# PREDICTIVE CONTROL WITH APPROXIMATELY GIVEN REFERENCE SIGNAL

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**Abstract:** This paper deals with two specific modifications of predictive control, which cope with approximately given reference signal. The both modifications consist in different definition of requirements (reference signals) for a system behavior. The first modification takes into account only permitted ranges (limits) of required reference signal. The second modification considers only initial and final system state (position) and system outputs can occur within whole system workspace (domain). In such cases, the real desired reference signal is not strictly given, but it follows from control design. Described modifications can solve e.g. manipulation tasks or stabilization of an output signal in some permitted range. The explanation of the both modifications is documented by several examples.

**Keywords:** Predictive state-space control, range-space control, approximate reference signal

## 1 INTRODUCTION

In industrial applications, there exist a lot of control operations, which provide different changes of working points, movements or interval stabilization. In these operations – processes, the accurate achievement of some predetermined trajectory is not in many cases important, however a fulfillment of some permitted output range or reaching of some point is required. The range or set of ranges are given by definite limits following from configuration of controlled system or used production technology. The points, which should be reached, can represent e.g. end-positions of manipulative operations or new working points of controlled system. Such defined requirements can be called as approximately given reference signals.

This paper concerns with a solution based on utilization of predictive control. Specifically, there are investigated its two possible modifications here. The both modifications consist in different definition of requirements for control design i.e. different use or processing of given reference signals. Usually, the reference signal is determined by sequence of known desired values, but it is not available here and the both modifications generate the control actions without this knowledge.

The first modification takes into account only ranges (limits) of required behavior, i.e. only ranges of reference signal. The second modification considers only initial and final system state (position) and system outputs can occur within whole system workspace – domain. In such cases, the real resultant behavior, i.e. desired reference signal is not strictly given, but it follows from control design. Described approach can solve e.g. manipulation tasks or a stabilization of output signal in some permitted range, where the achievement of exact values is not important, but the system has to reach or to remain in some state. The control actions are designed to meet such requirements. The both presented modifications are documented by several examples for simple theoretical system and for more sophisticated robotic system.

## 2 TASK FORMULATION

The information on desired signals can be defined by user in different ways according to given production technology, where the controlled system is used. As was mentioned, in this paper, there are investigated two specific tasks. The first is characterized by known permitted ranges (limits) of reference signal and the second task is determined only by defined end point. In general, these tasks represent control task with specific constraints on system output. The following subsections give the detail formulation of individual tasks from control design point of view.

## 2.1 Reference signal represented by permitted ranges

The task with the reference signal represented by permitted ranges can be defined as a controlled process: change or movement of controlled system through defined free ranges (see Figure 1).

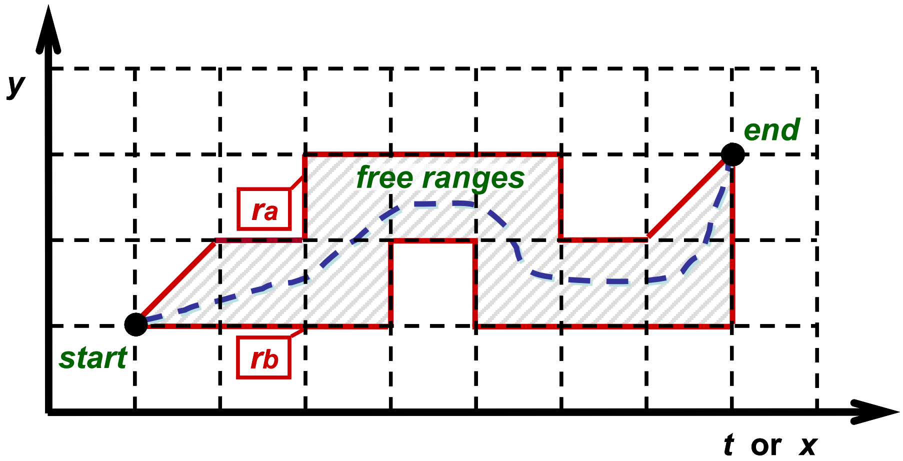



Figure 1 - Free ranges determining possible system change or movement

The system begins from start point and proceeds by suitable direction to the end point or to a new required output range taking into account local output ranges limiting the control process between start and end point. In Figure 1, the horizontal axis can represent either time axis for one system output or for some of system outputs in case of multi-output systems. In the former case, the Figure 1 represents time history of considered system output, and in the latter case, the Figure 1 represents dependence of some system output on other one.

## 2.2 Reference signal represented only by end-point

In this task, the reference signal is not defined in the right sense of the word, but only its direction is determined by position of the end point relative to start point, in which the system is located at the beginning of control process. In different way, the reference signal follows from topical system position and remaining distance from required end point. The task is illustrated in Figure 2.

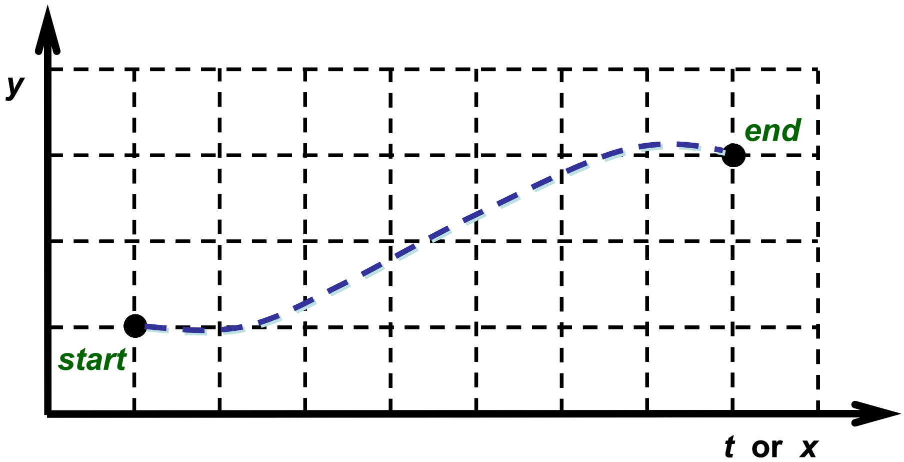



Figure 2 - Free ranges determining possible system change or movement

The Figure 2 as well as the Figure 1 can represent both time history of some system output and dependence of one system output on other one. The drawing of two outputs is advantageous, e.g. if individual outputs are Cartesian coordinates $x$ and $y$, then the figure represents usual $xy$-graph.

## 3 DEFINITION OF USED MODELS

The used models for mathematical description of controlled system are input-output autoregressive models with external input (ARX models) [Bobál et al., 2005]

$$y(k)=\sum_{i=1}^{n} b_i u(k-i)-\sum_{i=1}^{n} a_i y(k-i) \tag{1}$$

and state-space models

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\,\mathbf{x}(k)+\mathbf{B}\,u(k) \\ y(k) &= \mathbf{C}\,\mathbf{x}(k) \end{aligned} \tag{2}$$

where $\mathbf{x}(k)=\begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n+1) \\ y(k) \\ \vdots \\ y(k-n+1) \end{bmatrix}$, $\mathbf{A}=\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ b_2 & \cdots & b_n & -a_1 & \cdots & -a_n \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 1 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ 0 \end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \end{bmatrix}$

in case of pseudo state-space models, i.e. state-space modes with nonminimal state

or $\mathbf{x}(k)=\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}$, $\mathbf{A}=\begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$

in case of usual state-space models (general or in some canonical form), where the state $\mathbf{x}(k)$ is assumed to be known and available for control design. The internal structures of matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ indicated here have a Frobeniuns' transposed canonical form.

Due to digital character of automating devices, the discrete control techniques are preferred. Therefore, the models used in control design are considered to be discrete in spite of the facts that controlled systems are continuous. Discrete realization is useful, because naturally respects finite and predefined time for computation of control actions.

## 4 PRINCIPLE OF PREDICTIVE CONTROL AND ITS MODIFICATIONS

Predictive control is a multi-step control based on equations of predictions and the local minimization of quadratic criterion [Ordys et al., 1993]

$$J_k=\sum_{j=1}^{N}\left\{\|(\hat{y}(k+j)-w(k+j))\,Q_y\|^2+\|(u(k+j-1)\,Q_u\|^2\right\} \tag{3}$$

where $N$ is a horizon of predictions; $Q_y$ and $Q_u$ are penalizations – tuning parameters; and $w(k+j)$ are desired values. Obtained control actions $u(k),\cdots,u(k+N-1)$ from minimization are the control actions for whole horizon $N$. However, only the first appropriate action $u(k)$ is really used. This process is repeated in every time step for appropriately updated model.

The equations of predictions serve for the expression of feed-forward within the horizon $N$. Using discrete state-space form (2) (even for initial ARX model (1), transformed to it), the equations have the following basic form:

$$\begin{aligned}
\hat{\mathbf{x}}(k+1) &= \mathbf{A}\ \mathbf{x}(k) + \mathbf{B}u(k) \\
\hat{y}(k+1) &= \mathbf{C}\mathbf{A}\ \mathbf{x}(k) + \mathbf{C}\mathbf{B}u(k) \\
&\vdots \\
\hat{\mathbf{x}}(k+N) &= \mathbf{A}^N\mathbf{x}(k) + \mathbf{A}^{N-1}\mathbf{B}u(k) + \dots + \mathbf{B}u(k+N-1) \\
\hat{y}(k+N) &= \mathbf{C}\mathbf{A}^N\mathbf{x}(k) + \mathbf{C}\mathbf{A}^{N-1}\mathbf{B}u(k) + \dots + \mathbf{C}\mathbf{B}u(k+N-1)
\end{aligned} \tag{4}$$

The system (4) can be expressed in matrix notation

$$\hat{\mathbf{y}} = \mathbf{f} + \mathbf{G}\,\mathbf{u}, \quad \text{where} \quad \mathbf{f} = \begin{bmatrix} \mathbf{CA} \\ \vdots \\ \mathbf{CA}^N \end{bmatrix} \mathbf{x}(k) \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} \mathbf{C}\quad\mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{C}\mathbf{A}^{N-1}\mathbf{B} & \cdots & \mathbf{CB} \end{bmatrix} \tag{5}$$

The equations are composed by repetitive substitution of unknown future states for states determined from one topical state and state-space model (2). The state-space model is considered to be constant for composition of the equations for considered time step and corresponding horizon $N$.

### 4.1 Range-space modification

To design adequate control actions via range-space modification [Belda et al., 2006], it is necessary to make the following steps:

- define free ranges splitting the system workspace or its topical working domain;
- select suitable penalty matrices balancing of importance of the ranges.

Then, it is possible to minimize modified quadratic criterion, which provides mentioned assignment. The predictive control is suitable to be used, because it includes system dynamics and predictions within defined horizon. By this, the optimal distribution of input energy can be provided. The quadratic criterion is modified relating to new required ranges, which substitute usual desired values

$$J_k = \sum_{j=1}^{N} \left\{ \| (\hat{y}(k+j) - r_a(k+j))\, Q_{ra} \|^2 + \| (\hat{y}(k+j) - r_b(k+j))\, Q_{rb} \|^2 + \| (u(k+j-1) Q_u \|^2 \right\} \tag{6}$$

In the criterion, there occur two new terms $r_a(\cdot)$ and $r_b(\cdot)$, which correspond to differences from the limits of free (permissible) ranges. Furthermore, there are also two new output penalizations $Q_{ra}$ and $Q_{rb}$. In usual control process with standard criterion, the only one appropriate penalization $Q_y$ is selected as identity matrix. However, the new penalizations need not be constant and equal identity matrices. This criterion can be expressed for square-root minimization [Lawson et al., 1974; Golub et al. 1989], which is more suitable way of minimization by the following matrix form

$$J_k = \mathbf{J}\,\mathbf{J} = \|\mathbf{J}\|^2 = \left\| \begin{bmatrix} \mathbf{Q}_{ra} & \cdots & \mathbf{0} \\ \vdots & \mathbf{Q}_{rb} & \vdots \\ \mathbf{0} & \cdots & \mathbf{Q}_{\mathbf{u}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{y}} - \mathbf{r}_a \\ \hat{\mathbf{y}} - \mathbf{r}_b \\ \mathbf{u} \end{bmatrix} \right\|^2 \tag{7}$$

The minimization of described criterion generates the control actions as usual, only matrices in it have different types. The equations of predictions stay the same.

### 4.2 End-point modification

The task of the reference signal given only by end-point can be formulated in that way: "Let two points (start and end point) and presumptive time be given. In this time, the system should move between those points. A path is free of hard constraints; only end-point should be achieved".

Predictive design can be used again. As was mentioned, it partly consists in the composition of the equations of predictions, which involve the system dynamics and partly consists in minimization of quadratic criterion (3). It can be reshaped in different way.

$$J_k = \sum_{j=No+1}^{N} \| (\hat{y}(k+j) - w(k+j))\, Q_y \|^2 + \sum_{j=1}^{Nu} \| (u(k+j-1)\, Q_u \|^2 \tag{8}$$

The criterion includes more adjustable parameters: horizon of prediction $N$, horizon of initial insensitivity $No$ and control horizon $Nu$, penalizations $Q_y$ and $Q_u$, and also the desired values $w(k+j)$, which determine the transition from start to end point in the criterion.

The values $w(k+j)$ considered here are included only for completeness of the criterion. In case of end-point modification, the value vector $\mathbf{w}$ corresponds to end state, respectively, to the end position, i.e. $\mathbf{w} = [w(k+j), \cdots w(k+N)] \,|\, w(k+j) = const., \ j = No+1, \cdots, N$.

To distribute the input energy in whole time interval determined by time $T$, i.e. to design the trajectory with suitably distributed energy, the described parameters of the criterion are used.

If horizons $N$ and $No$ are set as

$$N = N\,max = T / Ts\,, \qquad No = N - n \tag{9}$$

where $Ts$ is a suitably selected sampling, $n$ is an order of the system, then the quadratic criterion will consider only last $n$ differences among predicted end-point and its reference value. The horizon $No$ representing initial insensitivity determines number of free outputs, i.e. outputs without penalization, which enable the algorithm to shift the reaching the end point at later time (step), at time $T$, with appropriate distribution of input energy. Thus, the criterion (8) represented in square-root matrix form is expressed as follows

$$\mathbf{J} = \begin{bmatrix} \mathbf{Q_y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q_u} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{y}} - \mathbf{w} \\ \mathbf{u} \end{bmatrix} \text{ with substituting } \hat{\mathbf{y}} = \mathbf{f} + \mathbf{G}\,\mathbf{u} \ \Rightarrow \ \mathbf{J} = \begin{bmatrix} \mathbf{Q_y} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q_u} \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{w} - \mathbf{f} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ -\mathbf{I} \end{bmatrix} \tag{10}$$

where the matrix $\mathbf{G}$ and vector of differences $(\mathbf{w} - \mathbf{f})$ contain due to horizon $No$ only last $n$ rows

$$\left. \left[\begin{array}{c|c} \mathbf{Q}_y & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_u \end{array}\right] \left[\begin{array}{c|c} \mathbf{G} & \mathbf{w} - \mathbf{f} \\ \hline \mathbf{1} & \mathbf{0} \end{array}\right] \right|_{\mathbf{Q}_y = \mathbf{I}} = \left[\begin{array}{cccc|c} \mathbf{CA}^{No}\,\mathbf{B} & \cdots\ \mathbf{CB} & \cdots & \mathbf{0} & w(k+No+1) - \mathbf{CA}^{No+1}\,\mathbf{x}(k) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{CA}^{N-1}\,\mathbf{B} & \mathbf{CA}^{N-2}\mathbf{B} & \cdots & \mathbf{CB} & w(k+N) - \mathbf{CA}^{N}\,\mathbf{x}(k) \\ \hline Q_u & \cdots & & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & & Q_u & 0 \end{array}\right] \tag{11}$$

In case of nonlinear systems, the model parameters have to be changed according to current state with simultaneous progressively shortened horizon $N$ according to topical time instant $k$

$$N := Nmax,\ Nmax\text{ - }1, \cdots, Nmin+1,\ Nmin, \quad Nmin > n \tag{12}$$

The shortening provides that the time limit $T$ is not overrun.

## 5 EXAMPLES

The examples included in this section demonstrate theoretical results from section 4 applied both to simple linear single-input single-output system of second order given by differential equation

$$\begin{aligned} &\ddot{y}(t) + 2\dot{y}(t) + y(t) = u(t) \quad \text{discretized for} \quad Ts = 0.1s: \\ &y(k) = 0.0047\,u(k-1) + 0.0044\,u(k-2) + 1.8097\,y(k-1) - 0.8187\,y(k-2) \end{aligned} \tag{13}$$

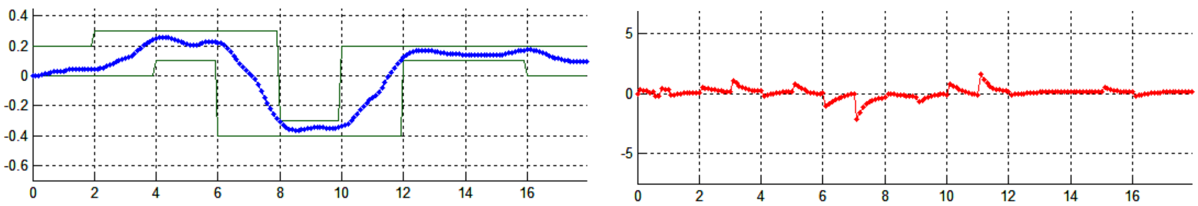
and to sophisticated nonlinear multi-input multi-output system – planar parallel robot described by nonlinear system of differential equations which are transformed into state-space form, linearized according to topical state and finally discretized as follows [Belda et al., 2005]

$$\ddot{\mathbf{y}} = \mathbf{f}(\mathbf{y},\dot{\mathbf{y}}) + \mathbf{g}(\mathbf{y})\mathbf{u} \;\Rightarrow\; \begin{matrix} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \\ \mathbf{y} = \mathbf{H}\,\mathbf{x} \end{matrix} \;\Rightarrow\; \begin{matrix} \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})\,\mathbf{x} + \mathbf{G}(\mathbf{x})\mathbf{u} \\ \mathbf{y} = \;\mathbf{H}\;\;\mathbf{x} \end{matrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix} \tag{14}$$

$$\Rightarrow \begin{matrix} \mathbf{x}(k+1) = \mathbf{A}(k)\,\mathbf{x}(k) + \mathbf{B}(k)\,\mathbf{u}(k) \\ \mathbf{y}(k) \;=\; \mathbf{C} \;\; \mathbf{x}(k) \end{matrix}, \quad \mathbf{x}(k) = \begin{bmatrix} \mathbf{y}(k) \\ \dot{\mathbf{y}}(k) \end{bmatrix} \tag{15}$$

### 5.1 Range-space modification

In Figure 3, there is an example for free ranges indicated by upper and lower angulate lines (Figure 3a) and appropriate control actions (Figure 3b). The ranges are defined by ordered pair of corresponding limits repeating in predefined required time interval – the time interval in Figure 3 is two seconds and its multiples.




(a) system output (b) system input (control actions)

Figure 3 - Example of range-space modification for single-input single output system (time histories)

The following Figure 4 represents $x$ and $y$ coordinates of the robot end-effector. In this case, the required ranges are defined for each coordinate separately, i.e. altogether four limits as against two limits for previous single-input single-output system.

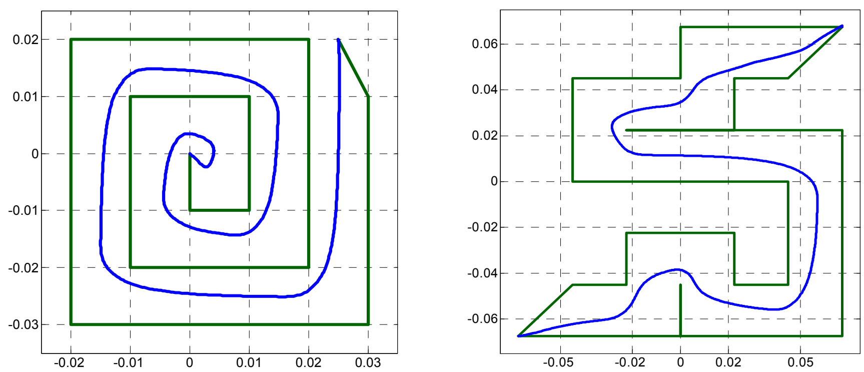
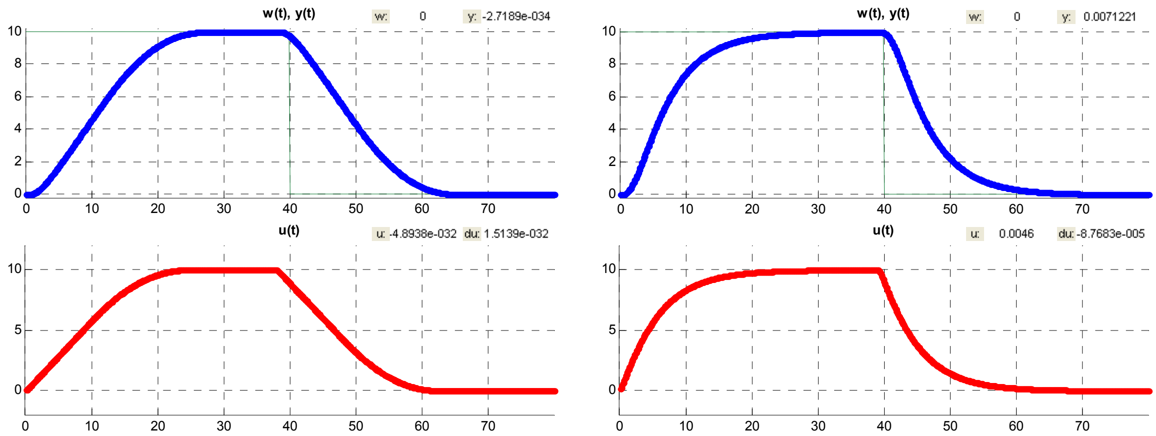



Figure 4 - Examples of range-space modification for 2D robotic application ( $xy$ -graphs)

### 5.2 End-point modification

The first figure – Figure 5 shows optimal distributions of input energy by control actions for required end point defined by pair of time and desired value of output $[t_1, w_1] = [35s, 10\, units]$ and $[t_2, w_2] = [70s, 0\, units]$. It can be understood as distribution of the energy for high-step change. There is a comparison with different solution via setting of higher values of penalization. However, the result seems to be the same, but the comparative case of higher penalization leads to slower system stabilization – slower reaching accurate level of the required end point. On the contrary, the considered modification has gradual increase of the control actions leading to accurate reaching of end level.




(a) end-point modification (b) comparative case with higher $Q_u$

Figure 5 - Comparison of end-point modification with different solution by higher penalization $Q_u$ for simple single-input single output system of second order (time histories)

The following Figure 6 represents, as well as Figure 4, $x$ and $y$ coordinates of the robot end-effector (continuous line). The dashed line indicates the direction from start to end point. The deflection of the end-effector from this expected direct direction is caused by presence of the gravitation affecting to the vertically configured robot structure.

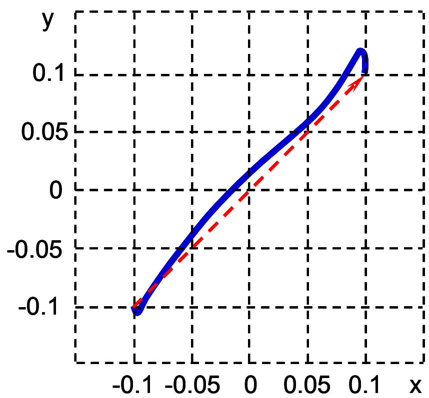



Figure 6 - Example of end-point modification for 2D (vertical) robotic application ($xy$ - graph)

### 6 CONCLUSION

The paper investigates two modifications of predictive control, which solve task of approximately given reference signal, either defined by limiting ranges or given only by requirement of reaching some specific end-point. Predictive control itself offers wide numbers of possibilities and modifications, and investigated modifications here belong to them.

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