form*

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***Abstract*—The almost linear form that is state and feedback equivalent to the dynamics of the so-called three-link (aka biped with torso) is derived and proved here. This result is then applied to the walking design with downward torso movement imitating balancing role of a hand. This motivates a challenging idea: the balancing role of hands in two-dimensional walking consists in synchronizing the hand angle with the hip angle in such a way that the resulting restricted dynamics is exact feedback linearizable. Results are demonstrated by the simulations of a single step including walking animations.**

***Index Terms*—mechanical systems, feedback linearization.**

I. Introduction

Planar underactuated walking, being a part of a more general study of the so-called underactuated mechanical systems [1], has been broadly and deeply studied during several decades. Please refer to [2], [3] and references within there for a brief and relatively recent picture, or [4] for systematic description of the area including one of the main tools of the walking design - the so-called **virtual holonomic constraints (VHC)**. The VHC used in [4] always include some absolute orientation angle and their number is equal to the number of independent actuators. In such a way, constrained dynamics is uncontrolled and VHC should be designed in such a way that constrained dynamics (called hybrid zero dynamics) contains hybrid cyclic stable trajectory.

In contrast to that, the so-called **collocated VHC (CVHC)** were studied in [5], [6] and further used in [7], [3] and some other references quoted there. CVHC include directly actuated angles only and their number is typically equal to the actuators' number minus one. Due to that, the CVHC restricted dynamics conserves the unactuated cyclic variable property, it has four states and one input and it is state and feedback equivalent to almost linear normal form presented by Olfati-Saber in [8] for some systems with two degrees of freedom and one actuator. Recall, that the mechanical system variable (*i.e.* the component of generalized coordinates) is called **cyclic** if the respective kinetic energy does not depend on that variable. The only nonlinearity in the Olfati-Saber normal form is due to $(1,1)$ entry of the inertia (aka mass) matrix. Natural idea is therefore to consider designing CVHC in such a way, that the mentioned entry is constant on the restricted submanifold thereby converting the respective restricted dynamics into an exact feedback linearizable one. Such an idea was first introduced in [7] and used in [3] to design the multi-step walking of the so-called **three-link (TL)** (aka biped with torso, or Compass Gait Walker with torso). These works, nevertheless, did use a certain two step procedure, first, enforcing the constraints using the torso torque, then controlling the restricted system using the hip torque simultaneously compensating any effect of the torso torque. It resulted in series of feedbacks and their regularity was difficult to asses.

The theoretical contribution of this paper is the derivation and the full proof of the state and feedback transformation taking TL to an almost linear form that becomes completely linear along conveniently designed CVHC. Moreover, CVHC in new coordinates becomes flat and dynamics of the deviation from that CVHC is the chain of two integrators. Restricted dynamics is then state and feedback equivalent to the chain of four integrators. Both chains are controlled by two independent virtual inputs decoupled by the feedback transformation from real inputs (torques) and invertibility of this feedback is explicitly characterized. Further novelty here is the design of the convenient CVHC around the **downward** position of the torso, while [7], [3] considered torso movement around the **upward** position only, see Fig. 1. Downward torso position provides more rich collection of the desired CVHC but it triggered yet another original novelty, due to its complex movement, a general type of CVHC is needed where neither of mutually constrained coordinates has the leading role.

These results are demonstrated by a single step walking desing and respective simulations. Note, that for the walking design, the chain of two integrators has to be stabilized to zero state, nevertheless, the chain of four integrators has to be stably steered from some initial state to some terminal state. Steering the integrators chain seems to be theoretically well understood and straightforward, yet, it is quite complex when practically viable solutions are needed, [9]–[11]. Respective design in this paper is therefore rather straightforward and may results *e.g.* in unrealistic torques, yet, it demonstrates a potential of the paper's theoretical results if a more sophisticated steering design is applied.

The downward torso may actually imitate the role of the

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hands during the walking. Unlike the design of VHC *e.g.* for the knees it is not so clear what should do the torso and/or hands during the step. In this context, the exact linearizable restricted dynamics appeared to be a challenging option.

The rest of the paper is organized as follows. Some definitions and known, or straightforward, results are repeated in the next section. Section III presents the main result, Section IV discusses the walking design including some simulations, while the final section presents some research outlooks.

Notations. For a smooth function $\phi(q)$, $q \in \mathbb{R}^n$, differential $d\phi = \partial\phi(q)/\partial q$ is the row vector of the partial derivatives, conveniently expressed by the well-known "nabla" operator $\nabla := [\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_1}]$ as $\nabla\phi$. In the same vein, the Hessian of ϕ can be expressed as $\nabla^\top\nabla\phi$. Finally, $0_{m\times p}$, $m,p \in \mathbb{Z}$, stands for $(m\times p)$ zero matrix, I_m for $(m\times m)$ the identity matrix, COM stands for the centre of mass, MI for the moment of inertia and DOF for the degree(s) of freedom.

II. Definitions and Preliminary Results

Consider the underactuated [1] Lagrangian system (LS):

$$\frac{d}{dt}\left[\frac{\partial\mathscr{L}}{\partial\dot q}\right]^\top - \left[\frac{\partial\mathscr{L}}{\partial q}\right]^\top = [0_k, u_{k+1}, \ldots, u_n]^\top, \qquad (1)$$

$$\mathscr{L}(q,\dot q) = K(q,\dot q) - V(q), \quad K(q,\dot q) = \frac{1}{2}\dot q^T D(q)\dot q, \qquad (2)$$

having n DOF, namely $q = (q_1, \ldots, q_n)^\top$, $\dot q = (\dot q_1, \ldots, \dot q_n)^\top$ are the generalized coordinates and velocities, $D(q) = D(q)^\top > 0$ is the inertia (aka mass) matrix, while K, V are the system kinetic and potential energy. Coordinate q_i is called **cyclic variable** if D does not depend on q_i. Integer $k \geq 1$ is called as the degree of the underactuation, while $u_{k+1}, \ldots, u_k$ are the input actuators (control inputs). The coordinates $q_{k+1}, \ldots, q_n$ are called **(directly) actuated** while $q_1, \ldots, q_k$ **unactuated**. The equations (1), (2) give

$$D(q)\ddot q + C(q,\dot q)\dot q + G(q) = [0_k, u_{k+1}, \ldots, u_n]^\top, \qquad (3)$$

$$G^\top = \frac{\partial V(q)}{\partial q}, \; C(q,\dot q)\dot q = \left[\sum_{i=1}^{n}\frac{\partial D(q)}{\partial q_i}\dot q_i\right]\dot q - C_s(q,\dot q), \qquad (4)$$

$$C_s^\top(q,\dot q) = [C_{s1}(q,\dot q), \ldots, C_{sn}(q,\dot q)] = \frac{\partial K(q,\dot q)}{\partial q}, \qquad (5)$$

$$C_{si}(q,\dot q) = \frac{1}{2}\dot q^\top\left[\frac{\partial D(q)}{\partial q_i}\right]\dot q, \; i = 1, \ldots, n. \qquad (6)$$

Here, $G(q)$ is the gravity vector while the Coriolis terms $C(q,\dot q)\dot q$ are expressed in (4)-(6) in a less usual way without introducing Coriolis matrix like *e.g.* in [12].

The following lemma establishes the important property of the unactuated cyclic variable, *e.g.* used by Olfati-Saber to derive normal forms of some underactuated systems [8].

Lemma 1: Consider (1)-(6) and its generalized momenta

$$\sigma_i := \frac{\partial\mathscr{L}}{\partial\dot q_i} = \frac{1}{2}\frac{\partial\dot q^\top D(q)\dot q}{\partial\dot q_i} = [0_{i-1}, 1, 0_{n-i}]D(q)\dot q,$$

$i \in \{1, \ldots n\}$. If q_i is unactuated cyclic then $\dot\sigma_i = -G_i(q)$.

Proof: Since (3)-(6) stem from (1), (2) it holds that

$$0 = \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot q_i} - \frac{\partial\mathscr{L}}{\partial q_i} = \dot\sigma_i - \frac{\partial[K(q,\dot q) - V(q)]}{\partial q_i} = \dot\sigma_i + G_i(q).$$

Here, the first equality is by q_i being unactuated (*i.e.* $i \in \{1, \ldots k\}$), while the last one is by (4) and q_i being cyclic (*i.e.* $\partial K(q,\dot q)(\partial q_i) \equiv 0$). ■

Planar **"three-link" (TL)** depicted in Fig. 1 is the underactuation degree 1 underactuated mechanical system having 3 degrees of freedom and 2 actuators. It mimics the pair of legs without knees and torso mounted at their hips. The i-th link ($i = 1,2,3$) is actually a thin homogeneous rod of mass μ_i with attached point mass M_i. It is equivalently modelled by a virtual one-dimensional mass-less rigid segment carrying the overall mass $m_i = \mu_i + M_i$ at its COM located at the black bold point. The virtual mass-less link moment of inertia I_i with respect to COM can be computed from l_i, μ_i, M_i.

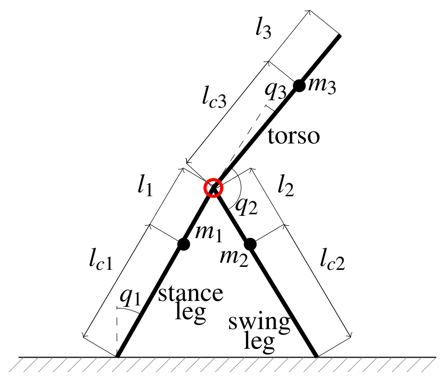

Fig. 1. **The "three-link" with upward torso position.**

Coordinates q_1, q_2, q_3 are the angles shown in Fig. 1 where the red circle locates two independent actuators providing torques u_2, u_3 acting on directly actuated angles q_2, q_3, respectively. Angle q_1 at the pivot point is the unactuated one. Computing properly K, V, equations (1), (2) give

$$D(q)\ddot q + C(q,\dot q)\dot q + G(q) = \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}, \quad q := \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \qquad (7)$$

$$D = [d_{ij}], \; i,j = 1,2,3, \; D^\top = D > 0, \; G = [G_1, G_2, G_3]^\top,$$

$$\begin{aligned}
&d_{11} = I_1 + I_2 + I_3 + l_1^2 m_2 + l_1^2 m_3 + l_{c1}^2 m_1 + \\
&l_{c2}^2 m_2 + l_{c3}^2 m_3 + 2l_1 l_{c2} m_2\cos q_2 + 2l_1 l_{c3} m_3 \cos q_3, \\
&d_{12} = m_2 l_{c2}^2 + l_1 m_2 \cos q_2 l_{c2} + I_2 \\
&d_{13} = m_3 l_{c3}^2 + l_1 m_3 \cos q_3 l_{c3} + I_3, \; d_{23} = 0, \\
&d_{22} = m_2 l_{c2}^2 + I_2, \; d_{33}(q_2,q_3) = m_3 l_{c3}^2 + I_3, \\
&G_1 = -g\,[l_1 m_2 \sin q_1 + l_1 m_3 \sin q_1 + l_{c1} m_1 \sin q_1 + \\
&l_{c2} m_2 \sin(q_1+q_2) + l_{c3} m_3 \sin(q_1+q_3)], \quad G_2 = \\
&-g l_{c2} m_2 \sin(q_1+q_2), \quad G_3 = -g l_{c3} m_3 \sin(q_1+q_3).
\end{aligned} \qquad (8)$$

Coriolis terms $C(q,\dot q)\dot q$ are straightforwardly determined by general expresion (6) taking $n = 3$ and $D(q)$ in (7), (8).

Later on, the dependence of $d_{11}(q_2,q_3)$ on q_2,q_3 will be thoroughly studied using a more convenient expression

$$d_{11}=\sum_{i=1}^{3}I_i+l_1^2m_2+l_1^2m_3+l_{c1}^2m_1+l_{c2}^2m_2+l_{c3}^2m_3 \\ +2l_1l_{c3}m_3\left[\beta\cos q_2+\cos q_3\right],\ \beta:=l_{c2}m_2/(l_{c3}m_3), \tag{9}$$

where $\beta\in\mathbb{R}$ will be called the **balancing factor**.

Definition 2: VHC for the system (1-7) are given by

$$\varphi_i(q)=0,\ [\mathrm{d}\varphi_i(q)]\dot q=0,\quad i=1,\dots,l, \tag{10}$$

where $\varphi_1,\dots,\varphi_l$ are smooth functions of the generalized coordinates having $\forall q\in\mathbb{R}^n$ satisfying (10) linearly independent differentials $\mathrm{d}\varphi_i(q), i=1,\dots,l$. The VHC are called **global** if the functions $\varphi_i(q), i=1,\dots,l$ in (10) can be completed to a diffeomorphism of $\mathbb{R}^n$. Furthermore, the VHC are called locally regular for (1-7) at some q^0, if it holds

$$\operatorname{rank}\left[\frac{\partial\varphi_1}{\partial q}^{\top},\dots\frac{\partial\varphi_l}{\partial q}^{\top}\right]^{\top}(q^0)D^{-1}(q^0)\begin{bmatrix}0_{k\times n}\\ I_{n-k}\end{bmatrix}=l. \tag{11}$$

The global VHC are called globally regular on some subset of $\mathbb{R}^n$ if they are locally regular at each its point. The VHC are called **flat** if there $\exists l$ mutually distinct integers $j_1,\dots,j_l\in\{1,\dots,n\}$ with $\varphi_1\equiv q_{j_1},\dots,\varphi_l\equiv q_{j_l}$. The VHC (10) are called **collocated VHC (CVHC)** if they depend only on directly actuated coordinates and the respective velocities.

III. Main result

Theorem 3: Let $q^0\in\mathbb{R}^3$ be such that $\det\mathscr{D}(q^0)\neq 0$,

$$\mathscr{D}(q)=\left[\begin{array}{c|c}\frac{\partial G_1}{\partial q_2}-\frac{d_{12}}{d_{11}}\frac{\partial G_1}{\partial q_1} & \frac{\partial G_1}{\partial q_3}-\frac{d_{13}}{d_{11}}\frac{\partial G_1}{\partial q_1}\\ \beta\sin q_2 & \sin q_3\end{array}\right]. \tag{12}$$

Then there exists a neighbourhood of q^0 denoted as $\mathscr{N}_{q^0}$ such that the mapping $(q_1,q_2,q_3,\dot q_1,\dot q_2,\dot q_3,u_2,u_3)^{\top}\in\mathbb{R}^8\mapsto(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,w_2,w_3)^{\top}\in\mathbb{R}^8$ given by

$$\begin{aligned}
\xi_1 &:= d_{11}(q_2^0,q_3^0)q_1+(m_2l_{c2}^2+I_2)q_2+l_1l_{c2}m_2\sin q_2\\
&\qquad +(m_3l_{c3}^2+I_3)q_3+l_1l_{c3}m_3\sin q_3,\\
\xi_2 &:= d_{11}(q_2,q_3)\dot q_1+d_{12}(q_2,q_3)\dot q_2+d_{13}(q_2,q_3)\dot q_3,\\
\xi_3 &= -G_1(q),\quad \xi_4:=-\nabla G_1(q)\dot q,\\
\xi_5 &:= \cos q_3+\beta\cos q_2-\cos q_3^0-\beta\cos q_2^0,\\
\xi_6 &:= -\dot q_3\sin q_3-\beta\dot q_2\sin q_2,\\
w_2 &:= -\nabla G_1D(q)^{-1}[[0,u_2,u_3]^{\top}-C(q,\dot q)\dot q-G(q)]\\
&\qquad -\dot q^{\top}\nabla^{\top}\nabla G_1(q)\dot q,\\
w_3 &:= \Big[0,-\beta\sin q_2,-\sin q_3\Big]D(q)^{-1}\Big[[0,u_2,u_3]^{\top}\\
&\qquad -C(q,\dot q)\dot q-G(q)\Big]-\dot q_3^2\cos q_3-\beta\dot q_2^2\cos q_2.
\end{aligned} \tag{13}$$

is smooth and one-to-one on the set $\mathscr{N}_{q^0}\times\mathbb{R}^5$ and transforms (7)-(8) into the following system

$$\dot\xi_1=\xi_2-2l_1l_{c3}m_3\dot q_1\xi_5,\ \dot\xi_2=\xi_3,\ \dot\xi_3=\xi_4,\ \dot\xi_4=w_2,\\ \dot\xi_5=\xi_6,\ \dot\xi_6=w_3. \tag{14}$$

Proof: First, check that (13) implies (14). Using (9) and definition of ξ_1,ξ_2,ξ_5 in (13) one has

$$\begin{aligned}
\dot\xi_1 &= d_{11}(q_2^0,q_3^0)\dot q_1+d_{12}(q_2,q_3)\dot q_2+d_{13}(q_2,q_3)\\
&= [d_{11}(q_2^0,q_3^0)-d_{11}(q_2,q_3)]\dot q_1+\xi_2\\
&= 2l_1l_{c3}m_3[\beta\cos q_2^0+\cos q_3^0-\beta\cos q_2-\cos q_3]\dot q_1\\
&\quad +\xi_2=\xi_2-2l_1l_{c3}m_3\dot q_1\xi_5,
\end{aligned}$$

which proves the first equality in (14). Next, to prove that $\dot\xi_2=\xi_3$, realize that $\xi_2=\sigma_1$ as in Lemma 1 and $\xi_3=-G_1(q)$ by definition of ξ_3 in (13), so that by Lemma 1 it holds $\dot\xi_2=\dot\sigma=-G_1(q)=\xi_3$. Equality $\dot\xi_3=\xi_4$ is straightforward due to definition of $\xi_4:=-\nabla G_1(q)\dot q$ in (13). Moreover

$$\dot\xi_4=-\nabla G_1\ddot q-\dot q^{\top}\nabla^{\top}\nabla G_1(q)\dot q$$

and by definition of w_2 in (13) and $\ddot q=D(q)^{-1}\left[[0,u_2,u_3]^{\top}-C(q,\dot q)\dot q-G(q)\right]$ obtained from (7) $\dot\xi_4=w_2$ holds. Finally, one has by definitions of ξ_5,ξ_6,w_3 in (13) and by (7)

$$\dot\xi_5=-\dot q_3\sin q_3-\beta\dot q_2\sin q_2=\xi_6,\\ \dot\xi_6=-\ddot q_3\sin q_3-\beta\ddot q_2\sin q_2-\dot q_3^2\cos q_3-\beta\dot q_2^2\cos q_2=w_3.$$

Secondly, to conclude the proof, one has to prove that (13) defines the mapping $(q_1,q_2,q_3,\dot q_1,\dot q_2,\dot q_3,u_2,u_3)^{\top}\in\mathbb{R}^8\mapsto(\xi_1,\xi_3,\xi_5,\xi_2,\xi_4,\xi_6,w_2,w_3)^{\top}\in\mathbb{R}^8$ which is smoothly invertible around a given q^0 and any $\dot q,u_2,u_3$. Its Jacobian is

$$\mathscr{J}(q^0,\dot q)=\begin{bmatrix}\overline{\mathscr{D}}(q^0) & 0_{3\times3} & 0_{3\times2}\\ * & \overline{\mathscr{D}}(q^0) & 0_{3\times2}\\ * & * & \widetilde{\mathscr{D}}(q^0)\end{bmatrix},\ \widetilde{\mathscr{D}}(q)= \tag{15}$$

$$\begin{bmatrix}\frac{\partial G_1}{\partial q_1} & \frac{\partial G_1}{\partial q_2} & \frac{\partial G_1}{\partial q_3}\\ 0 & \beta\sin q_2 & \sin q_3\end{bmatrix}D(q)^{-1}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}, \tag{16}$$

$$\overline{\mathscr{D}}(q^0)=\begin{bmatrix}d_{11}(q^0) & d_{12}(q^0) & d_{13}(q^0)\\ \frac{\partial G_1}{\partial q_1}(q^0) & \frac{\partial G_1}{\partial q_2}(q^0) & \frac{\partial G_1}{\partial q_3}(q^0)\\ 0 & \beta\sin q_2^0 & \sin q_3^0\end{bmatrix}. \tag{17}$$

Here, $\mathscr{J}$, $\overline{\mathscr{D}}$ and $\widetilde{\mathscr{D}}$ are (8×8), (3×3) and (2×2) matrices, respectively. Row elimination of $(2,1)$ entry of $\overline{\mathscr{D}}$ gives

$$\det\overline{\mathscr{D}}(q^0)=d_{11}(q^0)\det\mathscr{D}(q^0). \tag{18}$$

Let $\mathscr{S}$ be (2×2) Schur complement of (3×3) D in (7)

$$\mathscr{S}=\begin{bmatrix}d_{22}&d_{23}\\ d_{32}&d_{33}\end{bmatrix}-\frac{1}{d_{11}}\begin{bmatrix}d_{21}d_{12}&d_{21}d_{13}\\ d_{31}d_{12}&d_{31}d_{13}\end{bmatrix}, \tag{19}$$

$$D=D^{\top}>0\ \Rightarrow\ \mathscr{S}=\mathscr{S}^{\top}>0\ \Rightarrow\ \det\mathscr{S}>0. \tag{20}$$

Straightforward multiplication gives

$$\begin{bmatrix}d_{11}&d_{12}&d_{13}\\ d_{21}&d_{22}&d_{23}\\ d_{31}&d_{32}&d_{33}\end{bmatrix}\begin{bmatrix}-d_{11}^{-1}d_{12} & -d_{11}^{-1}d_{13}\\ 1&0\\ 0&1\end{bmatrix}=\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}\mathscr{S}$$

and therefore it holds (recall D in (7)) that

$$D^{-1}\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}=\begin{bmatrix}-d_{11}^{-1}d_{12} & -d_{11}^{-1}d_{13}\\ 1&0\\ 0&1\end{bmatrix}\mathscr{S}^{-1}. \tag{21}$$

Using (21) the definition of $\widetilde{\mathscr{D}}$ in (15), (16) gives

$$\widetilde{\mathscr{D}}(q) = \mathscr{D}(q)\mathscr{S}^{-1} \Rightarrow \det\widetilde{\mathscr{D}}(q) = \det\mathscr{D}(q)/\det\mathscr{S}. \quad (22)$$

Now, by (15), (18), (20), (22) and $d_{11}(q^0) > 0$ one has

$$\det\mathscr{J}(q^0,\dot{q}) = [d_{11}(q^0)]^2[\det\mathscr{D}(q^0)]^3/\det\mathscr{S} \neq 0$$

due to theorem assumption $\det\mathscr{D}(q^0) \neq 0$. Using Inverse Function Theorem [13] one concludes that the mapping (13) is a local diffeomorphism around the point in $\mathbb{R}^8$ given by q^0 considered in theorem formulation and any $\dot{q}, u_2, u_3$. ■

Remark 4: Note a theoretically remarkable, while practically quite a useful, feature that the local smooth invertibility of (13) depends only on the generalized coordinate q^0 while velocities and input values may be arbitrary. Moreover, respective regularity condition is reduced to $\det\mathscr{S}(q^0) \neq 0$ requiring to check (2×2) matrix only. Furthermore, (14) becomes completely linear for $\xi_5 = 0$ and ξ_5 has a simple linear dynamics with respect to virtual input w_3 and this input can be straightforwardly used to enforce the equality $\xi_5 = 0$ exponentially (or even in finite-time using [14]). The relation $\xi_5 = \xi_6 = 0$ actually defines general local CVHC as of Definition (2) where none of q_2, q_3 has a dominant role. The practically peculiar issue here is establishing that they are globally regular CVHC as well. Indeed, the basic assumption $\det\mathscr{S}(q^0) \neq 0$ of Theorem (3) is equivalent to those CVHC being locally regular at q^0 only. The global regularity is unrealistic, but for practical application regularity on the selected constrained set is sufficient.

IV. Walking design and simulations

A. Finding upward versus downward torso constraints

As already noted, downward torso case provides more rich collection of the desired CVHC. To be more specific, denote by q_1^0, q_2^0, q_3^0 and q_1^f, q_2^f, q_3^f the double support stance configuration at the beginning and the end of the step, respectively. By a simple triangularization it holds $q_3^0 = q_2^0/2 - \pi/2, q_2^0 \in (\pi, 2\pi)$, $q_3^f = q_2^f/2 - \pi/2, q_2^f \in (0,\pi)$ for the upward torso, while $q_3^0 = q_2^0/2 + \pi/2, q_2^0 \in (\pi, 2\pi)$, $q_3^f = q_2^f/2 + \pi/2, q_2^f \in (0,\pi)$ for the downward torso. Furthermore, $q_2 = \pi$ when the legs are passing by each other.

In such a way, the above desired collocated VHC is given by some level curve shown in Fig. 2 having intersection with blue (red) line for upward (downward) torso at some $q_2^0 \in (\pi, 2\pi)$ and some $q_2^f \in (0,\pi)$.

For the upward torso position $q_3 \in (-\pi/2, \pi/2)$, after fixing the balancing factor β, there is an unique level curve satisfying the above requirements, as illustrated by Fig. 2. Reason is that the blue line passes through the configuration $q_2 = \pi, q_3 = 0$ which is the **saddle** stationary point of the function $\cos q_3 + \beta\cos q_2$ and therefore there is a unique connected curve going from some $q_2^0 \in (\pi, 2\pi)$ to some $q_2^f \in (0,\pi)$. Changing the balancing factor β modifies the shapes of the level curves and their intersections with blue line change, yet there is unique one-to-one correspondence between β and $q_2^0 \in (\pi, 2\pi), q_2^f \in (0,\pi)$, *i.e.* for a fixed chosen β there is only one possible angle between legs at the initial double support stance position, and vice-verse (provided β is from some reasonable range). Due to that limited choice, only specific mechanical and geometric parameters, laboriously computed in [3] via numerical optimization procedure, were capable to provide a rather special multi-step, hybrid-cyclic walking-like trajectory.

For the downward torso position $q_3 \in (\pi/2, 3\pi/2)$ the configurations $q_2^0 \in (\pi, 2\pi), q_2^f \in (0,\pi)$, should be intersections of the red line and a level curve in Fig. 2. Red line passes through $q_3 = \pi, q_2 = \pi$ which is the local **minimum** stationary point of $\cos q_3 + \beta\cos q_2$, so there is continuum family of level curves circling that minimum. So, there is **always** a level curve passing through **any** $q_2^0 \in (\pi, 2\pi)$, $q_2^f \in (0,\pi)$ and the VHC design is much more flexible.

On the other hand, the unique and difficult to find CVHC for the upward torso can be always expressed as $q_2 = \phi(q_3)$ with ϕ being smooth and smoothly invertible. On the contrary, for the downward torso case, as it can be seen in Fig. 2, any of rich family of possible CVHC should stay in the general form $\varphi(q_2, q_3) = 0$, moreover, it is difficult to make it regular along all q_2, q_3 satisfying CVHC.

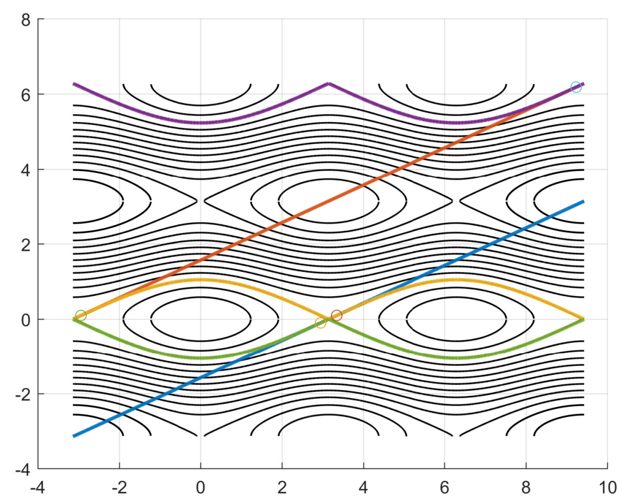

Fig. 2. Level curves (black, yellow, green) of the function $\cos q_3 + \beta\cos q_2$ plotted in (q_2, q_3) plane for β = 0.2506 used in [3]. Furthermore, the blue line is the plot of $(q_2, q_2/2 - \pi/2)$ while the red one of $(q_2, q_2/2 + \pi/2)$.

B. Sample single step simulations

To demonstrate both the potential and the peculiarities of the downward torso constrains used for the walking design, the simple design of the single step was performed using the chain of four integrators. It is the so-called pseudo-passive step where the virtual input $w_2 = 0$. This means that COM of the TL moves with constant velocity and the respective initial angular velocities are relatively easy to compute, while initial and terminal angles are given by selected double support configurations.

Two cases were simulated corresponding to different values of the balancing parameter β. All mechanical TL parameters are omitted for the space reason, besides, they are not so important to demonstrate the above mentioned pros and cons of the downward torso position. Both cases are in a sense dual each to other, one of them starts with swinging torso back and swing leg forward, while another one swing leg back and torso forward. Notably, neither of them reaches exactly the final double support position due to possible presence of the singularity where $\det\mathscr{D} = 0$. Red bold circles in Figs. 3 and 5 correspond to double support positions at the beginning and the end of the step, the blue line is the constrained set in (q_2, q_3)-plane along which TL is moving.

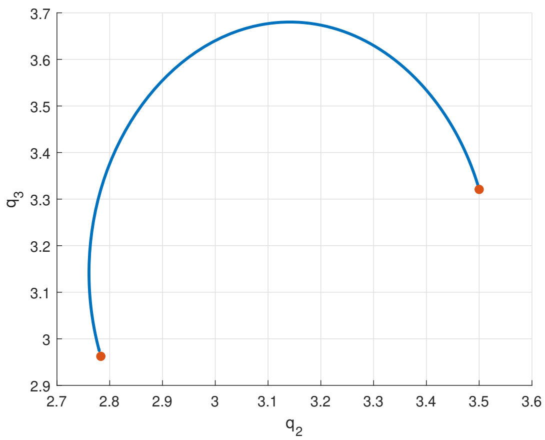

Fig. 3. The curves of (q_2, q_3) plane for β = 1.9757.

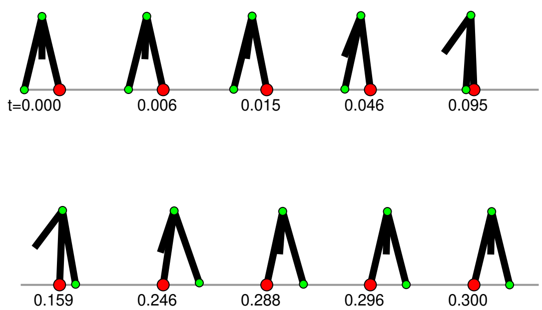

Fig. 4. Animation of the 3-link movement during one step for β = 1.9757.

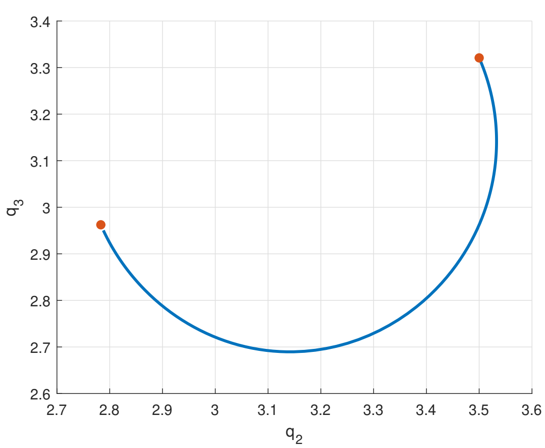

Fig. 5. The curves of (q_2, q_3) plane for β = 1.3286.

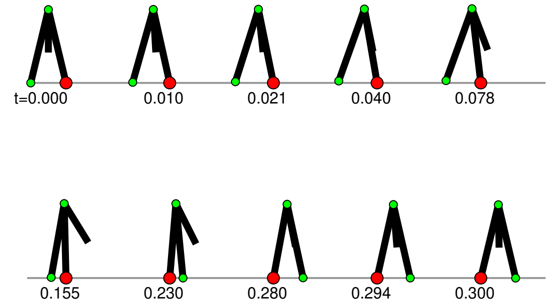

Fig. 6. Animation of the 3-link movement during one step for β = 1.3286.

V. Conclusions and Outlooks

The ongoing and future research is focused to considering a pair of downward torsa, so that mimicking the role of hands would be more adequate. Moreover, this seems to solve the singularity problem as the pair of "hands" need not to go along such a "curved" CVHC as in Figs. 3 and 5.

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