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On the coincidence of measure-based decomposition and superdecomposition integrals

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Abstract

This paper introduces two types of preorders on the system of all non-empty sets of collections (i.e., the set of all decomposition systems) based on a fixed monotone measure μ. Each of them refines the previous two kinds of preorders of decomposition systems. By means of these two new preorders of decomposition systems we investigate the coincidences of decomposition integrals and that of superdecomposition integrals, respectively. The *generalized integral equivalence theorem* is shown in the general framework involving an ordered pair of decomposition systems. This generalized theorem includes as special cases all the previous results related to the coincidences among the Choquet integral, the concave (or convex) integral and the pan-integrals. Thus, a unified approach to the coincidences of several well-known decomposition and superdecomposition integrals is presented.

Keywords: Monotone measure; Decomposition integral; Choquet integral; Concave integral; Pan-integral

1. Introduction

For any two types of integral, they differ once there is a measure μ and a function f leading to different integral outputs. As a typical example, recall the Choquet integral [1] and the Sugeno integral [25] applied to $[0, 1]$-valued monotone measures and functions. However, these integrals coincide once we consider an arbitrary $\{0, 1\}$-valued monotone measure and then, in the case of finite spaces, we obtain just lattice polynomials. More, in such a case, both these integrals coincide also with any copula-based integral introduced in [4]. In this paper, we focus on possible coincidences in the framework of integrals based on (sub-, super-) decompositions of the integrated functions.

The decomposition integral (Even and Lehrer [3]) forms a common framework for the well-known nonlinear integrals: the concave integral, the Choquet integral and the pan-integral from below, etc. As a counterpart of de-

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composition integrals, Mesiar et al. [15] introduced superdecomposition integral including the convex integral, the Choquet integral and the pan-integral from above, etc. The decomposition (resp. superdecomposition) integral is based on the system of collections and sub-(resp. super-)decompositions of function being considered. In general, two different decomposition systems induce different decomposition (or superdecomposition) integrals. Recall three important decomposition integrals: the Choquet integral, the concave integral and the pan-integral from below, including their corresponded superdecomposition integrals: the Choquet integral, the convex integral and the pan-integral from above, they are based on chains of sets, arbitrary finite set systems and finite partitions, respectively. These six types of integrals coincide with the Lebesgue integral for σ-additive measures. But for a general monotone measure, they are significantly different from each other.

In recent years the relationships among these three decomposition integrals (resp. superdecomposition integrals) were investigated and many interesting results were obtained. Lehrer and Teper [6] showed that the Choquet integral coincides with the concave integral if and only if the monotone measure μ is convex (or supermodular). Mesiar et al. [15] presented a corresponding result for the Choquet integrals and the convex integrals by using submodularity. In [17,20,22], by using the subadditivity, superadditivity and the characteristics of minimal atoms, we presented respectively some necessary and/or sufficient conditions that the concave integral coincides with the pan-integral from below, and that the convex integral coincides with the pan-integral from above. In [18,21] we showed that the (M)-property (which was proposed by Mesiar et al. [13]) is a sufficient condition that the Choquet integral coincides with the pan-integral from below. Note that in the case of finite spaces the condition is also necessary. Lv et al. [11] introduced the so-called *dual (M)-property* of monotone measures, and showed that it is sufficient for the coincidence of the Choquet integral and the pan-integral from above and, for finite space it is also necessary.

We will investigate the relationships between two decomposition integrals (resp. the superdecomposition integrals). Given a monotone measure space $(X, \mathcal{A}, \mu)$, we introduce two new types of preorders on the system of all non-empty sets of collections with respect to μ (a collection is a finite set systems from $\mathcal{A} \setminus \{\emptyset\}$, a set of collections is also called a decomposition system). Each of these two preorders refines the previous two kinds of preorders of decomposition systems: the standard inclusion ordering "$\subset$" and the preorder "$\preceq$" introduced by Mesiar and Stupňanová in [16]. By means of these two new preorders of decomposition systems we study the coincidences of decomposition integrals and of superdecomposition integrals. We present the so-called *generalized integral equivalence theorem* in the general framework relating to an ordered pair of decomposition systems. As special cases this generalized theorem includes all of the above mentioned results related to the coincidences among the Choquet integral, the concave (or convex) integral and the pan-integral from below (or from above). Thus, a unified approach to the coincidences of several well-known decomposition and superdecomposition integrals is presented.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X and $(X, \mathcal{A})$ denote a measurable space. Denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$. Let $\mathcal{F}^+$ be the set of all $\mathcal{A}$-measurable functions $f : X \to \mathbb{R}_+$, $\mathcal{F}_b^+$ be the set of all bounded $\mathcal{A}$-measurable functions $f : X \to \mathbb{R}_+$, and let χ_A denote the characteristic functions of $A \in \mathcal{A}$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{A}$.

2.1. Monotone measures

A set function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is called a *monotone measure* on $(X, \mathcal{A})$ if it satisfies the following conditions:
(i) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;
(ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.
The triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space [23].
Let $\mathcal{M}$ denote the set of all monotone measures defined on $(X, \mathcal{A})$.
A monotone measure μ is said to be
(i) *subadditive*, if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$;
(ii) *submodular* (or *concave*), if $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$;
(iii) *superadditive*, if $\mu(A \cup B) \geq \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;
(iv) *supermodular* (or *convex*), if $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$ holds for any $A, B \in \mathcal{A}$.
We recall the concept of (M)-property of monotone measure, its original idea was proposed by Mesiar, see [13].

A monotone measure μ is said to have (M)-property (resp. dual (M)-property), if for any $A, B \in \mathcal{A}$ with $A \subset B$, there exists $C \in \mathcal{A}$ such that $C \subset A$ (resp. $A \subset C \subset B$), $\mu(C) = \mu(A)$ and $\mu(B) = \mu(C) + \mu(B \setminus C)$.

The (M)-property implies superadditivity ([9,13]) and the dual (M)-property implies subadditivity ([11]).

2.2. Decomposition and superdecomposition integrals

We recall decomposition integral which was introduced by Even and Lehrer [3] and superdecomposition integral which was introduced by Mesiar et al. [15].

A non-empty set $\mathcal{H}$ of collections from $\mathcal{A} \setminus \{\emptyset\}$ is called a decomposition system on $(X, \mathcal{A})$. We denote by $\mathbb{X}$ the set of all such decomposition systems.

Let $\mathcal{H} \in \mathbb{X}$ be fixed. The mapping $I_{\mathcal{H}} : \mathcal{M} \times \mathcal{F}^+ \to \overline{\mathbb{R}}_+$ given by

$$I_{\mathcal{H}}(\mu, f) = \sup\left\{\sum_{i\in J} a_i \mu(A_i) : (A_i)_{i\in J} \in \mathcal{H}, \sum_{i\in J} a_i \chi_{A_i} \le f\right\}, \tag{2.1}$$

where all constants $a_i \ge 0$, is called a *decomposition integral* ([3]).

The mapping $I^{\mathcal{H}} : \mathcal{M} \times \mathcal{F}_b^+ \to \overline{\mathbb{R}}_+$ given by

$$I^{\mathcal{H}}(\mu, f) = \inf\left\{\sum_{i\in J} a_i \mu(A_i) : (A_i)_{i\in J} \in \mathcal{H}, \sum_{i\in I} a_i \chi_{A_i} \ge f\right\}, \tag{2.2}$$

where all constants $a_i \ge 0$, is called a *superdecomposition integral* ([15]).

Note that although the superdecomposition integral was introduced in a dual way, it has some properties that are similar and or dual with respect to decomposition integrals, but it also has some significant differences, i.e., it is not fully dual to the decomposition integrals (see [15]).

The decomposition and superdecomposition integrals depend on a decomposition system $\mathcal{H} \in \mathbb{X}$ (observe the formulas (2.1) and (2.2)), and, as we will see later, several well-known integrals are specific decomposition integrals or superdecomposition integrals.

Let $\mu \in \mathcal{M}$ be fixed and $f \in \mathcal{F}^+$ (in the case of superdecomposition integrals, the considered functions are always supposed to belong to $\mathcal{F}_b^+$).

(i) Let $\mathcal{H}_{Ch} = \{\mathcal{C} : \mathcal{C}$ is a finite chain in $\mathcal{A} \setminus \{\emptyset\}\}$. Then both $I_{\mathcal{H}_{Ch}}(\mu, f)$ and $I^{\mathcal{H}_{Ch}}(\mu, f)$, define the *Choquet integral* ([1,15]) of $f \in \mathcal{F}_b^+$ with respect to μ, i.e.,

$$I_{\mathcal{H}_{Ch}}(\mu, f) = I^{\mathcal{H}_{Ch}}(\mu, f) = \int_0^\infty \mu(\{x : f(x) \ge t\})\, dt.$$

Note that for $f \in \mathcal{F}^+$, still $I_{\mathcal{H}_{Ch}}(\mu, f)$ is the Choquet integral from f with respect to μ.

(ii) Let $\mathcal{H}_{pan}$ denote the set of all finite measurable partitions of X. Then $I_{\mathcal{H}_{pan}}(\mu, f)$ is the pan-integral from below, while $I^{\mathcal{H}_{pan}}(\mu, f)$ is the pan integral from above, based on the pair of standard addition and multiplication $(+, \cdot)$), see [27,28].

(iii) Let $\mathcal{H}_{cav} = \{\mathcal{B} : \mathcal{B}$ is a finite subset of $\mathcal{A} \setminus \{\emptyset\}\}$. Then $I_{\mathcal{H}_{cav}}(\mu, f)$ is the concave integral of f with respect to μ ([6,7]), $I^{\mathcal{H}_{cav}}(\mu, f)$ is the convex integral of f with respect to μ, see [15].

The basic properties of these types of integrals can be found in [2,3,5–8,12,14–16,19,26–28].

3. Relations between the decomposition systems

In order to further study the relationship among the different decomposition integrals (resp. superdecomposition integrals), we introduce two types of preorders on $\mathbb{X}$ with respect to a fixed monotone measure $\mu \in \mathcal{M}$.

Definition 3.1. Let $\mu \in \mathcal{M}$ be fixed, and let $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$.

(i) If for every $(A_i)_{i=1}^n \in \mathcal{G}$ with $\lambda_i \ge 0, i = 1, 2, \ldots, n$, there is a $(B_j)_{j=1}^m \in \mathcal{H}$ with $\delta_j \ge 0, j = 1, 2, \ldots, m$, such that

$$\sum_{i=1}^{n}\lambda_i\chi_{A_i} \geq \sum_{j=1}^{m}\delta_j\chi_{B_j} \tag{3.1}$$

and

$$\sum_{i=1}^{n}\lambda_i\mu(A_i) \leq \sum_{j=1}^{m}\delta_j\mu(B_j), \tag{3.2}$$

then we say that *the system $\mathcal{G}$ is smaller than the system $\mathcal{H}$ with respect to μ in the sense of sub-decomposition*, and denoted by $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$.

(ii) If the inequalities in formulas (3.1) and (3.2) are converse, i.e.,

$$\sum_{i=1}^{n}\lambda_i\chi_{A_i} \leq \sum_{j=1}^{m}\delta_j\chi_{B_j} \tag{3.3}$$

and

$$\sum_{i=1}^{n}\lambda_i\mu(A_i) \geq \sum_{j=1}^{m}\delta_j\mu(B_j), \tag{3.4}$$

then we say that *the system $\mathcal{G}$ is bigger than the system $\mathcal{H}$ with respect to μ in the sense of super-decomposition*, and denoted by $\mathcal{G} \succeq_{super} \mathcal{H}[\mu]$.

It is easy to see that the both relations "$\preceq_{sub}$" and "$\succeq_{super}$" are reflexive and transitive. In general, they are neither symmetric nor antisymmetric, therefore, they are preorders.

There are other two kinds of relations for decomposition systems from $\mathbb{X}$: the standard set inclusion relation "$\subset$" and the refinement relation "$\preceq$" (which was introduced by Mesiar and Stupňanová in [16], as follows: for $(\mathcal{G},\mathcal{H}) \in \mathbb{X}\times\mathbb{X}$, $\mathcal{G}$ is a refinement of $\mathcal{H}$, denoted by "$\mathcal{G} \preceq \mathcal{H}$", if for each $(A_i)_{i=1}^{n} \in \mathcal{G}$, there is $(B_j)_{j=1}^{m} \in \mathcal{H}$ such that $\{A\}_{i=1}^{n} \subset \{B\}_{j=1}^{m}$). Note that they are not related to any fixed monotone measures. When $\mathcal{G} \subset \mathcal{H}$ or $\mathcal{G} \preceq \mathcal{H}$, for any $(\mu, f) \in \mathcal{M}\times\mathcal{F}^+$, it holds $I_{\mathcal{G}}(\mu, f) \leq I_{\mathcal{H}}(\mu, f)$, see [16].

For $(\mathcal{G},\mathcal{H}) \in \mathbb{X}\times\mathbb{X}$, it is transparent that $\mathcal{G} \subset \mathcal{H}$ implies $\mathcal{G} \preceq \mathcal{H}$, but not vice-versa. For the relations "$\preceq$", "$\preceq_{sub}$" and "$\succeq_{super}$", we have the following results.

Proposition 3.2. *Let $\mu \in \mathcal{M}$ be fixed and let $(\mathcal{G},\mathcal{H}) \in \mathbb{X}\times\mathbb{X}$. If $\mathcal{G} \preceq \mathcal{H}$, then $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$ and $\mathcal{G} \succeq_{super} \mathcal{H}[\mu]$.*

Note that, in general, the converse implications in Proposition 3.2 are not true, see the following Example 3.5 (iii) and (iv). This shows that the relation "$\preceq_{sub}$" is a preorder on decomposition systems from $\mathbb{X}$ refining both of the standard inclusion ordering and the preorder "$\preceq$". Moreover, we define a relation "$\approx_{sub}$": given $\mu \in \mathcal{M}$, for $(\mathcal{G},\mathcal{H}) \in \mathbb{X}\times\mathbb{X}$, $\mathcal{G} \approx_{sub} \mathcal{H}[\mu]$ iff $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$ and $\mathcal{H} \preceq_{sub} \mathcal{G}[\mu]$. The relation "$\approx_{sub}$" is an equivalence relation and thus the space $\mathbb{X}$ of decomposition systems can be partitioned into equivalence class $[\mathcal{G}]_{\approx_{sub}}$. For any $\mathcal{E} \in [\mathcal{G}]_{\approx_{sub}}$, it holds $I_{\mathcal{E}}(\mu, f) = I_{\mathcal{G}}(\mu, f)$ for any $f \in \mathcal{F}^+$ (see the later Proposition 4.2). Similarly, we can define the equivalence relation "$\approx_{super}$" and obtain the equivalence class $[\mathcal{H}]^{\approx_{super}}$, such that for any $\mathcal{D} \in [\mathcal{H}]^{\approx_{super}}$, it holds $I^{\mathcal{D}}(\mu, f) = I^{\mathcal{H}}(\mu, f)$ for any $f \in \mathcal{F}_b^+$ (see the later Proposition 4.2).

In the following we consider respectively the preorders "$\preceq_{sub}$" and "$\succeq_{super}$" among the three types of decomposition systems: $\mathcal{H}_{Ch}$, $\mathcal{H}_{pan}$ and $\mathcal{H}_{cav}$. Note that there are six kinds of ordered pairs: $(\mathcal{H}_{cav}, \mathcal{H}_{pan})$, $(\mathcal{H}_{cav}, \mathcal{H}_{Ch})$, $(\mathcal{H}_{Ch}, \mathcal{H}_{pan})$, $(\mathcal{H}_{pan}, \mathcal{H}_{Ch})$, $(\mathcal{H}_{Ch}, \mathcal{H}_{cav})$ and $(\mathcal{H}_{pan}, \mathcal{H}_{cav})$.

For the ordered pairs $(\mathcal{H}_{Ch}, \mathcal{H}_{cav})$ and $(\mathcal{H}_{pan}, \mathcal{H}_{cav})$, it is obvious from Proposition 3.2 that $\mathcal{H}_{Ch} \preceq_{sub} \mathcal{H}_{cav}[\mu]$, $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{cav}[\mu]$, $\mathcal{H}_{Ch} \succeq_{super} \mathcal{H}_{cav}[\mu]$ and $\mathcal{H}_{pan} \succeq_{super} \mathcal{H}_{cav}[\mu]$ hold for any given $\mu \in \mathcal{M}$. For other four cases, we will see that they are closely related to the structure characteristics of monotone measure μ: subadditivity, superadditivity, submodularity, supermodularity, (M)-property and dual (M)-property, etc.

Proposition 3.3. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and let $\mu \in \mathcal{M}$ be fixed.*

(i) If μ is subadditive, then $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$.

(ii) If μ is superadditive, then $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$.
(iii) If μ has (M)-property, then $\mathcal{H}_{Ch} \preceq_{sub} \mathcal{H}_{pan}[\mu]$.
(iv) If μ has dual (M)-property, then $\mathcal{H}_{Ch} \succeq_{super} \mathcal{H}_{pan}[\mu]$.
(v) μ is superadditive if and only if $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$.
(vi) μ is subadditive if and only if $\mathcal{H}_{pan} \succeq_{super} \mathcal{H}_{Ch}[\mu]$.
(vii) μ is supermodular if and only if $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$.
(viii) μ is submodular if and only if $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{Ch}[\mu]$.

Proof. The properties (i) and (ii) can be verified with the methods we used in Theorem 9 in [20]. Nonetheless, for readers convenience, we provide proof of (ii) (it is postponed to the Appendix).

(iii) It can be proved with the methods we used the proof of Theorem 4.1 in [21]), therefore we omit its details.

(iv) It is similar to the proof of Theorem 3.2 in [11].

(v) and (vi) See the later Propositions 4.12 and 4.13.

(vii) and (viii) See the later Propositions 4.4 and 4.5. □

Note 3.4. The converse implication of each of Proposition 3.3 (i)-(iv) may not be true.

Example 3.5. (i) Let $X = [0, 1]$ and $\mathcal{A}$ be the Borel σ-algebra over X. $\mu : \mathcal{A} \to [0, 1]$ is defined by

$$\mu(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{if } A \neq X. \end{cases}$$

Then, obviously, μ is not subadditive. We show $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$.

Let $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ with $\lambda_i \geq 0, i = 1, 2, \dots n$. There are two cases: (1) For each $1 \leq i \leq n$, $A_i \neq X$. In this case, we take $B_1 = \cup_{i=1}^n A_i$, $\delta_1 = \min\{\lambda_i : i = 1, 2, \dots n\}$, then $\sum_{i=1}^n \lambda_i \chi_{A_i} \geq \delta_1 \chi_{B_1}$ and $0 = \sum_{i=1}^n \lambda_i \mu(A_i) \leq \delta_1 \mu(B_1)$. (2) There is some i_0, $A_{i_0} = X$, take $B_1 = X$, $\delta_1 = \sum_{A_i = X} \delta_i$. Thus, for $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ with $\lambda_i \geq 0$ $(i = 1, 2, \dots n)$, the above chosen B_1 and δ_1 satisfy the formulas (3.1) and (3.2).

(Note that μ is superadditive, and hence $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$ and $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$.)

(ii) Let $X = \{1, 2, 3, 4\}$ and $\mu : 2^X \to [0, 1]$ be defined by

$$\mu(A) = \begin{cases} 1 & \text{if } \{1, 2\} \subset A \text{ or } \{3, 4\} \subset A, \\ 0 & \text{else.} \end{cases}$$

Then, $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$. In fact, for any $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ with $\lambda_i \geq 0, i = 1, 2, \dots, n$, there are $\delta_j \geq 0$, $j = 1, 2, 3, 4$, such that

$$\sum_{i=1}^n \lambda_i \chi_{A_i} = \sum_{j=1}^4 \delta_j \chi_{\{j\}}.$$

Note that $\mu(\{j\}) = 0$, $j = 1, 2, 3, 4$, then

$$\sum_{i=1}^n \lambda_i \mu(A_i) \geq \sum_{j=1}^4 \delta_j \mu(\{j\}).$$

Therefore, the formulas (3.3) and (3.4) hold.

Since $\mu(\{1, 2\} \cup \{3, 4\}) = 1 < \mu(\{1, 2\}) + \mu(\{3, 4\}) = 2$, this shows that μ is not superadditive.

(Note that μ is not subadditive ($\mu(\{1, 3\} \cup \{2, 4\}) = 1 > \mu(\{1, 3\}) + \mu(\{2, 4\}) = 0$). Thus, μ has neither the (M)-property nor the dual (M)-property, see [11,13].)

(iii) Let $X = \{a, b\}$ and $\mu : 2^X \to [0, 1]$ be defined by

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Then $\mathcal{H}_{Ch} \preceq_{sub} \mathcal{H}_{pan}[\mu]$. Indeed, for the chain $C_1 = \{a\}$ (or $\{b\}$ or $\{a, b\}$) with $\lambda_1 \geq 0$, we take $B_1 = C_1$, $\delta_1 = \lambda_1$, then B_1 and δ_1 satisfy the requirements of the formulas (3.1) and (3.2); for the chain $C_1 = \{a\}, C_2 = \{a, b\}$ (resp.

$C_1 = \{b\}, C_2 = \{a, b\}$) with $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we take $B_1 = \{a\}$, $B_2 = \{b\}$, $\delta_1 = \lambda_1 + \lambda_2$, $\delta_2 = \lambda_2$ (resp. $\delta_1 = \lambda_2$, $\delta_2 = \lambda_1 + \lambda_2$), then $\lambda_1\chi_{C_1} + \lambda_2\chi_{C_2} = \delta_1\chi_{B_1} + \delta_2\chi_{B_2}$ and $\lambda_1\mu(C_1) + \lambda_2(C_2) \leq \delta_1\mu(B_1) + \delta_2\mu(B_2)$, i.e., the formulas (3.1) and (3.2) hold.

However, μ is not superadditive ($\mu(\{a\} \cup \{b\}) < \mu(\{a\}) + \mu(\{b\})$), and hence μ has not the (M)-property (see [13]).

(iv) Let $X = \{a, b\}$ and $\mu : 2^X \to [0, 1]$ be defined by

$$\mu(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{if } A \neq X. \end{cases}$$

Then, similar to the discussion of (iii) we can show $\mathcal{H}_{Ch} \succeq_{super} \mathcal{H}_{pan}[\mu]$. Obviously, μ is not subadditive, so it has not the dual (M)-property (see [11]).

4. The coincidences of decomposition integrals and of superdecomposition integrals

Let $\mu \in \mathcal{M}$ be fixed. Given a pair of decomposition systems $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$ we use the relations "$\preceq_{sub}$" and "$\succeq_{super}$" to discuss the coincidences of the decomposition integrals $I_{\mathcal{G}}(\mu, \cdot)$ and $I_{\mathcal{H}}(\mu, \cdot)$, and of the superdecomposition integrals $I^{\mathcal{G}}(\mu, \cdot)$ and $I^{\mathcal{H}}(\mu, \cdot)$.

4.1. The general results on coincidence of integrals

In the following we investigate the coincidence between the decomposition integrals (resp. superdecomposition integrals) in the framework related to the ordered pair of decomposition systems from $\mathbb{X}$.

Proposition 4.1. *Let $\mu \in \mathcal{M}$ be fixed and let $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$.*

(i) If $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$, then for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{G}}(\mu, f) \leq I_{\mathcal{H}}(\mu, f).$$

(ii) If $\mathcal{G} \succeq_{super} \mathcal{H}[\mu]$, then for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{G}}(\mu, f) \geq I^{\mathcal{H}}(\mu, f).$$

Proof. We only prove(i), the proof of (ii) is similar.

Assume $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$ and let $f \in \mathcal{F}^+$ be fixed.

For any given $\mathcal{G}$-sub-decomposition of f, $\sum_{i=1}^n \lambda_i\chi_{A_i}$ (i.e., $\sum_{i=1}^n \lambda_i\chi_{A_i} \leq f$, $(A_i)_{i=1}^n \in \mathcal{G}$, $\lambda_i \geq 0, i = 1, 2, \ldots, n, n \in \mathbb{N}$), from the condition $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$, then there is $(B_j)_{j=1}^m \in \mathcal{H}$ with $\delta_j \geq 0$, $j = 1, 2, \ldots, m$ such that the finite summation $\sum_{i=1}^n \lambda_i\chi_{A_i}$ corresponds to

$$\sum_{i=1}^n \lambda_i\chi_{A_i} \geq \sum_{j=1}^m \delta_j\chi_{B_j}$$

and

$$\sum_{i=1}^n \lambda_i\mu(A_i) \leq \sum_{j=1}^m \delta_j\mu(B_j).$$

Thus, the finite summation $\sum_{i=1}^m \delta_j\chi_{B_j}$ is a $\mathcal{H}$-sub-decomposition of f (i.e., $\sum_{j=1}^m \delta_j\chi_{B_j} \leq f$, $(B_j)_{j=1}^m \in \mathcal{H}$, $\delta_j \geq 0$, $j = 1, 2, \ldots, m, m \in \mathbb{N}$). Therefore, we have

$$\begin{aligned} I_{\mathcal{G}}(\mu, f) &= \sup\left\{\sum_{i=1}^n \lambda_i\mu(A_i) : \sum_{i=1}^n \lambda_i\chi_{A_i} \leq f, (A_i)_{i=1}^n \in \mathcal{H}, \lambda_i \geq 0\right\} \\ &\leq \sup\left\{\sum_{j=1}^m \delta_j\mu(B_j) : \sum_{j=1}^m \delta_j\chi_{B_j} \leq \sum_{i=1}^n \lambda_i\chi_{A_i} \leq f\right\} \end{aligned}$$

$$\le \sup\left\{\sum_{k=1}^{s} a_k\mu(E_k) : \sum_{k=1}^{s} a_k\chi_{E_k} \le f, (E_k)_{k=1}^{s} \in \mathcal{H}, a_k \ge 0\right\}$$
$$= I_{\mathcal{H}}(\mu, f).$$

The proof is now complete. □

From Proposition 4.1, we obtain the following result.

Proposition 4.2. *Let $\mu \in \mathcal{M}$ be fixed and let $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$.*

(i) If for all $f \in \mathcal{F}^+$, $I_{\mathcal{G}}(\mu, f) \ge I_{\mathcal{H}}(\mu, f)$ and $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$, then for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{G}}(\mu, f) = I_{\mathcal{H}}(\mu, f).$$

(ii) If for all $f \in \mathcal{F}^+$, $I^{\mathcal{G}}(\mu, f) \le I^{\mathcal{H}}(\mu, f)$ and $\mathcal{G} \succeq_{super} \mathcal{H}[\mu]$, then for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{G}}(\mu, f) = I^{\mathcal{H}}(\mu, f).$$

We call the following theorem *the generalized integrals equivalence theorem*.

Theorem 4.3. *Let $\mu \in \mathcal{M}$ be fixed and let $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$.*

(i) If $\mathcal{G} \approx_{sub} \mathcal{H}[\mu]$, i.e., $\mathcal{G} \preceq_{sub} \mathcal{H}[\mu]$ and $\mathcal{H} \preceq_{sub} \mathcal{G}[\mu]$, then for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{G}}(\mu, f) = I_{\mathcal{H}}(\mu, f).$$

(ii) If $\mathcal{G} \approx_{super} \mathcal{H}[\mu]$, i.e., $\mathcal{G} \preceq_{super} \mathcal{H}[\mu]$ and $\mathcal{H} \preceq_{super} \mathcal{G}[\mu]$, then for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{G}}(\mu, f) = I^{\mathcal{H}}(\mu, f).$$

Observe that Theorem 4.3, Propositions 4.1 and 4.2 involve the ordered pairs of decomposition systems. In such the framework we will discuss the coincidences among three types of important decomposition integrals: the Choquet integral [1], the concave integral [3,6] and the pan-integral from below [28], including their counterparts: the Choquet integral, the convex integral and the pan-integral from above.

In general, the converse of Theorem 4.3 is not true. For decomposition systems $\mathcal{H}_{cav}, \mathcal{H}_{Ch}$ and $\mathcal{H}_{pan}$ we have some special results, see the later Propositions 4.19 and 4.20.

4.2. *The coincidence of the Choquet integrals and concave (convex) integrals*

For any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$, we have

$$I^{cav}(\mu, f) \le I^{Ch}(\mu, f) = I_{Ch}(\mu, f) \le I_{cav}(\mu, f).$$

Lehrer and Teper [6,7] showed that the concave integral coincides with the Choquet integral if an only if the underlying monotone measure is supermodular. As a counterpart of Lehrer's result, Mesiar et al. [15] showed submodularity is a necessary and sufficient condition for the equivalence of the convex integral and the Choquet integral.

Considering the ordered pair $(\mathcal{H}_{cav}, \mathcal{H}_{Ch})$, we have the following further results.

Proposition 4.4. *Let $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*

(i) For all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f);$$

(ii) μ is supermodular, i.e., for all $A, B \in \mathcal{A}$,

$$\mu(A \cup B) + \mu(A \cap B) \geqslant \mu(A) + \mu(B);$$

(iii) $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$.

Proposition 4.5. *Let $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*

(i) For all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{Ch}}(\mu, f);$$

(ii) μ is submodular, i.e., for all $A, B \in \mathcal{A}$,

$$\mu(A \cup B) + \mu(A \cap B) \leqslant \mu(A) + \mu(B);$$

(iii) $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{Ch}[\mu]$.

Proof. We only prove Proposition 4.4.

$(i) \Leftrightarrow (ii)$: It has been proved by Lehrer and Teper in [6] (see also [7]).

$(i) \Rightarrow (iii)$: Suppose that $I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f)$ holds for all $f \in \mathcal{F}^+$.

Given $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ with $\lambda_i \geq 0,\ i = 1, 2, \ldots, n$. Put $g = \sum_{i=1}^n \lambda_i \chi_{A_i}$, then Ran_g, the range of g is a finite set. We suppose that $Ran_g = \{r_1, r_2, \ldots, r_k\}$, where $r_1 < r_2 < \cdots < r_k$. Let $\delta_1 = r_1, \delta_2 = r_2 - r_1, \ldots, \delta_k = r_k - r_{k-1}$ and $C_j = \{x : g(x) \geq \lambda_j\}$, $j = 1, 2, \ldots, k$. Then $(C_j)_{j=1}^k \in \mathcal{H}_{Ch}$ and

$$\sum_{i=1}^n \lambda_i \chi_{A_i} = \sum_{j=1}^k \delta_j \chi_{C_j}.$$

Since for all $f \in \mathcal{F}^+$, $I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f)$ holds, therefore we have

$$\begin{aligned}\sum_{i=1}^n \lambda_i \mu(A_i) &\leq I_{\mathcal{H}_{cav}}\big(\mu, \sum_{i=1}^n \lambda_i \chi_{A_i}\big)\\ &= I_{\mathcal{H}_{cav}}\big(\mu, \sum_{j=1}^k \delta_j \chi_{C_j}\big)\\ &= I_{\mathcal{H}_{Ch}}\big(\mu, \sum_{j=1}^k \delta_j \chi_{C_j}\big)\\ &= \sum_{j=1}^k \delta_j \mu(C_j).\end{aligned}$$

Therefore $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$.

$(iii) \Rightarrow (i)$: Note that $I_{\mathcal{H}_{cav}}(\mu, f) \geq I_{\mathcal{H}_{Ch}}(\mu, f)$ holds for all $f \in \mathcal{F}^+$. By the assumption $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$, then it follows from Proposition 4.2 that

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f)$$

for all $f \in \mathcal{F}^+$.

The proof is now complete. □

Note that the submodularity (resp. supermodularity) implies subadditivity (resp. superadditivity), but not vice versa. Thus, the superadditivity (resp. subadditivity) is a necessary condition that the concave (resp. convex) integral coincides with the Choquet integral.

4.3. The coincidences of the concave (convex) and pan-integrals from below (or from above)

For any $(\mu, f) \in \mathcal{M} \times \mathcal{F}^+$, we have the following

$$I^{\mathcal{H}_{cav}}(\mu, f) \leq I^{\mathcal{H}_{pan}}(\mu, f) \quad \text{and} \quad I_{\mathcal{H}_{pan}}(\mu, f) \leq I_{\mathcal{H}_{cav}}(\mu, f).$$

In general, the above inequalities cannot be replaced by equality.

In [20] we have shown that the subadditivity is a sufficient condition that the concave integral coincides with the pan-integral from below, while as a counterpart of this result, the superadditivity is sufficient for the equivalence of the convex integral and the pan-integral from above [22]. In this subsection, we further generalize these results.

Considering the ordered pair $(\mathcal{G}, \mathcal{H}) = (\mathcal{H}_{cav}, \mathcal{H}_{pan})$, the following is a special case of Proposition 4.2.

Proposition 4.6. *Let $\mu \in \mathcal{M}$ be fixed.*

(i) If $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$, then for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f).$$

(ii) If $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$, then for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f).$$

Proof. For readers convenience, we still want to render a proof in details, as follows:

We only prove (ii). Since for all $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_{cav}}(\mu, f) \leq I^{\mathcal{H}_{pan}}(\mu, f)$ holds. It suffices to prove that $I^{\mathcal{H}_{cav}}(\mu, f) \geq I^{\mathcal{H}_{pan}}(\mu, f)$ holds for all $f \in \mathcal{F}_b^+$.

Suppose $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$ and let $f \in \mathcal{F}_b^+$ be fixed. For any given $\mathcal{H}_{cav}$-super-decomposition of f, $\sum_{i=1}^n \lambda_i \chi_{A_i}$ (i.e., $\sum_{i=1}^n \lambda_i \chi_{A_i} \geq f$, $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$, $\lambda_i \geq 0, i = 1, 2, \ldots, n, n \in \mathbb{N}$), from $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$, then the finite summation $\sum_{i=1}^n \lambda_i \chi_{A_i}$ corresponds to $(B_j)_{j=1}^m \in \mathcal{H}_{pan}$ with $l_j \geq 0$, $j = 1, 2, \ldots, m$, such that

$$\sum_{i=1}^n \lambda_i \chi_{A_i} \leq \sum_{j=1}^m l_j \chi_{B_j} \quad \text{and} \quad \sum_{i=1}^n \lambda_i \mu(A_i) \geq \sum_{j=1}^m l_j \mu(B_j).$$

Thus, the finite summation $\sum_{i=1}^m l_j \chi_{B_j}$ is a $\mathcal{H}_{pan}$-super-decomposition of f (i.e., $\sum_{j=1}^m l_j \chi_{B_j} \geq f$, $(B_j)_{j=1}^m \in \mathcal{H}_{pan}$, $l_i \geq 0, i = 1, 2, \ldots, m, m \in \mathbb{N}$). Therefore, we have

$$\begin{aligned}
I^{\mathcal{H}_{cav}}(\mu, f) &= \inf\left\{\sum_{i=1}^n \lambda_i \mu(A_i) : \sum_{i=1}^n \lambda_i \chi_{A_i} \geq f, (A_i)_{i=1}^n \in \mathcal{H}_{cav}, \lambda_i \geq 0\right\} \\
&\geq \inf\left\{\sum_{i=1}^m l_j \mu(B_j) : \sum_{j=1}^m l_j \chi_{B_j} \geq \sum_{i=1}^n \lambda_i \chi_{A_i} \geq f\right\} \\
&\geq \inf\left\{\sum_{i=1}^s a_k \mu(E_k) : \sum_{k=1}^s a_k \chi_{E_k} \geq f, (E_i)_{k=1}^s \in \mathcal{H}_{pan}, a_k \geq 0\right\} \\
&= I^{\mathcal{H}_{pan}}(\mu, f).
\end{aligned}$$

The proof is complete. □

Example 3.5(i) and (ii) show that the conditions $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$ and $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$ are really weaker than the subadditivity and superadditivity of μ, respectively. Thus, Proposition 4.6 generalizes the previous results we obtained in [20,22]. As special results of Propositions 3.3 and 4.6, we have the following corollary:

Corollary 4.7. *(Theorem 9 in [20], Corollary 4.2 in [22]) Let $\mu \in \mathcal{M}$ be fixed.*

(i) If μ is subadditive, then for any $f \in \mathcal{F}^+$, $I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f)$.

(ii) If μ is superadditive, then for any $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f)$.

Example 4.8. Let $(X, \mathcal{A}, \mu)$ be the monotone measure space in Example 3.5(i). Then, $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$ and hence for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f) = \inf\{f(x)|x \in X\}$$

(note that μ is not subadditive). Note that $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$, therefore, for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f) = 0.$$

When X is finite we presented respectively a set of necessary and sufficient conditions that the concave integral coincides with the pan-integral from below (see Theorem 4.1 in [17]), and that the convex integral coincides with the pan-integral from above (see Theorem 5.2 in [22]), by using the characterizations of minimal atoms of monotone measures and of atoms of algebra $\mathcal{A}$ of subsets of X. For readers convenience, we provide these results in Appendix (see Theorems A and B in the Appendix).

Proposition 4.6 shows that the condition $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$ (resp. $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$) is a sufficient condition that the concave integral (resp. convex integral) coincides with the pan-integral from below (resp. from above). Now we show that if X is finite, then the condition $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$ (resp. $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$) is not only sufficient, but also necessary for the equivalence of these two integrals. Thus we present respectively a new sufficient and necessary condition for the equivalence of each of these two pairs of the integrals.

Proposition 4.9. *Let X be a finite space, and $\mu \in \mathcal{M}$ be fixed and finite. Then,*

(i) $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$ if and only if for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f).$$

(ii) $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$ if and only if for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f).$$

Proof. (i) The necessity is clear by Proposition 4.6, we only need to prove the sufficiency.

Sufficiency: Suppose for all $f \in \mathcal{F}^+$, $I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f)$. Then the both conditions (i) and (ii) in Theorem 4.1 in [17] are satisfied (see Theorem A in the Appendix).

Now given $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ with $\lambda_i \geq 0, i = 1, 2, \ldots, n$ (we can assume $\mu(A_i) > 0, i = 1, 2, \ldots, n$, without loss of generality). Since X is a finite space, then every A_i $(i = 1, 2, \ldots, n)$ can be expressed as

$$A_i = A_i^{(1)} \cup A_i^{(2)} \cup \cdots \cup A_i^{(k_i)} \cup \widetilde{A}_i^{(0)},$$

$i = 1, 2, \ldots, n$, where for every fixed i, $\{A_i^{(j)}\}_{j=1}^{k_i}$ is the set of all minimal atoms contained in A_i and $\mu(\widetilde{A}_i^{(0)}) = 0$ (note that for fixed i, $\{A_i^{(j)}\}_{j=1}^{k_i}$ is a family of pairwise disjoint sets). Since μ is subadditive w.r.t. minimal atoms, then

$$\mu(A_i) \leq \mu(A_i^{(1)}) + \mu(A_i^{(2)}) + \cdots + \mu(A_i^{(k_i)}),$$

for every $i = 1, 2, \ldots, n$. On the other hand, let $\{B_j\}_{j=1}^{m-1}$ be the set of all pairwise disjoint minimal atoms contained in $\bigcup_{i=1}^n \{A_i^{(1)}, A_i^{(2)}, \cdots, A_i^{(k_i)}\}$ $(B_{j_1} \cap B_{j_2} = \emptyset$ when $j_1 \neq j_2, 1 \leq j_1, j_2 \leq m-1)$, and $B_m = \bigcup_{i=1}^n \widetilde{A}_i^{(0)}$, then $(B_j)_{j=1}^m \in \mathcal{H}_{pan}$. Noting that μ satisfies the condition that μ possesses the minimal atoms disjointness property, then, it is not difficult to verify that there are nonnegative numbers $\delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m$ $(\delta_m = \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\})$, such that

$$\sum_{i=1}^n \lambda_i \chi_{A_i} \geq \sum_{j=1}^m \delta_j \chi_{B_j} \quad \text{and} \quad \sum_{i=1}^n \lambda_i \mu(A_i) \leq \sum_{j=1}^m \delta_j \mu(B_j).$$

This shows $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{pan}[\mu]$.

(ii) By Proposition 4.6 the necessity is obvious. Now we prove the sufficiency.

We recall the concept of *atom* of a σ-algebra of subsets of X. Let X be a finite set and $\mathcal{A}$ be an arbitrary algebra over X. A nonempty set $E \in \mathcal{A}$ is called *an atom of* $\mathcal{A}$ [24], if $\emptyset$ and A are the only $\mathcal{A}$-measurable subsets of A. The atoms of $\mathcal{A}$ possess some of basic properties, as follows: (a) Every two distinct atoms of $\mathcal{A}$ are disjoint; (b) Let $E_1, \ldots, E_k$ be all of atoms of $\mathcal{A}$. Then $E_1, \ldots, E_k$ are pairwise disjoint and $X = E_1 \cup E_2 \cup \cdots \cup E_k$, and hence $\{E_1, \ldots, E_k\}$ is a measurable partition of X; (c) Every nonempty set $A \in \mathcal{A}$ is the union of some atoms of $\mathcal{A}$.

Sufficiency: Suppose that for all $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f)$ holds. Then, μ satisfies one of the conditions (i) and (ii) in Theorem 5.2 in [22] (see Theorem B in the Appendix). We consider two cases:

(1) Assume that μ satisfies the condition (i) in Theorem 5.2 in [22], i.e., μ is superadditive w.r.t. atoms of $\mathcal{A}$ (for this concept, see [22]). Let $E_1, \ldots, E_k$ be all of the atoms of $\mathcal{A}$. Then $E_1, \ldots, E_k$ are pairwise disjoint and $X = E_1 \cup E_2 \cup \cdots \cup E_k$. Given $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ with $\lambda_i \geq 0, i = 1, 2, \ldots, n$. Since X is a finite space, then for every $i = 1, 2, \ldots, n$, A_i is the union of some atoms of $\mathcal{A}$, i.e.,

$$A_i = E^{(i)}_{i_1} \cup E^{(i)}_{i_2} \cup \cdots \cup E^{(i)}_{i_s},$$

where for every fixed i. $\{E^{(i)}_{i_1}, \ldots, E^{(i)}_{i_s}\} \subset \{E_1, E_2, \ldots, E_k\}$. Therefore,

$$\bigcup_{l=1}^{s}\{E^{(i)}_{i_1}, \ldots, E^{(i)}_{i_s}\} = \{B_1, B_2, \ldots, B_m\},$$

where $B_1, B_2, \ldots, B_m$ are some pairwise disjoint atoms of $\mathcal{A}$. Let $B_{m+1} = X \setminus \bigcup_{j=1}^{m} B_j$, then $(B_j)_{j=1}^{m+1} \in \mathcal{H}_{pan}$. Thus, it is easy to see that there are $\delta_j \geq 0$, $j = 1, 2, \ldots, m$ and $\delta_{m+1} = 0$ such that

$$\sum_{i=1}^{n} \lambda_i \chi_{A_i} = \sum_{j=1}^{m+1} \delta_j \chi_{B_j},$$

and noting that μ is superadditive w.r.t. atoms of $\mathcal{A}$, then

$$\sum_{i=1}^{n} \lambda_i \mu(A_i) \geq \sum_{j=1}^{m+1} \delta_j \mu(B_j).$$

This shows $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$.

(2) For the case satisfying the condition (ii) in Theorem 5.2 in [22], it is similar to proofs of Theorems 4.7 and 5.2 in [22], we can prove $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$.

The proof is now complete. □

4.4. The coincidence of the Choquet integrals and the pan-integrals

For some $\mu \in \mathcal{M}$, $I_{\mathcal{H}_{Ch}}(\mu, f)$ do not coincide with $I_{\mathcal{H}_{pan}}(\mu, f)$. Similarly, $I^{\mathcal{H}_{Ch}}(\mu, f)$ and $I^{\mathcal{H}_{pan}}(\mu, f)$ need not coincide. However, if the considered monotone measure μ is σ-additive, then all four integrals coincide with the Lebesgue integral, see [10].

Considering the ordered pair $(\mathcal{G}, \mathcal{H}) = (\mathcal{H}_{Ch}, \mathcal{H}_{pan})$, then the following is a direct result of Proposition 4.1.

Proposition 4.10. *Let $\mu \in \mathcal{M}$ be fixed.*

(i) If $\mathcal{H}_{Ch} \preceq_{sub} \mathcal{H}_{pan}[\mu]$, then for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{Ch}}(\mu, f) \leq I_{\mathcal{H}_{pan}}(\mu, f).$$

(ii) If $\mathcal{H}_{Ch} \succeq_{super} \mathcal{H}_{pan}[\mu]$, then for all $f \in \mathcal{F}^+_b$,

$$I^{\mathcal{H}_{Ch}}(\mu, f) \geq I^{\mathcal{H}_{pan}}(\mu, f).$$

Considering the ordered pair $(\mathcal{G}, \mathcal{H}) = (\mathcal{H}_{pan}, \mathcal{H}_{Ch})$, we get the following stronger results.

Proposition 4.11. *Let $\mu \in \mathcal{M}$ be fixed.*

(i) $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$ if and only if for all $f \in \mathcal{F}^+$ it holds

$$I_{\mathcal{H}_{pan}}(\mu, f) \leq I_{\mathcal{H}_{Ch}}(\mu, f).$$

(ii) $\mathcal{H}_{pan} \succeq_{super} \mathcal{H}_{Ch}[\mu]$ if and only if for all $f \in \mathcal{F}^+$ it holds

$$I^{\mathcal{H}_{pan}}(\mu, f) \geq I^{\mathcal{H}_{Ch}}(\mu, f).$$

Proof. We only prove (i), and the proof of (ii) is similar.

Necessity: This is a direct result of Proposition 4.1 considering the ordered pair $(\mathcal{G}, \mathcal{H}) = (\mathcal{H}_{pan}, \mathcal{H}_{Ch})$.

Sufficiency: Suppose for all $f \in \mathcal{F}^+$ it holds $I_{\mathcal{H}_{pan}}(\mu, f) \leq I_{\mathcal{H}_{Ch}}(\mu, f)$. For any given collection $(A_i)_{i=1}^{n} \in \mathcal{H}_{pan}$ with $\lambda_i \geq 0, i = 1, 2, \ldots, n$ (we can assume $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, without loss of generality), we take

$$C_i = A_i \cup A_{i+1} \cup \cdots \cup A_n$$

$i = 1, 2, \ldots, n$ and let $c_i = \lambda_i - \lambda_{i-1}$ for $i = 1, 2, \ldots, n$, $\lambda_{i-1} = 0$. Then $(C_i)_{i=1}^n \in \mathcal{H}_{Ch}$ with $c_i \geq 0, i = 1, 2, \ldots, n$, $\sum_{i=1}^n a_i \chi_{A_i} = \sum_{i=1}^n c_i \chi_{C_i}$. Noting that, from Lemma 3.1 in [13], μ is superadditive, therefore

$$\begin{aligned}\sum_{i=1}^n c_i \mu(C_i) &= \sum_{i=1}^n c_i \mu(A_i \cup A_{i+1} \cup \cdots \cup A_n)\\ &\geq \sum_{j=1}^n \left(\lambda_i - \lambda_{i-1}\right)\Big(\mu(A_i) + \mu(A_{i+1}) + \cdots + \mu(A_n)\Big)\\ &= \sum_{i=1}^n \lambda_i \mu(A_i).\end{aligned}$$

This shows $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$.

The proof is now complete. □

Combining Lemma 3.1 in [13] and Proposition 4.11(i), and Lemma 3.1 in [11] and Proposition 4.11(ii), respectively, we have the following results.

Proposition 4.12. *Let $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*

(i) μ is superadditive;

(ii) $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$;

(iii) $I_{\mathcal{H}_{pan}}(\mu, f) \leq I_{\mathcal{H}_{Ch}}(\mu, f)$ holds for all $f \in \mathcal{F}^+$.

Proposition 4.13. *Let $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*

(i) μ is subadditive;

(ii) $\mathcal{H}_{pan} \succeq_{super} \mathcal{H}_{Ch}[\mu]$;

(iii) $I^{\mathcal{H}_{pan}}(\mu, f) \geq I^{\mathcal{H}_{Ch}}(\mu, f)$ holds for all $f \in \mathcal{F}_b^+$.

In [21] we showed that *(M)-property* is a sufficient condition that the Choquet integral is equivalent to the pan-integral from below, and Lv et al. [11] introduced the so-called *dual (M)-property* of monotone measures, and showed that it is sufficient for the coincidence of the Choquet integral and the pan-integral from above. Considering both the ordered pairs $(\mathcal{H}_{Ch}, \mathcal{H}_{pan})$ and $(\mathcal{H}_{pan}, \mathcal{H}_{Ch})$ in Proposition 4.3, we obtain a new sufficient condition that the Choquet integral coincides with the pan-integral from below (resp. the pan-integral from above).

Proposition 4.14. *Let $\mu \in \mathcal{M}$ be fixed.*

(i) If $\mathcal{H}_{Ch} \approx_{sub} \mathcal{H}_{pan}[\mu]$, then for all $f \in \mathcal{F}^+$, $I_{\mathcal{H}_{Ch}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f)$.

(ii) If $\mathcal{H}_{Ch} \approx_{super} \mathcal{H}_{pan}[\mu]$, then for all $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_{Ch}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f)$.

Note that the (M)-property of μ implies $\mathcal{H}_{Ch} \approx_{sub} \mathcal{H}_{pan}[\mu]$ (in fact, from Proposition 3.3(iii), (M)-property implies $\mathcal{H}_{Ch} \preceq_{sub} \mathcal{H}_{pan}[\mu]$; (M)-property implies superadditivity, see [13], and hence $\mathcal{H}_{pan} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$, see Proposition 3.3(iv)). Similarly, the dual (M)-property implies $\mathcal{H}_{Ch} \approx_{super} \mathcal{H}_{pan}[\mu]$ (see [11] and Proposition 3.3). Thus, from Proposition 4.14 we obtain our previous results:

Corollary 4.15. *(Theorem 4.1, [21]; Theorem 3.2, [11]) Let $\mu \in \mathcal{M}$ be fixed.*

(i) If μ has (M)-property, then for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{Ch}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f).$$

(ii) If μ μ has dual (M)-property, then for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{Ch}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f).$$

Note 4.16. A natural open problem arises: *whether $\mathcal{H}_{Ch} \approx_{sub} \mathcal{H}_{pan}[\mu]$ implies the (M)-property of μ, and similarly, whether $\mathcal{H}_{Ch} \approx_{super} \mathcal{H}_{pan}[\mu]$ implies the dual (M)-property of μ.*

When X is finite, we showed that the (M)-property is not only necessary, but also sufficient for the equivalence of the Choquet integral and the pan-integral from below (see Theorem 4.6 in [18]), for the dual (M)-property, there is a similar result (see Theorem 3.4 in [11]). Combining these results in [11,18] and Proposition 4.14 we have the following propositions.

Proposition 4.17. *Let X be a finite space and $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*
(i) μ has (M)-property;
(ii) $\mathcal{H}_{Ch} \approx_{sub} \mathcal{H}_{pan}[\mu]$;
(iii) for all $f \in \mathcal{F}^+$, $I_{\mathcal{H}_{Ch}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f)$.

Proposition 4.18. *Let X be a finite space and $\mu \in \mathcal{M}$ be fixed. Then the following are equivalent:*
(i) μ has dual (M)-property;
(ii) $\mathcal{H}_{Ch} \approx_{super} \mathcal{H}_{pan}[\mu]$;
(iii) for all $f \in \mathcal{F}_b^+$, $I^{\mathcal{H}_{Ch}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f)$.

4.5. A summary of the main results and notes

We present a summary of the main results we have obtained in the above subsections. In general, for any $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$, $\mathcal{G} \approx_{sub} \mathcal{H}[\mu]$ implies $I_{\mathcal{G}}(\mu, f) = I_{\mathcal{H}}(\mu, f)$, and $\mathcal{G} \approx_{super} \mathcal{H}[\mu]$ implies $I^{\mathcal{G}}(\mu, f) = I^{\mathcal{H}}(\mu, f)$, but not vice-versa (Theorem 4.3, see also Propositions 4.1 and 4.2). When considering the ordered pairs $(\mathcal{G}, \mathcal{H}) = (\mathcal{H}_{cav}, \mathcal{H}_{Ch})$, $(\mathcal{H}_{cav}, \mathcal{H}_{pan})$ and $(\mathcal{H}_{Ch}, \mathcal{H}_{pan})$, we have the following further results (see Propositions 4.4, 4.5, 4.6 and 4.14).

Proposition 4.19. *Let $\mu \in \mathcal{M}$ be fixed. Then,*
(i) $\mathcal{H}_{cav} \preceq_{sub} \mathcal{H}_{Ch}[\mu]$ (in fact, $\mathcal{H}_{cav} \approx_{sub} \mathcal{H}_{Ch}[\mu]$) if and only if for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{Ch}}(\mu, f).$$

(ii) $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{Ch}[\mu]$ (in fact, $\mathcal{H}_{cav} \approx_{super} \mathcal{H}_{Ch}[\mu]$) if and only if for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{Ch}}(\mu, f).$$

Proposition 4.20. *Let X be a finite space, and $\mu \in \mathcal{M}$ be fixed and finite. Then,*
(i) $\mathcal{H}_{cav} \approx_{sub} \mathcal{H}_{pan}[\mu]$ if and only if for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f);$$

(ii) $\mathcal{H}_{cav} \approx_{super} \mathcal{H}_{pan}[\mu]$ if and only if for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f).$$

(iii) $\mathcal{H}_{Ch} \approx_{sub} \mathcal{H}_{pan}[\mu]$ if and only if for all $f \in \mathcal{F}^+$,

$$I_{\mathcal{H}_{Ch}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f);$$

(iv) $\mathcal{H}_{Ch} \approx_{super} \mathcal{H}_{pan}[\mu]$ if and only if for all $f \in \mathcal{F}_b^+$,

$$I^{\mathcal{H}_{Ch}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f).$$

Note 4.21. We don't know whether for general spaces (not necessarily finite) the sufficiencies in Proposition 4.20 (i)-(iv) remain valid, respectively, as Proposition 4.19. This is a subject of our further research.

5. Conclusions

We have introduced two types of preorders on the set consisting of all decomposition systems on $(X, \mathcal{A})$, and by means of these two preorders of decomposition systems we have presented the *generalized integral equivalence theorem* in the general framework involving an ordered pair of decomposition systems (Theorem 4.3, including Propositions 4.1 and 4.2). As we have seen, the previous results related to the coincidences among the Choquet integral, the concave (or convex) integral and the pan-integral from below (or from above), are special cases of this generalized integral equivalence theorem. Thus, we have presented a unified approach to the coincidences of several well-known decomposition and superdecomposition integrals.

On the other hand, in previous study the coincidences among the different decomposition integrals (or superdecomposition integrals), the concave integrals, the Choquet integrals and the pan-integrals, etc., were characterized from two aspects: (1) using the structure characteristics of monotone measures, such as, *subadditivity, superadditivity, submodularity, supermodularity, (M)-property and dual (M)-property*, etc., see [6,11,15,20,21]; (2) using the characteristics of measurable sets, such as, *minimal atoms*, *atoms of σ-algebra $\mathcal{A}$*, etc., see [17,18,22]. In this paper, as a third aspect, by using the characteristics of preorder of decomposition systems we have provided a way of studying measure-dependent coincidences of decomposition (superdecomposition) integrals.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Proof of Proposition 3.3(ii). We only prove (ii), the proof of (i) is similar. Suppose that μ is superadditive. Let $(A_i)_{i=1}^n \in \mathcal{H}_{cav}$ and $\lambda_i \geq 0, i = 1, 2, \ldots, n$.

For $n = 1$, it is obvious, for $n = 2$, observe that

$$\lambda_1 \chi_{A_1} + \lambda_2 \chi_{A_2} = \lambda_1 \chi_{A_1 - (A_1 \cap A_2)} + \lambda_2 \chi_{A_2 - (A_1 \cap A_2)} + (\lambda_1 + \lambda_2) \chi_{A_1 \cap A_2}.$$

If we let

$$l_1 = \lambda_1,\ l_2 = \lambda_2,\ l_3 = \lambda_1 + \lambda_2$$

and

$$B_1 = A_1 - (A_1 \cap A_2),\ B_2 = A_2 - (A_1 \cap A_2),\ B_3 = A_1 \cap A_2,$$

then

$$\sum_{i=1}^{2} \lambda_i \chi_{A_i} = \sum_{j=1}^{3} l_j \chi_{B_j}.$$

Moreover, it is from superadditivity of μ that

$$\begin{aligned} &\lambda_1 \mu(A_1) + \lambda_2 \mu(A_2) \\ \geq\ &\lambda_1 (\mu(B_1) + \mu(B_3)) + \lambda_2 (\mu(B_2) + \mu(B_3)) \\ =\ &l_1 \mu(B_1) + l_2 \mu(B_2) + l_3 \mu(B_3). \end{aligned}$$

Now suppose that the formulas (3.3) and (3.4) hold for $n = k$, we need to verify that they are also true for $n = k + 1$. For $\sum_{i=1}^{k+1} \lambda_i \chi_{A_i}$, we have

$$\begin{aligned}\sum_{i=1}^{k+1}\lambda_i\chi_{A_i} &= \sum_{i=1}^{k}\lambda_i\chi_{A_i}+\lambda_{k+1}\chi_{A_{k+1}}\\ &= \sum_{j=1}^{n'}\alpha_j\chi_{D_j}+\lambda_{k+1}\chi_{A_{k+1}},\end{aligned}$$

where D_j, $j=1,2,\ldots,n'$ are pairwise disjoint subsets of X, $\alpha_j \geq 0$ with $\sum_{i=1}^{k}\lambda_i\mu(A_i) \geq \sum_{j=1}^{n'}\alpha_j\mu(D_j)$. Observe the facts that

$$D_j = \big(D_j - (D_j \cap A_{k+1})\big)\bigcup(D_j \cap A_{k+1})$$

and

$$A_{k+1} = \left(A_{k+1} - \bigcup_{j=1}^{n'}(A_{k+1}\cap D_j)\right)\bigcup\left(\bigcup_{j=1}^{n'}(A_{k+1}\cap D_j)\right).$$

If we let

$$\begin{aligned} B_j &= D_j - (D_j \cap A_{k+1}),\quad j=1,2,\ldots,n'\\ B_{n'+j} &= D_j \cap A_{k+1},\ \ j=1,2,\ldots,n',\\ B_{2n'+1} &= A_{k+1} - \bigcup_{j=1}^{n'}(A_{k+1}\cap D_j)\end{aligned}$$

and let

$$l_j = \alpha_j,\ l_{n'+j} = \alpha_j + \lambda_{k+1},\ j=1,2,\ldots,n',\ l_{2n'+1} = \lambda_{k+1},$$

then $(B_j)_{j=2n'+1}^{n} \in \mathcal{H}_{pan}$,

$$\sum_{i=1}^{k+1}\lambda_i\chi_{A_i} = \sum_{j=1}^{2n'+1} l_j\chi_{B_j}$$

and

$$\begin{aligned}&\sum_{i=1}^{k+1}\lambda_i\mu(A_i)\\ &\geq \sum_{j=1}^{n'}\alpha_j\mu(C_j) + \lambda_{k+1}\mu(A_{k+1})\\ &\geq \sum_{j=1}^{n'}\alpha_j\left(\mu(B_j)+\mu(B_{n'+j})\right) + \lambda_{k+1}\left(\mu(B_{2n'+1}) + \sum_{j=1}^{n'}\mu(B_{n'+j})\right)\\ &= \sum_{j=1}^{n'}\alpha_j\mu(B_j) + \sum_{j=1}^{n'}(\alpha_j+\lambda_{k+1})\mu(B_{n'+j}) + \lambda_{k+1}\mu(B_{2n'+1})\\ &= \sum_{j=1}^{2n'+1} l_j\mu(B_j).\end{aligned}$$

Thus we have proved $\mathcal{H}_{cav} \succeq_{super} \mathcal{H}_{pan}[\mu]$. $\square$

Theorem A. *(Theorem 4.1 in [17]) Let X be a finite space, and $\mu \in \mathcal{M}$ be fixed and be finite. Then, for all $f \in \mathcal{F}^+$,*

$$I_{\mathcal{H}_{cav}}(\mu, f) = I_{\mathcal{H}_{pan}}(\mu, f) \tag{A.1}$$

if and only if the following two conditions hold:

(i) μ possesses the minimal atoms disjointness property, i.e., for every pair of minimal atoms A and B of μ, $A \neq B$ implies $A \cap B = \emptyset$;

(ii) μ is subadditive w.r.t. minimal atoms, i.e., for every set $A \in \mathcal{A}$ with $\mu(A) > 0$, we have

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i),$$

where $(A_i)_{i=1}^n$ is the set of all minimal atoms contained in A.

Note: A set $A \in \mathcal{A}$ is called a minimal atom of monotone measure μ if $\mu(A) > 0$ and for every $B \in \mathcal{A}$ and $B \subset A$ holds either (i) $\mu(B) = 0$, or (ii) $A = B$ (see [17]).

Theorem B. *(Theorem 5.2 in [22]) Let $(X, \mathcal{A})$ be a finite measurable space and $\mu \in \mathcal{M}$ be fixed. Then, for all $f \in \mathcal{F}_b^+$,*

$$I^{\mathcal{H}_{cav}}(\mu, f) = I^{\mathcal{H}_{pan}}(\mu, f), \tag{A.2}$$

if and only if one of the following conditions (i) and (ii) is satisfied:

(i) μ is superadditive w.r.t. atoms of $\mathcal{A}$.

(ii) For every minimal strictly μ-subadditive set A w.r.t. atoms of $\mathcal{A}$, we have $\mu(A_i) = \mu([A])$, where A_i is an arbitrary atom of $\mathcal{A}$ and $A_i \subset \bigcup_{B \in [A]} B$.

Note: A nonempty set $A \in \mathcal{A}$ is called *an atom of* $\mathcal{A}$ [24], if $\emptyset$ and A are the only $\mathcal{A}$-measurable subsets of A, i.e., there is no nonempty proper subset B of A such that $B \in \mathcal{A}$. The above other undefined concepts and symbols can be found in [22].

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