STABILITY OF SSD EFFICIENCY - MONTHLY VERSUS YEARLY RETURNS

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Abstract. This paper deals with second-order stochastic dominance portfolio efficiency. In existing portfolio efficiency tests with respect to the second-order stochastic dominance (SSD) criterion, the scenario approach for random returns is assumed. We analyse the stability of SSD portfolio efficient classification with respect to two possible set of historical scenarios: monthly returns and yearly returns. In both cases, 20 years history is considered. For both sets of scenarios we test SSD efficiency of almost one hundred thousand portfolios that can be formed from ten US industry representative portfolios. For each portfolio, we compare the monthly returns results with yearly return results.
Keywords. Second-order stochastic dominance, portfolio inefficiency, scenario approach.

1 Introduction

When solving portfolio selection problem several approaches can be used: mean-risk models, maximising expected utility problems, stochastic dominance criteria, etc. If the information about the risk attitude of a decision maker is not perfectly known one may adopt stochastic dominance approach to test an efficiency of a given portfolio with respect to considered set of utility functions. The second-order stochastic dominance is the most common stochastic dominance relations because of its risk aversion interpretation and relation to Conditional Value-at-Risk, see e.g. Ogryczak and Ruszczynski (2002), Levy (2006) or Kopa and Chovanec (2008).

In the context of portfolio selection problem, Post (2003) and Kuosmanen (2004) develop linear programming tests for testing if a given portfolio is SSD efficient relative to all possible portfolios formed from a set of assets. Using the formulation in terms of concave utility functions and the first-order condition for portfolio optimization, Post derives a computationally efficient LP test. A limitation of this test is that it focuses exclusively on the efficiency classification of the evaluated portfolio and gives minimal information about directions for improved allocation if the portfolio is SSD inefficient. Moreover the Post test is derived for strict SSD portfolio efficiency instead of general SSD portfolio efficiency as it is defined in Kuosmanen (2004) and Ruszczynski and Vanderbei (2003). Using the formulation in terms of second quantile functions, Kuosmanen derives a test that identifies another, SSD efficient portfolio that dominates the evaluated portfolio (if the latter is inefficient). This test involves solving two linear problems; one for a necessary condition and one for a sufficient condition. Unfortunately, the problem for the sufficient condition is large and introduces substantial additional computational burden. And therefore Kopa and Chovanec (2008) derived a new linear programming test. This test is approximately 6-times faster than the Kuosmanen test and identifies another, SSD efficient portfolio that dominates the evaluated portfolio, too.

In all these SSD portfolio efficiency tests, a scenario approach is assumed, that is, asset returns have a discrete probabilistic distribution. To analyse the stability of SSD portfolio efficiency classification with respect to changes in scenarios, Kopa (2009) suggested subsampling methods. Using bootstrap techniques SSD inefficiency of the US market portfolio with high confidence level was shown.

In this paper we compare SSD portfolio efficiency based on monthly returns (240 scenarios) with that based on yearly returns (20 scenarios). We apply the test derived in Kopa and Chovanec (2008) for almost one hundred thousand portfolios that can be formed from ten US industry representative portfolios. For each portfolio, we compare the results.

The reminder of this text is structured as follows. Section 2 introduces notations and basic definitions. In Section 3, a test for testing SSD portfolio efficiency is recalled. Section 4 presents an empirical application to compare the SSD portfolio efficiency classification for both considered sets of scenarios. Finally, Section 5 summaries the results and discuss the ideas for future research.

2 Preliminaries

Consider a random vector $\mathbf{r} = (r_1, r_2, ..., r_N)'$ of returns of N assets and T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \ldots, x_N^t)$ is the t-th row of matrix X. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_N)'$ for a vector of portfolio weights and the portfolio possibilities are given by

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \ \lambda_n \geq 0, \ \ n = 1, 2, \ldots, N\}.$$

The tested portfolio is denoted by $\boldsymbol{\tau} = (\tau_1, \tau_2, ..., \tau_N)'$. Following Ruszczynski and Vanderbei (2003), Kuosmanen (2004), Kopa and Chovanec (2008), we define second-order stochastic dominance relation in a strict form. Let $F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$. The *twice cumulative probability distribution function* of returns of portfolio $\boldsymbol{\lambda}$ is given by:

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(t) = \int_{-\infty}^{t} F_{\mathbf{r}'\boldsymbol{\lambda}}(x)dx \tag{1}$$

Definition 1: *Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by second-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$) if and only if*

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(t) \leq F^{(2)}_{\mathbf{r}'\boldsymbol{\tau}}(t) \qquad \forall t \in \mathbb{R}$$

with at least one strict inequality.

The equivalent definition, presented in e.g. Levy (2006) or Kopa and Chovanec (2008) is based on comparison of expected utility of portfolio returns:

$$\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau} \Longleftrightarrow \mathrm{E}u(\mathbf{r}'\boldsymbol{\lambda}) \geq \mathrm{E}u(\mathbf{r}'\boldsymbol{\tau})$$

for all concave utility functions u with strict inequality for at least some concave utility function.

Definition 2: *A given portfolio $\boldsymbol{\tau} \in \Lambda$ is SSD inefficient if and only if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ such that $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$. Otherwise, portfolio $\boldsymbol{\tau}$ is SSD efficient.*

This definition classifies portfolio $\boldsymbol{\tau} \in \Lambda$ as SSD efficient if and only if no other portfolio is better for all risk averse and risk neutral decision makers.

We follow Pflug (2000) in defining *conditional value-at-risk* (CVaR) for portfolio losses ($-\mathbf{r}'\boldsymbol{\lambda}$).

Definition 3: *Let $\alpha \in \langle 0, 1\rangle$. Conditional value-at-risk of portfolio $\boldsymbol{\lambda} \in \Lambda$ at level α is the optimal value of objective function of the following optimization problem:*

$$\mathrm{CVaR}_\alpha(\boldsymbol{\lambda}) = \min_{a \in \mathbb{R}} \{a + \frac{1}{1-\alpha} \mathbb{E}[-\mathbf{r}'\boldsymbol{\lambda} - a]^+\}$$

where $[x]^+ = \max(x, 0)$.

It was shown in Uryasev and Rockafellar (2002) that the $\mathrm{CVaR}_\alpha(\boldsymbol{\lambda})$ can be also defined as the conditional expectation of $-\mathbf{r}'\boldsymbol{\lambda}$, given that $-\mathbf{r}'\boldsymbol{\lambda} > F^{(-1)}_{-\mathbf{r}'\boldsymbol{\lambda}}(\alpha)$, i.e.

$$\mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) = \mathbb{E}(-\mathbf{r}'\boldsymbol{\lambda} | -\mathbf{r}'\boldsymbol{\lambda} > F^{(-1)}_{-\mathbf{r}'\boldsymbol{\lambda}}(\alpha)),$$

where

$$F^{(-1)}_{-\mathbf{r}'\boldsymbol{\lambda}}(\alpha) = \min\{u : F_{-\mathbf{r}'\boldsymbol{\lambda}}(u) \geq \alpha\}.$$

Since we apply scenario approach, following Rockafellar and Uryasev (2002) and Pflug (2000), it can be rewritten as a linear programming problem:

$$\begin{aligned} \mathrm{CVaR}_\alpha(\boldsymbol{\lambda}) = \min_{a,w_t} \quad & a + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T} w_t \\ s.t. \quad & w_t \geq -\mathbf{x}^t\boldsymbol{\lambda} - a \\ & w_t \geq 0. \end{aligned} \tag{2}$$

There exists a Fenchel duality connection between SSD relation and CVaR proved in Ogryczak and Ruszczynski (2002) and Kopa and Chovanec (2008).

Theorem 1: *Let $\boldsymbol{\lambda}, \boldsymbol{\tau} \in \Lambda$. Then*

$$\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau} \quad \Leftrightarrow \quad \mathrm{CVaR}_\alpha(\boldsymbol{\lambda}) \leq \mathrm{CVaR}_\alpha(\boldsymbol{\tau}) \qquad \forall \alpha \in \{0, \frac{1}{T}, \frac{2}{T}, \ldots, \frac{T-1}{T}\}.$$

with strict inequality for at least some $\alpha \in \{0, \frac{1}{T}, \frac{2}{T}, \ldots, \frac{T-1}{T}\}$.

3 SSD portfolio efficiency test

In this section we present the SSD portfolio efficiency linear programming test in the form of a necessary and sufficient condition derived in Kopa and Chovanec (2008). Let

$$D^*(\boldsymbol{\tau}) = \max_{D_k, \lambda_n, b_k, w_k^t} \sum_{k=1}^{T} D_k \tag{3}$$

$$\begin{aligned} \text{s.t.} \quad \mathrm{CVaR}_{\frac{k-1}{T}}(\boldsymbol{\tau}) - b_k - \tfrac{1}{(1-\frac{k-1}{T})T}\textstyle\sum_{t=1}^{T} w_k^t &\geq D_k, & k = 1,2,\ldots,T \\ w_k^t &\geq -\mathbf{x}^t\boldsymbol{\lambda} - b_k, & t,k = 1,2,\ldots,T \\ w_k^t &\geq 0, & t,k = 1,2,\ldots,T \\ D_k &\geq 0, & k = 1,2,\ldots,T \\ \boldsymbol{\lambda} &\in \Lambda. \end{aligned}$$

Theorem 2: *Let $D^*(\boldsymbol{\tau})$ be given by (3). If $D^*(\boldsymbol{\tau}) > 0$ then $\boldsymbol{\tau}$ is SSD inefficient and $\mathbf{r}'\boldsymbol{\lambda}^* \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$. Otherwise, $D^*(\boldsymbol{\tau}) = 0$ and $\boldsymbol{\tau}$ is SSD efficient.*

The alternative SSD portfolio efficiency tests were suggested by Post (2003) and Kuosmanen (2004). The advantages and disadvantages of these three tests were discussed in Kopa and Chovanec (2008).

4 Empirical application

We consider ten US industry portfolios which represents our basic assets. Using a regular grid on set Λ with step size 0.1, we create 92378 portfolios from these assets. In general the number of portfolios from the regular grid with step size s is given by formula:

$$\prod_{i=1}^{N-1}(1 + \frac{1}{si})$$

where N is the number of assets. For each of these portfolios we apply the SSD portfolio efficiency test, solving (3), for 240 monthly excess returns scenarios and then for 20 yearly excess returns scenarios. We consider historical scenarios from September 1987 to August 2007. Our aim is to compare the set of SSD efficient portfolios using monthly excess returns with that using yearly excess returns.

Using monthly excess returns, 94.1% portfolios from the grid were classified as SSD inefficient and 5.9% as SSD efficient. It means that SSD criteria dramatically reduced the set of all possible portfolios and all risk averse investors will chose their optimal allocations from this reduced set.

		Monthly returns	
		SSD efficient portfolios	SSD inefficient portfolios
Yearly returns	SSD efficient portfolios	1.1	1.4
	SSD inefficient portfolios	4.8	92.7

Table 1. Percentage comparison of SSD portfolio efficiency sets - monthly versus yearly excess returns.

	Monthly returns	Yearly returns
SSD efficient portfolios	19	44
SSD inefficient portfolios	99	95

Table 2. Relative percentage levels of results matching.

Using yearly excess returns, 97.5% portfolios from the grid were classified as SSD inefficient and 2.5% as SSD efficient. Again, we can see large portfolio set reduction.

The following table presents the full comparisons of these two cases.

From Table 1, we can conclude that 93.8% portfolios were equally classified in both cases. This high level of coinciding is probably caused by quite large number of SSD inefficient portfolios. If we limit our attention to SSD efficient portfolios then we can see that only 19% of SSD efficient portfolios using monthly scenarios is equally classified in the case of yearly returns. Similarly, 44% of SSD efficient portfolios using yearly scenarios is equally classified in the case of monthly returns. The same analysis can be done for comparing SSD inefficient portfolios. Table 2 summarizes these relative levels of SSD portfolio efficiency results matching. All computations were done in software GAMS 22.8 using solver CPLEX.

5 Summary and concluding remarks

In this study we analyzed the impact of chosen historical data frequency on SSD portfolio efficiency classification. We compared two cases: monthly returns versus yearly returns, both for 20 years history. We constructed almost one hundred thousand portfolios from ten US representative industry portfolios. We applied SSD portfolio efficiency test derived in Kopa and Chovanec (2008) for all portfolios and for both returns cases. Comparing the results, we concluded that 93.8 % portfolios were equally classified in both cases.

To improve the quality of these comparisons more historical scenarios can be used. However, it will increase the computational requests. In addition, including quarterly returns can make this analysis more complex.

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