From Probabilities to Belief Functions on MV-Algebras

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Abstract In this contribution we generalize belief functions to many-valued events represented by elements of the finite product of standard MV-algebras. Our definition is based on the mass assignment approach from Dempster-Shafer theory of evidence. The generalized belief function is totally monotone and it has Choquet integral representation w.r.t. a classical belief function.

Key words: Belief function, State, MV-algebra, Algebra of fuzzy sets.

1 Introduction

A main aim of this paper is to study belief functions in the more general setting than Boolean algebras of events. This effort is in the line with a growing interest in the generalization of classical probability towards "many-valued" events, such as those resulting from formulas in Łukasiewicz infinite-valued logic. An algebra of such many-valued events is called an MV-algebra (Definition 1). The counterpart of a probability on a Boolean algebra is a so-called state on an MV-algebra—see [10, 14, 11] for a detailed discussion of probability on MV-algebras including its interpretation in terms of bookmaking over many-valued events. The recent articles [7, 6, 9] focus on more general functionals on MV-algebras: namely, upper (lower) probabilities and possibility (necessity) measures. The presented paper is thus an attempt to fill the gap in the classification of uncertainty measures on MV-algebras.

Section 2 contains basic definitions related to MV-algebras and totally monotone functions. In Section 3 we will recall the notion of state and the integral representation of states (Theorem 1). The states on an MV-algebra of certain set functions will be of particular interest (Example 4). Section 4

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is devoted to belief functions. We will restrict their discussion only to the MV-algebra of all $[0,1]$-valued functions on a finite set. The case of belief functions on a general MV-algebra is left aside for future investigations since it involves intricate mathematical tools such as topologies on spaces of closed sets. Hence we develop a many-valued generalization of the usual notion of belief function in the finite setting [15]. In particular, Definition 3 of a belief function in the many-valued framework is based on a natural generalization of the notion of mass assignment. The properties of such belief functions are explored through Choquet integral representation (Proposition 1). This representation implies total monotonicity of belief functions and some other properties (Proposition 2). Finally, we give a complete description of the convex set of all belief functions by finding its extreme points (Proposition 3). Some proofs are omitted due to the lack of space.

2 Preliminaries

If X is any set, then $\mathscr{P}(X)$ denotes the set of all subsets of X. Put $\mathscr{P}_{\emptyset}(X) = \mathscr{P}(X) \setminus \{\emptyset\}$. For any $A \in \mathscr{P}(X)$, the *characteristic function* of the set A is given by $1_A(x) = 1$, if $x \in A$, and $1_A(x) = 0$, otherwise, for every $x \in X$.

A *fuzzy set* on X is a function $X \to [0,1]$. A set of fuzzy sets on X can be endowed with an algebraic structure to introduce the union, intersection, and other possible operations with fuzzy sets. This stream of research is based on so-called tribes of fuzzy sets parameterized by a t-norm [1]—see [12] for the latest exposition. Another motivation for investigating algebras of fuzzy sets stems from mathematical fuzzy logics. In this contribution we will tacitly confine to Łukasiewicz infinite-valued logic whose associated algebras of truth values are so-called MV-algebras—see [3] for their in-depth study. MV-algebras play the same role in Łukasiewicz logic as Boolean algebras in the classical two-valued logic.

Definition 1. An *MV-algebra* is an algebra $\langle M, \oplus, \neg, 0\rangle$ with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0 such that $\langle M, \oplus, 0\rangle$ is an abelian monoid and the following equations hold true for every $f, g \in M$: $\neg\neg f = f$, $f \oplus \neg 0 = \neg 0$, $\neg(\neg f \oplus g) \oplus g = \neg(\neg g \oplus f) \oplus f$.

On every MV-algebra M we define $1 = \neg 0$, $f \odot g = \neg(\neg f \oplus \neg g)$. For any two elements $f, g \in M$ we write $f \leq g$ if $\neg f \oplus g = 1$. The relation $\leq$ is in fact a partial order. Further, the operations $\vee, \wedge$ defined by $f \vee g = \neg(\neg f \oplus g) \oplus g$ and $f \wedge g = \neg(\neg f \vee \neg g)$, respectively, make the algebraic structure $\langle M, \wedge, \vee, 0, 1\rangle$ into a distributive lattice with bottom element 0 and top element 1.

Example 1. The most important example of an MV-algebra is the *standard MV-algebra,* which is the real unit interval $[0,1]$ equipped with operations $f \oplus g = \min(1, f+g)$ and $\neg f = 1 - f$. Note that we have $f \odot g = \max(0, f+g-1)$. The operations $\odot, \oplus$ are also known under the names *Łukasiewicz t-norm* and

Łukasiewicz t-conorm, respectively. The partial order $\leq$ on the MV-algebra $[0,1]$ coincides with the usual order of reals.

Example 2. More generally, the set $[0,1]^X$ of all fuzzy sets on a set X becomes an MV-algebra if the operations $\oplus$ and $\neg$ and the element 0 are defined pointwise. The corresponding lattice operations $\vee, \wedge$ are then the pointwise maximum and the pointwise minimum of two real functions, respectively.

Example 3. MV-algebras generalize Boolean algebras in the following sense. Every (Boolean) algebra of sets is an MV-algebra in which $\oplus$ coincides with $\vee$ and $\odot$ coincides with $\wedge$, where $\vee$ and $\wedge$ is the union and the intersection of two sets, respectively. The operation $\neg$ becomes the complement of a set.

We say that an MV-algebra is *semisimple* if it is (isomorphic to) an MV-algebra of continuous functions $[0,1]$ defined on some compact Hausdorff space. In particular, all the MV-algebras from Examples 1–3 are semisimple. Semisimple MV-algebras can be viewed as many-valued counterparts of algebras of sets.

Throughout the remainder we deal with real functions on an MV-algebra whose successive differences of all orders are nonnegative. This property (so-called total monotonicity) of real functions was studied already by Choquet in his foundational work about capacities [2]. Total monotonicity is the common property of belief functions studied in different settings such as a finite algebra of sets [15], any algebra of sets [16] or Borel σ-algebra of the real line [4]. We consider the difference operator with respect to the lattice operations of an MV-algebra M. This leads to the following definition. Let $b : M \to \mathbb{R}$ and put $\Delta_g b(f) = b(f) - b(f \wedge g)$, for every $f, g \in M$.

Definition 2. A function $b : M \to \mathbb{R}$ is *totally monotone* if

$$\Delta_{g_n} \cdots \Delta_{g_1} b(f) \geq 0, \quad \text{for every } n \geq 1 \text{ and every } f, g_1, \ldots, g_n \in M.$$

It is possible to show that b is totally monotone if and only if

(i) $b(f) \leq b(g)$ whenever $f \leq g$, for every $f, g \in M$,
(ii) for each $n \geq 2$ and every $f_1, \ldots, f_n \in M$:

$$b\left(\bigvee_{i=1}^{n} f_i\right) \geq \sum_{\substack{I \subseteq \{1,\ldots n\} \\ I \neq \emptyset}} (-1)^{|I|+1}\, b\left(\bigwedge_{i \in I} f_i\right).$$

3 Probabilities on MV-Algebras

A *state* on an MV-algebra M is a mapping $s : M \to [0,1]$ such that $s(1) = 1$ and $s(f \oplus g) = s(f) + s(g)$, for every $f, g \in M$ with $f \odot g = 0$. In case that M

is an algebra of sets, then the notion of state agrees with that of finitely additive probability measure. Properties of states are best analyzed through their correspondence to Borel probability measures: it turns out that every state on a semisimple MV-algebra is integral—see [8] or [13].

Theorem 1. *If s is a state on a semisimple MV-algebra M, then there exists a uniquely determined Borel probability measure μ on the compact Hausdorff space X such that $s(f) = \int f \, \mathrm{d}\mu$, for each $f \in M$.*

It is possible to show by using linearity of Lebesgue integral that s is a totally monotone function on M.

The following example is crucial for the investigation of belief functions in the next section. We will introduce an MV-algebra whose elements are set functions and single out a particular class of states for later use.

Example 4. Let X be a finite nonempty set. Consider the MV-algebra $[0,1]^{\mathscr{P}(X)}$ of all functions $\mathscr{P}(X) \to [0,1]$. We will deal only with those states s on $[0,1]^{\mathscr{P}(X)}$ for which $s(1_{\{\emptyset\}}) = 0$. Theorem 1 says that each such state s corresponds to a unique finitely additive probability μ on $\mathscr{P}(\mathscr{P}(X))$ satisfying $s(q) = \sum_{A \in \mathscr{P}(X)} q(A)\mu(\{A\})$ and $\mu(\{\emptyset\}) = 0$, for every $q \in [0,1]^{\mathscr{P}(X)}$. The set $\mathbf{S}$ of all states s on $[0,1]^{\mathscr{P}(X)}$ with $s(1_{\{\emptyset\}}) = 0$ can be identified with a convex subset of the $(2^{|X|} - 1)$-dimensional Euclidean space. Since the correspondence between $\mathbf{S}$ and the set of all probabilities μ on $\mathscr{P}(\mathscr{P}(X))$ with $\mu(\{\emptyset\}) = 0$ is a one-to-one affine mapping, the convex set $\mathbf{S}$ is in fact a $(2^{|X|} - 2)$-simplex. The extreme points of $\mathbf{S}$ are in one-to-one correspondence with the nonempty subsets of X: every state s_A, $A \in \mathscr{P}_{\emptyset}(X)$ such that $s_A(q) = q(A)$, for each $q \in [0,1]^{\mathscr{P}(X)}$, is an extreme point of $\mathbf{S}$. This characterization of state space and its extreme points is a consequence of the description of state space of any MV-algebra—see [10] or [8].

4 BFs on Finite Product of Standard MV-Algebras

The domain of belief functions introduced in this section is limited to those MV-algebras $[0,1]^X$ with X finite. Each such MV-algebra is in algebraic terms just a *finite product* of standard MV-algebras.

We will repeat basic definitions of Dempster-Shafer theory of belief functions [15]. Let X be a finite nonempty set. We say that a function $\beta : \mathscr{P}(X) \to [0,1]$ is a *belief function on* $\mathscr{P}(X)$ if there is a mapping $m : \mathscr{P}(X) \to [0,1]$ with $m(\emptyset) = 0$ and $\sum_{A \in \mathscr{P}(X)} m(A) = 1$ such that $\beta(A) = \sum_{B \subseteq A} m(B)$, for every $A \in \mathscr{P}(X)$. The function m is usually called a *basic assignment.* Observe that an equivalent description of a belief function β is possible by a finitely additive probability $\mu : \mathscr{P}(\mathscr{P}(X)) \to [0,1]$ with $\mu(\{\emptyset\}) = 0$ and such that

$$\beta(A) = \mu(\{B \in \mathscr{P}(X) \mid B \subseteq A\}), \quad \text{for every } A \in \mathscr{P}(X). \tag{1}$$

Every belief function β on $\mathscr{P}(X)$ is totally monotone on the lattice $\mathscr{P}(X)$.

A point of departure for the generalization of the notion of belief function to an MV-algebra $[0,1]^X$ of all $[0,1]$-valued functions from the finite set X is the introduction of the following operator. Let the operator $\rho : [0,1]^X \to [0,1]^{\mathscr{P}(X)}$ be defined for every $f \in [0,1]^X$ as

$$\rho(f)(B) = \begin{cases} \min\{f(x) \mid x \in B\}, & B \in \mathscr{P}_\emptyset(X), \\ 1, & B = \emptyset. \end{cases}$$

Given $A, B \in \mathscr{P}(X)$, observe that $\rho(1_A)(B) = 1$ if and only if $B \subseteq A$. This means that $\rho(1_A)$ is the characteristic function of $\{B \in \mathscr{P}(X) \mid B \subseteq A\}$. Thus, we can rewrite (1) with a slight abuse of notation as

$$\beta(A) = \mu(\rho(1_A)), \quad \text{for every } A \in \mathscr{P}(X). \tag{2}$$

The preceding considerations lead naturally to the following definition of belief function.

Definition 3. Let X be a finite nonempty set. A mapping $b : [0,1]^X \to [0,1]$ is called a *belief function on* $[0,1]^X$ if there is a state on the MV-algebra $[0,1]^{\mathscr{P}(X)}$ such that $s(1_{\{\emptyset\}}) = 0$ and $b(f) = s(\rho(f))$, for every $f \in [0,1]^X$. The state s is called a *state assignment.*

We are going to generalize the integral representation theorem for states (Theorem 1) to belief functions. This requires introduction of Choquet integral [5]. Although we are integrating only the functions defined on the finite set X, we keep the integral notation to emphasize the analogy with Theorem 1 in this setting.

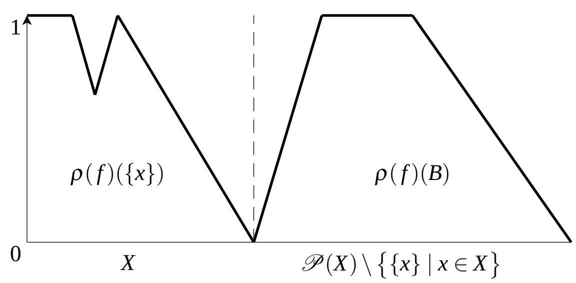


Fig. 1 Continuation of an element of the MV-algebra $[0,1]^X$ to $[0,1]^{\mathscr{P}(X)}$

If f is a function $X \to [0,1]$ and β is a set function $\mathscr{P}(X) \to [0,1]$ with $\beta(\emptyset) = 0$, then *Choquet integral* of f with respect to β is defined as $\oint f \, d\beta = \int_0^1 \beta\big(f^{-1}([t,1])\big) \, dt$. Since X is finite, the Choquet integral $\oint f \, d\beta$ exists and takes the form of a finite sum. Indeed, assume the set X has n elements

$x_1,\dots,x_n$ indexed in such a way that the numbers $y_i = f(x_i), i = 1,\dots,n$ satisfy $y_1 \geq \dots \geq y_n$. Put $y_{n+1} = 0$ and $S_i = \{x_1,\dots,x_i\}, i = 1,\dots,n$. Then $\oint f \, \mathrm{d}\beta = \sum_{i=1}^{n} (y_i - y_{i+1}) \beta(S_i)$.

Proposition 1. *For every belief function* b *on* $[0,1]^X$ *there exists a unique belief function* β *on* $\mathscr{P}(X)$ *such that* $b(f) = \oint f \, \mathrm{d}\beta,\ f \in [0,1]^X$.

Proof. Let s be the state assignment on $[0,1]^{\mathscr{P}(X)}$ corresponding to b. According to Example 4 there is a unique probability μ on $\mathscr{P}(\mathscr{P}(X))$ such that $s(q) = \sum_{A \in \mathscr{P}(X)} q(A) \mu(\{A\})$ and $\mu(\{\emptyset\}) = 0$, for every $q \in [0,1]^{\mathscr{P}(X)}$. This means that b can be expressed as

$$b(f) = s(\rho(f)) = \sum_{A \in \mathscr{P}(X)} \rho(f)(A) \mu(\{A\}). \tag{3}$$

For every $A \in \mathscr{P}_{\emptyset}(X)$ and $B \in \mathscr{P}(X)$, let $\varepsilon_A(B) = 1$, whenever $A \subseteq B$, and $\varepsilon_A(B) = 0$, otherwise. Then $\rho(f)(A) = \min\{f(x) \mid x \in A\} = \oint f \, \mathrm{d}\varepsilon_A$. The equality (3) together with linearity of Choquet integral with respect to the integrating set functions ε_A yield

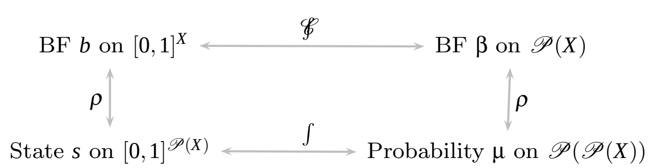
$$b(f) = \sum_{A \in \mathscr{P}_{\emptyset}(X)} \mu(\{A\}) \oint f \, \mathrm{d}\varepsilon_A = \oint f \, \mathrm{d} \left(\sum_{A \in \mathscr{P}_{\emptyset}(X)} \mu(\{A\}) \varepsilon_A \right).$$

It suffices to show that the function $\beta = \sum_{A \in \mathscr{P}_{\emptyset}(X)} \mu(\{A\}) \varepsilon_A$ is a belief function on $\mathscr{P}(X)$. For each $B \in \mathscr{P}(X)$,

$$\beta(B) = \sum_{A \in \mathscr{P}_{\emptyset}(X)} \mu(\{A\}) \varepsilon_A(B) = \sum_{A \subseteq B} \mu(\{A\}) = \mu(\{A \in \mathscr{P}(X) \mid A \subseteq B\}).$$

□

Fig. 2 The relation between belief functions (BF), states, and probabilities



The derived Choquet integral representation coincides with the definition of a belief function on "formulas" of Łukasiewicz logic proposed in [9]. Due to Proposition 1 the properties of belief functions on $[0,1]^X$ are completely determined by the properties of Choquet integral. These are the most important among them—see [5].

Proposition 2. *Let b be a belief function on $[0,1]^X$. Then b is totally monotone and for every $f,g \in [0,1]^X$:*

(i) $b(0) = 0, b(1) = 1$
(ii) if $f \odot g = 0$, *then* $b(f \oplus g) \geq b(f) + b(g)$
(iii) $b(f) + b(\neg f) \leq 1$
(iv) b *is a state if the state assignment* s *satisfies* $s(q) = 0$ *for each* $q \in [0,1]^{\mathscr{P}(X)}$ *such that* $q(A) > 0$ *for some* $A \in \mathscr{P}(X)$ *with* $|A| > 1$
(v) $b(f) = \min\{s(f) \mid s$ *state on* $[0,1]^X$ *with* $s \geq b\}$

The property (ii) is so-called *superadditivity*. The condition (iv) is a generalization of the analogous fact about belief functions on $\mathscr{P}(X)$: a belief function β on $\mathscr{P}$ is a probability iff the corresponding basic assignment satisfies $m(A) = 0$ for each $A \in \mathscr{P}(X)$ with $|A| > 1$. The last property (v) means that b is a lower probability in the sense of [6, Definition 4.1], which enables interpreting the belief function b in the game-theoretical framework based on a notion of coherence.

The geometrical structure of the set of all belief functions on $[0,1]^X$ is fully determined by the associated simplex of state assignments on $[0,1]^{\mathscr{P}(X)}$. For each $A \in \mathscr{P}_\emptyset(X)$, a belief function $b_A(f) = \min\{f(x) \mid x \in A\}$, $f \in [0,1]^X$ corresponds to the state assignment s_A (see Example 4). Consequently, we obtain the following characterization of the set of all belief functions.

Proposition 3. *The set of all belief functions on* $[0,1]^X$ *is a* $(2^{|X|} - 2)$*-simplex whose set of extreme points is* $\{b_A \mid A \in \mathscr{P}_\emptyset(X)\}$.

Observe that every b_A preserves finite minima since for every $f,g \in [0,1]^X$ we have $b_A(f \wedge g) = b_A(f) \wedge b_A(g)$. In general, it can be shown that each minimum-preserving function $b : [0,1]^X \to [0,1]$ with $b(0) = 0, b(1) = 1$ is a belief function. Such functions are termed *necessity measures* and they were recently investigated on formulas of finitely-valued Łukasiewicz logic in [7].

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