# Applications of Special Functions to Approximate Stochastic Bi-Homomorphisms and Stochastic Bi-Derivations in FB-Algebras and FC- $\diamond$-Algebras of the Matrix Type 

Zahra Eidinejad ${ }^{1,+(\mathbb{D}}$, Reza Saadati ${ }^{1, *,+\oplus}$, Radko Mesiar ${ }^{2,3, *,+(\mathbb{D})}$ and Pandora Raja ${ }^{4, \dagger}$<br>1 School of Mathematics, Iran University of Science and Technology, Tehran 13114-16846, Iran<br>2 Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 81005 Bratislava, Slovakia<br>3 Institute of Information Theory and Automation, The Czech Academy of Sciences, Pod Vodárenskou věží 4, 18208 Praha, Czech Republic<br>4 Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran 19839-69411, Iran<br>* Correspondence: rsaadati@iust.ac.ir (R.S.); mesiar@math.sk (R.M.)<br>$\dagger$ These authors contributed equally to this work.

Citation: Eidinejad, Z.; Saadati, R.; Mesiar, R.; Raja, P. Applications of Special Functions to Approximate Stochastic Bi-Homomorphisms and Stochastic Bi-Derivations in FB-Algebras and FC- $\diamond$-Algebras of the Matrix Type. Mathematics 2023, 11, 1329. https://doi.org/10.3390/ math11061329

Academic Editor: Salvatore Sessa
Received: 6 February 2023
Revised: 6 March 2023
Accepted: 7 March 2023
Published: 9 March 2023


Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/).


#### Abstract

We apply special functions and use the concept of the aggregation function to introduce a new class of fuzzy control functions, and based on this, we obtain the best approximation for the stochastic bi-homomorphisms and stochastic bi-derivations in FB-algebras and FC-৫-algebras of matrix type associated with the bi-additive random operator inequality.


Keywords: stochastic bi-derivation; C-৫-algebras; stochastic bi-homomorphism; FB-algebra; optimal stability; fuzzy inequality

MSC: 39B52; 46L05; 47B47; 47H10; 46L57; 39B62

## 1. Introduction

In 1940, Ulam raised the important question of the stability of group homomorphisms in his lecture. This question was stated as follows: Consider $\mathbb{Q}$ and $\mathbb{Q}^{\prime}$ as a group and a metric group with metric $\delta$. We consider the function $\mathscr{A}: \mathbb{Q} \rightarrow \mathbb{Q}^{\prime}$ in such a way that for every $\epsilon>0$, there is a $\mathrm{r}>0$ such that for every $\theta, \varsigma \in \mathscr{Q}$ we have $\delta(\mathscr{A}(\theta \varsigma), \mathscr{A}(\theta) \mathscr{A}(\varsigma))<\mathrm{r}$. Then, is there a homomorphism $\mathscr{Z}: \mathbb{Q} \rightarrow \mathbb{Q}^{\prime}$ such that the following inequality holds?

$$
\delta(\mathscr{A}(\theta), \mathscr{Z}(\theta))<\epsilon .
$$

As a result of Rassias's expansion of the concept of sustainability due to Ulam, this stability became known as the Hayers-Ulam-Rassias stability [1]. Aoki, Bourgin, Gajda, etc. are among those who have worked on the issue of stability in different ways. Most of the research has examined Banach space. Therefore, researchers have turned to other spaces and have investigated stability in different spaces [2,3]. In addition, the stability of groups and different types of Banach algebras have been topics of interest to researchers.

In situations involving fractional theory and the solutions of partial differential equations, special functions are used. In checking the stability to choose the control function as the controller, these functions are very suitable options. Today, the application of these functions is very evident in various fields such as physical sciences, engineering, probability theory, etc. The Mittag-Leffler function, Wright function [4,5], Fox's $H$-function and Gauss hypergeometric function are among the most frequently used functions from the category of special functions. These functions were presented and introduced for the first time by Mittag-Leffler (Swedish mathematician), E. M. Wright (British mathematician),

Charles Fox (English mathematician), and Friedrich Gauss (German mathematician and physicist), respectively. To choose the controller in the issue of stability, the effort is to choose the best and most optimal function among the specific functions. For this purpose, we go to another special function, which is much more comprehensive than the previously introduced special functions.

The development of these functions, known as aggregation functions, is accelerating owing to their many applications in disciplines such as decision theory, artificial intelligence, pattern recognition and image processing, mathematics, etc.

Let $(\mathrm{Y}, \Theta, \mathfrak{W})$ be a random measure space. Throughout this paper, we consider Borel measurable spaces $(\mathscr{M}, \mathfrak{B} \mathscr{M})$ and $\left(\mathscr{N}, \mathfrak{B}_{\mathcal{N}}\right)$ on the MVFN-S, $\mathscr{M}$ and $\mathcal{N}$. By defining the random operator $\mathfrak{S}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathscr{N}$ and considering $0<\xi<1$, we have the BI-AROI as follows:

$$
\begin{align*}
& \mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}-\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \\
& \succeq \mathfrak{W}(\mathfrak{\xi}(2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \mathfrak{\omega}) \tag{1}
\end{align*}
$$

To obtain the best approximation for S-BI-D and S-BI-H in MVFB-A and MVFC-$\diamond-A$, we apply special functions and use the aggregation function to find the optimal control function.

The article consists of the following parts:
In the first section, we investigate the optimal stability for S-BI-D and S-BI-H in the spaces MVFB-A and MVFC- $\diamond$-A using the direct method. For this purpose, in this section, we introduce all the basic concepts needed for these proofs. First, we define the required spaces and present some proofs. Then, to check the optimal stability, we introduce the optimal control function. In the second section, using the fixed point method (FPM), we have proved the optimal stability of S-BI-D and S-BI-H in the spaces MVFB-A and MVFC- $\diamond$-A. In the third section, we have expressed a summary of the article in the form of a conclusion.

## 2. Direct Technique for Optimal Stability of S-BI-D and S-BI-H in MVFB-A and MVFC- - -A

In this section, we investigate the best approximation for S-BI-D and S-BI-H in MVFBA and MVFC $-\diamond$-A by direct technique. Therefore, before dealing with the main proofs, we first introduce the required spaces. These spaces are used in all parts of the article.

Definition 1. On the interval $[0,1]$, we define $\Delta$ as follows:

$$
\Delta=\operatorname{diag} \Lambda([0,1])=\left\{\operatorname{diag}\left[\Lambda_{1}, \cdots, \Lambda_{l}\right]=\left[\begin{array}{ccc}
\Lambda_{1} & & \\
& \ddots & \\
& & \Lambda_{1}
\end{array}\right], \Lambda_{1}, \ldots, \Lambda_{l} \in[0,1]\right\}
$$

where every square matrix is $1 \times 1$ and for any $\boldsymbol{\Lambda}, \boldsymbol{\phi} \in \Delta$, we have $\boldsymbol{\Lambda}=\operatorname{diag}\left[\Lambda_{1}, \cdots, \Lambda_{1}\right]$, $\boldsymbol{\phi}=\operatorname{diag}\left[\phi_{1}, \cdots, \phi_{1}\right], \operatorname{diag}[1, \ldots, 1]=\mathbf{1}$ and $\operatorname{diag}[0, \ldots, 0]=\mathbf{0}$. In addition, $\boldsymbol{\Lambda} \preceq \boldsymbol{\phi}$ means that $\Lambda_{\iota} \leq \phi_{\iota}$ for every $\iota=1, \ldots, 1$.

Definition 2. We consider the mapping $\circledast$ from $\Delta \times \Delta$ to $\Delta$. If for each $\boldsymbol{\Lambda}, \boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Delta$ we have $\boldsymbol{\Lambda} \circledast \mathbf{1}=\boldsymbol{\Lambda}, \boldsymbol{\Lambda} \circledast \boldsymbol{\phi}=\boldsymbol{\phi} \circledast \boldsymbol{\Lambda}, \boldsymbol{\Lambda} \circledast(\boldsymbol{\phi} \circledast \boldsymbol{\alpha})=(\boldsymbol{\Lambda} \circledast \boldsymbol{\phi}) \circledast \boldsymbol{\alpha}, \boldsymbol{\Lambda} \preceq \boldsymbol{\phi}$ and $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ implies that $\boldsymbol{\Lambda} \circledast$ $\boldsymbol{\alpha} \preceq \boldsymbol{\phi} \circledast \boldsymbol{\beta}$, we say that $\circledast$ is a generalized $t$-norm or briefly GTN. In addition, we consider sequences $\left\{\boldsymbol{\Lambda}_{1}\right\}$ and $\left\{\boldsymbol{\phi}_{1}\right\}$ that converge to $\boldsymbol{\Lambda}$ and $\boldsymbol{\phi}$. If we have $\lim _{1}\left(\boldsymbol{\Lambda}_{1} \circledast \boldsymbol{\phi}_{1}\right)=\boldsymbol{\Lambda} \circledast \boldsymbol{\phi}$, then $\circledast$ is a CGTN.

There are different types of CGTN: minimum CGTN, product CGTN and Lukasiewicz CGTN can be mentioned among the most important of them. In this work, we choose the minimum CGTN $\circledast_{M}: \Delta \times \Delta \rightarrow \Delta$, which is defined as follows:

$$
\boldsymbol{\Lambda} \circledast_{M} \boldsymbol{\phi}=\operatorname{diag}\left[\Lambda_{1}, \cdots, \Lambda_{1}\right] \circledast_{M} \operatorname{diag}\left[\phi_{1}, \cdots, \phi_{1}\right]=\operatorname{diag}\left[\min \left\{\Lambda_{1}, \phi_{1}\right\}, \cdots, \min \left\{\Lambda_{1}, \phi_{1}\right\}\right] .
$$

For more details about the introduced CGTN, refer to $[3,6]$. In the following, we define MVFF and MVFN-S [3]. The MVFF $\Xi:[0, u] \times(0,+\infty) \rightarrow \Delta$ is increasing and continuous, $\lim _{\mathscr{\omega} \rightarrow+\infty} \Xi(\theta, \omega)=\mathbf{1}$ for every $\theta \in[0, u]$ and $\omega \in(0,+\infty), \mathscr{K} \precsim \Xi$ if and only if $\mathscr{K}(\theta, \omega) \preceq \mathscr{K}(\theta, \omega)$, for all $\omega \in(0,+\infty)$ and $\theta \in[0, u]$ where $\mathscr{K}$ is the MVFF and $\preceq$ is the relation defined for this type of function.

Consider the linear space $\mathcal{S}, \mathrm{CGTN} \circledast$ and the MVFF $\mathfrak{W}: \mathcal{S} \times(0,+\infty) \rightarrow \Delta$, we define $(\mathcal{S}, \mathfrak{W}, \circledast)$, which is called an MVFN-S and has the following properties,

- $\mathfrak{W}(\theta, \omega)=1$ if and only if $\theta=0$ for $\omega \in(0,+\infty)$;
- $\mathfrak{W}(\gamma \theta, \omega)=\mathfrak{W}\left(\theta, \frac{\infty}{|\gamma|}\right)$ for all $\theta \in \mathcal{S}$ and $\gamma \neq 0 \in \mathbb{C}$;
- $\mathfrak{W}(\theta+\varsigma, \mathfrak{\omega}+\alpha) \succeq \mathfrak{W}(\theta, \mathfrak{\omega}) \circledast \mathfrak{W}(\theta, \alpha)$ for all $\theta \in \mathcal{S}$ and any $\omega, \alpha \in(0,+\infty)$;
- $\lim _{\omega \rightarrow+\infty} \mathfrak{W}(\theta, \omega)=\mathbf{1}$ for any $\omega \in(0,+\infty)$.

When an MVFN-S is complete, we denote it by MVFB-S. If for $(\mathcal{\delta}, \mathfrak{W}, \circledast)$ and CGTN
$\boldsymbol{\theta}$, we have

- $\mathfrak{W}(\theta \varsigma, \omega \alpha) \succeq \mathfrak{W}(\theta, \omega) \mathfrak{W}(\varsigma, \alpha)$, for all $\theta, \varsigma \in \mathcal{S}$ and all $\omega, \alpha \in(0, \infty)$.

Then, $(\mathcal{S}, \mathfrak{W}, \circledast, \boldsymbol{\theta})$ is called a matrix valued fuzzy normed-algebra. If $\left(\mathcal{S}, \mathfrak{W}, \circledast_{M}, \circledast_{P}\right)$ is complete, then it is called an MVFB-A.

Definition 3. We assume that $\mathscr{M}$ and $\mathcal{N}$ are MVFN-Ss. Considering random measure space $(\mathrm{Y}, \Theta, \mathfrak{W})$; we define $\mathfrak{S}: \mathrm{Y} \times \mathscr{M} \rightarrow \mathcal{N}$ with the following properties:

- $\mathfrak{S}$ is an $S-O$ if for all $\mathfrak{x}$ in $\mathscr{M}$ and $B \in \mathfrak{B}_{\mathcal{N}}$, we have

$$
\{\mathfrak{t} \mathfrak{S}(\mathfrak{t}, \mathfrak{x}) \in B\} \in \Theta
$$

- $\mathfrak{S}$ is bi-linear if for each $\mathfrak{x}_{1}, \mathfrak{x}_{2}$ in $\mathscr{M}$ and for every $\varrho, \xi$, we have

$$
\mathfrak{S}\left(\mathfrak{t}, \varrho \mathfrak{x}_{1}+\mathfrak{\zeta} \mathfrak{x}_{2}, \mathfrak{z}\right)=\varrho \mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}_{1}, \mathfrak{z}\right)+\mathfrak{\xi} \mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}_{2}, \mathfrak{z}\right)
$$

and

$$
\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}, \varrho \mathfrak{z}_{1}+\mathfrak{\xi} \mathfrak{z}_{2}\right)=\varrho \mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}, \mathfrak{z}_{1}\right)+\mathfrak{\xi} \mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}, \mathfrak{z}_{2}\right) .
$$

- If there exists a random variable $\Delta(\mathfrak{t})$, which is positive and real-valued, such that for each $\mathfrak{x}, \mathfrak{z}$ in $\mathscr{M}$ and $\omega>0$, we have

$$
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x})-\mathfrak{S}(\mathfrak{t}, \mathfrak{z}), \Delta(\mathfrak{t}) \mathfrak{\omega}) \succeq \mathfrak{W}(\mathfrak{x}-\mathfrak{z}, \mathfrak{\omega})
$$

then $\mathfrak{S}$ is bounded.
Definition 4. Consider the MVFB- $A(\mathscr{E}, \mathfrak{W}, \circledast, \boldsymbol{*})$. $\mathscr{E}$ is called an MVFC- $\diamond-A$ if we have $\mathfrak{x} \rightarrow \mathfrak{x}^{\diamond}$ on $\mathscr{E}$ with
$(\diamond-A 1) \quad \mathfrak{x}^{\diamond \infty}=\mathfrak{x}$ for any $\mathfrak{x} \in \mathscr{E}$;
$(\diamond-A 2) \quad(\varrho \mathfrak{x}+\mathfrak{\xi} \mathfrak{z})^{\diamond}=\bar{\varrho} \mathfrak{x}^{\diamond}+\bar{\xi} \mathfrak{z}^{\diamond}$;
$(\diamond-A 3) \quad(\mathfrak{x z})^{\diamond}=\mathfrak{z} \mathfrak{x}^{\diamond}$ for any $\mathfrak{x}, \mathfrak{z} \in \mathscr{E}$.
In addition, if for each $\mathfrak{x} \in \mathscr{E}$ and $0<\omega<1$, we have
$(\diamond-A 4) \quad \mathfrak{W}(\mathfrak{x} \stackrel{\mathfrak{x}}{\boldsymbol{x}} \boldsymbol{\omega})=\mathfrak{W}(\mathfrak{x}, \omega)$.
Definition 5. Consider the ring $\mathscr{E}$. If S-BI-AM $\mathscr{T}: \mathrm{Y} \times \mathscr{E} \times \mathscr{E} \rightarrow \mathscr{E}$ has the following properties for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$
(SS-D1) $\mathscr{T}(\mathfrak{t}, \mathfrak{x z}, \mathfrak{d})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{z}+\mathfrak{x} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$,
(SS-D2) $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{z})=\mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{x})$,
then $\mathscr{T}$ is called an S-BI-D.
Definition 6. Consider an S-BI-AM $\mathscr{T}: \mathrm{Y} \times \mathscr{E} \times \mathscr{E} \rightarrow \mathscr{E}$, where $\mathscr{E}$ is MVFB-A. If for every $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ we have
$(S-D 1) \quad \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{z}+\mathfrak{x} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$,
(S-D2) $\quad \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d r})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{r}+\mathfrak{d} \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r})$,
then $\mathscr{T}$ is an S-BI-D. For an S-BI-D, the following condition is always true

$$
\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{z} \mathfrak{r})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{r} \mathfrak{z}+\mathfrak{d} \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r}) \mathfrak{z}+\mathfrak{x} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}) \mathfrak{r}+\mathfrak{x d} \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r}) .
$$

Definition 7. Consider an S-BI-AM Y : $\mathrm{Y} \times \mathscr{E} \times \mathscr{E} \rightarrow \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are MVFB-As. If for any $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ we have
(S-H1) $\quad \mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x} \mathfrak{z}, \mathfrak{d}^{2}\right)=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$,
$(S-H 2) \quad \mathfrak{Y}\left(\mathfrak{t}, \mathfrak{d}^{2}, \mathfrak{d r}\right)=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r})$,
then $\mathfrak{Y}$ is called an S-BI-H.
To achieve the desired results in all parts, we assume $\circledast=\circledast_{M}$ and $\xi \in(0,1)$. In addition, the optimal stability is evaluated by considering the optimal control function.
2.1. Optimal Stability of S-BI-D in MVFB-A

Lemma 1 ([7], Lemma 2.1). Consider the stochastic operator $\mathfrak{S}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$, if we have (RO1) For each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$

$$
\mathfrak{S}(\mathfrak{t}, 0, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, 0)=0
$$

(RO2) For all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$

$$
\begin{aligned}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d} & +\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}-\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \omega) \\
& \succeq \mathfrak{W}(\xi(2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega),
\end{aligned}
$$

then $\mathfrak{S}$ is an S-BI-AM.
Theorem 1. Consider an MVFB-S $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$, for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$, we define the function $\Xi: \mathscr{M}^{4} \rightarrow \Delta$ and $\mathfrak{S}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$. If we have

$$
\begin{gather*}
\Xi\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{z}}{2}, \mathfrak{d}, \mathfrak{r}, \frac{\mathfrak{g}}{2} \mathfrak{\omega}\right) \succeq \Xi(\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}, \omega),  \tag{2}\\
\mathfrak{S}(\mathfrak{t}, 0, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, 0)=0, \\
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}-\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \omega) \\
\succeq \mathfrak{W}(\mathfrak{S}(2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega)  \tag{3}\\
\circledast \Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega),
\end{gather*}
$$

then for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, we can find a unique S-BI-AM $\mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ such that

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4(1-\mathfrak{\xi})}{\xi} \mathfrak{\infty}\right) \tag{4}
\end{equation*}
$$

Proof. Assuming $\mathfrak{r}=0$ and $\mathfrak{z}=\mathfrak{x}$ and placing these assumptions in (3), and considering $\mathfrak{S}(\mathfrak{t}, 0, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, 0)=0$ and the second condition of MVFB-S, we have
$\mathfrak{W}(2 \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \mathfrak{W}\left(2 \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \frac{\mathfrak{w}}{\mathfrak{\xi}}\right) \circledast \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \omega)$,
now, according to the definition of MVFB-S and knowing that $\xi \in(0,1)$ or equivalently $\frac{1}{\xi}>1$, we have

$$
\mathfrak{W}(2 \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\infty}) \preceq \mathfrak{W}\left(2 \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \frac{\mathfrak{w}}{\mathfrak{\xi}}\right)
$$

then

$$
\mathfrak{W}(2 \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \mathfrak{\infty})
$$

which according to the second condition of MVFB-S, we obtain

$$
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{w})=\mathfrak{W}(2 \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), 2 \mathfrak{w})
$$

therefore, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), 2 \mathfrak{\omega}) . \tag{5}
\end{equation*}
$$

Given that $2 \omega=\frac{\xi}{2}\left(\frac{4}{\xi} \omega\right)$ and considering (2), for every $\mathfrak{x} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$, we obtain

$$
\begin{equation*}
\mathfrak{W}\left(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right), \omega\right) \succeq \Xi\left(\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{x}}{2}, \mathfrak{d}, 0\right), 2 \omega\right) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4}{\tilde{\xi}} \omega\right) \tag{6}
\end{equation*}
$$

By placing $\frac{\mathfrak{x}}{2^{n-1}}$ instead of $\mathfrak{x}$ in (6) and using the second condition of MVFB-S and (2), we have

$$
\begin{align*}
& \mathfrak{W}\left(2^{n-1} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n-1}}, \mathfrak{d}\right)-2^{n} \mathfrak{S}\left(\mathfrak{S}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \mathfrak{\infty}\right)  \tag{7}\\
\succeq & \Xi\left(\left(\frac{\mathfrak{x}}{2^{n-1}}, \frac{\mathfrak{x}}{2^{n-1}}, \mathfrak{d}, 0\right), \frac{4}{2^{n-1} \mathfrak{\xi}} \mathfrak{\infty}\right) \\
\succeq & \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \times 2^{n-1}}{2^{n-1} \mathfrak{\zeta}^{n}} \mathfrak{\omega}\right), \\
= & \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4}{\mathfrak{\xi}^{n}} \mathfrak{\omega}\right),
\end{align*}
$$

for every $\mathfrak{x} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \boldsymbol{\omega} \in(0, \infty)$ and $n \in \mathbb{N}$. Now, relabeling gives

$$
\mathfrak{W}\left(2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-2^{n-1} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n-1}}, \mathfrak{d}\right), \frac{\xi^{n}}{4} \mathfrak{\infty}\right) \succeq \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \omega)
$$

According to the second condition of MVFB-S and $2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=$ $\sum_{k=1}^{n} 2^{k} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{k}}, \mathfrak{d}\right)-2^{k-1} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{k-1}}, p\right)$, we obtain

$$
\begin{align*}
& \mathfrak{W}\left(2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \sum_{k=1}^{n} \frac{1}{4} \xi^{k} \mathfrak{\infty}\right)  \tag{8}\\
\succeq & \prod_{k=1}^{n}\left[\mathfrak{W}\left(2^{k} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{k}}, \mathfrak{d}\right)-2^{k-1} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{k-1}}, \mathfrak{d}\right), \frac{1}{4} \xi^{k} \mathfrak{\infty}\right)\right] \\
\succeq & \prod_{k=1}^{n}[\Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \omega)] \\
\succeq & \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{r}, 0), \omega)
\end{align*}
$$

and by relabeling for any $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \mathfrak{\omega} \in(0, \infty)$ and $n \in \mathbb{N}$, we conclude that

$$
\begin{equation*}
\mathfrak{W}\left(2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}\right) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \omega}{\sum_{k=1}^{n} \xi^{k}}\right) \tag{9}
\end{equation*}
$$

Again, using the second condition of MVFB-S and (2) and placing $\frac{\mathfrak{x}}{2^{m}}$ instead of $\mathfrak{x}$ in (9), for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \boldsymbol{\omega} \in(0, \infty)$ and $n, m \in \mathbb{N}$, we have

$$
\begin{align*}
\mathfrak{W}\left(2^{n+m} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n+m}}, \mathfrak{d}\right)-2^{m} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{m}}, \mathfrak{d}\right), \omega\right) & \succeq \Xi\left(\left(\frac{\mathfrak{x}}{2^{m}}, \frac{\mathfrak{x}}{2^{m}}, \mathfrak{d}, 0\right), \frac{4}{2^{m} \sum_{k=1}^{n} \xi^{k}} \omega\right),  \tag{10}\\
& \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \times 2^{m}}{2^{m} \tilde{\xi}^{m} \sum_{k=1}^{n} \xi^{k}} \omega\right) \\
& =\Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4}{\sum_{k=m+1}^{n+m} \xi^{k}} \omega\right) .
\end{align*}
$$

In the obtained inequality (10), we assume that $m, n$ tend to $\infty$. Given that $\xi \in(0,1)$, for each $\omega \in(0, \infty)$, we have

$$
\Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{o}, 0), \frac{4}{\sum_{k=m+1}^{n+m} \xi^{k}} \mathfrak{\omega}\right) \rightarrow \mathbf{1} .
$$

Then, for $\mathfrak{x} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$, the sequence $\left\{2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)\right\}$ is Cauchy in complete set $\mathscr{M}$, and then $\left\{2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)\right\}$ is convergent. Now, for each $\mathfrak{x} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$, we define the $\mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ function as follows:

$$
\begin{equation*}
\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \tag{11}
\end{equation*}
$$

To continue the proof, we consider (4) and set $m=0$ in (10) and also assume that $n$ tends to $\infty$. According to (3), (2) and using the second condition of MVFB-S, for any $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$, we have

$$
\begin{aligned}
& \mathfrak{W}(\mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}-\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{r}, \mathfrak{d}), \boldsymbol{w}) \\
& =\lim _{n \rightarrow \infty} \mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{\omega}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right)+\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}}, \mathfrak{d}-\mathfrak{r}\right)+\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}-\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right)+\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}-\mathfrak{z}}{2^{n}}, \mathfrak{d}-\mathfrak{r}\right)\right. \\
& \left.-4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \frac{\mathfrak{\infty}}{2^{n}}\right) \\
& \succeq \lim _{n \rightarrow \infty}\left[\mathfrak{W}\left(\xi\left(2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}}, \mathfrak{d}-\mathfrak{r}\right)+2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}-\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right)-4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)+4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{r}\right)\right), \frac{\mathfrak{D}}{2^{n}}\right)\right. \\
& \left.\circledast \Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{d}, 0\right), \frac{\mathcal{W}}{2^{n}}\right)\right] \\
& \succeq \lim _{n \rightarrow \infty} \mathfrak{W}\left(\xi\left(2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}}, \mathfrak{d}-\mathfrak{r}\right)+2 \mathfrak{S}\left(t, \frac{\mathfrak{x}-\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right)-4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)+4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{r}\right)\right), \frac{\mathfrak{w}}{2^{n}}\right) \\
& \circledast \lim _{n \rightarrow \infty} \Xi\left((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, 0), \frac{\omega}{\mathfrak{\xi}^{n}}\right) \\
& \succeq \mathfrak{W}(\xi(2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})+4 \mathfrak{A}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega) \circledast \mathbb{1} \\
& =\mathfrak{W}(\xi(2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})+4 \mathfrak{A}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega) .
\end{aligned}
$$

Then, for every $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \mathfrak{\omega} \in(0, \infty)$, we obtain

$$
\begin{align*}
& \mathfrak{W}(\mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})  \tag{12}\\
+ & \mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}-\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{O}) \\
\succeq & \mathfrak{W}(\xi(2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})+4 \mathfrak{A}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \mathfrak{\infty}) .
\end{align*}
$$

Therefore, according to Lemma $1, \mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathscr{N}$ is an S-BI-AM. Then, we assume that $\mathscr{T}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ is an S-BI-AM that satisfies (4). Then

$$
\mathfrak{W}(\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\infty})=\lim _{n \rightarrow \infty} \mathfrak{W}\left(2^{n} \mathfrak{A}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-2^{n} \mathscr{T}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \infty\right)
$$

Using (4), (2), the second condition of MVFB-S, and the third condition of MVFB-S, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in U, \mathfrak{t} \in \mathfrak{S}, \mathfrak{\omega} \in(0, \infty)$, we have

$$
\begin{aligned}
& \mathfrak{W}\left(2^{n} \mathfrak{A}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-2^{n} \mathscr{T}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \mathfrak{\omega}\right) \\
& \succeq \mathfrak{W}\left(2^{n} \mathfrak{A}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \frac{\mathfrak{\omega}}{2}\right) \circledast \mathfrak{W}\left(2^{n} \mathscr{T}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)-2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right), \frac{\mathfrak{W}}{2}\right) \\
& \succeq \Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}, 0\right), \frac{4(1-\xi) \mathfrak{\omega}}{2^{n+1} \xi}\right) \circledast \Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}, 0\right), \frac{4(1-\xi) \mathfrak{\omega}}{2^{n+1} \xi}\right) \\
& \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{2(1-\xi) \mathscr{\omega}}{\xi^{n+1}}\right) \circledast \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{2(1-\xi) \mathscr{\omega}}{\xi^{n+1}}\right) \\
& \rightarrow \mathbf{1} .
\end{aligned}
$$

Therefore, the uniqueness of $\mathfrak{A}$ is proved: that is, for every $\mathfrak{x}, \mathfrak{d} \in U$ and $\mathfrak{t} \in \mathrm{Y}$, we obtain

$$
\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})
$$

Theorem 2. Consider MVFB-S $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ and for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \omega \in(0, \infty), \xi \in(0,1)$, we define $\Xi: \mathscr{M}^{4} \rightarrow \Delta$ and $\mathfrak{S}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathscr{N}$. For the function $\Xi$, we assume that the following condition holds

$$
\begin{equation*}
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), 2 \mathfrak{\xi} \mathfrak{\infty}) \succeq \Xi\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{z}}{2}, \mathfrak{d}, \mathfrak{r}, \mathfrak{\infty}\right) \tag{13}
\end{equation*}
$$

and for $\mathfrak{S}$, we suppose that conditions (RO1) and (3) are satisfied for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$. Then, for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \operatorname{tin} Y$ and $\omega \in(0, \infty)$, there exists an S-BI-AM $\mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ such that

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \varsigma, \mathfrak{d})-\mathfrak{A}(\mathfrak{t}, \varsigma, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4(1-\xi)}{\xi} \mathfrak{\omega}\right) \tag{14}
\end{equation*}
$$

Proof. Due to the (5), we have

$$
\mathfrak{W}\left(2\left(\frac{1}{2} \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})\right), \omega\right) \succeq \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), 2 \omega)
$$

also, using the second condition of MVFB-S, we obtain

$$
\mathfrak{W}\left(\frac{1}{2} \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \frac{\omega}{2}\right) \succeq \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), 2 \omega)
$$

and relabeling for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$ gives

$$
\begin{equation*}
\mathfrak{W}\left(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\frac{1}{2} \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}\right) \succeq \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), 4 \mathfrak{\infty}) \tag{15}
\end{equation*}
$$

By putting $2^{n} \mathfrak{x}$ instead of $\mathfrak{x}$ in (15), for any $\mathfrak{x} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \boldsymbol{\omega} \in(0, \infty)$ and $n \in \mathbb{N}$, we obtain

$$
\begin{align*}
\mathfrak{W}\left(\frac{1}{2^{n}} \mathfrak{S}\left(\mathfrak{t}, 2^{n} \mathfrak{x}, \mathfrak{d}\right)-\frac{1}{2^{n+1}} \mathfrak{S}\left(\mathfrak{\omega}, 2^{n+1} \mathfrak{x}, \mathfrak{d}\right), \mathfrak{\omega}\right) & \succeq \Xi\left(\left(2^{n} \mathfrak{x}, 2^{n} \mathfrak{x}, \mathfrak{d}, 0\right), 4 \times 2^{n} \mathfrak{\infty}\right)  \tag{16}\\
& \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4}{\mathfrak{\xi}^{n}} \mathfrak{\omega}\right) .
\end{align*}
$$

Since,

$$
\frac{1}{2^{n}} \mathfrak{S}\left(\mathfrak{t}, 2^{n} \mathfrak{x}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{s}, \mathfrak{d})=\sum_{k=1}^{n} \frac{1}{2^{k}} \mathfrak{S}\left(\mathfrak{t}, 2^{k} \mathfrak{x}, \mathfrak{d}\right)-\frac{1}{2^{k-1}} \mathfrak{S}\left(\mathfrak{t}, 2^{k-1} \mathfrak{x}, \mathfrak{d}\right)
$$

we have that,

$$
\begin{align*}
& \mathfrak{W}\left(\frac{1}{2^{n}} \mathfrak{S}\left(\mathfrak{t}, 2^{n} \mathfrak{x}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \frac{\sum_{k=1}^{n} \tilde{\xi}^{k}}{4} \mathfrak{o}\right)  \tag{17}\\
\succeq & \prod_{k=1}^{n}\left[\mathfrak{W}\left(\frac{1}{2^{k}} \Gamma\left(\mathfrak{t}, 2^{k} \mathfrak{x}\right)-\frac{1}{2^{k-1}} \Gamma\left(\mathfrak{t}, 2^{k-1} \mathfrak{x}, \mathfrak{d}\right), \frac{\xi^{k}}{4} \mathfrak{\infty}\right)\right] \\
\succeq & \Xi((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \mathfrak{\omega}) .
\end{align*}
$$

Then, for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \mathfrak{\omega} \in(0, \infty)$ and $n \in \mathbb{N}$,

$$
\begin{equation*}
\mathfrak{W}\left(\frac{1}{2^{n}} \mathfrak{S}\left(\mathfrak{t}, 2^{n} \mathfrak{x}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \infty\right) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \omega}{\sum_{k=1}^{n} \xi^{k}}\right) \tag{18}
\end{equation*}
$$

Now, by putting $2^{m} \mathfrak{x}$ instead of $\mathfrak{x}$ in (18), for every $\mathfrak{x} \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \omega \in \mathbb{J}^{\circ}$ and $n, m \in \mathbb{N}$, we have

$$
\begin{align*}
\mathfrak{W}\left(\frac{1}{2^{n+m}} \mathfrak{S}\left(\mathfrak{t}, 2^{n+m} \mathfrak{x}, \mathfrak{d}\right)-\frac{1}{2^{m}} \mathfrak{S} a\left(\mathfrak{t}, 2^{m} \mathfrak{x}, \mathfrak{d}\right), \mathfrak{\infty}\right) & \succeq \Xi\left(\left(2^{m} \mathfrak{x}, 2^{m} \mathfrak{x}, \mathfrak{d}, 0\right), \frac{4 \times 2^{m} \mathfrak{O}}{\sum_{k=1}^{n} \xi^{k}}\right),  \tag{19}\\
& \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \mathfrak{\infty}}{\sum_{k=m+1}^{n+m} \xi^{k}}\right)
\end{align*}
$$

In the obtained inequality (19), we assume that $m, n$ tend to $\infty$. Given that $\xi \in(0,1)$, for each $\omega \in(0, \infty)$, we have

$$
\Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{o}, 0), \frac{4 \mathfrak{\omega}}{\sum_{k=m+1}^{n+m} \xi^{k}}\right) \rightarrow \mathbf{1} .
$$

Then, for $\mathfrak{x} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$, the sequence $\left\{\frac{1}{2^{n}} \mathfrak{S}\left(\omega, 2^{n} \mathfrak{x}, \mathfrak{d}\right)\right\}$ is Cauchy in complete set $\mathscr{M}$, and then $\left\{\frac{1}{2^{n}} \mathfrak{S}\left(\omega, 2^{n} \mathfrak{x}, \mathfrak{d}\right)\right\}$ is convergent. Now, for each $\mathfrak{x} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$, we define the $\mathrm{S}-\mathrm{O} \mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ by

$$
\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=\lim _{n \rightarrow \infty} \frac{1}{2^{n}} \mathfrak{S}\left(\mathfrak{t}, 2^{n} \mathfrak{x}, \mathfrak{d}\right)
$$

To continue the proof, we consider (14) and set $m=0$ in (19) and also assume that $n$ tends to $\infty$ and use the proof of Theorem 1.

Lemma 2 ([8], Lemma 2.1). We consider the S-BI-AM S: Y $\times \mathscr{M}^{2} \rightarrow \mathscr{M}$ such that for all $\omega, \rho \in \mathbb{D}^{1}:=\{v \in \mathbb{C}:|v|=1\}$ and each $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathfrak{S}\left(\mathfrak{t}, \lambda \mathfrak{x}, \eta_{2} \mathfrak{d}\right)=$ $\varsigma v \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Then, $\mathfrak{S}$ is a C-BI-SO.

Theorem 3. Consider the $\operatorname{MVFB}-A(\mathscr{E}, \mathfrak{W}, \circledast, \circledast)$; we define the $M V F F \Xi: \mathscr{C}^{4} \rightarrow \operatorname{diag} M_{n}((0,1])$ and $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that they satisfy the (2) and (RO1) conditions, respectively. For each $\varsigma, v \in \mathbb{D}^{1}, \mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, we assume that the following condition also holds for $\mathfrak{S}$

$$
\begin{align*}
& \mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}+\mathfrak{z}), \rho(\mathfrak{d}+\mathfrak{r}))+\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}+\mathfrak{z}), \rho(\mathfrak{d}-\mathfrak{r}))+\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}-\mathfrak{z}), \rho(\mathfrak{d}+\mathfrak{r}))  \tag{20}\\
& +\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}-\mathfrak{z}), \rho(\mathfrak{d}-\mathfrak{r})-4 \omega \rho \mathfrak{S}(\mathfrak{t}, \mathfrak{r}, \mathfrak{d}), \omega) \\
& \succeq \mathfrak{W}(\mathfrak{S}(2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})+4 \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega) \\
& \circledast \Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega),
\end{align*}
$$

then we can find C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$, which satisfies condition (4) by placing this operator in place of $\mathfrak{A}$. If we assume that for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, the following conditions are true for $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$

$$
\begin{gather*}
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ; \\
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{z}, \mathfrak{d})-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{z}-\mathfrak{x} \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, 0), \mathfrak{\omega}),  \tag{21}\\
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d} \mathfrak{r})-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{r}-\mathfrak{d} \mathfrak{S}(\mathfrak{\omega}, \mathfrak{x}, \mathfrak{r}), \mathfrak{\omega}) \succeq \Xi(\mathfrak{x}, 0, \mathfrak{d}, \mathfrak{r}, \mathfrak{\omega}), \tag{22}
\end{gather*}
$$

then $\mathfrak{S}$ is an S-BI-D.

Proof. If we consider (20) and assume that $\omega=\rho=1$, according to Theorem 1, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, there is an S-BI-AM $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$, which is defined as follows:

$$
\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right),
$$

and satisfying (4). If we assume that $\mathfrak{z}=\mathfrak{r}=0$ and apply this assumption to (20), for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and all $\omega, \rho \in \mathbb{D}^{1}$, we have

$$
\mathfrak{S}(\mathfrak{t}, \omega \mathfrak{x}, \rho \mathfrak{d})=\omega \rho \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})
$$

and due to the Lemma $2, \mathscr{T}$ is a C-BI-SO.
We assume that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. For all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in Y$, it is easy to see that $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. From (21), for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in Y$ and $\mathfrak{\omega} \in(0, \infty)$, we have

$$
\begin{aligned}
& \mathfrak{W}(\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathfrak{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{z}-\mathfrak{x} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\infty}) \\
= & \lim _{n \rightarrow \infty} \mathfrak{W}\left(\Gamma\left(\mathfrak{t}, \frac{\mathfrak{x} \mathfrak{z}}{2^{n} \cdot 2^{n}}, \mathfrak{d}\right)-\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right) \frac{\mathfrak{z}}{2^{n}}-\frac{\mathfrak{x}}{2^{n}} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{d}\right), \frac{\mathfrak{\omega}}{4^{n}}\right) \\
\succeq & \lim _{n \rightarrow \infty} \Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{d}, 0\right), \frac{\mathfrak{\omega}}{4^{n}}\right)=\mathbf{1} .
\end{aligned}
$$

Thus, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}, \mathscr{T}(\mathfrak{t}, \mathfrak{x} \mathfrak{z}, \mathfrak{d})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{z}+\mathfrak{x} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})$. With the same process, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d r})=\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{r}+\mathfrak{d} \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r})$. Since $\mathfrak{S}(t, x, d)$ is an S-BI-AM, we conclude that $\mathfrak{S}$ is an S-BI-D.

In the following, we investigate an optimal stability by introducing a new optimal control function. For this purpose, we go to the definition of the aggregation function. In the following, we provide a brief introduction of the special functions used in the optimal control function [2].

Definition 8. If for any $\left(\theta_{1}, \cdots, \theta_{\ell}\right),\left(\varsigma_{1}, \cdots, \varsigma_{\ell}\right) \in \mathbb{R}^{\ell}$ and $\iota \in\{1, \cdots, \ell\}$, and an idempotent function $u^{(\ell)}: \mathbb{R}^{\ell} \longrightarrow \mathbb{R}$, we have $\theta_{\iota} \leq \varsigma_{\iota} \Longrightarrow u^{(\ell)}\left(\theta_{1}, \cdots, \theta_{\ell}\right) \leq u^{(\ell)}\left(\varsigma_{1}, \cdots, \varsigma_{\ell}\right)$, then the $\ell$-ary $u^{(\ell)}$ is a generalized aggregation function where $\ell \in \mathbb{N}$. For $\ell=1$ and each $\theta \in \mathbb{R}$, we have $u^{(1)}(\theta)=\theta$ and for the convenience of writing, we can remove $\ell(\ell$ indicates the number of function variables).

The famous functions, i.e., arithmetic mean function, projection function, order statistic function, median function, minimum and maximum functions are among the important functions of aggregation type. In [2], the authors showed that a control function made by the minimum aggregation function is the optimal controller. The minimum MIN is the smallest generalized aggregation function, and it is defined as follows:

$$
\begin{equation*}
\operatorname{MIN}(\theta)=\min \left\{\theta_{1}, \cdots, \theta_{1}\right\}=\bigwedge_{\iota=1}^{\ell} \theta_{\iota} . \tag{23}
\end{equation*}
$$

Therefore, by studying the mentioned references, we consider the following function as the optimal controller

$$
\begin{equation*}
\operatorname{MIN}(\mathscr{Y}(\theta, \omega))=\operatorname{diag}[\operatorname{MIN}(\mathscr{Y}(\theta, \omega)), \cdots, \operatorname{MIN}(\mathscr{Y}(\theta, \omega))] . \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathscr{y}(\theta, \omega)=\left(E_{\eta_{1}, \eta_{2}}\left(\frac{-\|\theta\|}{\omega}\right), W_{\eta_{1}, \eta_{2}}\left(\frac{-\|\theta\|}{\omega}\right),{ }_{2} F_{1}\left(u, v, w, \frac{-\|\theta\|}{\omega}\right), H_{\theta_{2}, \theta_{4}}^{\vartheta_{3}, \theta_{1}}\left(\frac{-\|\theta\|}{\omega}\right), \exp \left(\frac{-\|\theta\|}{\omega}\right)\right) . \tag{25}
\end{equation*}
$$

In the following, we introduce special functions used in $\mathscr{Y}(\theta, \infty)$ function.
(1) For $\eta_{1}, \eta_{2} \in \mathbb{C}, \operatorname{Re}\left(\eta_{1}\right), \operatorname{Re}\left(\eta_{2}\right)>0$, the Mittag-Leffler functions are defined as follows:

$$
E_{\eta_{1}}(\theta)=\sum_{1=0}^{\infty} \frac{\theta^{1}}{\Gamma\left(l \eta_{1}+1\right)}, \quad E_{\eta_{1}, \eta_{2}}(\theta)=\sum_{\mathrm{l}=0}^{\infty} \frac{\theta^{1}}{\Gamma\left(\mathrm{l} \eta_{1}+\eta_{2}\right)}
$$

where $\Gamma($.$) is the famous gamma function and E_{\eta_{1}}, E_{\eta_{1}, \eta_{2}}$ are the one- and twoparameter Mittag-Leffler functions, respectively.
(2) For $\eta_{1}>-1, \eta_{2}>0, \theta \in \mathbb{R}$, the Wright function is defined as follows:

$$
W_{\eta_{1}, \eta_{2}}(\theta)=\sum_{1=0}^{\infty} \frac{\theta^{1}}{1!\Gamma\left(\eta_{1} 1+\eta_{2}\right)},
$$

such that it is of the $1 /(1+\sigma)$ order.
(3) Considering $u, v, w>0$, the Gauss hypergeometric function ${ }_{2} F_{1}: \mathbb{R}^{3} \times[0, u] \longrightarrow(0, \infty)$ is defined as follows:

$$
{ }_{2} F_{1}(u, v, w ; \theta)=\sum_{1=0}^{\infty} \frac{(u)_{1}(v)_{1}}{(w)_{1}} \frac{\theta^{1}}{1!}=\frac{\Gamma(w)}{\Gamma(u) \Gamma(v)} \sum_{1=0}^{\infty} \frac{\Gamma(u+1) \Gamma(v+1)}{\Gamma(w+1)} \frac{\theta^{1}}{1!} .
$$

(4) $H$-Fox function for $0 \leq \vartheta_{1} \leq \vartheta_{2}, 1 \leq \vartheta_{3} \leq \vartheta_{4},\left\{z_{l}, s_{l}\right\} \in \mathbb{C}$ and $\left\{\theta_{l}, \varsigma_{l}\right\} \in \mathbb{R}^{+}$is defined as follows:

$$
H_{\vartheta_{2}, \vartheta_{4}}^{\vartheta_{3}, \vartheta_{1}}(\theta)=H_{\vartheta_{2}, \vartheta_{4}}^{\vartheta_{3}, \vartheta_{1}}\left[\theta \left\lvert\, \begin{array}{l}
\left(z_{l}, \epsilon_{l}\right)_{l=1, \cdots, \vartheta_{2}}  \tag{26}\\
\left(s_{l}, \rho_{l}\right)_{l=1}, \cdots, \vartheta_{2}
\end{array}\right.\right]=\frac{1}{2 \pi i} \int_{\mathscr{A}} H_{\vartheta_{2}, \vartheta_{4}}^{\vartheta_{3}, \vartheta_{1}}(\mathrm{~h}) \theta^{\mathrm{h}} d \mathrm{~h},
$$

where $\mathscr{A} \in \mathbb{C}$ is a path that is deleted and $\mathscr{Z}_{1}(\mathrm{~h})=\prod_{l=1}^{\vartheta_{1}} \Gamma\left(s_{l}-s_{l} \mathrm{~h}\right), \mathscr{Z}_{2}(\mathrm{~h})=$ $\prod_{l=1}^{\vartheta_{3}} \Gamma\left(1-z_{l}+\theta_{l} \mathrm{~h}\right), \mathscr{Z}_{3}(\mathrm{~h})=\prod_{l=\vartheta_{3}+1}^{\vartheta_{3}} \Gamma\left(1-s_{l}+s_{l} \mathrm{~h}\right), \mathscr{Z}_{4}(\mathrm{~h})=\prod_{l=\vartheta_{1}+1}^{\vartheta_{2}} \Gamma\left(z_{l}-\theta_{l} \mathrm{~h}\right)$ and $\theta^{\mathrm{h}}=\exp \{\mathrm{h}(\log |\theta|+i \arg \theta)\}$. For these functions, there is a condition that $\vartheta_{1}=0$ if and only if $\mathscr{Z}_{2}(\mathrm{~h})=1, \vartheta_{3}=\vartheta_{4}$ if and only if $\mathscr{E}_{3}(\mathrm{~h})=1$ and $\vartheta_{1}=\vartheta_{2}$ if and only if $\vartheta_{4}(\mathrm{~h})=1$. In addition, $H_{\vartheta_{2}, \vartheta_{4}}^{\vartheta_{3}, \vartheta_{1}}(\mathrm{~h})=\frac{\mathscr{I}_{1}(\mathrm{~h}) \mathscr{I}_{2}(\mathrm{~h})}{\mathscr{I}_{3}(\mathrm{~h}) \mathscr{I}_{4}(\mathrm{~h})}$.
These functions are used in all the examples presented in the article.
Example 1. Consider the MVFB-A $(\mathscr{E}, \mathfrak{W}, *, *)$ and assuming $\epsilon>2$ and $\digamma \in(0,1)$, we define the $S-O \mathfrak{S}: Y \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that it satisfies the condition (RO1). For all $\omega, \rho \in \mathbb{D}^{1}$ and all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0 . \infty)$, we have

$$
\begin{align*}
& \mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}+\mathfrak{z}), \rho(\mathfrak{d}+\mathfrak{r}))+\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}+\mathfrak{z}), \rho(\mathfrak{d}-\mathfrak{r}))+\mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}-\mathfrak{z}), \rho(\mathfrak{d}+\mathfrak{r}))  \tag{27}\\
+ & \mathfrak{S}(\mathfrak{t}, \omega(\mathfrak{x}-\mathfrak{z}), \rho(\mathfrak{d}-\mathfrak{r}))-4 \omega \rho \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \omega) \\
\succeq & \mathfrak{W}(\xi(2 \mathfrak{G}(\mathfrak{t}, \mathfrak{r}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{S}(\mathfrak{t}, \mathfrak{r}, \mathfrak{d})+4 \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega) \\
\circledast & \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}\right), \omega\right)\right) .
\end{align*}
$$

Then, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, there is a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \omega) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\|\mathfrak{x}\|^{\epsilon}\|\mathfrak{d}\|^{\epsilon},\left(2^{\epsilon}-2\right) \omega\right)\right) . \tag{28}
\end{equation*}
$$

In addition, if we assume that for $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$, the following conditions are true

$$
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ;
$$

$$
\begin{align*}
& \mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x} \mathfrak{z}, \mathfrak{d})-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{z}-\mathfrak{x} \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\omega}) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \mathfrak{\omega}\right)\right)  \tag{29}\\
& \mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d} \mathfrak{r})-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{r}-\mathfrak{d} \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r}), \mathfrak{\omega}) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\|\mathfrak{x}\|^{\epsilon}\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}\right), \mathfrak{\omega}\right)\right)  \tag{30}\\
& \quad \text { for all } \mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in Y \text { and } \mathfrak{\omega} \in(0, \infty), \text { then } \mathfrak{S} \text { is an } S-B I-D .
\end{align*}
$$

Proof. For all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{\omega} \in(0, \infty)$ and by placing

$$
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega)=\operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}\right), \omega\right)\right)
$$

in Theorem 3, also by considering $\xi=2^{1-\epsilon}$, the proof is complete.
Theorem 4. Consider the $\operatorname{MVFB}-A(\mathscr{E}, \mathfrak{W}, \circledast, \circledast)$; we define $M V F F \Xi: \mathscr{E}^{4} \rightarrow \Delta$ and $\mathfrak{S}$ : $\mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (13) and (20) and RO1, respectively. Then, we can conclude that there is a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ that satisfies (14). If $\mathfrak{S}$ also satisfies (21), (22) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\omega, \mathfrak{x}, \mathfrak{d})$ in addition to the mentioned conditions, then for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}, \mathfrak{S}$ is an S-BI-D.

Proof. The proof is exactly the same as the process of proving Theorem 3.
Example 2. Consider the $\operatorname{MVFB}-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ and $\epsilon<1$ and $\digamma \in(0, \infty)$, we define $\mathfrak{S}$ : $\mathrm{Y} \times \mathfrak{E}^{2} \rightarrow \mathfrak{E}$ such that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \omega \in \mathrm{Y}$, it satisfies conditions (27) and (RO1). Therefore, for every $\mathfrak{x}, \mathfrak{d} \in \mathfrak{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0,1)$, there is a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that

$$
\begin{equation*}
\left.\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\frac{\digamma\|\mathfrak{x}\|^{\epsilon}\|\mathfrak{d}\|^{\epsilon}}{\left(2^{2-\epsilon}-2\right) \omega}\right), \mathfrak{\omega}\right)\right) . \tag{31}
\end{equation*}
$$

In addition, if for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, $\mathfrak{S}$ satisfies conditions (29), (30) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=$ $2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-D.

Proof. If for every $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{\omega} \in(0,1)$ and $\xi=2^{\epsilon-1}$, we consider the function $\Xi$ as follows:

$$
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \mathfrak{\omega})=\operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}, \omega\right)\right)\right)
$$

and use this function in Theorem 3, the proof is complete.
In the rest of this section, we will investigate the best approximation for S-BI-Ds in the unital MVFC- $\diamond-$ A. For this purpose, we consider MVFC- $\diamond-\mathrm{A}(\mathscr{E}, \mathfrak{W}, \circledast, \circledast)$ along with unit member and the unit group. We show the unit member by $e$ and the unit group by $\mathscr{M}(\mathscr{E})=\left\{\varsigma \in \mathscr{E}: \varsigma^{\diamond} \varsigma=\varsigma \varsigma^{\diamond}=e\right\}$.

### 2.2. Optimal Stability of S-BI-D in MVFC- $\diamond-A$

Theorem 5. Consider the MVFC-৫-A $(\mathscr{E}, \mathfrak{W}, \circledast, \circledast)$; we define $M V F F \Xi: \mathscr{E}^{4} \rightarrow \Delta$ and $\mathfrak{S}$ : $\mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (2) and (20) and (RO1), respectively. If for all $\mathfrak{\varsigma}, \theta \in \mathscr{M}(\mathscr{E}), \mathfrak{\omega} \in(0,1), \mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}, \mathfrak{S}$ satisfies the following conditions

$$
\begin{gather*}
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ; \\
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \varsigma \mathfrak{z}, \mathfrak{d})-\mathfrak{S}(\mathfrak{t}, \varsigma, \mathfrak{d}) \mathfrak{z}-\varsigma \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi((\varsigma, \mathfrak{z}, \mathfrak{d}, 0), \mathfrak{\omega}) ; \tag{32}
\end{gather*}
$$

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d} \theta)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \theta-\mathfrak{d} \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \theta), \mathfrak{\omega}) \succeq \Xi((\mathfrak{x}, 0, \mathfrak{o}, \theta), \mathfrak{\omega}) \tag{33}
\end{equation*}
$$

then $\mathfrak{S}$ is an S-BI-D.
Proof. To prove this theorem, we use Theorem 3. Referring to this theorem, for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$, there is a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$, which is defined as follows:

$$
\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\omega, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right),
$$

and satisfies (4). If we assume that for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$, we have $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then we can easily conclude that for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in Y, \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Again, referring to Theorem 3 , for every $\varsigma \in \mathscr{M}(\mathscr{E})$ and all $\mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathscr{T}(\mathfrak{t}, \varsigma \mathfrak{z}, \mathfrak{d})=\mathscr{T}(\mathfrak{t}, \varsigma, \mathfrak{d}) \mathfrak{z}+\varsigma \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$. Now, we consider $\mathfrak{x}=\sum_{i=1}^{n} \omega_{i} \mathfrak{x}_{i}$, for every $\mathfrak{x} \in \mathscr{E}, \omega_{i} \in \mathbb{C}, \mathfrak{x}_{i} \in \mathscr{M}(\mathscr{E})$ and according to the fact that $\mathscr{T}$ is $\mathbb{C}$-bilinear, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$, we have

$$
\begin{aligned}
\mathscr{T}(\mathfrak{t}, \mathfrak{x} \mathfrak{z}, \mathfrak{d}) & =\mathscr{T}\left(\mathfrak{t}, \sum_{i=1}^{n} \omega_{i} \mathfrak{x}_{i \mathfrak{z}}, \mathfrak{d}\right)=\sum_{i=1}^{n} \omega_{i} \mathscr{T}\left(\mathfrak{t}, \mathfrak{x}_{i} \mathfrak{z}, \mathfrak{d}\right)=\sum_{i=1}^{n} \omega_{i}\left(\mathscr{T}\left(\mathfrak{t}, \mathfrak{x}_{i}, \mathfrak{d}\right) \mathfrak{z}+\mathfrak{x}_{i} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})\right) \\
& =\left(\sum_{i=1}^{n} \omega_{i}\right) \mathscr{T}\left(\mathfrak{t}, \mathfrak{x}_{i}, \mathfrak{d}\right) \mathfrak{z}+\left(\sum_{i=1}^{n} \omega_{i} \mathfrak{x}_{i}\right) \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})=\mathscr{T}(\mathfrak{t}, \mathfrak{\zeta}, \mathfrak{d}) \mathfrak{z}+\mathfrak{x} \mathscr{T}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})
\end{aligned}
$$

With the same process, for all $\mathfrak{x}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we can show that $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d r})=$ $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{r}+\mathfrak{d} \mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r})$. Then, $\mathfrak{S}$ is an S-BI-D.

In the following, we investigate the optimal stability in MVFC- $\diamond-\mathrm{A}$.
Example 3. Consider the MVFC- $-A(\mathscr{E}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0, \infty)$, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we define $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that satisfies (27), (RO1) and the following conditions

$$
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ;
$$

$\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{\mathfrak { z }}, \mathfrak{d})-\mathfrak{S}(\mathfrak{t}, \varsigma, \mathfrak{d}) \mathfrak{z}-\varsigma \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \omega) \succeq \operatorname{MIN}\left(\mathscr{y}\left(-\digamma\left(1+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \mathfrak{\omega}\right)\right)$
$\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d} \theta)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \theta-\mathfrak{d} \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \theta), \mathfrak{\omega}) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(1+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \mathfrak{\omega}\right)\right)$
for all $\mathfrak{s}, \theta \in \mathscr{M}(\mathscr{E}), \mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$. Then, the $\mathfrak{S}$ is an S-BI-D.
Theorem 6. Consider the MVFC-ऽ-A $(\mathscr{E}, \mathfrak{W}, \circledast, \circledast)$, we define MVFF $\Xi: \mathscr{E}^{4} \rightarrow \Delta$ and $\mathfrak{S}$ : $\mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (7) and (20) and (RO1), respectively. If for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, $\mathfrak{S}$ satisfies (32), (33) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-D.

Proof. To prove this theorem, we use the process of proving Theorem 4.
Example 4. Consider the MVFC- $-A(\mathscr{E}, \mathfrak{W}, \circledast, \circledast), \epsilon<1$ and $\digamma \in(0, \infty)$, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we define $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ that satisfies (27) and (RO1). If we assume that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, $\mathfrak{S}$ satisfies in conditions (34), (35) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-D.

In the next subsection, we seek to find the best approximation for S-BI-H in MVFB-A using the direct technique. Therefore, in the following, we present the relevant theorems to prove our result.

### 2.3. Optimal Stability of S-BI-H in MVFB-A

Theorem 7. Considering MVFB-A $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \mathscr{M}=\mathscr{E}, \mathcal{N}=\mathscr{F}$, we define MVFF $\Xi: \mathscr{M}^{4} \rightarrow$ $\Delta$ and $S-O \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (2), (20), and (RO1), respectively. Therefore, there exists a C-BI-SO $\mathfrak{Y}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that it satisfies (4). In addition, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0,1)$, if we assume $\mathfrak{S}$ with the following conditions

$$
\begin{gather*}
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ; \\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x} \mathfrak{z}, \mathfrak{d}^{2}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\omega}\right) \succeq \Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, 0), \mathfrak{\omega}) ;  \tag{36}\\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}^{2}, \mathfrak{d} \mathfrak{r}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r}), \mathfrak{\omega}\right) \succeq \Xi((\mathfrak{x}, 0, \mathfrak{d}, \mathfrak{r}), \mathfrak{\omega}), \tag{37}
\end{gather*}
$$

then $\mathfrak{S}$ is an S-BI-H.
Proof. To prove this theorem, we can follow the proof of Theorem 3. Then, for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$, there exists a C-BI-SO $\mathfrak{Y}: Y \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ in the form of $\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=$ $\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)$. For every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in Y$, we suppose that $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Then, for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Now, according to (2), for any $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0,1)$, we obtain

$$
\begin{aligned}
& \mathfrak{W}\left(\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x} z, \mathfrak{d}^{2}\right)-\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\infty}\right) \\
= & \lim _{n \rightarrow \infty} \mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n} \cdot 2^{n}}, \mathfrak{d}^{2}\right)-\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right) \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{d}\right), \frac{\mathfrak{o}}{4^{n}}\right) \\
\succeq & \lim _{n \rightarrow \infty} \Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{v}{2^{n}}, \mathfrak{d}, 0\right), \frac{\mathfrak{\infty}}{4^{n}}\right) \\
= & 1 .
\end{aligned}
$$

Therefore, for every $\mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}^{2}\right)=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$. In the same way, we can show that for every $\mathfrak{x}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}^{2}, \mathfrak{d r}\right)=$ $\mathfrak{Y}(\omega, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\omega, \mathfrak{x}, \mathfrak{r})$, and then $\mathfrak{S}$ is an S-BI-D.

Example 5. Consider the MVFB- $A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \mathscr{M}=\mathscr{E}, \mathcal{N}=\mathscr{F}, \epsilon<1$ and $\digamma \in(0, \infty)$, we define $\mathfrak{S}: Y \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ in such a way that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (27), and (RO1). Therefore, there exists a C-BI-SO $\mathfrak{Y}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that it satisfies (28). In addition, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0,1)$, if we assume $\mathfrak{S}$ with the following conditions

$$
\begin{gather*}
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ; \\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x} \mathfrak{z}, \mathfrak{d}^{2}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \omega\right) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \boldsymbol{\omega}\right)\right)  \tag{38}\\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}^{2}, \mathfrak{d r}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r}), \omega\right) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(\digamma\left(\|\mathfrak{x}\|^{\epsilon}\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}\right), \omega\right)\right)\right. \tag{39}
\end{gather*}
$$

then $\mathfrak{S}$ is an S-BI-H.

Theorem 8. Consider the $\operatorname{MVFB}-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \mathscr{M}=\mathscr{E}, \mathcal{N}=\mathscr{F}$; we define MVFF $\Xi:$ $M^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (13), (20), and (RO1), respectively. Therefore, there exists a C-BI-SO $\mathfrak{Y}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that it
satisfies (14). In addition, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$, if we assume $\mathfrak{S}$ satisfies conditions (36), (37) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

Proof. To prove this theorem, we use Theorem 7.
Example 6. Consider the $\operatorname{MVFB}-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \mathscr{M}=\mathscr{E}, \mathcal{N}=\mathscr{F}, \epsilon<1$ and $\digamma \in(0, \infty)$; we define $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ in such a way that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). Therefore, there exists a US-C-bilinear operator $\mathfrak{Y}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that it satisfies (31). In addition, for all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\boldsymbol{\omega} \in(0,1)$, if we assume $\mathfrak{S}$ satisfies conditions (38), (39) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

In the rest of this section, we prove our results in MVFC- $\diamond-\mathrm{A}$. Then, from here on, we consider $\mathscr{E}$ as a MVFC- $\diamond$-A with unit member $e$ and unit group $\mathscr{M}(\mathscr{E})=\left\{\varsigma \in \mathscr{E}: \varsigma^{\diamond} \varsigma=\right.$ $\left.\varsigma t^{\diamond}=e\right\}$.

### 2.4. Optimal Stability of S-BI-H in MVFC- $\diamond-A$

Definition 9. Consider the MVFC- $-A \mathscr{E}$ and $\mathscr{F}$; we say that a $\mathrm{C}-\mathrm{BI}-\mathrm{SO} \mathfrak{Y}: \mathrm{Y} \times \mathscr{E} \times \mathscr{E} \rightarrow \mathscr{F}$ is an S-BI-H if the following conditions hold for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$
(SH1) $\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}^{2}\right)=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$;
(SH2) $\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}^{2}, \mathfrak{d r}\right)=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{r})$;
(SH3) $\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}^{*}, \mathfrak{d}\right)=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})^{*}$.

Theorem 9. Consider the $\operatorname{MVFC-}-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \mathscr{M}=\mathscr{E}$, we define $M V F F \Xi: \mathscr{M}^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (2), (20), and (RO1), respectively. If we assume that for every $\varsigma, \theta \in \mathscr{M}(\mathscr{E})$ and all $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty), \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ satisfies the following conditions

$$
\begin{gather*}
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ; \\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{\mathfrak { z }}, \mathfrak{d}^{2}\right)-\mathfrak{S}(\omega, t, p) \mathfrak{S}(\omega, \mathfrak{z}, \mathfrak{d}), \mathfrak{\omega}\right) \succeq \Xi((0, \mathfrak{z}, \mathfrak{d}, 0), \mathfrak{\omega}),  \tag{40}\\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}^{2}, \mathfrak{d} \theta\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \theta), \mathfrak{\omega}\right) \succeq \Xi((\mathfrak{x}, 0,0, \theta), \omega),  \tag{41}\\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}^{*}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})^{*}, \mathfrak{\omega}\right) \succeq \Xi((\mathfrak{x}, 0, \mathfrak{d}, \theta), \omega), \tag{42}
\end{gather*}
$$

then $\mathfrak{S}$ is an S-BI-H.
Proof. To prove this theorem, we can follow the proof of Theorem 3. Then, for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$, there exists a C-BI-SO $\mathfrak{Y}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ in the form of $\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=$ $\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)$. For every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in Y$, we suppose that $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Then, for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in Y$, we have $\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Now, referring to Theorem 7, for every $\varsigma, \theta \in \mathscr{M}(\mathscr{E})$ and all $\theta, \mathrm{d} \in \mathscr{E}$ and $\mathfrak{\omega} \in \mathfrak{S}$, we have $\mathfrak{Y}\left(\mathfrak{t}, \varsigma \mathfrak{z}, \mathfrak{d}^{2}\right)=$ $\mathfrak{Y}(\mathfrak{t}, \mathfrak{\varsigma}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})$. Now, for every $\mathfrak{x} \in \mathscr{E}$ and $\omega_{i} \in \mathbb{C}, \mathfrak{x}_{i} \in \mathscr{M}(\mathscr{E})$, we consider $\mathfrak{x}=\sum_{i=1}^{n} \omega_{i} \mathfrak{x}_{i}$ and according to the fact that $\mathfrak{Y}$ is $\mathbb{C}$-linear, for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d} \in \mathscr{E}$ and $\omega \in \Omega$, we have

$$
\begin{aligned}
\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}^{2}\right) & =\mathfrak{Y}\left(\mathfrak{t}, \sum_{i=1}^{n} \omega_{i} \mathfrak{x}_{i} \theta, \mathfrak{d}^{2}\right)=\sum_{i=1}^{n} \omega_{i} \mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}_{i} \theta, \mathfrak{d}^{2}\right)=\sum_{i=1}^{n} \omega_{i}\left(\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}_{i}, \mathfrak{d}\right) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})\right) \\
& =\left(\sum_{i=1}^{n} \omega_{i}\right) \mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}_{i}, \mathfrak{d}\right) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d})=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}) .
\end{aligned}
$$

With the same process, for each $\mathfrak{x}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, we have $\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d r})=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{Y}$ $(\mathfrak{t}, \mathfrak{x}, \mathfrak{r})$. Then

$$
\begin{aligned}
\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}^{*}, \mathfrak{d}\right) & =\mathfrak{Y}\left(\mathfrak{t}, \sum_{i=1}^{n} \bar{\omega}_{i} u_{i}^{*}, p\right)=\sum_{i=1}^{n} \bar{\omega}_{i} \mathfrak{Y}\left(\mathfrak{t}, \mathfrak{x}_{i}^{*}, \mathfrak{d}\right)=\sum_{i=1}^{n} \bar{\omega}_{i}\left(\mathfrak{Y}\left(\mathfrak{t}, \mathfrak{d}_{i}, \mathfrak{d}\right)\right)^{*} \\
& =\mathfrak{Y}\left(\mathfrak{t}, \sum_{i=1}^{n} \bar{\omega}_{i} \mathfrak{x}_{i}, \mathfrak{d}\right)^{*}=\mathfrak{Y}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})^{*},
\end{aligned}
$$

and therefore, $\mathfrak{S}$ is an S-BI-H.
Example 7. Consider the MVFC- $--A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0, \infty)$, we define $S-O$ $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). If we assume that $\mathfrak{S}$ for each $\varsigma, \theta \in \mathscr{M}(\mathscr{E})$ and every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$ satisfies conditions

$$
\begin{gather*}
\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) ; \\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{\mathfrak { z }}, \mathfrak{d}^{2}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{\zeta}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{z}, \mathfrak{d}), \mathfrak{\omega}\right)  \tag{43}\\
\mathfrak{M I N}\left(\mathscr{Y}\left(-\digamma\left(1+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \mathfrak{\omega}\right)\right)  \tag{44}\\
\mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \mathfrak{x}^{2}, \mathfrak{d} \theta\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}) \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \theta), \mathfrak{\omega}\right)  \tag{45}\\
\mathfrak{W I N}\left(\mathscr{Y}\left(-\digamma\left(1+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \mathfrak{\omega}\right)\right) \\
\left.\mathfrak{W}\left(\mathfrak{t}, \mathfrak{x}^{*}, \mathfrak{d}\right)-\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})^{*}, \mathfrak{\omega}\right) \succeq \operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(1+\|\mathfrak{z}\|^{\epsilon}\right)\|\mathfrak{d}\|^{\epsilon}, \mathfrak{\omega}\right)\right)
\end{gather*}
$$

then $\mathfrak{S}$ is an S-BI-H.
Proof. For proof, we refer to Theorem 9. For each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{\omega} \in(0,1)$ and $\xi=2^{\iota-1}$, we consider the MVFF as

$$
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega)=\operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}\right), \omega\right)\right)
$$

and use it in Theorem 9.
Theorem 10. Consider MVFC-৫-A $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\mathscr{M}=\mathscr{E}$, we define MVFF $\Xi: \mathscr{M}^{4} \rightarrow$ $\Delta$ and S-O S : $\mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ in such a way that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (13), (20), and (RO1), respectively. If we assume that for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}, \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ satisfies conditions (40)-(42) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-D.

Proof. The proof process is similar to Theorem 9.
Example 8. Consider MVFC-৫-A $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0, \infty)$; we define $S$ - $O \mathfrak{S}$ : $\mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). If we assume that $\mathfrak{S}$ for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in Y$ satisfies conditions (43)-(45) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

## 3. FPM for Optimal Stability of S-BI-D and S-BI-H in MVFB-A and MVFC- $\diamond-$ A

In this section, using the FPM and considering inequality (1), we investigate the optimal stability for S-BI-D and S-BI-H in MVFB-A and MVFC- $\diamond-\mathrm{A}$. Therefore, before starting the main proofs, we refer the alternative FPT from Diaz-Margoliz ([9], Theorem).
3.1. Optimal Stability of S-BI-D in MVFB-A

Theorem 11. Consider the MVFB-A $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\xi \in(0,1)$; we define MVFF $\Xi$ : $\mathscr{M}^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: Y \times M^{2} \rightarrow \mathcal{N}$ that they satisfy in conditions

$$
\begin{equation*}
\Xi\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{z}}{2}, \mathfrak{d}, \mathfrak{r}, \frac{\xi}{2} \mathfrak{W}\right) \succeq \Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \mathfrak{w}) \tag{46}
\end{equation*}
$$

(RO1), and (3) for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{o}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, respectively. Therefore, there exists a $C-B I-S O \mathfrak{A}: Y \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ such that for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, the following inequality holds

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{w}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4(1-\xi)}{\xi} \mathfrak{\infty}\right) \tag{47}
\end{equation*}
$$

Proof. Referring to Theorem 1 and considering (6), also by placing $\mathfrak{r}=0$ and $\mathfrak{z}=\mathfrak{x}$ in (3), for any $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$, we have

$$
\begin{equation*}
\mathfrak{W}\left(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right), \omega\right) \succeq \Xi\left(\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{x}}{2}, \mathfrak{d}, 0\right), 2 \omega\right) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \mathfrak{\omega}}{\xi}\right) \tag{48}
\end{equation*}
$$

We consider the set $\mathscr{U}:=\left\{\mathscr{R}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathscr{N}, \mathscr{R}(\mathfrak{t}, \mathfrak{x}, 0)=\mathscr{R}(\mathfrak{t}, 0, \mathfrak{d})=0 \forall \mathfrak{x}, \mathfrak{d} \in\right.$ $\mathscr{M}, \mathfrak{t} \in \mathrm{Y}\}$ and define the following complete metric [10] on this set

$$
\delta(\mathfrak{S}, \mathscr{R})=\inf \left\{v \in \mathbb{R}_{+}: \mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathscr{R}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{\mathfrak{\omega}}{v}\right), \forall \mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}, \boldsymbol{\omega} \in(0, \infty)\right\}
$$

Now, for any $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in Y$, we define stochastic linear mapping $\Phi: \mathscr{U} \rightarrow \mathscr{U}$ as $\Phi \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right)$. If we assume $\delta(\mathfrak{S}, \mathscr{R})=\boldsymbol{7}$, for every $\mathfrak{S}, \mathscr{R} \in \mathscr{U}$, then for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in Y$, we have $\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathscr{R}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\infty}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{\mathscr{W}}{\boldsymbol{W}}\right)$. Considering this inequality, put $\left.\frac{\mathcal{D}}{2\rceil}=\frac{\xi}{2}\left(\frac{\mathcal{W}}{\xi}\right\rceil\right)$, use the second condition of MVFB-S and (46); then, for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$, we obtain

$$
\begin{aligned}
& \mathfrak{W}(\Phi \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\Phi \mathscr{R}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \\
= & \mathfrak{W}\left(2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right)-2 \mathscr{R}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right), \omega\right) \\
\succeq & \Xi\left(\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{x}}{2}, \mathfrak{d}, 0\right), \frac{\mathfrak{o}}{27}\right) \\
\succeq & \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{\mathfrak{\infty}}{\mathfrak{\xi}\rceil}\right),
\end{aligned}
$$

therefore, $\delta(\Phi \mathfrak{S}, \Phi \mathscr{R}) \leq \xi\rceil$ and as a result for each $\mathfrak{S}, \mathscr{R} \in \mathscr{U}, \delta(\Phi \mathfrak{S}, \Phi \mathscr{R}) \leq \xi \delta(\mathscr{F}, \mathscr{R})$. Now, using (48), for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, we obtain

$$
\mathfrak{W}\left(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right), \omega\right) \succeq \Xi\left(\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{x}}{2}, \mathfrak{d}, 0\right), 2 \omega\right) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4 \omega}{\mathfrak{\xi}}\right)
$$

and this means $\delta(\mathfrak{S}, \Phi \mathfrak{S}) \leq \frac{\xi}{4}$. Then, all the conditions of the theorem [9, Theorem] are satisfied, and this means that there is a $\mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ such that
(1) $\Phi$ has a fixed point such as $\mathfrak{A}$, which is $\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})=2 \mathfrak{A}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2}, \mathfrak{d}\right)$, and is in the set of $H=\{\mathscr{R} \in \mathscr{U}, \delta(\mathfrak{S}, \mathscr{R})<\infty\}$, for any $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$.
(2) When $n$ tends to $\infty$, we have $\delta\left(\Phi^{n} \mathfrak{S}, \mathfrak{A}\right) \rightarrow 0$, i.e., for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$, $\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)=\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$.
(3) For all $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$, we have $\delta(\mathfrak{S}, \mathfrak{A}) \leq \frac{1}{1-\tilde{\xi}} \delta(\mathfrak{S}, \Phi \mathfrak{S})$, that is

$$
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{w}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4(1-\mathfrak{\xi}) \mathfrak{w}}{\xi}\right)
$$

therefore, (47) is obtained. Now, using (3) and (46), for every $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\omega \in(0, \infty)$, we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \mathfrak{W}\left(\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right)+\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}}, \mathfrak{d}-\mathfrak{r}\right)+\mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}-\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right), \frac{\mathfrak{\infty}}{2^{n}}\right) \\
\succeq & \lim _{n \rightarrow \infty} \mathfrak{W}\left(\xi\left(2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}+\mathfrak{z}}{2^{n}} \mathfrak{d}-\mathfrak{r}\right)+2 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}-\mathfrak{z}}{2^{n}}, \mathfrak{d}+\mathfrak{r}\right)-4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)+4 \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{r}\right)\right), \frac{\mathfrak{o}}{2^{n}}\right) \\
\circledast & \Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{d}, 0\right), \frac{\mathfrak{o}}{2^{n}}\right) \\
\succeq & \mathfrak{W}(\xi(2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})+4 \mathfrak{A}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \omega) .
\end{aligned}
$$

On the other hand, considering that when $n$ tends to $\infty$, the inequality $\Xi\left(\left(\frac{\mathfrak{x}}{2^{n}}, \frac{\mathfrak{z}}{2^{n}}, \mathfrak{d}, 0\right), \frac{\omega}{2^{n}}\right) \succeq$ $\Xi\left((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, 0), \frac{\mathscr{W}}{\mathcal{\zeta}^{n}}\right)$ tends to $\mathbf{1}$ tends to $\mathbf{1}$, we have

$$
\begin{aligned}
& \mathfrak{W}(\mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})+\mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}-\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \omega) \\
\succeq & \mathfrak{W}(\xi(2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}+\mathfrak{z}, \mathfrak{d}-\mathfrak{r})+2 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}-\mathfrak{z}, \mathfrak{d}+\mathfrak{r})-4 \mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})+4 \mathfrak{A}(\mathfrak{t}, \mathfrak{z}, \mathfrak{r})), \boldsymbol{\infty}),
\end{aligned}
$$

for every $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$ and $\mathfrak{\omega} \in(0, \infty)$. Therefore, according to Lemma 1 , there exists a $\mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathscr{N}$ which is a C-BI-SO.

Theorem 12. Consider the MVFB-A $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\mathscr{M}=\mathscr{E}$, we define MVFF $\Xi$ : $\mathscr{M}^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (46), (20), and (RO1), respectively. Then, we can find a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ that satisfies (47). If we assume that for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}, \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ satisfies conditions (21), (22) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\omega, u, p)$, and then $\mathfrak{S}$ is an S-BI-D.

Proof. For proof, we consider (20) and put $\omega=\rho=1$ in it. Considering Theorem 11, for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, there exists a C-BI-SO $\mathscr{T}: Y \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ of the form $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=$ $\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)$,, which satisfies (47). Now, in (20), we put $\mathfrak{z}=\mathfrak{r}=0$ and for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$ and all $\omega, \rho \in \mathbb{D}^{1}$ obtain $\mathfrak{S}(\mathfrak{t}, \omega \mathfrak{x}, \rho \mathfrak{d})=\omega \rho \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$. Therefore, according to Lemma $2, \mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ is C-BI-SO.

Example 9. Considering $\operatorname{MVFB}-A(\mathbb{M}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0,1)$, we define $\mathfrak{S}: Y \times$ $\mathscr{E}^{2} \rightarrow \mathscr{F}$ in such a way that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). Therefore, there exists a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that it satisfies (28). In addition, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, if we assume $\mathfrak{S}$ satisfies conditions (29), (30) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

Proof. We consider the following function

$$
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega)=\operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}, \omega\right)\right)\right.
$$

where $\xi=2^{1-\epsilon}$ and $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}$. We consider Theorem 12 and use this function in this theorem.

In the following, we consider the MVFC- $\diamond$-A to prove the results.

### 3.2. Optimal Stability of S-BI-D in MVFC-®-A

Theorem 13. Consider the MVFC-৫-A $(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\xi \in(0,1)$; we define MVFF $\Xi: \mathscr{M}^{4} \rightarrow \Delta$ and S-O S : $\mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ that they satisfy conditions

$$
\begin{equation*}
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega) \succeq \Xi\left(\left(\frac{\mathfrak{x}}{2}, \frac{\mathfrak{z}}{2}, \mathfrak{d}, \mathfrak{r}\right), \frac{\mathscr{\omega}}{2 \mathfrak{\xi}}\right) \tag{49}
\end{equation*}
$$

(RO1), and (3) for each $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{M}, \mathfrak{t} \in \mathrm{Y}$, respectively. Therefore, there exists an C-BI-SO $\mathfrak{A}: \mathrm{Y} \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ such that for every $\mathfrak{x}, \mathfrak{d} \in \mathbb{M}, \mathfrak{t} \in \mathrm{Y}$ and $\boldsymbol{\omega} \in(0, \infty)$, the following inequality holds

$$
\begin{equation*}
\mathfrak{W}(\mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})-\mathfrak{A}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}), \mathfrak{\omega}) \succeq \Xi\left((\mathfrak{x}, \mathfrak{x}, \mathfrak{d}, 0), \frac{4(1-\mathfrak{\xi})}{\xi} \mathfrak{\omega}\right) . \tag{50}
\end{equation*}
$$

Proof. The proof is exactly the same as the process of proving Theorem 11. Here, considering the metric $(\mathscr{U}, \delta)$ introduced in Theorem 11, we define $\Phi: \mathscr{U} \rightarrow \mathscr{U}$ as follows:

$$
\Phi \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=\frac{1}{2} \mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})
$$

where $\mathfrak{x}, \mathfrak{d} \in \mathscr{M}$ and $\mathfrak{t} \in \mathrm{Y}$.
Theorem 14. Consider the MVFC- $-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\mathscr{M}=\mathscr{E}$; we define $M V F F \Xi$ : $\mathscr{M}^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ in such a way that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (49), (20), and (RO1), respectively. Then, we can find a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ that satisfies (50). If we assume that for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}, \mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ satisfy conditions (21) and (22) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-D.

Proof. The proof process is exactly the same as the proof process of Theorem 12.
Example 10. Considering the $\operatorname{MVFC-}-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0,1)$, we define $S-O$ $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). Therefore, there exists a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that it satisfies (31). In addition, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, if we assume $\mathfrak{S}$ satisfies conditions (29), (30) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-D.

Proof. We consider the following function

$$
\Xi((\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r}), \omega)=\operatorname{MIN}\left(\mathscr{Y}\left(-\digamma\left(\|\mathfrak{x}\|^{\epsilon}+\|\mathfrak{z}\|^{\epsilon}\right)\left(\|\mathfrak{d}\|^{\epsilon}+\|\mathfrak{r}\|^{\epsilon}, \omega\right)\right)\right.
$$

where $\mathfrak{\xi}=2^{\epsilon-1}$ and $\mathfrak{x}, \mathfrak{z}, \mathfrak{d}, \mathfrak{r} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}, \boldsymbol{\omega} \in(0, \infty)$. We consider Theorem 14 and use this function in this theorem.

### 3.3. Optimal Stability of S-BI-H in MVFC- $\diamond-A$

Theorem 15. Consider the MVFC- $-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\mathscr{M}=\mathscr{E}$, we define MVFF $\Xi$ : $\mathscr{M}^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: \mathrm{Y} \times \mathbb{M}^{2} \rightarrow \mathcal{N}$ such that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$, they satisfy in conditions (46), (20) and (RO1), respectively. Therefore, there exists a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that satisfying (47). If for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{U}$ and $\mathfrak{t} \in \mathrm{Y}$, we assume that $\mathfrak{S}$ satisfies (36), (37) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

Proof. To prove this theorem, we refer to Theorem 12. Using Theorem 12, for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, there is a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ in the form $\mathscr{T}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d}):=\lim _{n \rightarrow \infty} 2^{n} \mathfrak{S}\left(\mathfrak{t}, \frac{\mathfrak{x}}{2^{n}}, \mathfrak{d}\right)$. The continuation of the proof follows from Theorem 12.

Example 11. Consider the MVFC- $\diamond-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0,1)$, we define $S-O$ $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). Therefore, there exists a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that it satisfies (28). In addition, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, if we assume $\mathfrak{S}$ satisfies conditions (38), (39) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

Theorem 16. Consider the $M V F C-\diamond-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast)$ in which $\mathscr{M}=\mathscr{E}$; we define $M V F F \Xi$ : $M^{4} \rightarrow \Delta$ and $S-O \mathfrak{S}: Y \times \mathscr{M}^{2} \rightarrow \mathcal{N}$ in such a way that for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}, \mathfrak{t} \in \mathrm{Y}$, they satisfy conditions (49), (20) and (RO1), respectively. Therefore, there exists a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ that satisfies (50). If for every $\mathfrak{x}, \mathfrak{d} \in \mathscr{U}$ and $\mathfrak{t} \in Y$, we assume that $\mathfrak{S}$ satisfies (36), (37) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, and then $\mathfrak{S}$ is an S-BI-H.

Proof. The process of proving this theorem is similar to Theorem 15.
Example 12. Consider the MVFC- $\diamond-A(\mathscr{M}, \mathfrak{W}, \circledast, \circledast), \epsilon>2$ and $\digamma \in(0,1)$; we define $S-O$ $\mathfrak{S}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{F}$ such that for each $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$ satisfy conditions (27) and (RO1). Therefore, there exists a C-BI-SO $\mathscr{T}: \mathrm{Y} \times \mathscr{E}^{2} \rightarrow \mathscr{E}$ such that it satisfies (31). In addition, for all $\mathfrak{x}, \mathfrak{d} \in \mathscr{E}$ and $\mathfrak{t} \in \mathrm{Y}$, if we assume $\mathfrak{S}$ satisfies conditions (38), (39) and $\mathfrak{S}(\mathfrak{t}, 2 \mathfrak{x}, \mathfrak{d})=2 \mathfrak{S}(\mathfrak{t}, \mathfrak{x}, \mathfrak{d})$, then $\mathfrak{S}$ is an S-BI-H.

## 4. Conclusions

In this paper, we applied the special functions and used the concept of aggregation functions to obtain a new class of control functions. This new form of fuzzy control functions helped us to obtain an optimal approximation of S-BI-H and S-BI-D in MVFC- - -A.

Author Contributions: All authors contributed to the study's design and coordination, drafted the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.
Institutional Review Board Statement: This article contains no surveys on human or animal participants conducted by any of the authors. All authors read and approved the final version.

Informed Consent Statement: Informed consent was obtained from all the individuals who participated in this paper.

Data Availability Statement: Not applicable.
Conflicts of Interest: The authors declare no conflict of interest.

## Abbreviations

The following abbreviations are used in this manuscript:

| MVFN-S | matrix valued fuzzy normed spaces <br> BI-AROI |
| :--- | :--- |
| bi-additive random operator inequality |  |

## References

1. Hyers, D.H. On the stability of the linear functional equation. Proc. Nat. Acad. Sci. USA 1941, 27, 222-224. [CrossRef]
2. Eidinejad, Z.; Saadati, R.; Mesiar, R. Optimum Approximation for $\varsigma$-Lie Homomorphisms and Jordan $\varsigma$-Lie Homomorphisms in $\varsigma$-Lie Algebras by Aggregation Control Functions. Mathematics 2022, 10, 1704. [CrossRef]
3. Eidinejad, Z.; Saadati, R.; de la Sen, M. Radu-Mihet Method for the Existence, Uniqueness, and Approximation of the $\Xi$-Hilfer Fractional Equations by Matrix-Valued Fuzzy Controllers. Axioms 2021, 10, 63. [CrossRef]
4. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and applications of fractional differential equations. In North-Holland Mathematics Studies; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204, p. xvi+523.
5. Kiryakova, V. Some special functions related to fractional calculus and fractional (non-integer) order control systems and equations. Facta Univ. Ser. Autom. Control Robot. 2008, 7, 79-98.
6. Eidinejad, Z.; Saadati, R. Hyers-Ulam-Rassias-Wright stability for fractional oscillation equation. Discret. Dyn. Nat. Soc. 2022, 2022, 9412009. [CrossRef]
7. Park, C. Bi-Additive s-Functional Inequalities and Quasi-*-Multipliers on Banach Algebras. Mathematics 2018, 6, 31. [CrossRef]
8. Bae, J.-H.; Park, W.-G. Approximate bi-homomorphisms and bi-derivations in C*-ternary algebras. Bull. Korean Math. Soc. 2010, 47, 195-209. [CrossRef]
9. Diaz, J.B.; Margolis, B. A fixed point theorem of the alternative, for contractions on a generalized complete metric space. Bull. Amer. Math. Soc. 1968, 74, 305-309. [CrossRef]
10. Miheţ, D.; Radu, V. On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl. 2008, 343, 567-572. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.

