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Uncertainty merging with basic uncertain information in probability environment

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ABSTRACT

Basic uncertain information is a recently introduced and significant type of uncertainty that proves particularly valuable in decision-making environments with inherent uncertainties. In this study, we propose the concept of uncertainty cognition merging, which effectively combines basic uncertain information granules with probability measures to generate new probability measures within the same probability space. Additionally, we present a degenerated method that merges basic uncertain information granules with unit intervals to create new subintervals. We introduce four distinct uncertainty cognition merging methods and thoroughly compare and analyze their respective properties, limitations, and advantages. To demonstrate the practical application potential of our proposals, we provide numerical examples alongside further mathematical results.

1. Introduction

The introductory section initially discusses a comprehensive framework for uncertain evaluation, encompassing both primary uncertain information and supplementary reference information. Subsequently, we provide a concise overview of the main objective of this study: namely, merging basic uncertain information and probability information into a new probability distribution.

1.1. Two sources of evaluation information in uncertain decision making

In management and decision-making, a large amount of numerical evaluation or prediction is often required [1–5], which increasingly involves more uncertainty. As a result, the core decision-making group in companies or organizations often cannot provide definitive evaluation or prediction [6]. Therefore, they frequently seek external investigation and consulting agencies to obtain refer-

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ence information that can help them adjust their initial evaluation or prediction values. These external agencies often have access to statistical and probabilistic information based on history or big data, which generally gives them certain advantages such as appearing more objective and closer to reality and the market. However, there are also obvious drawbacks: these data are more general and may not be well-suited for specific decision-making scenarios; moreover, these data tend to be delayed, increasing the error rate of decisions. On the other hand, the core decision-making group within an organization is usually composed of internal experts or specially invited professionals who provide evaluations and predictions that are more specialized and relevant to the company's products. The downside of expert evaluations is that they tend to be subjective and with a significant amount of uncertainty. In summary, both internal evaluations and external consultations are important means of obtaining evaluation or prediction data.

If internal experts assess that there is no uncertainty, decision-makers often tend to forgo seeking external consulting firms, even though these assessments may have some subjectivity, due to the high consultancy fees charged by such firms. However, in many cases, constrained by factors such as individual knowledge, experience, abilities, time and dedication of each expert, expert assessments carry significant uncertainties. This necessitates the reliance on statistical or probabilistic information provided by external consulting firms in order to make more informed assessments and decisions. The integration and merging of this external consulting information with the initial uncertain evaluations made by internal experts to obtain a more reasonable assessment result is the issue that will be discussed in this article.

1.2. Using basic uncertain information with additional reference information

The uncertain information addressed in this study pertains to the recently proposed uncertain data paradigm, known as the Basic Uncertain Information (BUI) [7,8]. Several extended forms of BUI have soon been developed [9,10] and BUI is also applied to soft set environment [11] and aggregation theory [12,13]. BUI can be readily applied in practical decision-making and evaluation within companies, or after minor adaptation or standardization. Recall that a BUI granule is with a pair $(x, c) \in [0, 1]^2$ in which x is the evaluation value while c is the certainty degree (or certainty) of x; $1 - c$ is called the uncertainty degree (or uncertainty) of x. The restriction of the discourse of x to the compact unit interval $[0, 1]$ primarily stems from its practical advantages over non-compact sets and its relevance to fuzzy sets theory. For instance, when predicting the future market share of a product, it is customary to express it as a percentage value such as $100x\%$ with a certain level of confidence (certainty) denoted by c. Researchers also developed some applied forms for BUI in both theory and application, e.g., the basic uncertain linguistic information (BULI) [13–19]. BUI has been further studied and demonstrates significant application potential across various fields [20–22].

Several recent studies have investigated methodologies for aggregating and merging BUI inputs. Jin et al. [23] employed some special techniques to discuss the Choquet integral with BUI inputs. Jin et al. [24] proposed a heuristic approach to define a type of OWA operators with BUI inputs. Additionally, Jin et al. [25] introduced generalized Sugeno integrals of BUI inputs using distinct techniques and aggregation frames. Furthermore, Jin et al. [26] presented three distinct aggregation paradigms for BUI inputs.

A simple aggregation mechanism for BUI inputs is to firstly transform all BUI granules into some intervals and then take some representative real values for those intervals and finally take aggregation operators for the obtained real values, as discussed in the known literature [27,28]. A direct advantage of such mechanism lies in that it can apply all existing aggregation operators rather then the mere weighted averaging operators. Nevertheless, such BUI-interval transformation method cannot be well applied to the situation where additional probability considerations are introduced. For instance, in corporate decision-making processes, traditional aggregation theory can easily combine different opinions from managers or experts if they provide evaluations with full certainty (i.e., complete confidence). However, since opinions often involve uncertainties (allowing the use of BUI inputs), decision-makers frequently seek consultation from external agencies that analyze historical data or conduct thorough market surveys to provide statistical or probabilistic information. Probability information and related merging techniques are particularly important in numerous applications and theoretical studies [29–32]. Aggregation theory for both real and uncertain information is important [33–35] and some probability related theory has also been applied in aggregation theory [36].

Therefore, it is imperative to propose and analyze suitable approaches for aggregating that can effectively operate in a probabilistic environment while adhering to general human cognition principles. Additionally, it is intriguing to observe that if the given probability follows a uniform distribution, these approaches can seamlessly degenerate into corresponding aggregation methods without considering probability. Therefore, we will propose three BUI-probability merging methods and analyze their related mathematical properties.

The contribution of this work lies in the following aspects: We propose an uncertain evaluation framework that incorporates two sources of information, e.g., inner BUI and outer probability information. We present some approaches for generating and merging probability distributions within the uncertain evaluation framework with BUI and probability. The present study makes contributions to the fields of uncertain information fusion theory and uncertain decision making.

The remainder of this work is organized as follows. Section 2 presents some preliminary knowledge. In Section 3, we introduce the concept of uncertainty cognition merging and discuss its application in BUI environment without incorporating probability. Section 4 explores two methods for uncertainty cognition merging considering the influence of probabilistic environments. Section 5 proposes and analyzes the substitution uncertainty cognition merging method. Section 6 briefly discusses the proposed methods in uncertain and probabilistic decision-making environments. In Section 7, we provide further discussion and comparison of the proposed methods for uncertainty cognition merging. Finally, in Section 8, we conclude and remark on this work.

2. Preparations

Denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{R}$ the set of all real numbers, and define $[n] = \{1, 2, ..., n\}$. Sequence form $\mathbf{a} = (a_i)_{i=1}^n \in [0,1]^n$ denotes a vector of n elements which take values in unit interval. The characteristic function of A is denoted by $\chi_A : [0,1] \to \{0,1\}$ such that $\chi_A(y) = 1$ if and only if $y \in A$.

Basic uncertain information (BUI) is a recently proposed uncertain data type. A BUI granule is with a pair $(x,c) \in [0,1]^2$ in which x is the evaluation value while c is the certainty degree (or certainty) of x; $1-c$ is called the uncertainty degree (or uncertainty) of x. The set of all BUI granules is denoted by $\mathcal{B}$. A BUI vector (or a vector of BUI granules) is written in the form $(\mathbf{x}, \mathbf{c}) = \left((x_i, c_i)\right)_{i=1}^n$ where $\mathbf{x} = (x_i)_{i=1}^n \in [0,1]^n$ and $\mathbf{c} = (c_i)_{i=1}^n \in [0,1]^n$ are the vector of evaluation values (evaluation vector) and the vector of certainty degrees (certainty vector), respectively.

The interval values (closed intervals) considered in this work are closed subintervals $[a,b] \subseteq [0,1]$, and the set of all such interval values is denoted by $\mathcal{I}$. We may also consider some simple operations for closed interval. For any $a_1, a_2, b_1, b_2, k \in [0, +\infty)$ we consider the operation $k[a_1, b_1] = [ka_1, kb_1] = [a_1, b_1]k$ and the operation $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$; sometimes we may restrict the operation results to [0,1] for closeness. For the degenerated intervals, we write $[a,a] = a$ for all $a \in [0,1]$.

Throughout this work, the concerned probability spaces $(\mathbb{R}, \mathcal{M}, P)$ are equipped with σ-algebra $\mathcal{M}$ of the set of all Borel measurable subsets of $\mathbb{R}$, and P are probability measures that are concentrated on [0,1] (i.e., $P([0,1]) = 1$). The uniform distribution $U_{[a,b]} : \mathcal{M} \to [0,1]$ on an interval $[a,b]$ is defined such that $U_{[a,b]}(A) = \frac{\lambda(A \bigcap [a,b])}{\lambda([a,b])}$ where λ is the Lebesgue measure on $\mathbb{R}$. For a degenerated interval $[a,a] = \{a\}$, note that $U_{[a,a]} = U_{\{a\}} = \delta_a$, where δ_ω is the Dirac measure for a point $\omega \in [0,1]$, i.e., $\delta_\omega(A) = 1$ if $\omega \in A$ and $\delta_\omega(A) = 0$ otherwise.

Denote by M_1 the set of all probability measures on the measurable space $(\mathbb{R}, \mathcal{M})$, and denote by C_1 ($\mathrm{C}_1 \subset \mathrm{M}_1$) the set of all continuous probability measures on $(\mathbb{R}, \mathcal{M})$; that is, for any $P \in C_1$ and any singleton $\{x\}$, $P(\{x\}) = 0$ (note also that P need not be absolutely continuous). Denote by F_1 the set of all non-decreasing functions $F : \mathbb{R} \to [0,1]$ such that $F(-\infty) = 0$ and $F(+\infty) = 1$; denote by $\mathrm{CF}_1 (\subset \mathrm{F}_1)$ the set of all continuous non-decreasing functions $F : \mathbb{R} \to [0,1]$ such that $F(0-) = 0$ and $F(1) = 1$. For any probability measure $P \in \mathrm{M}_1$, recall its distribution function $F_P \in \mathrm{F}_1$ satisfying $F_P(t) = P((-\infty, t])$. Recall that not all of the functions in F_1 can be cumulative distribution functions, and a function in F_1 can derive a probability measure only when it is continuous from the right.

3. Uncertainty cognition merging methods for BUI without considering probability environments

In this section, we introduce the concept of uncertainty cognition merging (UCM), which has significant implications in probability environments and can be applied to all types of aggregation operators with obtained anticipated or expected values. Broadly speaking, given an uncertain environment (interval or probability information), we can merge any BUI granule into the uncertain environment and generate new resultant uncertain information that remains within the interval or probability framework. Conversely, when there is no uncertainty involved in a BUI granule (i.e., $c = 1$), then the entire merging process degenerates into the evaluation value x of the BUI granule; thus resulting in either a degenerated interval $[x,x] = \{x\}$ or Dirac distribution δ_x.

The philosophic principle of UCM is relatively direct but which may have more derivations and embodiments: if an evaluation value x is obtained or evaluated with uncertainty but it is sure that the evaluation value is contained in a larger set S (called uncertain environment, which for example may be an interval or a probability space $(S, \mathcal{F}, P)$), then we may merge this evaluation value (with uncertainty) into the uncertain environment S and obtain a resultant subset V of S or obtain a resultant new probability space $(S, \mathcal{F}, m)$ as the uncertainty cognition merging result. Note that in the later definitions we will still apply the probability spaces $(\mathbb{R}, \mathcal{M}, P)$ (or $(\mathbb{R}, \mathcal{M}, m)$) but with $S = [0,1] \subset \mathbb{R}$, $\mathcal{F}$ being the $\mathcal{M}$ restricted to [0,1] and P (or m) concentrated on [0,1].

For example, since the evaluation value x in a BUI granule (x,c) is known to be necessarily within the defining space [0,1] (an uncertain environment S), then if x is obtained or evaluated with full uncertainty (i.e., $c = 0$), it is natural that we can regard [0,1] as its UCM result; if x is not with full uncertainty (i.e., $c > 0$), then we can in general regard a subset $V \subseteq [0,1]$ as its UCM result (e.g., $I_{x,1-c} = c[x,x] + (1-c)[0,1] = [cx, cx + 1 - c]$ whose length is $1-c$). Furthermore, with considering probability environment, we can flexibly obtain some probability spaces $(\mathbb{R}, \mathcal{M}, m)$ as UCM results by some reasonable cognitions and techniques, as later analyzed. Subsequently, with such obtained subset V or probability spaces $(\mathbb{R}, \mathcal{M}, m)$, we can select, if necessary, some representative or anticipated value for V or take the expected value of the identity random variable X_{id} (i.e., $X_{id}(t) = t$) with $(\mathbb{R}, \mathcal{M}, m)$ as some comprehensive evaluation results that consider both BUI and probability environment in a merged manner.

Without considering additional probability information, for any BUI granule we may take the uncertain environment [0,1] and have the following UCM method.

UCM method 1 (without probability) Given a BUI granule $(x,c) \in \mathcal{B}$, we find a subinterval

$$I_{x,1-c} = c[x,x] + (1-c)[0,1] = [cx, cx + 1 - c] \tag{1}$$

of [0,1] as the UCM result according to both x and c of a BUI granule.

In the UCM method, our objective is to obtain a merged subinterval (or probability environment) as a whole in order to derive representative values from it later on. Therefore, if a BUI has smaller certainty c (larger uncertainty), it is natural for us to consider a wider subinterval (or broader domain for the probability environment), and thus we use $1-c$ (actually we can also further consider a monotonic mapping of $1-c$ but for simplicity in this work we only consider this most representative case).

With the obtained interval $I_{x,1-c}$ in decision making problems we can often take the middle point of it as the representative or anticipated value for x. Observe that the middle point of $I_{x,1-c}=[a,b]$ is $MID(I_{x,1-c})=(a+b)/2=(cx+cx+1-c)/2=cx+0.5(1-c)$. Hence, we have the following BUI aggregation method for BUI vectors which is applicable to all known aggregation operators [28, 29] and is based on the principle of UCM. For any aggregation operator $F:[0,1]^n\to[0,1]$, we define a BUI aggregation $H:\mathcal{B}^n\to[0,1]$ such that $H((\mathbf{x},\mathbf{c}))=F(\mathbf{z})$ where $\mathbf{z}=(z_i)_{i=1}^n\in[0,1]^n$ satisfying

$$z_i=c_ix_i+0.5(1-c_i) \tag{2}$$

That is, z_i is the middle point of $I_{x_i,1-c_i}$ for each $i\in[n]$.

When there is no uncertainty involved to x_i (i.e., $c_i=1$), then $z_i=x_i$. Moreover, it is easy to note that the map $x\mapsto MID(I_{x,1-c})$ is non-decreasing, and the two maps $x\mapsto MID(I_{x,1-c})$ and $c\mapsto MID(I_{x,1-c})$ are continuous.

Example 1. For BUI granules $(0,0.7),(0.2,0.5),(0.5,0.2),(0.8,1)\in\mathcal{B}$, we have

$$I_{0,0.3}=[0,0.3] \text{ and } MID(I_{0,0.3})=0.15;\; I_{0.2,0.5}=[0.1,0.6] \text{ and } MID(I_{0.2,0.5})=0.35;$$

$$I_{0.5,0.8}=[0.1,0.9] \text{ and } MID(I_{0.5,0.8})=0.5;\; I_{0.8,0}=[0.8,0.8] \text{ and } MID(I_{0.8,0})=0.8.$$

4. Some UCM methods for BUI in probability environments

This section discusses two UCM methods considering the influence of probability environments.

4.1. Two UCM methods considering probability information

This segment proposes two UCM methods to generate new merged probability measure with the influence of given additional probability measure.

The two methods are both grounded in the cognitive principle: greater uncertainty inherent in the initial evaluation (i.e., BUI granule with smaller certainty degree c) prompts decision makers to seek more objective information (i.e., additional probability information referenced from external investigative institutions). The subsequent UCM method employs a direct merging style through a simple convex combination.

UCM method 2 With the additional probability information $(\mathbb{R},\mathcal{M},P)$ (P concentrated on [0,1]), for any BUI granule $(x,c)\in\mathcal{B}$, we form a new probability measure m (defined on $(\mathbb{R},\mathcal{M})$ and concentrated on [0,1]) as the UCM result:

$$m=c\delta_x+(1-c)P \tag{3}$$

Remark. The obtained new probability measure is formed by two measures and related to x, c, and P. Precisely we should write it as the parameterized form $m^{(x,c,P)}$. For convenience, sometimes when there is no confusion arises we apply the concise form without parameters.

Example 2. Let P be defined such that $P=0.5P_1+0.5\delta_{0.5}$ where P_1 has density function $p(t)=2t$. For BUI granules $(0,0.7),(0.2,0.5)$, $(0.5,0.2),(0.8,1)\in\mathcal{B}$, we have

$$m^{(0.0.7,P)}=0.15P_1+0.15\delta_{0.5}+0.7\delta_0;\, m^{(0.2.0.5,P)}=0.25P_1+0.25\delta_{0.5}+0.5\delta_{0.2};$$

$$m^{(0.5,0.2,P)}=0.4P_1+0.6\delta_{0.5};\, m^{(0.8,1,P)}=\delta_{0.8}.$$

In comparison to the first combination method, the next alternative method is based on the following cognition: we actually only wish to consider “a part” of P on the neighborhood $I_{x,1-c}$ of x; it is plausible that the mass of the P on $I_{x,1-c}$ has more influence on the evaluation value x than those mass that is far from it. Moreover, normalization is often needed since the “part” of P may only be a positive measure which might not be a probability measure. Hence, conditional probability will be applied, and uniform distributions will also be used when conditional probability cannot be defined.

UCM method 3 With the additional probability information $(\mathbb{R}, \mathcal{M}, P)$, for any BUI granule $(x, c) \in \mathcal{B}$, we form a new probability measure m as the UCM result such that

$$m(A) = \frac{P(A \bigcap I_{x,1-c})}{P(I_{x,1-c})} \ (A \in \mathcal{M})) \text{ when } P(I_{x,1-c}) > 0; \ m = U_{I_{x,1-c}} \text{ when } P(I_{x,1-c}) = 0; \tag{4}$$

where $U_{I_{x,1-c}}$ is the uniform distribution on $I_{x,1-c}$.

Remark. When $P(I_{x,1-c}) = 0$ and $c = 1$, note that $m = \delta_x$.

Remark. For the definition "$m = U_{I_{x,1-c}}$, when $P(I_{x,1-c}) = 0$ and $c < 1$", we actually applied some cognitive "substitution"; that is, if $P(I_{x,1-c}) = 0$, then there is no "mass" of P concentrated on $I_{x,1-c}$ and hence we may substitute it with the uniform distribution $U_{I_{x,1-c}}$. This substitution is "full" and "immediate", and does not depend on the certainty degree c and the length of the interval $I_{x,1-c}$. In the next section we will discuss another substitution method that depends on certainty degree c and the length of the interval $I_{x,1-c}$.

Remark. Note that when $P(I_{x,1-c}) > 0$, $m(A) = \frac{P(A \bigcap I_{x,1-c})}{P(I_{x,1-c})} = P(A|I_{x,1-c})$ is the conditional probability of A given $I_{x,1-c}$ and hence we may also alternatively have a concise form $m = P(\cdot|I_{x,1-c})$.

With the merged probability measure m, for a BUI granule (x, c) we can take the expected value of the identity map $X_{id} : \mathbb{R} \to \mathbb{R}$ ($X_{id}(t) = t$) w.r.t. $(\mathbb{R}, \mathcal{M}, m)$ as a more reasonable evaluation result than the original evaluation value x. That is, we can take $E(X_{id}) = \int X_{id} dm = \int_{[0,1]} X_{id} dm$. For a BUI vector $(\mathbf{x}, \mathbf{c}) = ((x_i, c_i))_{i=1}^n$ under aggregation, we may respectively obtain $m^{(x_i,c_i,P)}$ and calculate $E_i = \int_{[0,1]} X_{id} dm^{(x_i,c_i,P)}$, and then we can apply any existing aggregation operator to aggregate the obtained vector $(E_i)_{i=1}^n$.

In view of the related expected value, for both UCM method 2 and UCM method 3 the map $x \mapsto \int_{[0,1]} X_{id} dm^{(x,c,P)}$ can be regarded as a generalization of the map $x \mapsto MID(I_{x,1-c})$. That is, the map $x \mapsto cx + 0.5(1 - c)$ can be regarded as the special case when $P = U_{[0,1]}$ being the uniform distribution on [0,1].

Proposition 1. *$\int_{[0,1]} X_{id} dm^{(x,c,U_{[0,1]})} = cx + 0.5(1 - c)$ holds for both UCM method 2 and UCM method 3.*

Proof. (i) (with UCM method 2) Observe that for all $x, c \in [0, 1]$, $m^{(x,c,U_{[0,1]})} = c\delta_x + (1 - c)U_{[0,1]}$,

$$\int_{[0,1]} X_{id} dm^{(x,c,U_{[0,1]})} = c \int_{[0,1]} X_{id} d\delta_x + (1 - c) \int_{[0,1]} X_{id} dU_{[0,1]} = cx + 0.5(1 - c).$$

(ii) (with UCM method 3) Note that when $c = 1$, then $P(I_{x,1-c}) = 0$, and that when $c < 1$, then $P(I_{x,1-c}) > 0$. Therefore, when $c < 1$, we have

$$\int_{[0,1]} X_{id} dm^{(x,c,U_{[0,1]})} = \int_{[0,1]} X_{id} dU_{I_{x,1-c}} = E(X_{id}|I_{x,1-c}) = MID(I_{x,1-c}) = cx + 0.5(1 - c)$$

(where $E(X_{id}|I_{x,1-c})$ is the conditional expectation of X_{id} given $I_{x,1-c}$).

When $c = 1$, we have

$$\int_{[0,1]} X_{id} dm^{(x,c,U_{[0,1]})} = \int_{[0,1]} X_{id} d\delta_x = x = MID(I_{x,0}).$$

Example 3. Let P be defined such that $P = 0.5P_1 + 0.5\delta_{0.5}$ where P_1 has density function $p(t) = 2t$. For BUI granules $(0, 0.7), (0.2, 0.5), (0.5, 0.2), (0.8, 1) \in \mathcal{B}$, we have

$m^{(0.0.7,P)}$ being with density $q(t) = \frac{t}{0.045} \cdot \chi_{[0,0.3]}$;

$m^{(0.2.0.5,P)} = \frac{1}{0.675} 0.175Q + \frac{1}{0.675} 0.5\delta_{0.5} = \frac{7}{27} Q + \frac{20}{27} \delta_{0.5}$ where Q has density $q(t) = \frac{t}{0.175} \cdot \chi_{[0.1,0.6]}$;

$m^{(0.5,0.2,P)} = \frac{1}{0.9} 0.4Q + \frac{1}{0.9} 0.5\delta_{0.5} = \frac{4}{9} Q + \frac{5}{9} \delta_{0.5}$ where Q has density $q(t) = \frac{t}{0.4} \cdot \chi_{[0.1,0.9]}$;

$m^{(0.8,0.1,P)} = \delta_{0.8}$.

4.2. Some properties and analyses for the two UCM methods

We mainly discuss continuities and monotonicities related to the two UCM methods. For each $P \in \mathrm{M}_1$, recall its distribution function $F_P \in \mathrm{F}_1$ such that $F_P(t) = P((-\infty, t])$. We use the Levy's metric as the distance $d : {\mathrm{M}_1}^2 \to [0, +\infty)$ for the discussion of continuity. Recall that for any two probability measures $\mu, \nu \in \mathrm{M}_1$,

$$d(\mu, \nu) = \inf\{\varepsilon > 0 : F_\mu(t) \le F_\nu(t+\varepsilon) + \varepsilon \text{ and } F_\nu(t) \le F_\mu(t+\varepsilon) + \varepsilon \text{ for all } t \in \mathbb{R}\}.$$

Recall that a sequence $(P_r)_{r=1}^\infty$ in M_1 is said to converges weakly to $P \in \mathrm{M}_1$ (i.e., $\mathrm{w}\!-\!\lim_{r\to\infty} P_r = P$ if $\int f\, dP = \lim_{r\to\infty} \int f\, dP_r$ for all bounded continuous functions $f : \mathbb{R} \to [0, +\infty)$. Then, we know that if $\mathrm{w}\!-\!\lim_{r\to\infty} P_r = P$ then (i) $\lim_{r\to\infty} d(P_r, P) = 0$ and (ii) $F_P(t) = \lim_{r\to\infty} F_{P_r}(t)$ whenever F_P is continuous.

Proposition 2. *Let $m^{(x,c,P)}$ be the probability measure obtained in UCM method 2 and Eq. (3), then*

(i) *the map $x \mapsto m^{(x,c,P)}$ is continuous on [0,1];*
(ii) *the map $c \mapsto m^{(x,c,P)}$ is continuous on [0,1];*
(iii) *the map $P \mapsto m^{(x,c,P)}$ is continuous on M_1.*

Proof. (i) This holds simply because $x \mapsto \delta_x$ is continuous on [0,1].

(ii) For each $c \in [0,1]$ and any sequence $(c_r)_{r=1}^\infty$ with $\lim\limits_{r\to\infty} c_r = c$, we have for all $t \in \mathbb{R}$

$$\begin{aligned}\lim_{r\to\infty} F_{m^{(x,c_r,P)}}(t) &= \lim_{r\to\infty} F_{c_r\delta_x + (1-c_r)P}(t) = \lim_{r\to\infty}\Big(c_r F_{\delta_x}(t) + F_P(t) - c_r F_P(t)\Big) = \Big(F_{\delta_x}(t) - F_P(t)\Big)\lim_{r\to\infty} c_r + F_P(t)\\ &= \Big(F_{\delta_x}(t) - F_P(t)\Big)c + F_P(t) = cF_{\delta_x}(t) + F_P(t) - cF_P(t) = F_{c\delta_x + (1-c)P}(t) = F_{m^{(x,c,P)}}(t)\end{aligned}$$

and hence $\lim_{r\to\infty} d(m^{(x,c_r,P)}, m^{(x,c,P)}) = 0$.

(iii) For each $P \in \mathrm{M}_1$ and any sequence $(P_r)_{r=1}^\infty$ with $\mathrm{w}\!-\!\lim_{r\to\infty} P_r = P$, we have $\lim_{r\to\infty} F_{P_r}(t) = F_P(t)$ whenever F_P is continuous at *t*. Since F_P is continuous at *t* if and only if $F_{m^{(x,c,P)}}$ is continuous at *t*, then

$$\begin{aligned}\lim_{r\to\infty} F_{m^{(x,c,P_r)}}(t) &= \lim_{r\to\infty} F_{c\delta_x + (1-c)P_r}(t) = cF_{\delta_x}(t) + (1-c)\lim_{r\to\infty} F_{P_r}(t)\\ &= cF_{\delta_x}(t) + (1-c)F_P(t) = F_{c\delta_x + (1-c)P}(t) = F_{m^{(x,c,P)}}(t)\end{aligned}$$

whenever $F_{m^{(x,c,P)}}$ is continuous at *t*. Hence, $\lim_{r\to\infty} d(m^{(x,c_r,P)}, m^{(x,c,P)}) = 0$.

Remark. Viewing $m^{(x,c,P)}$ as parameterized with *x*, *c* and *P*, the weak-convergence obtained in the above proposition immediately implies that the expectation related maps $x \mapsto \int X_{id}\, dm^{(x,c,P)}$, $c \mapsto \int X_{id}\, dm^{(x,c,P)}$ and $P \mapsto \int X_{id}\, dm^{(x,c,P)}$ are all continuous.

Proposition 3. *For any two BUI granules $(x_1, c), (x_2, c) \in \mathcal{B}$ with $x_1 < x_2$, let $m^{(x_1,c,P)}$ and $m^{(x_2,c,P)}$ be the probability measures obtained using UCM method 2 and Eq. (3), respectively. Then, $E_1 = \int X_{id}\, dm^{(x_1,c,P)} \le \int X_{id}\, dm^{(x_2,c,P)} = E_2$.*

Proof.

$$\begin{aligned}E_1 &= \int X_{id}\, dm^{(x_1,c,P)} = \int X_{id}\, d\Big(c\delta_{x_1} + (1-c)P\Big) = c\int X_{id}\, d\delta_{x_1} + (1-c)\int X_{id}\, dP\\ &= cx_1 + (1-c)\int X_{id}\, dP \le cx_2 + (1-c)\int X_{id}\, dP = \int X_{id}\, dm^{(x_2,c,P)} = E_2.\end{aligned}$$

As we showed UCM method 2 has some properties, the drawback of it lies in that the probability distribution *P* itself plays little role in the measure merging process while it is its related expected value that matters. Indeed, for any two different probability distributions $P_1, P_2 \in \mathrm{M}_1$, if $\int X_{id}\, dP_1 = \int X_{id}\, dP_2$, then we have $\int X_{id}\, dm^{(x,c,P_1)} = \int X_{id}\, dm^{(x,c,P_2)}$ for any BUI granule (x, c), irrespective of the detailed distributions of P_1 and P_2; it will be easy to observe that this is not the case of the UCM method 3.

Although some continuity properties hold for UCM method 2, they do not hold for UCM method 3. Let $m^{(x,c,P)}$ be the probability measure obtained in UCM method 3 and Eq. (4), and we consider the following examples to show discontinuities: (i) When $P = U_{[0.5,1]}$, $x \mapsto m^{(x,0.5,P)}$ is not continuous at 0 and $c \mapsto m^{(0,c,P)}$ is not continuous at 0.5. (ii) When $x = 0$ and $c = 0.5$, $P \mapsto m^{(x,c,P)}$ is not continuous at $U_{[0.5,1]}$.

We can prove that the desired monotonicity also holds for UCM method 3.

Proposition 4. *For any two BUI granules $(x_1, c), (x_2, c) \in \mathcal{B}$ with $x_1 < x_2$, let $m^{(x_1,c,P)}$ and $m^{(x_2,c,P)}$ be the probability measures obtained using UCM method 3 and Eq. (4), respectively. Then, for any $P \in \mathrm{M}_1$, $E_1 = \int_{[0,1]} X_{id}\, dm^{(x_1,c,P)} \le \int_{[0,1]} X_{id}\, dm^{(x_2,c,P)} = E_2$.*

Proof. For any $c \in [0,1]$, note that $\sup I_{x_1,1-c} \le \sup I_{x_2,1-c}$ and $\inf I_{x_1,1-c} \le \inf I_{x_2,1-c}$.

When $c = 1$, $E_1 = \int_{[0,1]} X_{id} d\delta_{x_1} = x_1 < x_2 = \int_{[0,1]} X_{id} d\delta_2 = E_2$.

For any $c < 1$, note that $\lambda(I_{x_1,1-c}) = \lambda(I_{x_2,1-c}) > 0$. If $I_{x_1,1-c} \bigcap I_{x_2,1-c} = \emptyset$, then we immediately know $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$ since $\sup I_{x_1,1-c} \leq \inf I_{x_2,1-c}$. If $I_{x_1,1-c} \bigcap I_{x_2,1-c} = B \neq \emptyset$ (which is a closed interval), for convenience we take the notations $I_{x_1,1-c} \backslash I_{x_2,1-c} = A$ and $I_{x_2,1-c} \backslash I_{x_1,1-c} = C$. Then, we discuss the following five situations, respectively.

(i) If $P(I_{x_1,1-c}) = P(I_{x_2,1-c}) = 0$, then $m^{(x_1,c,P)} = U_{I_{x_1,1-c}}$ and $m^{(x_2,c,P)} = U_{I_{x_2,1-c}}$. Therefore, it is evident that $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$.

(ii) If $P(I_{x_1,1-c}) > 0$ while $P(I_{x_2,1-c}) = 0$, then we immediately know that $P(B) = 0$ and then $m^{(x_1,c,P)}$ is concentrated on A while $m^{(x_2,c,P)}$ is concentrated on $B \bigcup C$. Hence, $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$.

(iii) If $P(I_{x_2,1-c}) > 0$ while $P(I_{x_1,1-c}) = 0$, then with a similar manner to (ii) we also have $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$.

(iv) If $P(I_{x_1,1-c}) \geq P(I_{x_2,1-c}) > 0$, then firstly we easily note that $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in A \bigcup C$ (since $m^{(x_1,c,P)}$ is concentrated on $A \bigcup B$ while $m^{(x_2,c,P)}$ is concentrated on $B \bigcup C$). Secondly, for any $t \in B$ note that $m^{(x_1,c,P)}([\inf I_{x_2,1-c}, t]) = \frac{P([\inf I_{x_2,1-c}, t])}{P(I_{x_1,1-c})}$, $m^{(x_2,c,P)}([\inf I_{x_2,1-c}, t]) = \frac{P([\inf I_{x_2,1-c}, t])}{P(I_{x_2,1-c})}$. Then, we also note that for any $t \in B$,
$m^{(x_1,c,P)}([\inf I_{x_2,1-c}, t]) = \frac{P(I_{x_2,1-c})}{P(I_{x_1,1-c})} m^{(x_2,c,P)}([\inf I_{x_2,1-c}, t])$, which implies for any $t \in B$,
$m^{(x_1,c,P)}(B \backslash [\inf I_{x_2,1-c}, t]) = \frac{P(I_{x_2,1-c})}{P(I_{x_1,1-c})} m^{(x_2,c,P)}(B \backslash [\inf I_{x_2,1-c}, t])$. Hence, for any $t \in B$,

$$\begin{aligned} m^{(x_1,c,P)}([\inf I_{x_1,1-c}, t]) &= 1 - m^{(x_1,c,P)}(B \backslash [\inf I_{x_2,1-c}, t]) \geq 1 - m^{(x_2,c,P)}(B \backslash [\inf I_{x_2,1-c}, t]) \\ &= m^{(x_2,c,P)}([\inf I_{x_2,1-c}, t] \bigcup C) \geq m^{(x_2,c,P)}([\inf I_{x_2,1-c}, t]) = m^{(x_2,c,P)}([\inf I_{x_1,1-c}, t]). \end{aligned}$$

Consequently, for all $t \in B$ we have $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$. In summary, $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$.

(v) If $P(I_{x_2,1-c}) \geq P(I_{x_1,1-c}) > 0$, then with a similar manner to (iv) we also have $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$.

Finally, since $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$, then from measure theory we know

$$\begin{aligned} E_1 &= \int_{[0,1]} X_{id} dm^{(x_1,c,P)} = 1 - \frac{1}{2} \int_{[0,1]} (F_{m^{(x_1,c,P)}}(t) + F_{m^{(x_1,c,P)}}(t-))\lambda(dt) = 1 - \int_{[0,1]} F_{m^{(x_1,c,P)}} d\lambda \\ &\leq 1 - \int_{[0,1]} F_{m^{(x_2,c,P)}} d\lambda = \int_{[0,1]} X_{id} dm^{(x_2,c,P)} = E_2, \end{aligned}$$

where for any probability distribution function F, $F(t-) = \lim_{y \to t^-} F(y) = \sup\{F(y) : y \in (-\infty, t)\}$.

5. The substitution UCM method

After analysis, it can be observed that UCM method 2 possesses some related continuity properties which other UCM methods may not have; however, the method is just with a simple combination, making it easier to overlook the distinctive "shapes" of different distributions when calculating expected values. On the other hand, UCM method 3 may lack some continuity properties but places more emphasis on considering the role played by the unique "shapes" of various distributions while obtaining merged measures.

Actually, we have applied a simple "substitution" in UCM method 3 with "$m = U_{I_{x,1-c}}$, when $P(I_{x,1-c}) = 0$". As discussed, this substitution is too direct and sensitive, which makes it lack continuity properties. That is, even if $P(I_{x,1-c}) = \varepsilon > 0$ is very small but positive, the merged measure $m^{(x,c,P)}$ will have the similar "shape" to P on $I_{x,1-c}$, i.e., $m^{(x,c,P)}$ is a multiple of P on $I_{x,1-c}$. Nevertheless, when value P vanishes at $I_{x,1-c}$ (i.e., $P(I_{x,1-c}) = 0$), the merged measure $m^{(x,c,P)}$ abruptly becomes a uniform one $U_{I_{x,1-c}}$.

In consideration of the characteristics exhibited by the aforementioned two methods, this section proposes an alternative UCM method employing substitution that not only ensures some continuity properties but also takes into account the distinct roles played by different "shapes" within various distributions.

UCM method 4 With an additional probability information $(\mathbb{R}, \mathcal{M}, P)$, for any BUI granule $(x, c) \in \mathcal{B}$, we form a new probability measure m as the UCM result such that

$$\begin{aligned} &m = P(\cdot | I_{x,1-c}) \text{ when } P(I_{x,1-c}) \geq \lambda(I_{x,1-c}) > 0; \\ &m = \frac{P(I_{x,1-c})}{\lambda(I_{x,1-c})} P(\cdot | I_{x,1-c}) + \frac{\lambda(I_{x,1-c}) - P(I_{x,1-c})}{\lambda(I_{x,1-c})} U_{I_{x,1-c}} \text{ when } \lambda(I_{x,1-c}) > P(I_{x,1-c}) > 0; \\ &m = U_{I_{x,1-c}} \text{ when } P(I_{x,1-c}) = 0 \text{ and } \lambda(I_{x,1-c}) > 0; \\ &m = \delta_x \text{ when } \lambda(I_{x,1-c}) = 0 \text{ and } P(I_{x,1-c}) \geq 0. \end{aligned} \tag{5}$$

Remark. One can check that the above four situations about values of $P(I_{x,1-c})$ and $\lambda(I_{x,1-c})$ together form the total of all possible situations; that is, given any $P(I_{x,1-c})$, there is one and only one situation among the four situations for it to be related to $\lambda(I_{x,1-c})$.

Remark. Recall the probability measure $P(\cdot|I_{x,1-c}) : \mathcal{M} \to [0,1]$ ($P(I_{x,1-c}) > 0$) is defined by $P(A|I_{x,1-c}) = \frac{P(A \bigcap I_{x,1-c})}{P(I_{x,1-c})}$ ($A \in \mathcal{M}$). Recall also that $\lambda(I_{x,1-c}) = 1 - c$.

Remark. Note that different from the simple substitution in UCM method 3, the substitution applied in UCM methods depends on certainty degree c.

Remark. There have some equivalent definitions for UCM method 4. For example, we may also equivalently define for the third and fourth conditions: $m = U_{I_{x,1-c}}$ when $P(I_{x,1-c}) = 0$ and $\lambda(I_{x,1-c}) \geq 0$; $m = \delta_x$ when $\lambda(I_{x,1-c}) = 0$ and $P(I_{x,1-c}) > 0$.

Example 4. Let P be defined such that $P = 0.5P_1 + 0.5\delta_{0.5}$ where P_1 has density function $p(t) = 2t$. For BUI granules $(0, 0.7), (0.2, 0.5), (0.5, 0.2), (0.8, 1) \in \mathcal{B}$, we have the following computations, respectively.

(i) $I_{0,0.3} = [0, 0.3]$; since $\lambda(I_{x,1-c}) = \lambda([0, 0.3]) = 0.3 > P(I_{x,1-c}) = P([0, 0.3]) = 0.045 > 0$ then

$$\begin{aligned} m^{(0.0.7,P)} &= \frac{P([0,0.3])}{\lambda([0,0.3])} P(\cdot|[0,0.3]) + \frac{\lambda([0,0.3]) - P([0,0.3])}{\lambda([0,0.3])} U_{[0,0.3]} \\ &= \frac{0.045}{0.3} P(\cdot|[0,0.3]) + \frac{0.265}{0.3} U_{[0,0.3]}. \end{aligned}$$

Since $P(\cdot|[0, 0.3])$ has density $q(t) = \frac{t}{0.045} \cdot \chi_{[0,0.3]}(t)$, and $U_{[0,0.3]}$ has density $s(t) = \frac{1}{0.3}\chi_{[0,0.3]}(t)$, then $m^{(0.0.7,P)}$ is absolutely continuous with density $p_1(t) = \left[\frac{10}{3}t + \frac{53}{18}\right] \cdot \chi_{[0,0.3]}(t)$.

(ii) $I_{0.2,0.5} = [0.1, 0.6]$; since $P(I_{x,1-c}) = 0.675 \geq \lambda(I_{x,1-c}) = 0.5 > 0$ then $m^{(0.2.0.5,P)} = P(\cdot|[0.1, 0.6]) = \frac{7}{27}Q + \frac{20}{27}\delta_{0.5}$ where Q has density $q(t) = \frac{t}{0.175} \cdot \chi_{[0.1,0.6]}(t)$ as computed in Example 3.

(iii) $I_{0.5,0.8} = [0.1, 0.9]$; since $P(I_{x,1-c}) = 0.9 \geq \lambda(I_{x,1-c}) = 0.8 > 0$, then $m^{(0.5,0.2,P)} = \frac{4}{9}Q + \frac{5}{9}\delta_{0.5}$ where Q has density $q(t) = \frac{t}{0.4} \cdot \chi_{[0.1,0.9]}(t)$ as computed in Example 3.

(iv) $I_{0.8,0} = [0.8, 0.8]$; $\lambda(I_{x,1-c}) = 0$ and $P(I_{x,1-c}) \geq 0$, then $m^{(0.8,0.1,P)} = \delta_{0.8}$.

We find UCM method 4 also degenerates into UCM method 1 in sense of calculating expectation.

Proposition 5. *$\int_{[0,1]} X_{id}\, dm^{(x,c,U_{[0,1]})} = cx + 0.5(1 - c)$ holds for UCM method 4.*

Proof. Note that $U_{[0,1]}(I_{x,1-c}) = \lambda(I_{x,1-c})$. When $U_{[0,1]}(I_{x,1-c}) > 0$ (i.e., $c < 1$), we have $m^{(x,c,U_{[0,1]})} = U_{[0,1]}(\cdot|I_{x,1-c}) = U_{I_{x,1-c}}$; when $U_{[0,1]}(I_{x,1-c}) = 0$ (i.e., $c = 1$), then $m^{(x,c,U_{[0,1]})} = U_{I_{x,1-c}} = \delta_x$. Using the same deduction to Proposition 1 (ii), the result holds.

The maps related to UCM method 4 still do not have continuity properties in all situations as the following examples indicate. When $c = 0.5$ and $P = \delta_0$, it is easy to observe that the map $x \mapsto m^{(x,c,P)}$ is not continuous at 0; when $x = 0$ and $P = \delta_{0.5}$, the map $c \mapsto m^{(x,c,P)}$ is not continuous at 0.5; when $x = 0$ and $c = 0.5$, the map $P \mapsto m^{(x,c,P)}$ is not continuous at $\delta_{0.5}$.

However, unlike UCM method 3 which lacks all continuity properties, UCM method 4 retains certain desirable continuity properties (when the considered probability measure is continuous). Recall that C_1 ($\mathrm{C}_1 \subset \mathrm{M}_1$) is the set of all continuous probability measures on $(\mathbb{R}, \mathcal{M})$.

Proposition 6. *Let $m^{(x,c,P)}$ be the probability measure obtained in UCM method 4 and Eq. (5), then the map $x \mapsto m^{(x,c,P)}$ is continuous on [0,1] when $P \in \mathrm{C}_1$.*

Proof. Fixing $c \in [0, 1]$ and $P \in \mathrm{C}_1$, let $(x_r)_{r=1}^{\infty}$ be an arbitrary sequence with $\lim_{r\to\infty} x_r = x$; we discuss four situations, respective. (Firstly, since all the concerned probability measures are concentrated on [0,1], then it is direct that $\lim_{r\to\infty} F_{m^{(x_r,c,P)}}(t) = F_{m^{(x,c,P)}}(t)$ holds for all $t \in (-\infty, 0) \bigcup (1, \infty)$.)

(a) If $P(I_{x,1-c}) > \lambda(I_{x,1-c}) > 0$, then from the continuity of P we easily observe that for any $t \in [0, 1]$,

$$\begin{aligned} \lim_{r\to\infty} F_{m^{(x_r,c,P)}}(t) &= \lim_{r\to\infty} P([0,t]|I_{x_r,1-c}) = \lim_{r\to\infty} \frac{P([0,t] \bigcap I_{x_r,1-c})}{P(I_{x_r,1-c})} = \frac{\lim_{r\to\infty} P([0,t] \bigcap I_{x_r,1-c})}{\lim_{r\to\infty} P(I_{x_r,1-c})} \\ &= \frac{\lim_{r\to\infty} F_P(t) - \lim_{r\to\infty} F_P(\inf I_{x_r,1-c})}{\lim_{r\to\infty} F_P(\sup I_{x_r,1-c}) - \lim_{r\to\infty} F_P(\inf I_{x_r,1-c})} = \frac{F_P(t) - F_P(\inf I_{x,1-c})}{F_P(\sup I_{x,1-c}) - F_P(\inf I_{x,1-c})} = \frac{P([0,t] \bigcap I_{x,1-c})}{P(I_{x,1-c})} \\ &= P([0,t]|I_{x,1-c}) = F_{m^{(x,c,P)}}(t). \end{aligned}$$

(b) If $\lambda(I_{x,1-c}) > P(I_{x,1-c}) > 0$, then for any $t \in [0,1]$,

$$\begin{aligned}
\lim_{r\to\infty} F_{m^{(x_r,c,P)}}(t) &= \lim_{r\to\infty} P([0,t]|I_{x_r,1-c}) \\
&= \lim_{r\to\infty} \frac{P(I_{x_r,1-c})}{\lambda(I_{x_r,1-c})} \cdot \frac{P([0,t]\bigcap I_{x_r,1-c})}{P(I_{x_r,1-c})} + \lim_{r\to\infty} \frac{\lambda(I_{x_r,1-c}) - P(I_{x_r,1-c})}{\lambda(I_{x_r,1-c})} \cdot \frac{\lambda([0,t]\bigcap I_{x_r,1-c})}{\lambda(I_{x_r,1-c})} \\
&= \lim_{r\to\infty} \frac{P([0,t]\bigcap I_{x_r,1-c})}{1-c} + \lim_{r\to\infty} \frac{1-c-P(I_{x_r,1-c})}{1-c} \cdot \frac{\lambda([0,t]\bigcap I_{x_r,1-c})}{1-c} \\
&= \frac{\lim_{r\to\infty} P([0,t]\bigcap I_{x_r,1-c})}{1-c} + \frac{1-c-\lim_{r\to\infty} P(I_{x_r,1-c})}{1-c} \cdot \frac{\lim_{r\to\infty} \lambda([0,t]\bigcap I_{x_r,1-c})}{1-c} \\
&= \frac{P([0,t]\bigcap I_{x,1-c})}{1-c} + \frac{1-c-P(I_{x,1-c})}{1-c} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{1-c} \\
&= \frac{P(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{P([0,t]\bigcap I_{x,1-c})}{P(I_{x,1-c})} + \frac{\lambda(I_{x,1-c}) - P(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{\lambda(I_{x,1-c})} \\
&= P([0,t]|I_{x,1-c}) = F_{m^{(x,c,P)}}(t).
\end{aligned}$$

(c) If $\lambda(I_{x,1-c}) = P(I_{x,1-c}) > 0$, then there always exists a $N > 0$ such that $(x_r)_{r=N}^{\infty}$ is the possible union of one, two or three of the subsequences: $(x_{\sigma_1(r)})_{r=1}^{\infty}$, $(x_{\sigma_2(r)})_{r=1}^{\infty}$ and $(x_{\sigma_3(r)})_{r=1}^{\infty}$ such that $\lim_{r\to\infty} x_{\sigma_1(r)} = x$ and $P(I_{x_{\sigma_1(r)},1-c}) > \lambda(I_{x_{\sigma_1(r)},1-c}) > 0$ for all $r \in \mathbb{N}$, $\lim_{r\to\infty} x_{\sigma_2(r)} = x$ and $\lambda(I_{x_{\sigma_2(r)},1-c}) > P(I_{x_{\sigma_2(r)},1-c}) > 0$ for all $r \in \mathbb{N}$, and $\lim_{r\to\infty} x_{\sigma_3(r)} = x$ and $P(I_{x_{\sigma_3(r)},1-c}) = \lambda(I_{x_{\sigma_3(r)},1-c}) > 0$ for all $r \in \mathbb{N}$, respectively. (That is, for each $r \geq N$, r satisfies one and only one of the following seven conditions: $r \in \sigma_1(\mathbb{N})$, $r \in \sigma_2(\mathbb{N})$, $r \in \sigma_3(\mathbb{N})$, $r \in \sigma_1(\mathbb{N})\bigcup\sigma_2(\mathbb{N})$, $r \in \sigma_2(\mathbb{N})\bigcup\sigma_3(\mathbb{N})$, $r \in \sigma_1(\mathbb{N})\bigcup\sigma_3(\mathbb{N})$, $r \in \sigma_1(\mathbb{N})\bigcup\sigma_2(\mathbb{N})\bigcup\sigma_3(\mathbb{N})$.) Note that for the possible three subsequences, we can apply the same deduction in (a) to prove for any $t \in \mathbb{R}$, $\lim_{r\to\infty} F_{m^{(x_{\sigma_1(r)},c,P)}}(t) = F_{m^{(x,c,P)}}(t)$ and $\lim_{r\to\infty} F_{m^{(x_{\sigma_3(r)},c,P)}}(t) = F_{m^{(x,c,P)}}(t)$, and apply the same deduction in (b) to prove for any $t \in \mathbb{R}$, $\lim_{r\to\infty} F_{m^{(x_{\sigma_2(r)},c,P)}}(t) = F_{m^{(x,c,P)}}(t)$. Consequently, for any $t \in \mathbb{R}$, $\lim_{r\to\infty} F_{m^{(x_r,c,P)}}(t) = F_{m^{(x,c,P)}}(t)$.

(d) If $P(I_{x,1-c}) = 0$ and $\lambda(I_{x,1-c}) > 0$, then there always exists a $N > 0$ such that $(x_r)_{r=N}^{\infty}$ is the possible union of one or two of the subsequences: $(x_{\sigma_1(r)})_{r=1}^{\infty}$ and $(x_{\sigma_2(r)})_{r=1}^{\infty}$ such that $\lim_{r\to\infty} x_{\sigma_1(r)} = x$ and $\lambda(I_{x_{\sigma_1(r)},1-c}) > P(I_{x_{\sigma_1(r)},1-c}) > 0$ for all $r \in \mathbb{N}$, and $\lim_{r\to\infty} x_{\sigma_2(r)} = x$ and $\lambda(I_{x_{\sigma_2(r)},1-c}) > P(I_{x_{\sigma_2(r)},1-c}) = 0$ for all $r \in \mathbb{N}$, respectively. (It is impossible that there exists a subsequence $(x_{\tau(r)})_{r=1}^{\infty}$ with $\lim_{r\to\infty} x_{\tau(r)} = x$ and $P(I_{x_{\tau(r)},1-c}) \geq \lambda(I_{x_{\tau(r)},1-c}) > 0$. Indeed, if there exists, then we have $0 < \lim_{r\to\infty} P(I_{x_{\tau(r)},1-c}) \leq \lim_{r\to\infty} P(I_{x,1-c} \bigcup I_{x_{\tau(r)},1-c}) = P\left(\bigcap_{r=1}^{\infty}(I_{x,1-c} \bigcup I_{x_{\tau(r)},1-c})\right) = P(I_{x,1-c}) = 0$, which leads to contradiction.) Note that for the subsequence $(x_{\sigma_1(r)})_{r=1}^{\infty}$, we apply a similar but different deduction in (b); for any $t \in [0,1]$,

$$\begin{aligned}
\lim_{r\to\infty} F_{m^{(x_{\sigma_1(r)},c,P)}}(t) &= \lim_{r\to\infty} P([0,t]|I_{x_{\sigma_1(r)},1-c}) \\
&= \lim_{r\to\infty} \frac{P(I_{x_{\sigma_1(r)},1-c})}{\lambda(I_{x_{\sigma_1(r)},1-c})} \cdot \frac{P([0,t]\bigcap I_{x_{\sigma_1(r)},1-c})}{P(I_{x_{\sigma_1(r)},1-c})} + \lim_{r\to\infty} \frac{\lambda(I_{x_{\sigma_1(r)},1-c}) - P(I_{x_{\sigma_1(r)},1-c})}{\lambda(I_{x_{\sigma_1(r)},1-c})} \cdot \frac{\lambda([0,t]\bigcap I_{x_{\sigma_1(r)},1-c})}{\lambda(I_{x_{\sigma_1(r)},1-c})} \\
&= \frac{\lim_{r\to\infty} P([0,t]\bigcap I_{x_{\sigma_1(r)},1-c})}{1-c} + \frac{1-c-\lim_{r\to\infty} P(I_{x_{\sigma_1(r)},1-c})}{1-c} \cdot \frac{\lim_{r\to\infty} \lambda([0,t]\bigcap I_{x_{\sigma_1(r)},1-c})}{1-c} \\
&= \frac{P([0,t]\bigcap I_{x,1-c})}{1-c} + \frac{1-c-P(I_{x,1-c})}{1-c} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{1-c} \\
&= \frac{0}{1-c} + \frac{1-c-0}{1-c} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{1-c} = \frac{\lambda([0,t]\bigcap I_{x,1-c})}{\lambda(I_{x,1-c})} \\
&= U_{I_{x,1-c}}([0,t]) = F_{U_{I_{x,1-c}}}(t) = F_{m^{(x,c,P)}}(t).
\end{aligned}$$

For the subsequence $(x_{\sigma_2(r)})_{r=1}^{\infty}$, note that $m^{(x_{\sigma_2(r)},c,P)} = U_{I_{x_{\sigma_2(r)},1-c}}$ for each $r \in \mathbb{N}$, and it is easy to know that $w - \lim_{r\to\infty} m^{(x_{\sigma_2(r)},c,P)} = w - \lim_{r\to\infty} U_{I_{x_{\sigma_2(r)},1-c}} = U_{I_{x,1-c}} = m^{(x,c,P)}$.

(e) If $\lambda(I_{x,1-c}) = 0$ and $P(I_{x,1-c}) \geq 0$, then $m^{(x_r,c,P)} = \delta_{x_r}$ for each $r \in \mathbb{N}$, and it is evident that $w - \lim_{r\to\infty} m^{(x_r,c,P)} = w - \lim_{r\to\infty} \delta_{x_r} = \delta_x = m^{(x,c,P)}$.

In summary, in any situation we have $w - \lim_{r\to\infty} m^{(x_r,c,P)} = m^{(x,c,P)}$.

Proposition 7. *Let $m^{(x,c,P)}$ be the probability measure obtained in UCM method 4 and Eq. (5), then the map $c \mapsto m^{(x,c,P)}$ is continuous on [0,1] when $P \in \mathrm{C}_1$.*

Proof. Fixing $x \in [0,1]$ and $P \in \mathrm{C}_1$, let $(c_r)_{r=1}^{\infty}$ be an arbitrary sequence with $\lim_{r\to\infty} c_r = c$; we discuss five situations, respectively.

If (a) $P(I_{x,1-c}) > \lambda(I_{x,1-c}) > 0$, or (b) $\lambda(I_{x,1-c}) > P(I_{x,1-c}) > 0$, or (c) $\lambda(I_{x,1-c}) = P(I_{x,1-c}) > 0$, or (d) $P(I_{x,1-c}) = 0$ and $\lambda(I_{x,1-c}) > 0$, then using the very similar deductions in the (a), (b), (c) and (d) of the proof of Proposition 6, respectively, we also have $w - \lim_{r\to\infty} m^{(x,c_r,P)} = m^{(x,c,P)}$. But when (e) $\lambda(I_{x,1-c}) = 0$ and $P(I_{x,1-c}) \ge 0$, we observe that for any $t \in (-\infty, x) \bigcup (x, \infty)$, $\lim_{r\to\infty} F_{m^{(x_r,c,P)}}(t) = F_{m^{(x,c,P)}}(t)$.

In summary, in any situation we have $w - \lim_{r\to\infty} m^{(x_r,c,P)} = m^{(x,c,P)}$.

Proposition 8. *Let $m^{(x,c,P)}$ be the probability measure obtained in UCM method 4 and Eq. (5), then the map $P \mapsto m^{(x,c,P)}$ is continuous on C_1 (with respect to Levy's metric).*

Proof. Fixing $x \in [0,1]$ and $c \in [0,1]$, let $(P_r)_{r=1}^{\infty}$ be an arbitrary sequence with $w - \lim_{r\to\infty} P_r = P$; we discuss five situations, respectively.

(a) If $P(I_{x,1-c}) > \lambda(I_{x,1-c}) > 0$, then we observe that for any $t \in [0,1]$,

$$\begin{aligned}
\lim_{r\to\infty} F_{m^{(x,c,P_r)}}(t) &= \lim_{r\to\infty} P_r([0,t]|I_{x,1-c}) = \lim_{r\to\infty} \frac{P_r([0,t]\bigcap I_{x,1-c})}{P_r(I_{x,1-c})} = \frac{\lim_{r\to\infty} P_r([0,t]\bigcap I_{x,1-c})}{\lim_{r\to\infty} P_r(I_{x,1-c})} \\
&= \frac{\lim_{r\to\infty} F_{P_r}(t) - \lim_{r\to\infty} F_{P_r}(\inf I_{x,1-c})}{\lim_{r\to\infty} F_{P_r}(\sup I_{x,1-c}) - \lim_{r\to\infty} F_{P_r}(\inf I_{x,1-c})} = \frac{F_P(t) - F_P(\inf I_{x,1-c})}{F_P(\sup I_{x,1-c}) - F_P(\inf I_{x,1-c})} = \frac{P([0,t]\bigcap I_{x,1-c})}{P(I_{x,1-c})} \\
&= P([0,t]|I_{x,1-c}) = F_{m^{(x,c,P)}}(t).
\end{aligned}$$

(b) If $\lambda(I_{x,1-c}) > P(I_{x,1-c}) > 0$, then for any $t \in [0,1]$,

$$\begin{aligned}
\lim_{r\to\infty} F_{m^{(x,c,P_r)}}(t) &= \lim_{r\to\infty} P_r([0,t]|I_{x,1-c}) \\
&= \lim_{r\to\infty} \frac{P_r(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{P_r([0,t]\bigcap I_{x,1-c})}{P_r(I_{x,1-c})} + \lim_{r\to\infty} \frac{\lambda(I_{x,1-c}) - P_r(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{\lambda([0,t]\bigcap I_{x_r,1-c})}{\lambda(I_{x_r,1-c})} \\
&= \frac{P(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{P([0,t]\bigcap I_{x,1-c})}{P(I_{x,1-c})} + \frac{\lambda(I_{x,1-c}) - P(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{\lambda(I_{x,1-c})} \\
&= P([0,t]|I_{x,1-c}) = F_{m^{(x,c,P)}}(t).
\end{aligned}$$

(c) If $\lambda(I_{x,1-c}) = P(I_{x,1-c}) > 0$, then there always exists a $N > 0$ such that $(P_r)_{r=N}^{\infty}$ is the possible union of one, two or three of the subsequences: $(P_{\sigma_1(r)})_{r=1}^{\infty}$, $(P_{\sigma_2(r)})_{r=1}^{\infty}$ and $(P_{\sigma_3(r)})_{r=1}^{\infty}$ such that $w - \lim_{r\to\infty} P_{\sigma_1(r)} = P$ and $P_r(I_{x,1-c}) > \lambda(I_{x,1-c}) > 0$ for all $r \in \mathbb{N}$, $w - \lim_{r\to\infty} P_{\sigma_2(r)} = P$ and $\lambda(I_{x,1-c}) > P_r(I_{x,1-c}) > 0$ for all $r \in \mathbb{N}$, and $w - \lim_{r\to\infty} P_{\sigma_3(r)} = P$ and $P(I_{x,1-c}) = \lambda(I_{x,1-c}) > 0$ for all $r \in \mathbb{N}$, respectively. (That is, for each $r \ge N$, r satisfies one and only one of the following seven conditions: $r \in \sigma_1(\mathbb{N})$, $r \in \sigma_2(\mathbb{N})$, $r \in \sigma_3(\mathbb{N})$, $r \in \sigma_1(\mathbb{N}) \bigcup \sigma_2(\mathbb{N})$, $r \in \sigma_2(\mathbb{N}) \bigcup \sigma_3(\mathbb{N})$, $r \in \sigma_1(\mathbb{N}) \bigcup \sigma_3(\mathbb{N})$, $r \in \sigma_1(\mathbb{N}) \bigcup \sigma_2(\mathbb{N}) \bigcup \sigma_3(\mathbb{N})$.) Note that for the possible three subsequences, we can apply the same deduction in (a) to prove for any $t \in \mathbb{R}$, $\lim_{r\to\infty} F_{m^{(x,c,P_{\sigma_1(r)})}}(t) = F_{m^{(x,c,P)}}(t)$ and $\lim_{r\to\infty} F_{m^{(x,c,P_{\sigma_3(r)})}}(t) = F_{m^{(x,c,P)}}(t)$, and apply the same deduction in (b) to prove for any $t \in \mathbb{R}$, $\lim_{r\to\infty} F_{m^{(x_{\sigma_2(r)},c,P)}}(t) = F_{m^{(x,c,P)}}(t)$. Consequently, for any $t \in \mathbb{R}$, $\lim_{r\to\infty} F_{m^{(x,c,P_r)}}(t) = F_{m^{(x,c,P)}}(t)$.

(d) If $P(I_{x,1-c}) = 0$ and $\lambda(I_{x,1-c}) > 0$, then there always exists a $N > 0$ such that $(x_r)_{r=N}^{\infty}$ is the possible union of one or two of the subsequences: $(P_{\sigma_1(r)})_{r=1}^{\infty}$ and $(P_{\sigma_2(r)})_{r=1}^{\infty}$ such that $w - \lim_{r\to\infty} P_{\sigma_1(r)} = P$ and $\lambda(I_{x,1-c}) > P_{\sigma_1(r)}(I_{x,1-c}) > 0$ for all $r \in \mathbb{N}$, and $w - \lim_{r\to\infty} P_{\sigma_2(r)} = P$ and $\lambda(I_{x,1-c}) > P_{\sigma_2(r)}(I_{x,1-c}) = 0$ for all $r \in \mathbb{N}$, respectively. (It is impossible that there exists a subsequence $(P_{\tau(r)})_{r=1}^{\infty}$ with $w - \lim_{r\to\infty} P_{\tau(r)} = P$ and $P_{\tau(r)}(I_{x,1-c}) \ge \lambda(I_{x,1-c}) > 0$. Suppose there exists; since F_P is continuous on $\mathbb{R}$, then $\lim_{r\to\infty} F_{P_{\tau(r)}}(t) = F_P(t)$ holds for all $t \in I_{x,1-c}$, and therefore $0 = \lim_{r\to\infty} F_{P_{\tau(r)}}(\sup I_{x,1-c}) - F_P(\sup I_{x,1-c}) \ge \lambda(I_{x,1-c}) > 0$, which leads to contradiction.) Note that for the subsequence $(P_{\sigma_1(r)})_{r=1}^{\infty}$, we apply a similar but different deduction in (b); for any $t \in [0,1]$,

$$\begin{aligned}
\lim_{r\to\infty} F_{m^{(x,c,P_{\sigma_1(r)})}}(t) &= \lim_{r\to\infty} P([0,t]|I_{x,1-c}) \\
&= \lim_{r\to\infty} \frac{P_{\sigma_1(r)}(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{P_{\sigma_1(r)}([0,t]\bigcap I_{x,1-c})}{P_{\sigma_1(r)}(I_{x,1-c})} + \lim_{r\to\infty} \frac{\lambda(I_{x,1-c}) - P_{\sigma_1(r)}(I_{x,1-c})}{\lambda(I_{x,1-c})} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{\lambda(I_{x,1-c})} \\
&= \frac{\lim_{r\to\infty} P_{\sigma_1(r)}([0,t]\bigcap I_{x,1-c})}{1-c} + \frac{1-c-\lim_{r\to\infty} P_{\sigma_1(r)}(I_{x,1-c})}{1-c} \cdot \frac{\lim_{r\to\infty} \lambda([0,t]\bigcap I_{x,1-c})}{1-c} \\
&= \frac{P([0,t]\bigcap I_{x,1-c})}{1-c} + \frac{1-c-P(I_{x,1-c})}{1-c} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{1-c} \\
&= \frac{0}{1-c} + \frac{1-c-0}{1-c} \cdot \frac{\lambda([0,t]\bigcap I_{x,1-c})}{1-c} = \frac{\lambda([0,t]\bigcap I_{x,1-c})}{\lambda(I_{x,1-c})}
\end{aligned}$$

$$= U_{I_{x,1-c}}([0,t]) = F_{U_{I_{x,1-c}}}(t) = F_{m^{(x,c,P)}}(t).$$

For the subsequence $(P_{\sigma_2(r)})_{r=1}^{\infty}$, note that $m^{(x,c,P_{\sigma_2(r)})} \equiv U_{I_{x,1-c}} = m^{(x,c,P)}$ for each $r \in \mathbb{N}$.

(e) If $\lambda(I_{x,1-c}) = 0$ and $P(I_{x,1-c}) \geq 0$, then $m^{(x,c,P_r)} \equiv \delta_x = m^{(x,c,P)}$ for each $r \in \mathbb{N}$.
In summary, in any situation we have $w - \lim_{r\to\infty} m^{(x_r,c,P)} = m^{(x,c,P)}$.
The corresponding monotonicity still holds for UCM method 4.

Proposition 9. *For any two BUI granules $(x_1,c),(x_2,c) \in \mathcal{B}$ with $x_1 < x_2$, let $m^{(x_1,c,P)}$ and $m^{(x_2,c,P)}$ be the probability measures obtained using UCM method 4 and Eq. (5), respectively. Then, for any $P \in \mathrm{M}_1$, $E_1 = \int_{[0,1]} X_{id}\, dm^{(x_1,c,P)} \leq \int_{[0,1]} X_{id}\, dm^{(x_2,c,P)} = E_2$.*

Proof. For any $c \in [0,1]$, note that $\sup I_{x_1,1-c} \leq \sup I_{x_2,1-c}$ and $\inf I_{x_1,1-c} \leq \inf I_{x_2,1-c}$.

When $c = 1$, $E_1 = \int_{[0,1]} X_{id}\, d\delta_{x_1} = x_1 < x_2 = \int_{[0,1]} X_{id}\, d\delta_2 = E_2$.

For any $c < 1$, note that $\lambda(I_{x_1,1-c}) = \lambda(I_{x_2,1-c}) > 0$. If $I_{x_1,1-c} \bigcap I_{x_2,1-c} = \emptyset$, then we immediately know $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$ since $\sup I_{x_1,1-c} \leq \inf I_{x_2,1-c}$. If $I_{x_1,1-c} \bigcap I_{x_2,1-c} = B \neq \emptyset$ (which is a closed interval), for convenience we take the notations $I_{x_1,1-c} \backslash I_{x_2,1-c} = A$ and $I_{x_2,1-c} \backslash I_{x_1,1-c} = C$. Then, we discuss the following nine situations, respectively.

(1A) If $P(I_{x_1,1-c}) \geq P(I_{x_2,1-c}) \geq \lambda(I_{x_1,1-c}) = 1-c$, then note that $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t) = 0$ for all $t \in A$; $1 = F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in C$. For any $t \in B$, we observe that

$$\begin{aligned}
m^{(x_1,c,P)}((t,\infty)) &= P((t,\infty)|I_{x_1,1-c}) = \frac{P((t,\infty)\bigcap I_{x_1,1-c})}{P(I_{x_1,1-c})} = \frac{P((t,\sup B)\bigcap I_{x_1,1-c})}{P(I_{x_1,1-c})} = \frac{P((t,\sup B))}{P(I_{x_1,1-c})} \\
&\leq \frac{P((t,\sup C))}{P(I_{x_2,1-c})} = \frac{P((t,\infty)\bigcap I_{x_2,1-c})}{P(I_{x_2,1-c})} = P((t,\infty)|I_{x_2,1-c}) \leq m^{(x_2,c,P)}((t,\infty)).
\end{aligned}$$

Therefore, $F_{m^{(x_1,c,P)}}(t) = 1 - m^{(x_1,c,P)}((t,\infty)) \geq 1 - m^{(x_2,c,P)}((t,\infty)) = F_{m^{(x_2,c,P)}}(t)$ for all $t \in B$. Consequently, $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in \mathbb{R}$.

(1B) If $P(I_{x_2,1-c}) > P(I_{x_1,1-c}) \geq 1-c$, then we can apply a similar deduction in (1A) to show $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in \mathbb{R}$.

(2A) If $P(I_{x_1,1-c}) \geq 1-c > P(I_{x_2,1-c}) > 0$, then it is easy to note that $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t) = 0$ for all $t \in A \bigcup C$. For any $t \in B$, we have the following deduction

$$\begin{aligned}
m^{(x_1,c,P)}((t,\infty)) &= m^{(x_1,c,P)}((t,\sup B]) = P((t,\sup B]|I_{x_1,1-c}) = \frac{P((t,\sup B]\bigcap I_{x_1,1-c})}{P(I_{x_1,1-c})} \\
&= \frac{P((t,\sup B])}{P(I_{x_1,1-c})} \leq \frac{P((t,\sup B])}{\lambda(I_{x_1,1-c})} \leq \frac{P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})}\frac{P((t,\sup B])}{P(I_{x_2,1-c})} + \frac{\lambda(I_{x_2,1-c}) - P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})}\frac{\lambda((t,\sup B])}{\lambda(I_{x_2,1-c})} \\
&= \frac{P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})}P((t,\sup B]|I_{x_2,1-c}) + \frac{\lambda(I_{x_2,1-c}) - P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})}U_{I_{x_2,1-c}}((t,\sup B]) \\
&= m^{(x_2,c,P)}((t,\sup B])) \leq m^{(x_2,c,P)}((t,\infty)).
\end{aligned}$$

Therefore, $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in \mathbb{R}$.

(2B) If $P(I_{x_2,1-c}) \geq 1-c > P(I_{x_1,1-c}) > 0$, then we can apply a similar deduction in (2A).

(3A) If $1-c > P(I_{x_1,1-c}) \geq P(I_{x_2,1-c}) > 0$, it is easy to note that $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t) = 0$ for all $t \in A \bigcup C$. For any $t \in B$, we have

$$\begin{aligned}
m^{(x_1,c,P)}((t,\infty)) &= m^{(x_1,c,P)}((t,\sup B]) \\
&= \frac{P(I_{x_1,1-c})}{\lambda(I_{x_1,1-c})}P((t,\sup B]|I_{x_1,1-c}) + \frac{\lambda(I_{x_1,1-c}) - P(I_{x_1,1-c})}{\lambda(I_{x_1,1-c})}U_{I_{x_1,1-c}}((t,\sup B]) \\
&= \frac{P(I_{x_1,1-c})}{\lambda(I_{x_1,1-c})}\frac{P((t,\sup B]\bigcap I_{x_1,1-c})}{P(I_{x_1,1-c})} + \frac{\lambda(I_{x_1,1-c}) - P(I_{x_1,1-c})}{\lambda(I_{x_1,1-c})}\frac{\lambda((t,\sup B])}{\lambda(I_{x_1,1-c})} \\
&= \frac{P((t,\sup B])}{1-c} + \frac{1-c-P(I_{x_1,1-c})}{1-c}\frac{\lambda((t,\sup B])}{1-c} \\
&\leq \frac{P(I_{x_2,1-c})}{1-c}\frac{P((t,\sup C])}{P(I_{x_2,1-c})} + \frac{1-c-P(I_{x_2,1-c})}{1-c}\frac{\lambda((t,\sup C])}{1-c} \\
&= \frac{P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})}\frac{P((t,\sup C]\bigcap I_{x_2,1-c})}{P(I_{x_2,1-c})} + \frac{\lambda(I_{x_2,1-c}) - P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})}\frac{\lambda((t,\sup C])}{\lambda(I_{x_2,1-c})}
\end{aligned}$$

$$= \frac{P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})} P((t,\sup C]|I_{x_2,1-c}) + \frac{\lambda(I_{x_2,1-c}) - P(I_{x_2,1-c})}{\lambda(I_{x_2,1-c})} U_{I_{x_2,1-c}}((t,\sup C])$$
$$= m^{(x_2,c,P)}((t,\sup C]) = m^{(x_2,c,P)}((t,\infty)).$$

Therefore, $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in \mathbb{R}$.

(3B) If $1-c > P(I_{x_2,1-c}) > P(I_{x_1,1-c}) > 0$, then we can apply a similar deduction in (3A).

(4A) If $P(I_{x_1,1-c}) > P(I_{x_2,1-c}) = 0$, then we immediately observe that $m^{(x_1,c,P)}$ is concentrated on A while $m^{(x_2,c,P)}$ is concentrated on $B \bigcup C$. Hence, $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in \mathbb{R}$.

(4B) If $P(I_{x_2,1-c}) > P(I_{x_1,1-c}) = 0$, the result holds using similar observation in (4A).

(5) If $P(I_{x_1,1-c}) = P(I_{x_2,1-c}) = 0$, then $m^{(x_1,c,P)} = U_{I_{x_1,1-c}}$ and $m^{(x_2,c,P)} = U_{I_{x_2,1-c}}$, and it trivially holds that $F_{m^{(x_1,c,P)}}(t) \geq F_{m^{(x_2,c,P)}}(t)$ for all $t \in \mathbb{R}$.

Finally, since $F_{m^{(x_1,c,P)}} \geq F_{m^{(x_2,c,P)}}$, then from the same analysis as discussed earlier

$$E_1 = \int_{[0,1]} X_{id}\, dm^{(x_1,c,P)} = 1 - \frac{1}{2}\int_{[0,1]} (F_{m^{(x_1,c,P)}}(t) + F_{m^{(x_1,c,P)}}(t-))\lambda(dt) = 1 - \int_{[0,1]} F_{m^{(x_1,c,P)}}\, d\lambda$$
$$\leq 1 - \int_{[0,1]} F_{m^{(x_2,c,P)}}\, d\lambda = \int_{[0,1]} X_{id}\, dm^{(x_2,c,P)} = E_2.$$

6. UCM methods in uncertain and probabilistic decision environment

Assuming a high-end vehicle manufacturer is considering increasing production of its branded electric bicycles, the decision-making process depends on the predicted repurchase rate of these bicycles. If the forecasted repurchase rate exceeds a certain threshold K_1, then mass production will occur; if it reaches K_2 but is less than K_1, then limited production will take place; otherwise, no additional production will be made. The manufacturer has four internal experts $(E_i)_{i=1}^4$ who each predict the repurchase rate with some uncertainty (forming a 4-dimensional BUI vector $(\mathbf{x},\mathbf{c}) = ((x_i,c_i))_{i=1}^4$). To eliminate this uncertainty in their predictions, the manufacturer decides to consult a professional market research firm that can provide a probability distribution P based on historical data and market research to estimate the repurchase rate. Using UCM method, the manufacturer can fuse four BUI granules and probability distribution P to generate a 4-dimensional real prediction vector which can be weighted averaged to obtain an ultimate comprehensive result as reasonable predicted value for repurchase rates and be applied to make further decisions accordingly.

For each BUI granule $(x(i), c(i))$ $(i \in [4])$, $100x(i)\%$ represents the predicted repurchase rate by E_i and $c(i)$ is his/her certainty about the prediction. Suppose $P = 0.5P_1 + 0.5\delta_{0.5}$ and $(\mathbf{x},\mathbf{c}) = ((0,0.7),(0.2,0.5),(0.5,0.2),(0.8,1))$ as in Example 4. We next compute the expected repurchase rates respectively.

$$E_1 = \int X_{id}\, dm^{(0.0.7,P)} = \int t \cdot p_1(t)dt = \int_{[0,0.3]} t \cdot \left[\frac{10}{3}t + \frac{53}{18}\right] dt$$
$$= \frac{10}{3}\int_{[0,0.3]} t^2 dt + \frac{53}{18}\int_{[0,0.3]} t dt = \frac{10}{3}\frac{0.3^3}{3} + \frac{53}{18}\frac{0.3^2}{2} = 0.03 + 0.1325 = 0.1625;$$
$$E_2 = \int X_{id}\, dm^{(0.2.0.5,P)} = \int X_{id}\, d\left[\frac{7}{27}Q + \frac{20}{27}\delta_{0.5}\right] = \frac{7}{27}\int_{[0,1]} X_{id}\, dQ + \frac{20}{27}\int_{[0,1]} X_{id}\, d\delta_{0.5}$$
$$= \frac{7}{27}\int_{[0.1,0.6]} t\frac{t}{0.175}dt + \frac{20}{27}\cdot 0.5 = \frac{7}{27(0.175)}\left[\frac{0.6^2}{2} - \frac{0.1^2}{2}\right] + \frac{20}{27}\cdot 0.5 \doteq 0.26 + 0.37 = 0.63;$$
$$E_3 = \int X_{id}\, dm^{(0.5,0.2,P)} = \int X_{id}\, d\left[\frac{4}{9}Q + \frac{5}{9}\delta_{0.5}\right] = \frac{4}{9}\int X_{id}\, dQ + \frac{5}{9}\cdot 0.5$$
$$= \frac{4}{9}\int_{[0.1,0.9]} t\frac{t}{0.4}dt + \frac{5}{9}\cdot 0.5 \doteq \frac{10}{9}\left[\frac{0.9^2}{2} - \frac{0.1^2}{2}\right] + 0.278 \doteq 0.444 + 0.278 = 0.722;$$
$$E_4 = \int X_{id}\, dm^{(0.8,0.1,P)} = \int X_{id}\, d\delta_{0.8} = 0.8.$$

With the obtained four expected repurchase rates, we take their mean $E = (1/4)\sum_{i=1}^4 E_i = 0.25(0.1625 + 0.63 + 0.722 + 0.8) = 0.578625$. That means the resultant comprehensive expected repurchase rate is 57.8625%. If we set $K_1 = 70\%$ and $K_2 = 50\%$, then the recommended course of action is to manufacture a limited quantity of the branded electric bicycles.

This uncertainty merging method reflects the decision makers' preference order, which prioritizes the predictions of internal experts (as they are more trusted). If the experts hold a certain attitude, their predictions are fully believed (i.e., BUI degenerates into real value). If the experts are partially certain, their predictions still dominate (this can be seen from some conditional probability definitions in UCM method), while the predictions from external market research agencies serve as supplementary information. However, if the experts are completely uncertain, we have no choice but to rely entirely on data from market research agencies. The evaluation and decision model based on UCM method is more practical for real-world decision-making problems.

7. Some discussion and comparison of the three UCM methods

The proposed three UCM methods (in probability environment) have their respective characteristics which may matter in the choice of them in decision making.

For any fixed $x \in [0,1]$ and $c \in [0,1]$, let $H^{<j>}_{(x,c)} : \mathrm{M}_1 \to \mathrm{M}_1$ be related to UCM method j with $H^{<j>}_{(x,c)}(P) = m^{(x,c,P)}$ and $j = 2,3,4$, respectively. We first note that for any $x \in [0,1]$, $H^{<j>}_{(x,1)}(P) = \delta_x$ for all $P \in \mathrm{M}_1$ and all $j \in \{2,3,4\}$. We also note that for any $x \in [0,1]$, $H^{<j>}_{(x,0)}(P) = P$ for all $P \in \mathrm{M}_1$ and all $j \in \{2,3,4\}$. Next we discuss some related "linearity".

Proposition 10. *UCM method 2 has the following property*

$$H^{<2>}_{(x,c)}(\alpha P_1 + (1-\alpha)P_2) = \alpha H^{<2>}_{(x,c)}(P_1) + (1-\alpha) H^{<2>}_{(x,c)}(P_2)$$

for any $\alpha \in [0,1]$.

Proof.

$$\begin{aligned} H^{<2>}_{(x,c)}(\alpha P_1 + (1-\alpha)P_2) &= c\delta_x + (1-c)(\alpha P_1 + (1-\alpha)P_2) = \alpha\left(c\delta_x + (1-c)P_1\right) + (1-\alpha)\left(c\delta_x + (1-c)P_2\right) \\ &= \alpha H^{<2>}_{(x,c)}(P_1) + (1-\alpha) H^{<2>}_{(x,c)}(P_2). \end{aligned}$$

The UCM method 3 and UCM method 4 do not have the "linearity" property. For example, consider $P_1 = U_{[0,0.5]}$, $P_2 = \delta_{0.5}$, $\alpha = 0.5$ and $(x,c) = (1,0.5)$; then $H^{<3>}_{(x,c)}(\alpha P_1 + (1-\alpha)P_2) = H^{<4>}_{(x,c)}(\alpha P_1 + (1-\alpha)P_2) = \delta_{0.5}$ but

$$\alpha H^{<3>}_{(x,c)}(P_1) + (1-\alpha) H^{<3>}_{(x,c)}(P_2) = \alpha H^{<4>}_{(x,c)}(P_1) + (1-\alpha) H^{<4>}_{(x,c)}(P_2) = 0.5U_{[0.5,1]} + 0.5\delta_{0.5}.$$

The linearity of UCM method 2 can sometimes be effectively demonstrated and applied in decision-making processes. For instance, if a probability distribution P_1 is obtained early on from historical statistics, we can merge it with another probability distribution $H^{<2>}_{(x,c)}(P_1)$ to calculate an expected value $E_1 = \int X_{id}\, d H^{<2>}_{(x,c)}(P_1)$. Subsequently, if a new probability distribution P_2 is obtained later using updated statistics, we can create an updated merged probability distribution $H^{<2>}_{(x,c)}(P_2)$ with a new expected value $E_2 = \int X_{id}\, d H^{<2>}_{(x,c)}(P_2)$. By assigning equal weights to the initial and updated distributions and deriving an updated probability distribution $P = 0.5P_1 + 0.5P_2$, we obtain the corresponding updated expected value $E = \int X_{id}\, d H^{<2>}_{(x,c)}(0.5P_1 + 0.5P_2)$. In fact, if linearity holds true in this context, then we only need to calculate $H^{<2>}_{(x,c)}(P_2)$ to obtain $E = 0.5 \int X_{id}\, d H^{<2>}_{(x,c)}(P_1) + 0.5 \int X_{id}\, d H^{<2>}_{(x,c)}(P_2) = 0.5E_1 + 0.5E_2$ instead of recalculating everything from scratch. This showcases evaluation consistency that may be sought by rigorous evaluators; however, it does not imply that linearity is universally required across all evaluators or decision-making scenarios.

The following property we discuss appears to be purely theoretical at present and may have limited relevance to practical decision-making problems.

For j=2, 3, 4, the map $\left(H^{<j>}_{(x,c)}\right)^{(n)} : \mathrm{M}_1 \to \mathrm{M}_1$ ($n \in \mathbb{N}$) is conventionally defined such that $\left(H^{<j>}_{(x,c)}\right)^{(n+1)}(P) = H^{<j>}_{(x,c)}\left(\left(H^{<j>}_{(x,c)}\right)^{(n)}(P)\right)$ with $\left(H^{<j>}_{(x,c)}\right)^{(1)} = H^{<j>}_{(x,c)}$.

Proposition 11.

(i) *For any fixed* $x \in [0,1]$, $c \in (0,1]$ *and* $P \in \mathrm{M}_1$, $w - \lim_{n\to\infty} \left(H^{<2>}_{(x,c)}\right)^{(n)}(P) = \delta_x$; $\left(H^{<2>}_{(x,0)}\right)^{(n)}(P) = P$ *for all* $n \in \mathbb{N}$.

(ii) *For any fixed* $x \in [0,1]$, $c \in [0,1]$ *and* $P \in \mathrm{M}_1$, $\left(H^{<3>}_{(x,c)}\right)^{(n)}(P) = H^{<3>}_{(x,c)}(P)$ *for all* $n \in \mathbb{N}$. *(That is, if* $P(I_{x,1-c}) > 0$ *then* $\left(H^{<3>}_{(x,c)}\right)^{(n)}(P) = P(\cdot | I_{x,1-c})$ *for all* $n \in \mathbb{N}$*, and if* $P(I_{x,1-c}) = 0$ *then* $\left(H^{<3>}_{(x,c)}\right)^{(n)}(P) = U_{I_{x,1-c}}$ *for all* $n \in \mathbb{N}$*.)*

(iii) *For any fixed* $x \in [0,1]$, $c \in [0,1]$ *and* $P \in \mathrm{M}_1$, $\left(H^{<4>}_{(x,c)}\right)^{(n)}(P) = H^{<4>}_{(x,c)}(P)$ *for all* $n \in \mathbb{N}$.

Proof. (i) We firstly show that $\left(H^{<2>}_{(x,c)}\right)^{(n)}(P) = (1-(1-c)^n)\,\delta_x + (1-c)^n P$ for all $n \in \mathbb{N}$. Using induction, on the one hand, suppose $\left(H^{<2>}_{(x,c)}\right)^{(n)}(P) = (1-(1-c)^n)\,\delta_x + (1-c)^n P$, then

$$\left(H^{<2>}_{(x,c)}\right)^{(n+1)}(P) = H^{<2>}_{(x,c)}\left(\left(H^{<2>}_{(x,c)}\right)^{(n)}(P)\right) = c\delta_x + (1-c)\left((1-(1-c)^n)\,\delta_x + (1-c)^n P\right)$$
$$= c\delta_x + (1-c)\delta_x - (1-c)^{n+1}\delta_x + (1-c)^{n+1}P = \left(1-(1-c)^{n+1}\right)\delta_x + (1-c)^{n+1}P.$$

On the other hand, $\left(H^{<2>}_{(x,c)}\right)^{(1)}(P) = H^{<2>}_{(x,c)}(P) = c\delta_x + (1-c)P = \left(1-(1-c)^1\right)\delta_x + (1-c)^1 P$ holds true for $n = 1$.

Therefore, if $c \in (0,1]$, then $w - \lim_{n\to\infty}\left(H^{<2>}_{(x,c)}\right)^{(n)}(P) = w - \lim_{n\to\infty}\left((1-(1-c)^n)\,\delta_x + (1-c)^n P\right) = \delta_x$. (Indeed, it is easy to observe $\left(H^{<2>}_{(x,1)}\right)^{(n)}(P) = \delta_x$ for all $n \in \mathbb{N}$.) It is also easy to observe $\left(H^{<2>}_{(x,0)}\right)^{(n)}(P) = P$ for all $n \in \mathbb{N}$.

(ii) If $P(I_{x,1-c}) > 0$, then $H^{<3>}_{(x,c)}(P) = P(\cdot|I_{x,1-c})$; using induction, $\left(H^{<3>}_{(x,c)}\right)^{(n)}(P) = P(\cdot|I_{x,1-c})$ implies $\left(H^{<3>}_{(x,c)}\right)^{(n+1)}(P) = H^{<3>}_{(x,c)}\left(\left(H^{<3>}_{(x,c)}\right)^{(n)}(P)\right) = H^{<3>}_{(x,c)}\left(P(\cdot|I_{x,1-c})\right) = \left(P(\cdot|I_{x,1-c})\right)(\cdot|I_{x,1-c}) = P(\cdot|I_{x,1-c})$. If $P(I_{x,1-c}) = 0$, then $H^{<3>}_{(x,c)}(P) = U_{I_{x,1-c}}$; using induction, $\left(H^{<3>}_{(x,c)}\right)^{(n)}(P) = U_{I_{x,1-c}}$ implies $\left(H^{<3>}_{(x,c)}\right)^{(n+1)}(P) = H^{<3>}_{(x,c)}\left(\left(H^{<3>}_{(x,c)}\right)^{(n)}(P)\right) = H^{<3>}_{(x,c)}\left(U_{I_{x,1-c}}\right) = U_{I_{x,1-c}}(\cdot|I_{x,1-c}) = U_{I_{x,1-c}}$. Hence, $\left(H^{<3>}_{(x,c)}\right)^{(n)}(P) = H^{<3>}_{(x,c)}(P)$ for all $n \in \mathbb{N}$.

(iii) If $P(I_{x,1-c}) \geq \lambda(I_{x,1-c}) > 0$, then $H^{<4>}_{(x,c)}(P) = P(\cdot|I_{x,1-c})$; applying the similar induction in (ii), we have $\left(H^{<4>}_{(x,c)}\right)^{(n)}(P) = H^{<4>}_{(x,c)}(P) = P(\cdot|I_{x,1-c})$. If $\lambda(I_{x,1-c}) > P(I_{x,1-c}) > 0$, then $H^{<4>}_{(x,c)}(P) = \frac{P(I_{x,1-c})}{\lambda(I_{x,1-c})}P(\cdot|I_{x,1-c}) + \frac{\lambda(I_{x,1-c})-P(I_{x,1-c})}{\lambda(I_{x,1-c})}U_{I_{x,1-c}}$ (note that it is concentrated on $I_{x,1-c}$ and with $\left(H^{<4>}_{(x,c)}(P)\right)(I_{x,1-c}) \geq \lambda(I_{x,1-c}) > 0$); using induction, $\left(H^{<4>}_{(x,c)}\right)^{(n)}(P) = \frac{P(I_{x,1-c})}{\lambda(I_{x,1-c})}P(\cdot|I_{x,1-c}) + \frac{\lambda(I_{x,1-c})-P(I_{x,1-c})}{\lambda(I_{x,1-c})}U_{I_{x,1-c}}$ implies $\left(H^{<4>}_{(x,c)}\right)^{(n+1)}(P) = H^{<4>}_{(x,c)}\left(\left(H^{<4>}_{(x,c)}\right)^{(n)}(P)\right) = H^{<4>}_{(x,c)}\left(H^{<4>}_{(x,c)}(P)\right) = \left(H^{<4>}_{(x,c)}(P)\right)(\cdot|I_{x,1-c}) = H^{<4>}_{(x,c)}(P)$. If $P(I_{x,1-c}) = 0$ and $\lambda(I_{x,1-c}) > 0$, applying the similar induction in (ii), we have $\left(H^{<4>}_{(x,c)}\right)^{(n)}(P) = H^{<4>}_{(x,c)}(P) = U_{I_{x,1-c}}$. If $\lambda(I_{x,1-c}) = 0$ and $P(I_{x,1-c}) \geq 0$, then $H^{<4>}_{(x,c)}(P) = \delta_x$; using induction, $\left(H^{<4>}_{(x,c)}\right)^{(n)}(P) = \delta_x$ implies $\left(H^{<4>}_{(x,c)}\right)^{(n+1)}(P) = H^{<4>}_{(x,c)}\left(\left(H^{<4>}_{(x,c)}\right)^{(n)}(P)\right) = H^{<4>}_{(x,c)}\left(\delta_x\right) = \delta_x = H^{<4>}_{(x,c)}(P)$. Hence, $\left(H^{<4>}_{(x,c)}\right)^{(n)}(P) = H^{<4>}_{(x,c)}(P)$ for all $n \in \mathbb{N}$.

8. Conclusions

We proposed, analyzed, and compared four UCM methods for merging BUI granules with probability measures. UCM method 1 is purely interval-based and defined simply by combining intervals. The other three UCM methods operate within probability environments. UCM method 2 transforms a BUI granule into Dirac measure and generates a new probability measure using a combination form that has good continuity properties but lacks distinguishability of probability measures. Although lacking in continuity properties, UCM method 3 offers better distinguishability than the previous method. Finally, substitution UCM method (UCM method 4) is mainly defined based on well-defined combinations of conditional probabilities and uniform distributions; it exhibits better continuity properties than UCM method 3. We demonstrate that all three UCM methods result in Dirac measure when there is no uncertainty involved in the given BUI granules.

The expectation values associated with the four UCM methods are as follows: UCM 1 merely yields its representative value $cx + 0.5(1-c)$, while all the expectation values (of identity random variable) for the other three UCM methods, $\int X_{id}\,dm$, degenerate into $cx + 0.5(1-c)$ when the given probability measure is uniform distribution. Furthermore, all four UCM methods exhibit desirable monotonicities.

We present practical applications in business decision-making and provide several numerical examples to demonstrate their utility and potential. Furthermore, we conduct further discussions to highlight the distinctive characteristics of the three UCM methods (in a probabilistic environment), which may be considerations for decision-making.

Although we employ conditional probability and conditional distribution, our methodology diverges from Bayesian analysis. For instance, in UCM method 4, if $\lambda(I_{x,1-c}) > P(I_{x,1-c})$, the fused probability takes into account both $P(\cdot|I_{x,1-c})$ and $U_{I_{x,1-c}}$ simultaneously based on a specific ratio. Our approach also exhibits connections with Dempster-Shafer theory [37,38]. For example, while determining the ultimate fused probability distribution, we explore various techniques for allocating mass with a quantity of $1-c$. In future research endeavors, we will delve deeper into more suitable UCM methods and further investigate their associations with Bayesian analysis, Dempster-Shafer theory, and mass allocation [37–40].

CRediT authorship contribution statement

LeSheng Jin: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Yi Yang:** Writing – review & editing, Methodology. **Zhen-Song Chen:** Writing – review & editing, Project administration, Funding acquisition. **Muhammet Deveci:** Writing – review & editing. **Radko Mesiar:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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