Akademie věd České republiky
Ústav teorie informace a automatizace, v.v.i.

Academy of Sciences of the Czech Republic
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RESEARCH REPORT

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GA 19-07635S
Outputs and Results

No. 2396 January 6, 2023

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Abstract

This manuscript aims to deliver a survey of results obtained during the solution of the project No. GA19-07635S of the Czech Science Foundation. The timespan dedicated to the work on this project was 1.3.2019 - 30.6.2022. The main area dealt with were nonlinear multi-agent systems and their synchronization, further, attention was paid to some auxiliary results in the area of nonlinear observers.

This Report briefly introduces the Project, provides a summary of the results obtained and also sketches an outline how these results will be applied and extended in future.

1 Introduction

1.1 Goals of the Project

The purpose of the project was the design of algorithms for synchronization of nonlinear multi-agent systems based on fully nonlinear approaches like the exact feedback linearization. Several variants of the problems were to be studied, such as state-feedback as well as dynamic feedback. Thus, some auxiliary results in the nonlinear observers theory were also included into the project.

1.2 Knowledge Gap to be Filled

Up to the start of the Project, theory of linear multi-agent systems was fairly well-developed. However, this cannot be said about theory of nonlinear multi-agent systems, in spite of numerous papers dealing with this topic. The reason is that a majority of these works dealt with nonlinearities in the agents in a very simple way: they assumed the Lipschitz property of the nonlinearities. Then, these nonlinearities were estimated, usually using the Young inequality. This converted the problem of synchronization of a nonlinear multi-agent system into the problem of sychronization of a linear uncertain multi-agent system. This problem was extensively elaborated at those time. On the other hand, the drawback of this method was a relatively high conservativeness of the results. Namely, the synchronizing control law was able to deal not only with a given nonlinear system but rather with a class of different systems satisfying a common estimate by the aforementioned Young inequality. Hence there was a need for a more precise controller design that would have promised larger applicability combined with more suitable properties of the resulting control law.

To accomplish this task and to deliver practically applicable results, systems with other features were also studied. This concerns multi-agent systems with delays in the control signals as this feature is often encountered in system controlled by communication networks - a typical case of control of multi-agent systems. Moreover, as, in practical applications, not all quantities are measurable. For instance, in case of mechanical systems, measurement of velocity is often connected with difficulties. Hence, it is necessary to reconstruct values of these quantities from the set of the measurable quantities (e.g. the positions or accelerations). To achieve this goal, observers have to be used; in the connection with nonlinear systems, the observers must be able to deal with nonlinearities in the system. Theory of nonlinear observers was not sufficiently elaborated, with many results applicable only under too restrictive assumptions. Hence, extension of the nonlinear observer theory and achievement of practically viable method was also one of the goals set in this Project.

Finally, the ultimate goal was to jointly apply the theory of synchronization of nonlinear multi-agent systems and the theory of nonlinear observers so as to solve the problem of synchronization of a nonlinear multi-agent system with output measurements.

2 A Brief Description of Publications Dedicated to the Project

In the list of references, only the publications supported by the Project are listed.

References

[1] B. Rehák and V. Lynnyk. Consensus of a multi-agent system with heterogeneous delays. *Kybernetika*, 56:363–381, 2020.

[2] B. Rehák and V. Lynnyk. Synchronization of multi-agent systems with directed topologies and heterogeneous delays. In *2020 European Control Conference (ECC)*, pages 1671–1676. IEEE, 2020.

[3] B. Rehák and V. Lynnyk. Synchronization of nonlinear multi-agent systems via exact feedback linearization. In *2020 20th International Conference on Control, Automation and Systems (ICCAS)*, pages 670–677, 2020.

[4] B. Rehák and V. Lynnyk. Leader-following synchronization of a multi-agent system with heterogeneous delays. *Frontiers of Information Technology & Electronic Engineering*, 22(1):97–106, 2021.

[5] Branislav Rehák. Wirtinger inequality-based control design for an interconnected large-scale system with sampled controls. In *2019 Chinese Control Conference (CCC)*, pages 1009–1014, 2019.

[6] Branislav Rehak. Finite element-based observer design for nonlinear systems with delayed measurements. *Kybernetika*, 55:1050–1069, 03 2020.

[7] Branislav Rehák and Petr Augusta. A comparison of two approaches to the control of a system with distributed parameters. In *2019 22nd International Conference on Process Control (PC19)*, pages 43–48, 2019.

[8] Branislav Rehak and Volodymyr Lynnyk. Design of a nonlinear observer using the finite element method with application to a biological system. *Cybernetics and Physics*, 8:292–297, 12 2019.

[9] Branislav Rehák and Volodymyr Lynnyk. Robust stabilization of a discrete-time large-scale interconnected system composed of identical subsystems. In *2019 24th International Conference on Methods and Models in Automation and Robotics (MMAR)*, pages 179–184, 2019.

[10] Branislav Rehák and Volodymyr Lynnyk. Consensus of a nonlinear multi-agent system with output measurements. *IFAC-PapersOnLine*, 54(14):400–405, 2021. 3rd IFAC Conference on Modelling, Identification and Control of Nonlinear Systems MICNON 2021.

[11] Branislav Rehák and Volodymyr Lynnyk. Consensus synchronization of underactuated systems. *IFAC-PapersOnLine*, 54(19):275–280, 2021. 7th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2021.

[12] Branislav Rehák and Volodymyr Lynnyk. Event-triggered luenberger observer for nonlinear systems. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 4701–4706, 2021.

[13] Branislav Rehák and Volodymyr Lynnyk. A functional equation-based computational method for the discrete-time nonlinear observer∗∗supported by the czech science foundation through the research grant no. ga19-07635s. *IFAC-PapersOnLine*, 54(14):120–125, 2021. 3rd IFAC Conference on Modelling, Identification and Control of Nonlinear Systems MICNON 2021.

[14] Branislav Rehák and Volodymyr Lynnyk. Nonlinear luenberger observer for systems with quantized and delayed measurements. *2021 25th International Conference on Methods and Models in Automation and Robotics (MMAR)*, pages 29–34, 2021.

[15] Branislav Rehák and Volodymyr Lynnyk. Stabilization of large-scale systems and consensus of multi-agent systems with time delay - common features and differences. In *2021 23rd International Conference on Process Control (PC)*, pages 13–18, 2021.

[16] Branislav Rehák and Volodymyr Lynnyk. Synchronization of a linear multi-agent system with sampled control and a jointly connected topology. In *2021 9th International Conference on Systems and Control (ICSC)*, pages 210–215, 2021.

[17] Branislav Rehák and Volodymyr Lynnyk. Synchronization of a network composed of Hindmarsh-Rose neurons with stochastic disturbances. *IFAC-PapersOnLine*, 54(17):65–70, 2021. 6th IFAC Conference on Analysis and Control of Chaotic Systems CHAOS 2021.

[18] Branislav Rehák and Volodymyr Lynnyk. Synchronization of a network composed of stochastic Hindmarsh-Rose neurons. *Mathematics*, 9, 2021.

[19] Branislav Rehak and Volodymyr Lynnyk. Synchronization of a nonlinear multi-agent system with application to a network composed of Hindmarsh-Rose neurons. In *2020 28th Mediterranean Conference on Control and Automation (MED)*, pages 1086–1091, 06 2021.

The publications can be grouped into the following groups:

1. Synchronization of multi-agent systems with delays: journal papers [1], [4], conference papers [2], [16].

2. Stabilization of large-scale interconnected systems: conference papers [5], [7], [9].

3. Nonlinear observers: journal papers [6], [8], conference papers [12],[13], [14], a conference contribution at the PANM 2019 meeting.

4. Synchronization of nonlinear multi-agent systems and its application to synchronization of neurons: journal paper [18], conference papers [3], [10], [11], [17], [19].

5. Other results: PANM (two contributions: one devoted to generalized synchronization of chaotic systems and one dealt with the small gain theorem for systems described by quasilinear parabolic equations).

In the subsequent section, the results are described in detail. For results of greater importance, the abstract of the publication is presented.

The above list of publications splits as follows:

- J_{imp}: 4 papers,
- J_{sc}: 1 paper,
- D: 14 papers.

The number of results in the D category significantly exceeded the planned number. All contributions were presented at meetings co-sponsored by IFAC or IEEE.

2.1 Goals of the Project and their Fulfillment

The goals of the project were:

1. Design of networked control algorithms of nonlinear large-scale interconnected systems.
2. Design of networked control algorithms of nonlinear multi-agent systems, for both the leader-following as well as consensus problems.
3. Observer-based control for nonlinear large-scale and multi-agent systems, especially polynomial large-scale systems.
4. All algorithms will be studied for systems with quantization, time delays and also for the case of event-triggered control.

2.1.1 Large-scale Interconnected Systems

The primary concern was to adopt new, efficient methods for the control of large-scale interconnected systems. This is a prerequisite for application to nonlinear large-scale systems as nonlinearities usually require to apply efficient computational methods, otherwise feasibility of the problem is compromised.

To achieve this goal, publications [5] and [9] were elaborated.

2.1.2 Nonlinear Multi-Agent Systems and Multi-Agent Systems with Delays

This is the topic most attention was devoted to. Publications concerning nonlinear multi-agent systems include [3], [10], [11]. Moreover, application of the theory of the nonlinear multi-agent systems to the synchronization of a neural network composed of Hindmarsh-Rose neurons also belongs to this part of the Project. This area is covered by [18], [17] and the preliminary result published in [19].

Since the delays are inevitable in practical control of multi-agent systems, it was also necessary to prepare theoretical foundations for application of control signals that are delayed to the control of multi-agent systems. Moreover, both these areas were unified into the theory of control of nonlinear multi-agent systems with delayed controls. This covers many real-world applications such as control of unmanned aerial or underwater vehicles. Control of multi-agent systems with delayed control signals is studied in [1], [4] or [2].

2.1.3 Observer-Based Control

Here, the primary goal was to improve results in the field of nonlinear observers before they are applied to the synchronization of nonlinear multi-agent systems. This area includes the paper [6] whose results were applied in [8], further [13]. Nonlinear observers with event-triggering were studied in [12], with quantization in [14]. Finally, application of these results to the synchronization problem of multi-agent systems was presented in [10], thus fully satisfying this goal.

3 Synchronization of Multi-Agent Systems

3.1 Paper [1]

The paper dealing with the problem of consensus synchronization of linear multi-agent systems with time delays was [1].

Abstract: *The paper presents an algorithm for the solution of the consensus problem of a linear multi-agent system composed of identical agents. The control of the agents is delayed, however, these delays are, in general, not equal in all agents. The control algorithm design is based on the H_∞-control, the results are formulated by means of linear matrix inequalities. The dimension of the resulting convex optimization problem is proportional to the dimension of one agent only but does not depend on the number of agents, hence this problem is computationally tractable. It is shown that heterogeneity of the delays in the control loop can cause a steady error in the synchronization. Magnitude of this error is estimated. The results are illustrated by two examples.*

The most important result proved in this paper is existence of an upper bound for multi-agent systems with heterogeneous delays in the control loop.

To comment on the main result of that paper, it is necessary to note that the multi-agent system considered there attainsis composed of N agents in the form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad u_i(t) = K \sum_{neighbors\ of\ i} (x_j(t-\tau_j) - x_i(t-\tau_i)) \tag{1}$$

where τ_i, τ_j are communication delays (uniformly bounded) and the graph concerning the interconnection topology is undirected and contains a spanning tree. Moreover, matrices Q_i,$i = 1, 2$, R, S and Y of suitable dimensions appear in the main result as well as matrix function Σ. To be precise, this functions is defined as follows: Assume first matrices $Q_1, Q_2, S \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ are given. Then one can define functions $\sigma_{11}, \sigma_{12} : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\sigma_{13}, \sigma_{23} : \mathbb{R} \to \mathbb{R}^{n\times m}$ and matrices σ_{16}, σ_{33} as

$$\begin{aligned}
\sigma_{11}(d) =& AQ_2 + Q_2^T A^T + d(BY + Y^T B^T),\\
\sigma_{12}(d) =& Q_1 - Q_2 + \varepsilon Q_2^T A^T + dY^T B^T,\\
\sigma_{13}(d) =& \bar{\tau} dBY,\\
\sigma_{16} =& Q_2^T C^T,\\
\sigma_{22} =& -\varepsilon(Q_2 + Q_2^T - \bar{\tau} S)
\end{aligned}$$

Using these functions, let us define matrix-valued function $\Sigma : \mathbb{R} \to \mathbb{R}^{(4n+m+p)\times(4n+m+p)}$ by

$$\Sigma(d) = \begin{pmatrix} \sigma_{11}(d) & \sigma_{12}(d) & \sigma_{13}(d) & I_n & 0 & \sigma_{16} \\ * & \sigma_{22} & \varepsilon\sigma_{13}(d) & 0 & I_n & 0 \\ * & * & \sigma_{33} & 0 & 0 & 0 \\ * & * & * & -\sigma I_n & 0 & 0 \\ * & * & * & * & -\frac{\gamma}{\varepsilon} I_n & 0 \\ * & * & * & * & * & -I_p \end{pmatrix}$$

It is denoted as (here, a simplified version is presented)

Theorem 4.1: *Consider the multi-agent system (1) satisfying Assumption 2.1.*[1] *Assume also the minimal nonzero eigenvalue of the Laplacian matrix is equal to d_1 and maximal eigenvalue of the Laplacian matrix equals d_{N-1}. Let there exist $n \times n$-dimensional matrices $Q_1 > 0$, Q_2 nonsingular, $S > 0$, a $m \times n$-dimensional matrix Y and scalars $\gamma > 0$, $\varepsilon > 0$ such that*

$$\Sigma(d_1) < 0, \ \Sigma(d_{N-1}) < 0. \tag{2}$$

holds. Then

1. *if*
$$w_1 = \cdots = w_N, \ \tau_1 = \cdots = \tau_N \quad \text{for every } t \geq 0 \tag{3}$$
then consensus is achieved;

2. *if condition (3) is not satisfied, then there exists a constant $c > 0$ such that*
$$\limsup_{t\to\infty} \sum_{i=1}^{N} \|x_i - \bar{x}\| \leq c\|\omega_2\| \tag{4}$$
and the function $\omega_2 = \begin{pmatrix} \int_{t-\tau_1}^{t} \dot{\bar{x}}(s)ds \\ \vdots \\ \int_{t-\tau_N}^{t} \dot{\bar{x}}(s)ds \end{pmatrix}$ *where $\bar{x}$ is the average dynamics.*

The paper highlights a not well known feature preventing the synchronization to appear. Even if heterogeneous delays are much more natural, attention was almost exclusively paid to multi-agent systems with equal delays before. Moreover, a bound on the steady synchronization error was derived, hence inducing a bound on the maximal difference in the delays of different agents.

3.2 Paper [4]

This is a counterpart of the previously mentioned paper for the case of a leader-following synchronization.

Abstract: *An algorithm for leader-following synchronization of a multi-agent system that is composed of linear agents with a time delay is presented. The presence of different delays*

[1] Assumption about connectedness and existence of a spanning tree of the graph describing the interconnection topology

in various agents can cause a synchronization error that does not converge to zero. However, the norm of this error can be bounded and this bound is presented. The proof of the main results is formulated by means of linear matrix inequalities and the size of this problem is not dependent on the number of agents. Results are illustrated through examples highlighting the fact that the steady error is caused by heterogeneous delays and demonstrate the capability of the proposed algorithm to achieve synchronization up to a certain error.

The multi-agent system is composed of the following agents

$$\dot{x}_0 = Ax_0,\ \dot{x}_i = Ax_i + Bu_i,\ i = 1, \ldots, N, \tag{5}$$
$$x_i(0) = x_{i,0}, i = 0, \ldots, N \tag{6}$$

and the control of the ith agent is given by

$$u_i = -Kd_i(x_{0,\tau_0} - x_{i,\tau_i}) - \sum_{j=1}^{N} E_{ij}K(x_{j,\tau_j} - x_{i,\tau_i}). \tag{7}$$

The time delays are uniformly bounded as in the previous paper. The graph describing the interconnection topology is supposed to contain a spanning tree, however, the graph can be directed.

The main result uses a matrix function Σ defined in the Appendix of the paper. It is defined in a manner similar to the matrix function Σ introduced in the previously mentioned paper.

Theorem 5: *Let constants $\lambda_1, \lambda_2 \in \mathbb{R}$, $\iota_1, \ldots, \iota_k \in \mathbb{C}$ such that all eigenvalues of $L + D$*[2] *lie in the convex hull of $\lambda_1, \lambda_2, \iota_1, \bar{\iota}_1, \ldots, \iota_k, \bar{\iota}_k$. Assume that $n \times n$-dimensional matrices Q, S and W exist such that $S > 0$ and $W > 0$, Q is nonsingular, a $m \times n$-dimensional matrix Y and scalars $\varepsilon > 0$ and $\gamma > 0$ and $K = YQ^{-1}$ so that $\Sigma(A, \lambda_i B, Y, Q, S, W) < 0$ for all $i = 1, 2$ and $\Sigma(I_2 \otimes A, J'(\iota) \otimes B, I_2 \otimes Y, I_2 \otimes Q, I_2 \otimes S, I_2 \otimes W, \varepsilon, \gamma) < 0$ for all $i = 1, \ldots, k$. Then a constant $\varkappa$ exists such that for all $T' > 0$ holds*

$$\int_0^{T'} \|\xi\| dt \leq \varkappa \int_0^{T'} \|\omega_2\|^2 dt \tag{8}$$

if $\xi(t) = 0$ for $t \in [-\bar{\tau}, 0]$ and $\omega_2 = \begin{pmatrix} \int_{t-\tau_1}^{t-\tau_0} \dot{x}_1(s)ds \\ \vdots \\ \int_{t-\tau_N}^{t-\tau_0} \dot{x}_N(s)ds \end{pmatrix}.$

A generalization of the results described in the previous papers.

3.3 Conference paper [2]

A preliminary version of the aforementioned journal paper, with reduced amount of information, was presented at the ECC 2020 conference.

[2] matrix L is the Laplacian matrix, D is the pinning matrix

3.4 Conference paper [16]

A multi-agent system with a jointly connected topology (the interconnection topology of the agents periodically switches among a finite number of interconnection topologies, however, the union of these interconnection topologies is strongly connected) was considered here. Moreover, the system was assumed to have a sampled control. A synchronization algorithm for such a system was developed.

Abstract: *An algorithm for synchronization of a linear multi-agent system with jointly connected topology is presented.The system is composed of identical agents. It is assumed that the control of the agents is sampled. The resulting control is obtained as a solution of a set of linear matrix inequalities originating from a suitable Lyapunov-Krasovskii functional in connection with the Barbalat lemma. An example illustrates the results.*

It is assumed the agents are identical, described by the equation $\dot{x}_i = Ax_i + Bu_i$, the control u_i is sampled with period $\bar{\tau}$, the interconnection topologies are denoted by $\mathcal{G}_i$, $i = 1, \ldots, \nu$; it is assumed all topologies are exchanged during any interval of length T_1, the maximal time for any topology to stay unchanged is $T_2 > 0$, the minimal time between topology switches is $T_3 > 0$.

The results are summarized by Theorem 7:

Theorem *Let there exist $n \times n$-dimensional matrices $Q_1 > 0$, Q_2 non-singular, $R > 0$, a $m \times n$-dimensional matrix Y and scalars $\alpha > 0$, $\beta \in (0, 1)$ and $\varepsilon > 0$ so that, with matrix-valued function Σ defined below, with some positive constant c and κ (depending on the system, see the paper) and $K = YQ_2^{-1}$ the following holds: $\Sigma(d_1) < 0$, $\Sigma(d_{N-1}) < 0$, $(T_2\alpha\|Q_2^{-1}\|^2\nu(1-\beta) - \nu T_3\kappa(\frac{1}{\beta}+1)c^2) > 0$. Then the multi-agent system is synchronized.*

Here,

$$\Sigma(d) = \begin{pmatrix} \varepsilon((AQ_2 + dBY) + (AQ_2 + dBY)^T) & Q_1 - Q_2 + \varepsilon(Q_2^T A^T + dY^T B^T) & \bar{\tau}dBY \\ Q_1 - Q_2^T + \varepsilon(AQ_2 + dBY) & -\varepsilon(Q_2 + Q_2^T) + \bar{\tau}R & \varepsilon\bar{\tau}dBY \\ \bar{\tau}dY^T B^T & \varepsilon\bar{\tau}dY^T B^T & -\bar{\tau}R \end{pmatrix} \tag{9}$$

The result is more efficient than existing approaches as it uses a method specifically tailored to handle sampled multi-agent systems with jointly connected topology.

Note also that systems with jointly connected topology can be used as a model for multi-agent systems with partially disturbed communication, e.g. by malicious attacks. This result can thus be a base for a development of a control law resilient against these attacks.

4 Stabilization of Large-Scale Systems

4.1 Conference paper [5]

This paper improves results obtained before by using a more powerful matrix inequalities based on the Wirtinger inequality. Moreover, the interconnection of the subsystems is more general (the previous paper whose results are generalized, namely the paper Bakule et al., Decentralized H-infinity control of complex systems with delayed feedback, Automatica, 67: 127 – 131, 2016 is focused on systems where all subsystems are coupled).

The system considered is composed of subsystems

$$\dot{x}_i = Ax_i + BKx_i + BK(x_{i,\tau_i} - x_i) + \sum_{j=1}^{N} l_{ij}\tilde{A}(x_j - x_i),\ x_i(0) = x_{i,0},\quad i = 1, \ldots, N. \tag{10}$$

The control actions are given by $u_i(t) = Kx_i(t - \tau_i)$.

The most important result is as follows:

Assume $\gamma > 0$ is a scalar and let there be $n \times n$-dimensional matrices $\bar{P} > 0$, $\bar{R} > 0$, $\bar{S} > 0$, $\overline{W} > 0$ and $\bar{Z} > 0$ and $Y \in R^{n\times m}$. Let us also define matrix Ψ by

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} & \psi_{16} & 0 \\ * & \psi_{22} & \psi_{23} & \varepsilon\psi_{14} & 0 & 0 & \sqrt{\varepsilon}\psi_{16} \\ * & * & \psi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \psi_{44} & 0 & 0 & 0 \\ * & * & * & * & -I_n & 0 & 0 \\ * & * & * & * & * & -\gamma I_n & 0 \\ * & * & * & * & * & * & -\gamma I_n \end{pmatrix}$$

where

$$\begin{aligned}
\psi_{11} =& Q^T(A + d\tilde{A})^T + (A + d\tilde{A})Q + \bar{R} - \bar{S} \\
\psi_{12} =& \bar{P} - Q + \varepsilon(Q^T A^T + dQ^T \tilde{A}^T) \\
\psi_{13} =& S + BY \\
\psi_{14} =& BY \\
\psi_{15} =& Q^T C^T \\
\psi_{16} =& G \\
\psi_{22} =& -\varepsilon(Q + Q^T) + \tau'^2\bar{R} + (\bar{\tau}^2 - \tau'^2)\overline{W} + \varepsilon Z \\
\psi_{23} =& \varepsilon BY \\
\psi_{33} =& -\bar{R} - \bar{S} \\
\psi_{44} =& -\frac{\pi^2}{4}\overline{W}
\end{aligned}$$

Theorem: *Assume there exist scalars $\varepsilon > 0$, $\gamma > 0$ and matrices $\bar{P} > 0$, $\overline{W} > 0$, $\bar{R} > 0$, $S > 0$, Q nonsingular and Y so that*

$$\Psi(d) < 0 \tag{11}$$

holds for $d = \lambda_{\min}(L)$ as well as for $d = \lambda_{\max}(L)$. Let also $K = YQ^{-1}$. Then system (1) with control given by (13) is stable. This paper also shows how much improvement was achieved in comparison with previously obtained results.

A less conservative method for finding the stabilizing control of an interconnected large-scale system was obtained, its properties were illustrated. A thorough comparison with methods developed earlier was presented.

4.2 Conference paper [9]

Here, the attention is paid to discrete-time large-scale systems with delays in the control loop. To be specific, we consider the interconnected system composed of N subsystems

$$x_i(k+1) = (A + DF_i(k)E)x_i(k) + Bu_i(k) + \sum_{j=1}^{N} l_{ij}\tilde{A}x_j(k), \tag{12}$$

where $i = 1, \ldots, N$, $x_i : \mathbb{N} \to \mathbb{R}^n$, $A, \tilde{A} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the control $u_i : \mathbb{N} \to \mathbb{R}^m$ is given by

$$u_i(k) = Kx_i(k - \tau(k)) \tag{13}$$

with a matrix $K \in \mathbb{R}^{m\times n}$ to be designed.

It is assumed the interconnection matrix is symmetric and its smallest eigenvalue is denoted by $d_{\min}$, the maximal one is $d_{\max}$ and the time delays are uniformly bounded between d_m and d_M.

First, for given matrices P, X, Q_i, $i = 1, 2, 3$, M_i, N_i, R_i, L_i, Y_i, $i = 1, 2$ one constructs matrix $\Omega(d)$ as follows:

$$\begin{aligned}
\Omega(d) &= \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14}(d) \\ * & \omega_{22} & \omega_{23} & \omega_{24} \\ * & * & \omega_{33} & 0 \\ * & * & * & \omega_{44} \end{pmatrix}, \\
\omega_{11} =& - P + (1 + d_M - d_m)Q_1 + Q_2 + Q_3 + \mathcal{H}(N_1), \\
\omega_{12} =& (L_1 - M_1 + N_2^T - N_1 \quad M_1 \quad - L_1), \\
\omega_{13} =& (\sqrt{d_M - d_m}L_1 \quad \sqrt{d_M - d_m}M_1 \quad \sqrt{d_M}N_1), \\
\omega_{14}(d) =& (\omega_{14,1}(d) \quad \omega_{14,2}(d) \quad \omega_{14,3}(d)) \\
\omega_{14,1}(d) =& A^T + d\tilde{A}^T, \\
\omega_{14,2}(d) =& \sqrt{d_M - d_m}(A^T + d\tilde{A}^T - I), \\
\omega_{14,3}(d) =& \sqrt{d_M}(A^T + d\tilde{A}^T - I)^T\Big),
\end{aligned}$$

$$\begin{aligned}
\omega_{22} &= \begin{pmatrix} -Q_1 + \mathcal{H}(L_2 - M_2 - N_2) & M_2 & -L_2 \\ * & -Q_2 & 0 \\ * & * & -Q_3 \end{pmatrix}, \\
\omega_{23} &= \begin{pmatrix} \sqrt{d_M - d_m}L_2 & \sqrt{d_M - d_m}M_2 & \sqrt{d_M}N_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\omega_{24} &= \begin{pmatrix} K^TB^T & \sqrt{d_M - d_m}K^TB^T & \sqrt{d_M}K^TB^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\omega_{33} =& \operatorname{diag}(-R_1 - R_2, -R_1, -R_2), \\
\omega_{44} =& \operatorname{diag}(X, Y_1, Y_2).
\end{aligned}$$

The paper presents a proof of the statement that, if there exist matrices $P > 0$, $Q_i > 0$, M_i etc. such that the following relations hold

$$\begin{aligned} 0 >& \Omega(d_{\min}), && (14)\\ 0 >& \Omega(d_{\max}), && (15)\\ I_n =& PX, && (16)\\ I_n =& R_1 Y_1, && (17)\\ I_n =& R_2 Y_2 && (18) \end{aligned}$$

then the discrete-time large-scale system composed of N subsystems (12) is stable.

The resulting set of matrix inequalities is not linear. An iterative procedure was proposed that renders the problem into a subsequent solution of a set of linear matrix inequalities. This procedure enables us to use effective LMI solvers to obtain the result.

Discrete-time complex systems are being studied less frequently than systems with continuous time, thus this paper aims to fill this gap.

5 Nonlinear Observers

5.1 Paper [6]

Abstract: *This paper presents a computational procedure for the design of an observer of a nonlinear system. Outputs can be delayed, however, this delay must be known and constant. The characteristic feature of the design procedure is computation of a solution of a partial differential equation. This equation is solved using the finite element method. Conditions under which existence of a solution is guaranteed are derived. These are formulated by means of theory of partial differential equations in L^2-space. Three examples demonstrate viability of this approach and provide a comparison with the solution method based on expansions into Taylor polynomials.*

Here, a construction of an observer for the systems

$$\dot{x} = F(x) \text{ for } t \geq -\tau,\ y = h(x_\tau) \text{ for } t \geq 0 \tag{19}$$

with initial conditions

$$x(-\tau) = x_0 \in \mathbb{R}^n. \tag{20}$$

is described. As noted in the references to this paper, the observer is given by

Preliminaries: We aim to find an observer defined as

$$\dot{\hat{x}} = F(\hat{x}) + L(\hat{x})(y_\tau - \hat{y}_\tau),\ \hat{y} = h(\hat{x}). \tag{21}$$

The crucial object is the observer gain $L : \mathbb{R}^n \to \mathbb{R}^n$ designed so that the observation error $e(t) = x(t) - \hat{x}(t)$ converges to zero for $t \to \infty$.

To find the observer gain, one has to solve the equation for a smooth function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$:

$$\frac{\partial \Phi}{\partial x} F(x) = \tilde{A}\Phi(x) + bh(x_\tau) \tag{22}$$

where matrix $\tilde{A}$ has sufficiently stable eigenvalues and the pair $(\tilde{A}, b)$ is controllable. Then, the control gain is given by

$$L = \Big(\frac{\partial \Phi}{\partial x}(\hat{x})\Big)^{-1} b \tag{23}$$

The main contribution of the paper was twofold: first, it delivered a proof of existence of equation (21) on a predefined domain (as opposed to the proof of existence using Taylor series obtained before). The other one is a successful application of the Finite Element Method (FEM) to the solution of (21).

The existence result (in this report, it is presented in a rather simplified version) is formulated as (with A being the linearization of function F at the origin)

Theorem:

Assume $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, is a bounded domain such that

$$a_i - \frac{1}{2}\Big(\mathrm{div} F(x)\Big) > 0 \tag{24}$$

holds for all $x \in \Omega$ and all $i = 1, \ldots, n$. Let also $\Gamma_i^- = \{x \in \partial\Omega | n(x).(F(x)) < 0\}$. Then

- *for every $i \in \{1, \ldots, N\}$ there exist uniquely determined functions $\phi_i \in L^2(\Omega)$, $\phi_i = 0$ on Γ_i^-, such that $\phi = (\phi_1, \ldots, \phi_n)^T$ solves (22).*
- *the observer gain L given by*

$$L(\hat{x}) = \Big(\bar{\Phi} + \frac{\partial \phi}{\partial x}(\hat{x})\Big)^{-1} b \tag{25}$$

  is such that observer (21) guarantees $\lim_{t\to\infty} \|e(t)\| = 0$.

This is a point that justifies the application of the finite element method.

This paper generalizes some assumptions guaranteeing existence of the observer gain, thus increasing number of potential applications of this method. Note that before, these assumptions were too restrictive to be applicable to many real-world systems. Moreover, implementation details and a brief tutorial for a numerical solution by the Finite Element Method were presented.

5.2 Conference paper [13]

This conference paper can be regarded as a discrete-time counterpart to the paper described in the previous subsection. The methodology is analogous, albeit instead of a partial differential equation, a functional equation is solved. As there is not a software for solution of this kind of equations, it was necessary to develop algorithms including implementation of this method from scratch.

The most important contribution of this paper is twofold: first, the method developed here enables to remove the requirement for asymptotic stability of the observed system (that was imposed on the observed system earlier) and an effective numerical method for computation of the observer was proposed.

We consider the discrete-time system

$$x(k+1) = F(x(k)),\ x(0) = x_0, \tag{26}$$
$$y(k) = h(x(k)) \tag{27}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ is the state, $y : \mathbb{N} \to \mathbb{R}$ is the output, $F : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

An observer is constructed for this system. It was known previously that the observer is constructed using the solution T of the following equation

$$T(F(x)) = Ax + bh(x),\ T(0) = 0 \tag{28}$$

on a pre-defined domain. Solution of this equation was conducted numerically, using the finite difference method. This means, values of the solution were evaluated on a fixed discrete mesh. The values in other points were approximated using polynomial interpolation.

Numerical methods (apart from those using the expansion into Taylor polynomials) were not proposed for this kind of observers for discrete-time systems. The proposed method allows to find a solution under milder assumptions than in the pioneering paper.

5.3 Journal article [8]

The paper presents an application of the aforementioned method to the problem of state estimation of a biological system - a bioreactor.

Abstract: *An observer for a nonlinear biological system — biomass production in a bioreactor — is proposed. The specific growth rate is estimated. The key point of the observer design is finding a solution of a certain partial differential equation. Conditions guaranteeing existence of its solution are presented. The solution is approximated using finite element method. The results are illustrated by a numerical example.*

The problem concerned is estimation of the state of the biomass growth in a bioreactor. This process can be described by equations (see references in the paper):

$$\dot{x} = (\mu - D(x,t))x, \tag{29}$$
$$\dot{\mu} = \rho(x, \mu, t). \tag{30}$$

The first equation describes changes in the biomass concentration x dependent on the dilution rate D while the second equation shows how the specific growth rate changes. The function ρ attains a specific form according to the kinetic model used - the kinetics is reformulated using the Monod or Haldane curves.

This problem poses several challenges connected mainly with the fact that the system exhibits a dynamics whose parts are very fast and, simultaneously, other parts are very slow. Still, it was demonstrated that the proposed algorithms copes well with the problem. The results were illustrated by examples.

This paper demonstrates how the finite element-based method for a construction of a nonlinear observer is capable of handling a real-world problem. This result is interesting also since the observed system exhibits dynamics with two time scales - parts of its dynamics are much faster than other parts. Even this challenge was successfully resolved.

5.4 Conference paper [14]

In this paper, the results concerning the Luenberger observer were extended to the case when the measurements are quantized. Quantization of the measured data in combination with the Luenberger-like nonlinear observer were not investigated before.

Abstract: *A Luenberger-like observer for nonlinear systems with quantized measurements is proposed. The observer design is based on the solution of a certain partial differential equation that is solved numerically. Then, stability of this observer is proved even in presence of quantized measurements and delayed measurements. The results are illustrated by an example.*

Quantization often occurs when data is transmitted through communication networks. Therefore the need for investigation of effects of quantization on the precision of the Luenberger observer for nonlinear systems. In the paper, the observer investigated in [6] was augmented by the quantization mechanism: the measured data were quantized by a logoarithmic quantized, resulting in the fact that the measured data attained only a discrete set of values $\{\sigma\rho^j\} \cup \{-\sigma\rho^j\} \cup \{0\}$, $j \in N$ with $\sigma > 0$ and $\rho > 0$ were parameters of the quantizer.

The scheme of the observer with quantized measurements is shown in Fig. 5.4. The transmitted value of the output of the quantizer is transmitted through the channel.

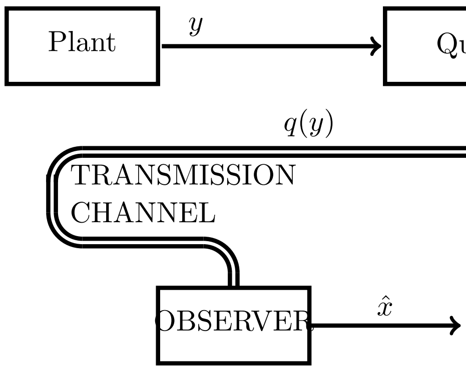

Figure 1: Observer with quantized measurements.

It was shown that the problem of observation with a quantized observer can be converted into a problem of observation with an observer with uncertainty. The major tool to derive the results were also Lyapunov functions in connection with linear matrix inequalities theory. The main results (Theorem 5) reads **Theorem** *Consider system (as in [6]) with the time delay $\tau > 0$. Let there exist $n \times n$-dimensional matrices $P > 0$, $Q > 0$ and a scalar $h > 0$ so* *that the following matrix inequalities hold:*

$$\tilde{A}^T P + P\tilde{A} + \tau\Big(2Q + (M\delta)^2 P + (M\delta)^4 P\Big) < 0, \tag{31}$$

$$h\tilde{A}^T\tilde{A} < P, \tag{32}$$

$$hI_n < P, \tag{33}$$

$$\begin{pmatrix} Q & P \\ * & hI_n \end{pmatrix} > 0, \tag{34}$$

Then the observation error converges to zero. (The constant $M > 0$ describes the robustness required due to the application of the quantized control; more to be found in the original paper.)

To our best knowledge, this is the first attempt to combine the Luenberger-like observer design with quantization.

5.5 Conference paper [12]

This paper is another continuation of the effort in investigation of the Luenberger-like nonlinear observers. Here, the nonlinear observer is combined with the event-triggering mechanism; these two areas have not been combined before.

Abstract: *A Luenberger-like observer for nonlinear systems combined with the event-triggering mechanism is proposed. A proof of stability of the proposed scheme is presented. Moreover, it is proved that the Zeno behavior does not happen. The results are illustrated by an example.*

The paper was presented at the CDC 2021 conference, this is one of most important events in the area of the control theory and cybernetics.

The observer is the same as considered in paper [6]. However, here, the measurements are not obtained continuously in time but rather in certain time instants when the so-called event-triggering condition is satisfied. The observer with the event-triggering mechanism can be seen in Fig. 5.5. Note that the data is transmitted only at the discrete time instants t_k rather than continuously in time, thus economizing the capacity of the transmission channel.

The event-triggering condition reads

$$t_{k+1} = \inf\{t > t_k \mid |y(t) - y(t_k)| \geq \delta(|y(t)| + d)\}. \tag{35}$$

with user-defined parameters d and δ.

The main idea is to design a Lyapunov function $V = e^T Pe$ (e being the observation error and P is a symmetric positive definite matrix) for the observer without the event-triggering mechanism, then, this Lyapunov function is used as a tool for analysis of the event-triggered observer. Existence of the event-triggering mechanism yields that the observation error does not converge to zero but there is a steady error that can be bounded as

$$\|e\|^2 \leq \bar{K}\frac{\beta\delta^2}{\lambda_{\min}(P)\gamma}(M + d)^2 \tag{36}$$

with $\bar{K}$, β, γ and M are certain positive constants.

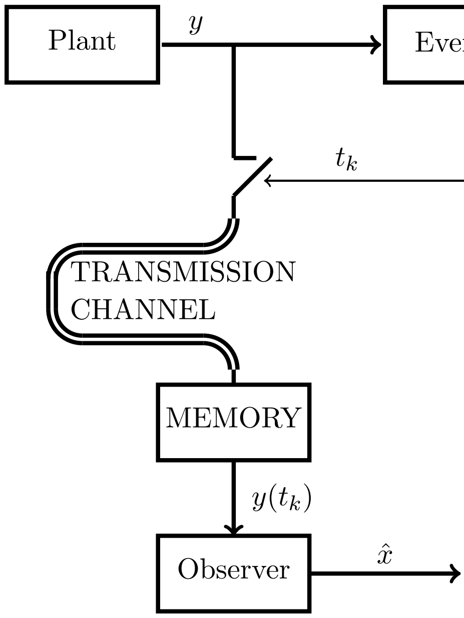

Figure 2: Event-triggered mechanism and observer.

Moreover, it has been demonstrated that the minimal inter-event time is positive, hence, the so-called Zeno behavior (infinite number of events during a finite time interval) does not occur. The lower bound on the minimal time interval between two events is also derived and presented in the paper.

The Luenberger observer with event-triggering mechanism was not studied before. Convergence of the estimate together with absence of the Zeno behavior (that means, existence of a nonzero minimal interval between events) has been proved.

5.6 Conference Contribution - A Numerical Method for the Solution of the Nonlinear Observer Problem, Programs and Algorithms of Numerical Mathematics

Moreover, numerical aspects of the finite element-based method for the solution of the nonlinear observer problem were discussed at the PANM 2020 meeting at Hejnice, July 2020. Issues concerning convergence of the finite element method were discussed in a more detail than in the aforementioned papers.

This contribution has not been included into the proceedings.

Reference to this contribution: : A Numerical Method for the Solution of the Nonlinear Observer Problem, Programs and Algorithms of Numerical Mathematics. Proceedings of Seminar, p. 110-119 , Eds: Chleboun J., Kůs P., Přikryl P., Rozložník M., Segeth K., Šístek J., Programy a algoritmy numericke matematiky 2020 (PANM 2020), (Hejnice, CZ, 20200621)

6 Nonlinear Multi-Agent Systems

6.1 Conference paper [3]

The synchronization problem for a nonlinear multi-agent system is investigated in this paper. Both the leader-following as well as the consensus problem are investigated. The most important result is a procedure that guarantees how to change the control signal in order to change the resulting average dynamics to be equal to the dynamics of one agent.

Abstract: *Synchronization of a multi-agent system composed of identical nonlinear agents admitting exact feedback linearization is presented. The agents may not be fully linearizable, however, if they are minimum-phase systems, synchronization is achieved. Both leader-following as well as the consensus problem are treated. In the latter case, it is guaranteed that the average dynamics equals to the dynamics of one agent. The results are illustrated by an example.*

Application of the exact feedback linearization, as described in several papers before, is a nonlinear transformation of all agents into a linear form. The resulting linear form is useful for effective design of the synchronizing control, however, some undesirable effects arise. In the case of the consensus problem, if the consensus is achieved, the resulting average dynamics (sometimes called the consensus dynamics) is not identical to the dynamics of one agent (which is nonlinear) but rather to the linearized dynamics of the agent. (Still, one can say that the consensus problem has been solved as the synchronization error converges to zero.) In the case of the leader-following problem, the situation is more complicated - the followers do not track the dynamics of the leader but rather the linearized dynamics of the leader, hence the leader following problem has not been solved successfully.

A remedy is found for both problems by adding some terms to the feedback. Addition of these terms requires to use the robust control theory. Then, using a suitable Lyapunov function in connection with the small gain theorem one can infer that the synchronization in both the consensus problem as well as the leader-following problem is guaranteed while the dynamics of the leader is exactly followed in the latter case and the average dynamics is identical to the dynamics of a single agent in the first case.

This result eliminates some unwanted features of the method developed earlier (namely, alteration of the average dynamics), thus increasing its potential applicability.

6.2 Conference paper [10]

Consensus of nonlinear multi-agent systems with output feedback is solved. This result fuses both directions investigated in this Project - theory of synchronization of nonlinear multi-agent systems with theory of nonlinear observers.

Abstract *The consensus problem of a multi-agent system with nonlinear agents is solved. It is assumed only the outputs of the agents are measurable, in order to obtain the estimate of the state, the nonlinear Luenberger observer is applied. The control of the agents is based on the exact feedback linearization. The method is illustrated by an example.*

The task is divided into two subproblems: finding an observer for the nonlinear agents and finding the synchronizing control law with state feedback. Then, both problems are combined, it is proved that application of the state estimate instead of the true state of the agents also leads to synchronization.

The observer is constructed using the same method as in paper [6], with application of the finite elements. Thus, details are omitted here. Moreover, with the designed observer, one can construct a Lyapunov function V (depending on the observation error) such that, for each agent, this Lyapunov function converges to zero. Such a function exists since the observer was constructed so that the observation error converges to the origin.

The synchronizing controller was designed using the exact feedback linearization, with the assumption of the state feedback in mind. As such a control renders the synchronization error to zero, one can again construct a Lyapunov function W that converges to zero for time growing to infinity.

The last task is to investigate the sum of both Lyapunov functions under the condition that, for the feedback, the output of the observer was used instead of the true state. This causes appearance of an additional term that intercouples both Lyapunov functions. The main result, formulated by means of **Theorem 5.1**, states that, if the Lyapunov functions are designed so that certain inequalities hold, then this control law guarantees synchronization of the nonlinear multi-agent system under the output feedback.

This is the most important result obtained in this Project as it merges two important research lines this Project was focused at - theory of nonlinear observers and nonlinear multi-agent systems. It was presented at the Micnon 2021 meeting, which is one of important meetings in the area of nonlinear control systems.

6.3 Conference paper [11]

In this paper, a special class of nonlinear agents - so-called underactuated systems - were studied. Such systems are often met in practice, e.g. in robotics.

Abstract *Consensus synchronization problem was solved for a multi-agent system composed of identical underactuated system. The method is based on using a collocated feedback combined with a choice of the fictitious output so that the agents are minimum-phase with this output. Then, a synchronizing control is found based on the transformed variables. The results are illustrated on an example.*

The dynamics of the underactuated system was supposed to attain the form

$$D(q_i)\ddot{q}_i + C(q_i, \dot{q}_i)\dot{q}_i + G(q_i) = B(q_i)u_i \tag{37}$$

with q_i being the state and u_i being the control.

Before the result about the synchronization of the nonlinear multi-agent systems can be applied, it is necessary to ensure that the agents have an exponentially stable zero dynamics. Fortunately, there is a result that enables to redefine the output of an underactuated system so that this property is established. This procedure in combination with the algorithm for synchronization of nonlinear multi-agent systems yield the final result.

This paper, thanks to the practical importance of the underactuated systems, might potentially have a major impact of the project in real-world applications.

7 Applications to the Synchronization of Hindmarsh-Rose Neurons

The results obtained in the field of synchronization of nonlinear multi-agent systems were applied to the synchronization of a network composed of Hindmarsh-Rose neurons. These networks have the favourable property that the agents - in this case, the neurons - admit the exact feedback linearization and, moreover, their zero dynamics is asymptotically stable.

7.1 Paper [18]

This is the most important result in this part of the Project. Here, the expertise gained in the field of synchronization of nonlinear multi-agent systems was applied to the problem of synchronization of a network composed of Hindmarsh-Rose neurons. Moreover, the aforementioned methodology was extended by investigating the network with noise in agents (neurons) which is a very practical extension.

Abstract: *An algorithm for synchronization of a network composed of interconnected Hindmarsh-Rose neurons is presented. Delays are present in the interconnections of the neurons. Noise is added to the controlled input of the neurons. The synchronization algorithm is designed using convex optimization and is formulated by means of linear matrix inequalities via the stochastic version of the Razumikhin functional. The recovery and the adaptation variables are also synchronized, this is demonstrated with the help of the minimum-phase property of the Hindmarsh-Rose neuron. The results are illustrated by an example.*

In this paper, it is shown that the model of the Hindmarsh-Rose neuron, described by the following set of equations, the HR neuron is defined by the following equations:

$$\dot{x}_1 = ax_1^2 - x_1^3 + x_2 - x_3 + I + \bar{\sigma}(x_1)dw(t), \tag{38}$$

$$\dot{x}_2 = 1 + bx_1^2 - x_2, \tag{39}$$

$$\dot{x}_3 = c(x_1 + 1.56) - 0.006x_3. \tag{40}$$

satisfies the requiements necessary to apply the exact feedback linearization-based method for synchronization of nonlinear multi-agent systems. Moreover, this system has a nontrivial asymptotically stable zero dynamics. Hence all features of the theory developed in this Project can be applied to this system.

Moreover, as the noise is inevitable in practical applications, it was decided to include the stochastic properties into the agents. This does not pose much difficulties, however, the theory had to be adapted.

The main result is (with L being the Laplacian matrix of the interconnection of the neurons and e being the synchronization error):

Theorem 2: *Let there exist the constants $r, \varkappa_1, \varkappa_2, \varkappa$ and y and symmetric positive definite matrices Q, Z, W, R, S and D (diagonal) so that the following LMIs hold:*

$$r > 1, \tag{41}$$

$$L^T DL \leq \varkappa_1 D, \tag{42}$$

$$L^T L \leq \varkappa_2 D, \tag{43}$$

$$DL + L^T D > \varkappa D, \tag{44}$$

$$\varkappa y + 2\varkappa_1 \bar{\tau}(Z + W) + 2(\varkappa_2 \bar{\tau} + \Sigma^2)Q + \Sigma^2 Q < 0, \tag{45}$$

$$\begin{pmatrix} Z & y \\ y & Q \end{pmatrix} \geq 0, \tag{46}$$

$$\begin{pmatrix} W & y \\ y & Q \end{pmatrix} \geq 0, \tag{47}$$

$$R + S < Q, \tag{48}$$

$$\begin{pmatrix} rQ & y \\ y & R \end{pmatrix} \geq 0, \tag{49}$$

$$\begin{pmatrix} r\Sigma^2 Q & Q\Sigma \\ \Sigma Q & S \end{pmatrix} \geq 0, \tag{50}$$

then $\lim_{t\to\infty} \mathbb{E}(\|e\|^2) = 0$ *for any initial condition* $e(t), t \in [-\bar{\tau}, 0]$.

The symbol $\mathbb{E}$ stands for the mean value.

As an illustration of the results, some of the figures showing the convergence of the states of the neurons are included. The detailed description of these figures can be found in [18], hence it is omitted here, together with values of the parameters of the model. Just note that the membrane potential is the controlled variable while the adaptation and recovery (not illustrated here) variables consitute the zero dynamics. From the figures, one can see the entire dynamics of one neuron and also the synchronization error in the membrane potential as well as in the adaptation variable. The most important result is that the latter two quantities converge to zero up to the effects of the noise.

This paper shows one of the potential of applications of the developed theory of synchronization of nonlinear multi-agent systems based on the exact feedback linearization. Moreover, effects of stochastic disturbances were studied here.

7.2 Conference paper [19]

Some preliminary results needed to show that the Hindmarsh-Rose neurons are satisfying requirements required to apply the synchronization method based on the exact feedback linearization are presented in this paper. This paper thus can be regarded as a preliminary version of the aforementioned journal article. However, one important difference is absence of noise (and thus, all terms involving the variance of the noise in the set of LMIs to be solved).

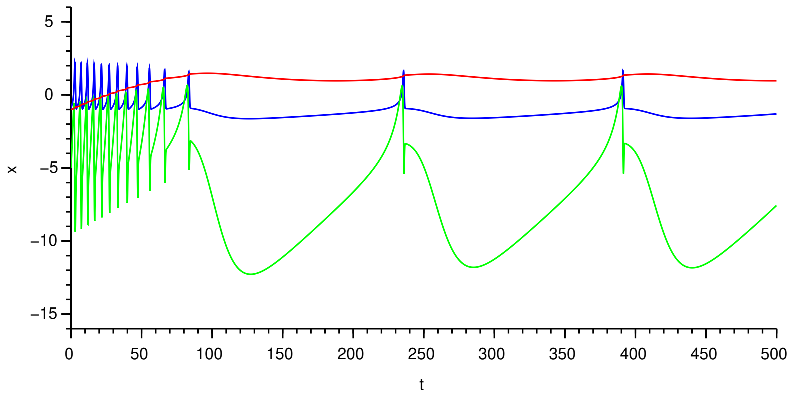

Figure 3: State of the leader neuron. Blue line: $x_{0,1}$, green line: $x_{0,2}$, red line: $x_{0,3}$.

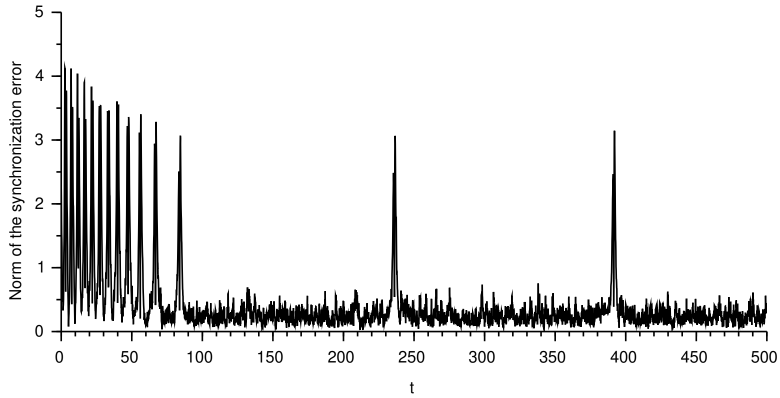

Figure 4: Norm of synchronization error in the membrane potential.

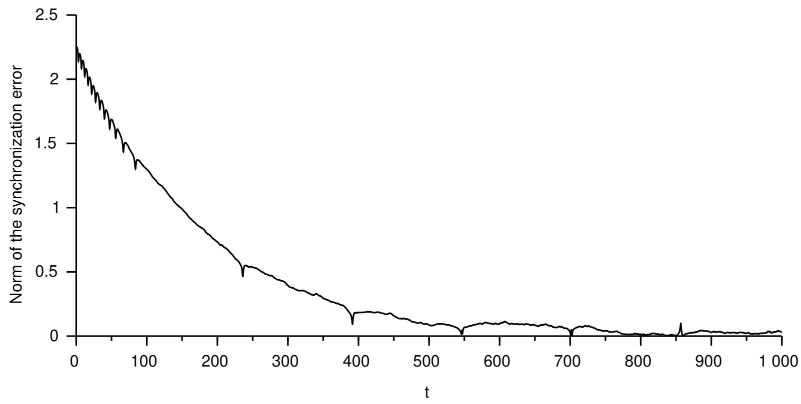

Figure 5: Norm of the synchronization error in the adaptation variable.

7.3 Conference paper [17]

This paper contains an abbreviated version of the journal article [18]. The results are presented in much more concise way, also the presentation of the theoretical results is brief and some of proofs were omitted.

8 Other Results

In this part of the Report, results that do not fit into the previous categories, even though they are related to the main problems solved in this Project. However, all of them have some, at least a tangential, connection to the main topic of the Project.

8.1 Conference paper [7]

The stabilization of the large-scale interconnected system can be adopted to the control of a system with distributed parameters, such as systems described. However, another method for control of these systems exists. This method is based on the z-transform of the temporal as well as spatial variable and subsequent analysis of this transformed equation, which is then an algebraic equation; to be precise, it is a multivariate polynomial equation. This can be used to design a controller.

Both methods have been compared in many ways. The controllers obtained guaranteed a satisfactory behavior. However, the time and memory requirements were found to be higher in the case of the method based on a control of large-scale interconnected systems. On the other hand, this method is applicable under less restrictive assumptions - the spatial domain needs not be the interval $(-\infty, +\infty)$ - a clearly unrealistic assumption, however, needed to apply the method based on the z-transform. Moreover, extension to nonlinear or uncertain systems would be more straightforward.

The results of the extensive tests and simulations can be found in this paper.

8.2 Conference paper [15]

The problem of control of a large-scale interconnected systems and the problem of synchronization of a multi-agent system share many similar features, e.g. using similar mathematical tools needed for their solution. However, they also exhibit several important differences. This paper aimed at highlighting the common as well as different characteristics of both problems.

The center of investigation were linear large-scale interconnected systems with delays and linear multi-agent systems, again with delay in the control loop. The most important result is that, even if the delays in different subsystems of the large-scale interconnected system are not equal, stabilization can be achieved. This stands in contrast with the fact that, if the delays in the control loops of the multi-agent system are not equal in all agents, a steady synchronization error can arise. This error does not, in general, converge to zero. On the other hand, its magnitude can be estimated by means of the robust control. This estimate was presented in this paper.

This paper initiated a lively discussion in the audience.

8.3 Conference Contribution - Small Gain Theorem for Systems Governed by Quasilinear Parabolic Equations

This is a problem whose solution might be useful for a future work. To be specific, if multi-agents composed of agents that are systems governed by quasilinear equations are considered, a small gain theorem would come handy as it would allow to handle uncertainties etc. in a straightforward way.

The small gain theorem for quasilinear equations was formulated and its properties were discussed. This contribution was presented in the poster form at the PANM 2021 meeting held in Jablonec nad Nisou. It was not included into proceedings, a paper is in preparation.

The proper reference is:
Rehák Branislav, Lynnyk Volodymyr: Small gain theorem for systems described by quasilinear parabolic equations, PANM 21 Programy a algoritmy numerické matematiky 21, Abstrakty, p. 22-22, PANM 21 - Programy a algoritmy numerické matematiky 21 (2022), (Jablonec nad Nisou, CZ, 20220619)

8.4 Conference Contribution - Detection of the Generalized Synchronization of Chaotic Systems: Auxiliary System Approach with Delayed Communication

The detection of the generalized synchronization of chaotic systems was also a topic of the Project, however, due to existence of another project with a similar topic, a relatively small attention was paid to this problem. However, this paper describes an important result in this area. It combines the previously obtained approaches concerning the general synchronization of chaotic systems with methods for delayed systems. Here, the communication between the chaotic systems is delayed, with the delays being equal and constant. It was demonstrated that this fact does not prevent the general synchronization from being established.

Further research in this area is to be conducted, e.g. focusing at systems with not equal delays or with quantized communication.

The proper reference is:
Lynnyk Volodymyr, Rehák Branislav: Detection of the generalized synchronization of chaotic systems: auxiliary system approach with delayed communication, PANM 21 Programy a

algoritmy numerické matematiky 21, Abstrakty, p. 17-17, PANM 21 - Programy a algoritmy numerické matematiky 21 (2022), (Jablonec nad Nisou, CZ, 20220619)

9 Future Research

The results of the Project will be applied in the future research. A hot topic nowadays is the development of decentralized algorithms for synchronization of the multi-agent systems resilient against malicious attacks. Several results achieved during the solution of the Project will serve as a suitable platform to the development of these algorithms. This can be said about the paper [16] where the synchronization of multi-agent systems with jointly connected topology is investigated. This approach can be adopted to the case of synchronization of the multi-agent system under the Denial-of-Service (DoS) attacks when the inter-agent communication is interrupted. Other results, such as observer-based synchronization of nonlinear multi-agent systems or multi-agent systems with delayed communication, in both cases under DoS attacks will also be investigated, making advantage of several results of this Project will also be developed. The last result of the Project, namely [12], will form a base for the development of an algorithm for the event-triggered synchronization of a nonlinear multi-agent system under the dynamic feedback.

Moreover, there exist several extensions of the synchronization problem for multi-agent systems. One of them is the containment problem, where the multi-agent system contains several leaders while the remaining agents should assume a position in the convex hull of these leaders. This problem has been solved for linear multi-agent systems but there is a lack of its solution for nonlinear systems. The main obstacle is probably the loss of the convex structure of the area to which the states of the agents converge.

10 Conclusions

The solution of the tasks set in the Project GA 19-07635S was described in this Report. Moreover, the results achieved were briefly described and their main contribution was highlighted. Further lines of the research in this area were outlined.

This report brings evidence that all themes supposed to be solved of the Project were addressed. All topics of the Project also resulted in journal papers or contributions at conferences. The research is advanced further and publication in journals is also expected in future.