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RESEARCH REPORT

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State estimation with missing data and bounded uncertainty

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Abstract
The paper deals with two problems in the state estimation: (i) bounded uncertainty and (ii) missing measurement data. An algorithm for the state estimation of the discrete-time state space model whose uncertainties are bounded is proposed here. The algorithm also copes with situations when some data for identification are missing. The Bayesian approach is used and maximum a posteriori probability estimates are evaluated in the discrete time instants. The proposed estimation algorithm is applied to the estimation of vehicle position when incomplete data from global positioning system together with complete data from the inertial measurement unit are at disposal.

Keywords: Bayesian learning, state-space models, filtering, bounded noise, incomplete data, vehicles, global positioning systems, position estimation algorithms, inertial measurement units

1 Introduction

A state space model is frequently used for a description of real systems where some variables are hidden and cannot be measured directly. Then, the task of the state estimation arises. The noises of state evolution as well as observation model are often supposed to have normal distribution and the problem is then solved by means of the Kalman filtering (KF), see e.g. [GA08].

However, the unbounded support of the Gaussian distribution can cause difficulties in a case when the estimated quantity is physically restricted, e.g., it may give unreasonable negative estimates of naturally non-negative variable. There are several ways how to deal with this drawback.

In the KF framework, the state estimates are projected onto the constraint surface via quadratic programming [Fle00]. Use of truncated probability density functions (pdf) is another way of solving this problem [SS10]. Here, constraints are incorporated by the cutting off that part the pdf describing the state estimate that violates the constraints. In both cases, the key drawback of KF – it works well only when noise processes covariances are well chosen – is enhanced.

Monte-Carlo sampling alias particle filtering [DDFG01] is another popular estimation technique. In [LCGU07], inequality constraints are imposed by accept/reject steps in the algorithm. The appropriate constrained prior distribution is truncated or modified to satisfy the constraints, which ensures that the posterior also satisfies the constraints. The Monte-Carlo methods are based on simulation. Therefore, a huge amount of data is required to obtain acceptable results.

Further, techniques dealing with unknown-but-bounded equation errors are used [TPV90]. The unknown states lie within a bounded set. The complexity of this set is very high so approximation is needed to obtain recursively feasible solution. The approximation by ellipsoid is proposed in [PNDW04]. An approximation by a multivariate box is proposed in [MB82] and by a union of non-overlapping boxes in [BFT04]. A recursive Kalman-like algorithm for the state estimation of linear models with disturbances bounded by ellipsoids is proposed in [BABD08]. These methods lack a stochastic interpretation of the noises.

Missing data represent another practical problem faced in real applications requiring a state estimation. It is addressed repeatedly.

The paper [SP97] considers the problem of missing data within the framework of a class of uncertain discrete-time systems with a deterministic description of noise and uncertainty. A recursive scheme for constructing an ellipsoidal state estimation set of all states consistent with the available measured output and the given noise and uncertainty description is proposed.

In [SSF$^+$04], the problem of Kalman filtering with intermittent observations is considered. There, the existence of a critical value for the arrival rate of the observations is shown, beyond which a transition to an unbounded state error covariance occurs.

In spite of the fact that bounds and missing measurement occur jointly in practice it seems that there is no established methodology coping practically with both bounded uncertainty and missing data. Here, a straightforward algorithm of this type is proposed.

A state space model with uniformly distributed state and measurement noises is used. For the state estimation, the Bayesian approach is applied. By restricting to maximum a posteriori probability (MAP) estimates and batch estimation directness of unknown-but-bounded errors techniques is preserved.

2 Preliminaries

2.1 Notation

Throughout, the following notation is used

$\equiv$	equality by definition
$\propto$	equality up to a constant factor (proportionality)
$\mathbf{z}^*$	a set of $\mathbf{z}$-values, $\mathbf{z} \in \mathbf{z}^*$, $\mathbf{z}$ is a column vector
$\mathbf{z}_t$	value of $\mathbf{z}$ in discrete time instant t; $t \in t^* \subset \{0, 1, 2, \ldots T\}$, $T < \infty$
$\hat{\mathbf{z}}_t$	estimate of $\mathbf{z}_t$;
$\mathbf{z}^{k:l}$	the ordered sequence; $\mathbf{z}^{k:l} \equiv [\mathbf{z}'_k, \mathbf{z}'_{k+1}, \ldots, \mathbf{z}'_l]'$, $0 \le k \le l$
$'$	transposition
$\underline{\mathbf{z}}$, $\overline{\mathbf{z}}$	lower and upper bound on $\mathbf{z}$, respectively; they are used entry-wise
$f(\cdot\|\cdot)$	probability density functions (pdf); respective pdfs are distinguished by the argument names; no formal distinction is made between a random variable, its realization and pdf argument

Integrals used are always definite and multivariate ones. The integration domain coincides with the support of the pdf in its argument.
Note that vectors are always columns.

2.2 Calculus with pdfs

Let us consider the joint pdf $f(a, b, c)$. For any $(a, b, c) \in (a, b, c)^*$, the following relationships between pdfs [KBG+05] are exploited in the paper:

Chain rule

$$f(a, b|c) = f(a|b, c)f(b|c) = f(b|a, c)f(a|c)$$

Bayes rule

$$f(b|a, c) = \frac{f(a|b, c)f(b|c)}{f(a|c)} = \frac{f(a|b, c)f(b|c)}{\int f(a|b, c)f(b|c)\, db} \propto f(a|b, c)f(b|c). \tag{1}$$

2.3 Basics of Bayesian estimation

In Bayesian view [Ber85], [KBG+05], the system is described by probability density functions (pdfs). The quantities describing the system consist generally of observable outputs $\mathbf{y}^{1:T}$, optional inputs $\mathbf{u}^{1:T}$ and internal quantities that are never observed directly. The internal quantities consist of system states $\mathbf{x}^{0:T}$ and a time invariant parameters θ. The collection of the outputs and inputs is called data and denoted $\mathbf{d}^{1:T}$, i.e. $\mathbf{d}_t = (\mathbf{y}_t, \mathbf{u}_t)$, $t \in t^*$. The joint pdf

$$f(\mathbf{d}^{1:T}, \mathbf{x}^{0:T}, \theta)$$

describing both observed and internal quantities can be decomposed onto a product of the following elements:

- observation model
$$\{f(\mathbf{y}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_t, \theta)\}_{t\in t^*} \tag{2}$$
- time evolution model
$$\left\{f(\mathbf{x}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_{t-1}, \theta)\right\}_{t\in t^*} \tag{3}$$
- controller
$$\{f(\mathbf{u}_t|\mathbf{d}^{1:t-1}) \equiv f(\mathbf{u}_t|\mathbf{d}^{1:t-1}, \mathbf{x}^{0:t-1}, \theta)\}_{t\in t^*} \tag{4}$$
here the validity of the natural conditions of control is supposed [KBG$^+$05], i.e., $\mathbf{x}^{0:t-1}$ and θ are unknown to the controller
- prior pdf
$$f(\mathbf{x}_0, \theta). \tag{5}$$

Under (2) – (5), it holds

$$f(\mathbf{d}^{1:T}, \mathbf{x}^{0:T}, \theta) = f(\mathbf{x}_0, \theta) \prod_{t=1}^{T} f(\mathbf{y}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_t, \theta)$$

$$\times f(\mathbf{x}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_{t-1}, \theta) f(\mathbf{u}_t|\mathbf{d}^{1:t-1}) \tag{6}$$

$$\propto f(\mathbf{x}_0, \theta) \prod_{t=1}^{T} f(\mathbf{y}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_t, \theta) f(\mathbf{x}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_{t-1}, \theta).$$

As the controller does not depend on the internal quantities $\mathbf{x}_t$, θ it plays no role in estimation. Therefore, the knowledge of the controller is not required. Only, the generated input values have to be known.
The Bayesian state estimation works with characteristics of the joint pdf (6). This pdf combines prior information in $f(\mathbf{x}_0, \theta)$, theoretical knowledge described by both observation (2) and time evolution (3) models and observed data $\mathbf{d}^{1:T}$ by using deductive rules of the calculus with pdfs (1).

2.4 State model with bounded noise

A discrete-time state space model is used. It describes a given system by the following state (7) and output (8) equations in the discrete time instants $t \in t^*$

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t \tag{7}$$
$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{e}_t, \tag{8}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are known time invariant model matrices of appropriate dimensions;
$\mathbf{w}_t$, $\mathbf{e}_t$ are the vectors of the state and output noises respectively, i.e., they are zero mean with constant variances, mutually conditionally independent and identically distributed. Here, they are assumed to have uniform distribution on the multivariate boxes with the center $\mathbf{0}$ and half-widths of the support interval equal to $\mathbf{p}$ and $\mathbf{r}$, respectively, i.e,.

$$f(\mathbf{w}_t|\mathbf{p}) = \mathcal{U}(\mathbf{0}, \mathbf{p}), \quad f(\mathbf{e}_t|\mathbf{r}) = \mathcal{U}(\mathbf{0}, \mathbf{r}), \tag{9}$$

where $\mathbf{0}$ is the vectors of zeros of the appropriate length.
Further, we suppose that $\mathbf{x}_0$, $\mathbf{p}$, and $\mathbf{r}$ are a priori mutually independent and that it holds

$$\underline{\mathbf{x}}_0 \leq \mathbf{x}_0 \leq \overline{\mathbf{x}}_0,\ \mathbf{0} \leq \mathbf{p} \leq \overline{\mathbf{p}},\ \mathbf{0} \leq \mathbf{r} \leq \overline{\mathbf{r}},\ \{\underline{\mathbf{x}} \leq \mathbf{x}_t \leq \overline{\mathbf{x}}\}_{t \in t^*}. \tag{10}$$

Note that restrictions in (10) are defined by the user so that they reflect the reality. These known optional values specify user's prior information.
Note that restriction in (10) are defined by user so that they reflect the reality. These values are known and optional and specify our prior information.
Equations (7) and (8) together with the assumptions (9) and (10)define the state uniform model (SU model).
We introduce the column vector $\mathbf{X}$ as follows

$$\mathbf{X} = \left[\ \left(\mathbf{x}^{0:T}\right)' \quad \mathbf{p}' \quad \mathbf{r}'\ \right]'. \tag{11}$$

The joint pdf (6) of data $\mathbf{d}^{1:T}$, the state trajectory $\mathbf{x}^{0:T}$, and unknown parameters $\theta = [\mathbf{p}', \mathbf{r}']'$ of the SU model takes the form

$$f\left(\mathbf{d}^{1:T}, \mathbf{x}^{0:T}, \theta\right) \propto \left[\prod_{i=1}^{m} p_i \prod_{j=1}^{n} r_j\right]^{-T} \chi(\mathcal{S}), \tag{12}$$

where m, n are the lengths of the state and output vector, respectively, $\chi(\mathcal{S})$ is the indicator of the support $\mathcal{S}$ of this pdf,

$$\mathcal{S} = \mathcal{S}0 \cap \mathcal{S}1 \tag{13}$$

where $\mathcal{S}0$ is a set of $\mathbf{X}$ (11) that meets requirements (10) and $\mathcal{S}1$ is a set of $\mathbf{X}$ (11) that fulfills (7) and (8) with innovations bounds implied by (9), i.e.,

$$\{-\mathbf{p} \leq \mathbf{x}_t - \mathbf{A}\mathbf{x}_{t-1} - \mathbf{B}\mathbf{u}_t \leq \mathbf{p},\ -\mathbf{r} \leq \mathbf{y}_t - \mathbf{C}\mathbf{x}_t \leq \mathbf{r}\}_{t \in t^*}. \tag{14}$$

For a possible generalization, see [Pav08]. There, a SU model with partially unknown model matrices is introduced. The model (12) is its simplified version.

3 Bounded state estimation with missing data

3.1 State and noise boundary estimation

We suppose that the considered system is described by the SU model (12). We aim to estimate states $\mathbf{x}^{0:T}$ and the noise bounds $\mathbf{p}, \mathbf{r}$, i.e. vector $\mathbf{X}$ (11).
The maximizer of the a posteriori pdf (MAP estimate, [Ber85]) is taken a point estimate of the unobserved quantity. The MAP estimate $\hat{\mathbf{X}}_{MAP}$ of $\mathbf{X}$ with linearized logarithm of a posteriori pdf has the following form [Pav08]

$$\hat{\mathbf{X}}_{MAP} = \arg\ \min_{\mathbf{X} \in \mathcal{S}} \left(\sum_{i=1}^{m} p_i + \sum_{j=1}^{n} r_j\right), \tag{15}$$

where m, n are the lengths of the state and output vector, respectively, and $\mathcal{S}$ is given by (13).

Thus, the state and parameter estimation becomes the linear programming (LP) task [Fle00]

$$\begin{aligned} &\text{Find a vector } \mathbf{X} \text{ such that } J \equiv \mathbf{c}'\mathbf{X} \\ &= \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} r_j \ \rightarrow \min \\ &\text{while } \mathcal{A}\mathbf{X} \le \mathbf{b},\ \underline{\mathbf{X}} \le \mathbf{X} \le \overline{\mathbf{X}}, t \in t^*, \end{aligned} \tag{16}$$

where
(i)

$$\mathbf{c}' \equiv [\mathbf{0}'_{((T+1)n)}, \mathbf{1}'_{(m+n)}] \tag{17}$$

$\mathbf{0}$, $\mathbf{1}$ are the vectors of zeros and ones of the indicated lengths, respectively;
(ii) $\mathcal{A}$ and $\mathbf{b}$ are known matrix and vector, respectively; they result from the inequalities describing the set $\mathcal{S}1$ (14) and have the form (16)

$$\mathcal{A}\mathbf{X} \le \mathbf{b} \quad \text{with} \quad \mathcal{A} = \begin{bmatrix} \mathcal{A}11 & \mathcal{A}12 \\ \mathcal{A}21 & \mathcal{A}22 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}1 \\ \mathbf{b}2 \end{bmatrix}, \tag{18}$$

with

$$\begin{aligned} \mathcal{A}11 &= -\mathcal{R}_m(\mathbf{I}_{(T)} \otimes \mathbf{K} \otimes \mathbf{A}) + \mathcal{L}_m(\mathbf{I}_{(T)} \otimes \mathbf{K} \otimes \mathbf{I}_{(m)}), \\ \mathcal{A}12 &= -\mathbf{1}_{(2T)} \otimes \mathcal{R}_n(\mathbf{I}_{(m)}), \\ \mathcal{A}21 &= \mathcal{L}_m(\mathbf{I}_{(T)} \otimes \mathbf{K} \otimes \mathbf{C}), \\ \mathcal{A}22 &= -\mathbf{1}_{(2T)} \otimes \mathcal{L}_m(\mathbf{I}_{(n)}), \\ \mathbf{b}1 &= \left[\mathbf{I}_{(T)} \otimes \mathbf{K} \otimes \mathbf{B}\right] \mathbf{u}^{1:T}, \\ \mathbf{b}2 &= \left[\mathbf{I}_{(T)} \otimes \mathbf{K} \otimes \mathbf{I}_{(n)}\right] \mathbf{y}^{1:T}, \end{aligned}$$

where $\mathbf{I}_{(\alpha)}$ is the square identity matrix of the order α.
$\otimes$ denotes Kronecker product defined as follows

$$\mathbf{G}_{(\alpha,\beta)} \otimes \mathbf{H} \equiv \begin{bmatrix} G_{11}\mathbf{H} & \dots & G_{1\beta}\mathbf{H} \\ \vdots & & \vdots \\ G_{\alpha 1}\mathbf{H} & \dots & G_{\alpha\beta}\mathbf{H} \end{bmatrix}.$$

$\mathcal{R}_{col}(M)$ and $\mathcal{L}_{col}(M)$ are operators adding *col* zero columns to the matrix M from the right and left, respectively. $\mathbf{K} \equiv [1 \ -1]'$.
The matrix $\mathcal{A}$ and vector $\mathbf{b}$ arisen from the inequalities describing $\mathcal{S}1$ (14) reorganized so that terms containing entries from $\mathbf{X}$ are on the left-hand side.
(iii) $\underline{\mathbf{X}}$, $\overline{\mathbf{X}}$ are known vectors; they stem from the set $\mathcal{S}0$ (10) and have the following form

$$\underline{\mathbf{X}} = \begin{bmatrix} \underline{\mathbf{x}}_0 \\ \mathbf{1}_{(2\,T\,m)} \otimes \underline{\mathbf{x}} \\ \mathbf{0}_{(m)} \\ \mathbf{0}_{(n)} \end{bmatrix}, \quad \overline{\mathbf{X}} = \begin{bmatrix} \overline{\mathbf{x}}_0 \\ \mathbf{1}_{(2\,T\,m)} \otimes \overline{\mathbf{x}} \\ \overline{\mathbf{p}} \\ \overline{\mathbf{r}} \end{bmatrix}. \tag{19}$$

3.2 State and noise boundary estimation with missing data

The problem of missing measurement data can be easily incorporated into the estimation algorithm. The missing measurement causes that the output equality (8) is missing in a given time instant. This fact influences the sizes of matrices $\mathcal{A}21$, $\mathcal{A}22$, $\mathbf{b}2$ in (18). If N measurement is missing, then $2Nn$ rows in each of the above mentioned matrices are omitted.

3.3 Algorithm

The batch estimate of unknown states with bounded model uncertainty and missing measurement data according to (16) is performed in the following way:

1. Choose the data for the estimation so that the measurements are at disposal both at the beginning and on the end of the chosen time interval.

2. Enter $\underline{\mathbf{x}}$, $\overline{\mathbf{x}}$ (10).
If the bounds (or some of their entries) are not specified, assign $\underline{\mathbf{x}} := -\infty$, $\overline{\mathbf{x}} := -\infty$.

3. Enter the prior information $\underline{\mathbf{x}}_0$, $\overline{\mathbf{x}}_0$, $\bar{\mathbf{p}}$, $\bar{\mathbf{r}}$ (10).

4. Construct $\mathbf{c}$ (17), $\underline{\mathbf{X}}$, $\overline{\mathbf{X}}$ (19).

5. FOR t=1:T

- read data $\mathbf{d}_t$
- add 2m rows into $\mathcal{A}11$, $\mathcal{A}12$, $\mathbf{b}1$ (18)
- IF $\mathbf{y}_t$ is at disposal

  add 2n rows into $\mathcal{A}21$, $\mathcal{A}22$, $\mathbf{b}2$ (18)
- END IF

END FOR

6. Construct $\mathcal{A}$, $\mathbf{b}$ (18).

7. Run LP (16).
IF LP fails

- increase $\overline{\mathbf{p}}$, $\bar{\mathbf{r}}$ and redefine bounds (19) and
- repeat **7**.

END IF

8. Provide the LP solution as the obtained estimate $\hat{\mathbf{X}}_{MAP}$ of $\mathbf{X}$.

To obtain an on-line version of the algorithm 3.3, a moving horizon estimation (MHE) approach [RRM03] is applied. Then, the estimation run on the sliding window of the specified length. In [Pav08], the on-line estimation in the case of bounded uncertainties is proposed, however, it considers complete measurement data.

4 Vehicle position estimation

Exact positioning of a vehicle with missing data represents practically significant case to which the developed algorithm can be directly applied. Its simplified presentation illustrates the use of the proposed algorithm.

4.1 Problem description

The position of a moving vehicle is determined by global positioning system (GPS). The GPS provides the position directly in the Cartesian coordinates but signal outages can occur in the data. These data fallouts are caused e.g. by big trees or buildings that inhibit the signal receiving. Further, the frequency of GPS signal may be insufficient for some application.
Inertial navigation system (INS) is a navigation aid placed inside a car that provides additional information on a vehicle movement. The INS data include a complete information about vehicle velocity, yaw rate and acceleration. The INS measurements are at disposal during the whole driving time and often with a higher frequency than GPS data. If a starting location is known, then, with the knowledge of INS data, the position of a moving vehicle can be estimated without the need for external references. But the INS data are relative and therefore, the estimation error has a cumulative character.
To use the advantages of both these data sources, the estimation methods combining the GPS/INS data are developed. GPS corrects the errors in INS, INS supplies data during GPS signal outages.
For positioning, a model is usually constructed that uses kinematics laws, i.e., it is not concerned with the causing forces. It exploits a dependency among the vehicle position, velocity and acceleration. The estimation algorithms often use the Kalman filter, see e.g. [VN99], [QM02], [WLC04].
Here, the proposed estimation with the SU model (12) is applied. This removes the need to know noise covariances and allows to respect physical bounds on the involved quantities.

4.2 State model of the vehicle motion

First, a state model (7) and (8) is constructed that describes the vehicle movement using GPS/INS data. The form of this model is inspired by [QM02]. The list of considered quantities is in Table 1.

Table 1: Considered and measured quantities

quantity (unit)	notation	source	in model as
true position (m)	(p_x, p_y)	-	state
true velocity (m/s)	(v_x, v_y)	-	state
measured position (m)	$(\tilde{p}_x, \tilde{p}_y)$	GPS	output
transformed acceleration (m/s^2)	(a_x, a_y)	INS	input

Altitude changes are neglected and the Cartesian coordinate system is used. Then, the vehicle movement is described by a position vector $\mathbf{p} = (p_x, p_y)'$, velocity $\mathbf{v} = (v_x, v_y)'$, and acceleration $\mathbf{a} = (a_x, a_y)'$ whereas the following relationships hold

$$v_x = \dot{p}_x, \ v_y = \dot{p}_y, \ a_x = \dot{v}_x, \ a_y = \dot{v}_y \tag{20}$$

where $\dot{z}$ expresses a time change of z, i.e., its time derivative.
The relations (20) cannot be used directly in a discrete state model (7) and (8). Therefore, the differential equations are approximated by the difference ones. The following equations are obtained

$$\begin{array}{l} p_{x;t} = p_{x;t-1} + h\ v_{x;t-1} + 0.5\ h^2 a_{x;t-1} + w_{x;t} \\ p_{y;t} = p_{y;t-1} + h\ v_{y;t-1} + 0.5\ h^2 a_{y;t-1} + w_{y;t} \\ v_{x;t} = v_{x;t-1} + h\ a_{x;t-1} + w_{p_x;t} \\ v_{y;t} = v_{y;t-1} + h\ a_{y;t-1} + w_{p_y;t} \end{array}, \tag{21}$$

where h is the length of the time step. It is chosen as the time difference between two sequential time instants of measurement labelled by t and $t-1$.
Using assignment from Table 1, the state equation (7) takes the following form

$$\begin{bmatrix} p_{x;t} \\ p_{y;t} \\ v_{x;t} \\ v_{y;t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{x;t-1} \\ p_{y;t-1} \\ v_{x;t-1} \\ v_{y;t-1} \end{bmatrix} + \tag{22}$$

$$+ \begin{bmatrix} h^2/2 & 0 \\ 0 & h^2/2 \\ h & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} a_{x;t} \\ a_{x;t} \end{bmatrix} + \begin{bmatrix} w_{x;t} \\ w_{y;t} \\ w_{p_x;t} \\ w_{p_y;t} \end{bmatrix}$$

with $m = 4$ dimensional state. The output equation (8), with $n = 2$, is as follows

$$\mathbf{i}_t \begin{bmatrix} \tilde{p}_{x;t} \\ \tilde{p}_{y;t} \end{bmatrix} = \mathbf{i}_t \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{x;t} \\ p_{y;t} \\ v_{x;t} \\ v_{y;t} \end{bmatrix} + \begin{bmatrix} e_{x;t} \\ e_{y;t} \end{bmatrix} \right), \tag{23}$$

where $\mathbf{i}_t$ is a measurement indicator; $\mathbf{i}_t = \mathbf{I}_{(n)}$ if the GPS data are available, $\mathbf{i}_t = \mathbf{0}_{(n)}$ otherwise.
Note that a simplified model is used here. In reality, the INS data have to be transformed from local coordinates to global ones. This transformation can be included directly into estimation scheme using additional data concerning a vehicle turning as azimuth measured by GPS and angular speed provided by INS. For the sake of presentation simplicity, the original input data are already preprocessed and transformed before the position estimation starts.

5 Illustrative experiment

The purpose of the presented experiment is to illustrate properties of the proposed estimation algorithm.

5.1 Data

A real GPS data are used that are provided with frequency 10 Hz. The original noisy INS data are provided with the frequency 50 Hz. In the experiment, preprocessed INS data are used. Prior to the estimation start, the original INS data are filtered, re-sampled accordingly to GPS data and transformed into global coordinates.

During the experiment, a complete GPS data set is at disposal. Data outages are simulated by an artificial omission of some GPS data items. Afterwards, the estimation is performed. At last, the estimates are compared with states within the time span where measured data were omitted.
The experiment run in the Matlab environment using LP from its optimization toolbox.

5.2 Evaluation

The estimation of the vehicle position is performed using the proposed algorithm described in Section 3.3. For comparison, the algorithm based on Kalman forward-backward filtering [Sim06] is run with the same data. The later algorithm solves the problem of missing data as follows. If the measurement data are at disposal, then both the time-update and data-update phases proceed. If the measurement data item is missing, then only time-update phase proceeds. The covariances were set manually to values giving the best results.
To evaluate the quality of the estimates, the absolute error of estimates $E[\mathbf{p}_t]$ is used. It is defined as the difference between the measured vehicle position $\tilde{\mathbf{p}}_t$ and estimated vehicle position $\hat{\mathbf{p}}_t$, i.e.,

$$E[\mathbf{p}_t] \equiv |\tilde{\mathbf{p}}_t - \hat{\mathbf{p}}_t|, \; t \in t^*. \tag{24}$$

The maximum entry of the sequence $\{E[\mathbf{p}_t]\}_{t\in t^*}$ is denoted by $\max(E)$.

5.3 Results

The estimation was performed with various lengths of measurement data outages.
Figure 1 shows a vehicle trajectory where the longest simulated outage of the GPS signal is marked. In the sequence of experiments, the starting point of data outages is always the same whereas the length of the outages is successively raising from minimal length up to the maximal one.
Figure 2 shows how the maximal estimation error $\max(E)$ of (24) evolves in dependence on the length of data outage. The results of proposed algorithm are compared with the algorithm based on Kalman filtering. It can be seen that both methods give comparable results (the proposed algorithm is slightly better).
Figure 3 shows the course of state estimation error $E[\mathbf{p}_t]$ for one selected data outage. Here, the length of data outage is 5 seconds, it starts in between time samples 50 and 100. This is a typical course with maximal error value in the middle of missing data.

6 Conclusions

In the paper, a simple state estimation algorithm is presented that copes with bounded model uncertainty, constraints on states, and missing measurement data. These problems are often met jointly in a reality. The algorithm utility is demonstrated on the problem of vehicle position estimation and compared with Kalman filtering (KF) based algorithm. Results with the adopted SU model are comparable to KF or slightly better. The main advantage of SU model in comparison with KF is that just a very rough knowledge of the actual value of noise boundaries is only required and state constraints are “naturally” incorporated. The very rough prior guess on the noise boundary is estimated together with states. The estimation

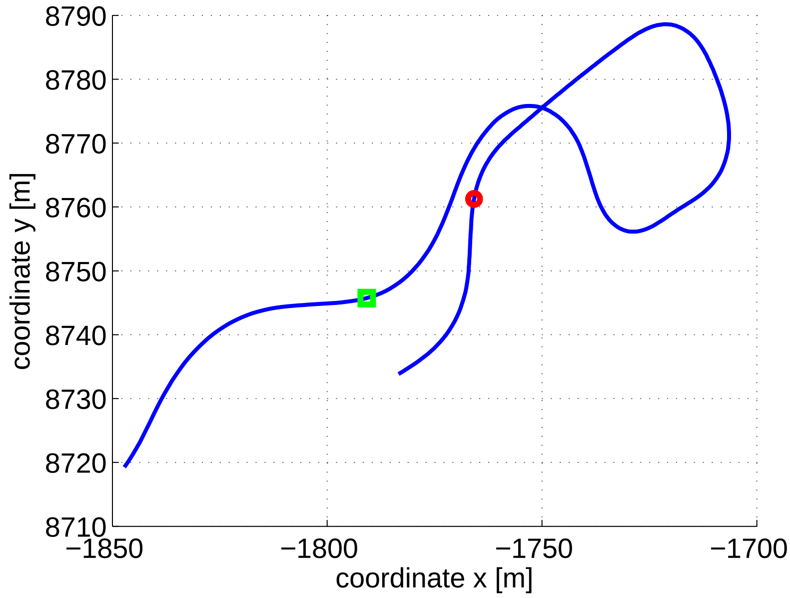

Figure 1: The trajectory of a moving vehicle in Cartesian coordinates with the marked beginning (bullet) and end (square) of the GPS signal outage

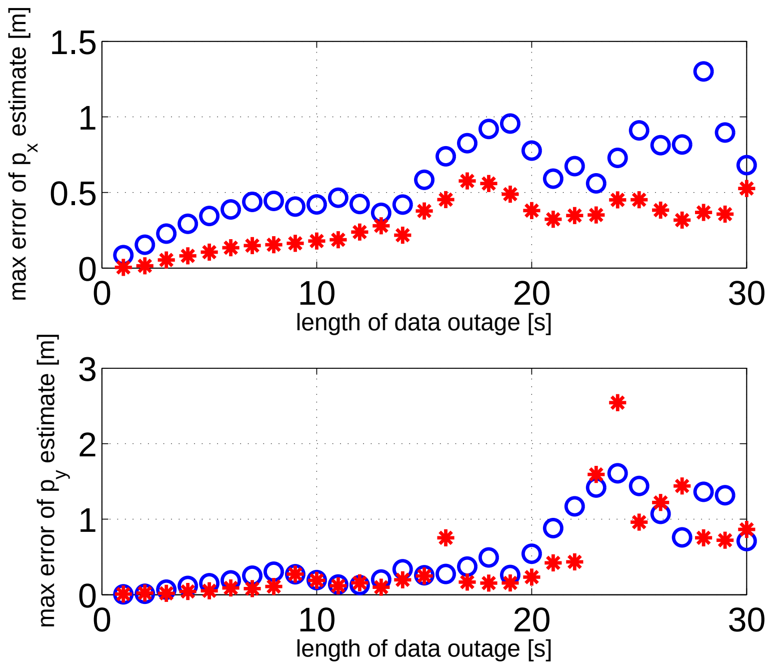

Figure 2: Evolution of the maximum estimation error $\max(E)$ in dependence on the length of data outage for SU model (*) and KF (o)

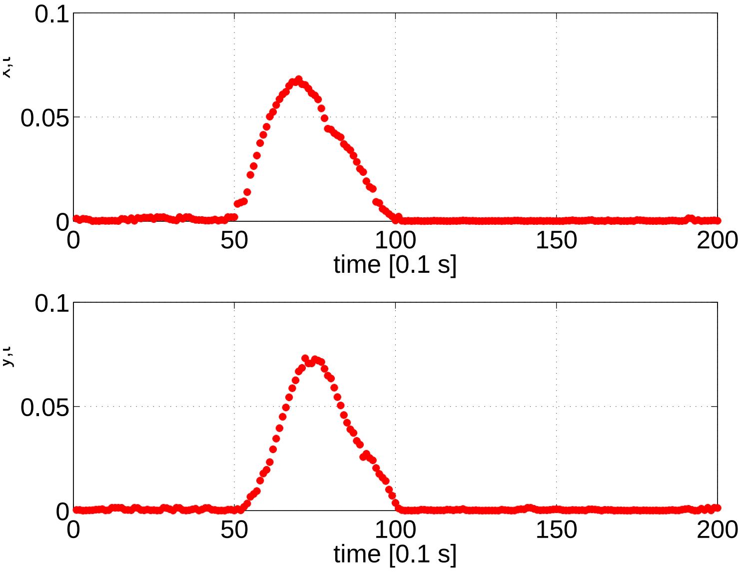

Figure 3: Evolution of estimation error $\{E[\mathbf{p}_t]\}_{t \in t^*}$ for one selected data outage spanning 5 seconds

algorithm is straightforward application of linear programming and dealing with the missing measurements is simple.

Joint parameter and state estimation can be addressed as well, see [Pav08]. To obtain an on-line version of the algorithm 3.3, a moving horizon estimation (MHE) approach [RRM03] is used and the estimation on the sliding window of the specified length is performed. In [Pav08], the on-line estimation in the case of bounded uncertainties is proposed, however, it considers complete measurement data. The application of this algorithm to the case with missing measurement data requires an introduction of the varying length of the mentioned sliding window.

The proposed basic scheme can be further improved. In the future, we will focus on extension to bounded but non-uniform noises.

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