Information Sciences 181 (2011) 4101–4111

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Ultramodular aggregation functions

Erich Peter Klement [a,*], Maddalena Manzi [b], Radko Mesiar [c,d]

[a] Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria
[b] Department of Pure and Applied Mathematics, University of Padua, Padova, Italy
[c] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
[d] Institute of Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic

ARTICLE INFO

Article history:
Received 12 May 2010
Received in revised form 6 March 2011
Accepted 26 May 2011
Available online 7 June 2011

Keywords:
Aggregation function
Ultramodular function

ABSTRACT

Ultramodular aggregation functions are investigated and discussed, including a study of structural properties and the proposal of some construction methods.

1. Introduction

The convexity of real functions in one variable is an interesting and fundamental analytical property, playing an important role in several mathematical fields and applications, especially when solving optimization problems [25,26,30,31].

Definition 1.1. Let I be a subinterval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a real function.

(i) f is said to be *convex* if, for all $x, y \in I$ and for all $\lambda \in [0,1]$,

$$\lambda \cdot f(x) + (1-\lambda) \cdot f(y) \geqslant f(\lambda \cdot x + (1-\lambda) \cdot y); \tag{1}$$

(ii) f is said to be Jensen convex if, for all $x,y \in I$,

$$\frac{f(x)+f(y)}{2} \geqslant f\left(\frac{x+y}{2}\right). \tag{2}$$

Trivially, each convex function is also Jensen convex. There are a number of conditions which are equivalent to the convexity (1):

Remark 1.2. Let I be a subinterval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a real function. Then we have:

(i) f is convex if and only if, for all $x,y \in I$ and for all $\varepsilon > 0$ such that $x < y$ and $y + \varepsilon \in I$,

$$f(y+\varepsilon) - f(y) \geqslant f(x+\varepsilon) - f(x). \tag{3}$$

* Corresponding author.
E-mail addresses: ep.klement@jku.at (E.P. Klement), mmanzi@math.unipd.it (M. Manzi), mesiar@math.sk (R. Mesiar).

0020-0255/$ - see front matter
doi:10.1016/j.ins.2011.05.021

(ii) If f is a continuous function then f is convex if and only if it is Jensen convex.
(iii) If f is a monotone function then f is convex if and only if it is Jensen convex.
(iv) If f is a bounded function then f is convex if and only if it is Jensen convex.

However, for real functions defined on subsets of $\mathbb{R}^n$ with $n > 1$, the corresponding extensions of these definitions of convexity are no more equivalent, in general. In this paper, we will discuss ultramodular aggregation functions, i.e., we focus on an approach to convexity of special functions based on a generalization of (3), called *ultramodularity*. Ultramodular modular functions are also called *directionally convex functions* or *functions having increasing increments*, and they are described and studied in detail in [20]. Additional constraints considered in this paper allow us to find several new results which are particularly important for copulas where the ultramodularity has a statistical counterpart, namely, stochastic decreasingness.

The paper is organized as follows. In the following section, modular, supermodular and ultramodular aggregation functions are introduced and some basic results are recalled. Section 3 is devoted to some constructions of ultramodular aggregation functions, especially those based on the composition of appropriate functions. Finally, the structure of ultramodular functions is discussed in Section 4.

2. Modular, supermodular, and ultramodular aggregation functions

The concept of modularity and supermodularity was introduced for functions from a lattice L into $\mathbb{R}$. In our paper, we mostly will restrict our attention to *n-ary aggregation functions*, i.e., to non-decreasing functions $A:[0,1]^n \to [0,1]$ satisfying $A(0,0,\ldots,0) = 0$ and $A(1,1,\ldots,1) = 1$ (see [1,13]). For the arguments of such a function in n variables we shall use the notations $\mathbf{x}$ and $(x_1,\ldots,x_n)$ synonymously.

Definition 2.1. Let $(L,\wedge,\vee)$ be a lattice.

(i) A function $f: L \to \mathbb{R}$ is called *modular* if, for all $x,y \in L$,

$$f(x \vee y) + f(x \wedge y) = f(x) + f(y). \tag{4}$$

(ii) A function $f: L \to \mathbb{R}$ is called *supermodular* if, for all $x,y \in L$,

$$f(x \vee y) + f(x \wedge y) \geqslant f(x) + f(y). \tag{5}$$

In [2] modular functions were called *valuations*. This terminology is also used in the context of measures, i.e., when L is a σ-algebra of subsets of a universe.

In the context of aggregation functions, the following characterization of modularity is easily obtained (see [29, Theorem 3.3]):

Proposition 2.2. *An n-ary aggregation function $A:[0,1]^n \to [0,1]$ is modular if and only if there are non-decreasing functions $f_1,f_2,\ldots,f_n:[0,1] \to [0,1]$ with $\sum_{i=1}^n f_i(0) = 0$ and $\sum_{i=1}^n f_i(1) = 1$ such that, for all $(x_1,\ldots,x_n) \in [0,1]^n$,*

$$A(x_1,\ldots,x_n) = \sum_{i=1}^n f_i(x_i).$$

For non-decreasing functions $f:[0,1]^2 \to [0,1]$, supermodularity can be reformulated as

$$f(x^*,y^*) - f(x^*,y) - f(x,y^*) + f(x,y) \geqslant 0 \tag{6}$$

for all $x,x^*,y,y^* \in [0,1]$ with $x \leqslant x^*$ and $y \leqslant y^*$. Two-dimensional aggregation functions satisfying (6) are said to be 2-*increasing* [23] or of *moderate growth* [16].

The following characterization of supermodular functions $f:[0,1]^n \to [0,1]$ is due to [3,15]:

Proposition 2.3. *An n-ary function $f:[0,1]^n \to [0,1]$ is supermodular if and only if each of its two-dimensional sections is supermodular, i.e., for each $\mathbf{x} \in [0,1]^n$ and all $i,j \in \{1,2,\ldots,n\}$ with $i \neq j$, the function $f_{\mathbf{x},i,j}:[0,1]^2 \to [0,1]$ given by $f_{\mathbf{x},i,j}(u,v) = f(\mathbf{y})$, where $y_i = u$, $y_j = v$ and $y_k = x_k$ for $k \in \{1,2,\ldots,n\}\setminus\{i,j\}$, is supermodular.*

The supermodularity of a function $f:[0,1]^n \to [0,1]$ is preserved if the arguments are distorted, i.e., if $g_1,\ldots,g_n:[0,1] \to [0,1]$ are non-decreasing functions, then the function $h:[0,1]^n \to [0,1]$ given by $h(\mathbf{x}) = f(g_1(x_1),\ldots,g_n(x_n))$ is supermodular (if f is a supermodular aggregation function with $f(g_1(0),\ldots,g_n(0)) = 0$ and $f(g_1(1),\ldots,g_n(1)) = 1$ then h is also a supermodular aggregation function).

Well-known examples of supermodular n-dimensional aggregation functions (with $n \geqslant 2$) are *modular aggregation functions* as characterized in Proposition 2.2 and *copulas* as introduced in [27] (see also [14,23]).

In the case $n = 2$, the supermodularity is even used as an axiom for copulas:

Definition 2.4. An aggregation function $C:[0,1]^2 \to [0,1]$ is called a 2-*copula* (or, briefly, a *copula*) if it is supermodular and has 1 as neutral element, i.e., if $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$.

Copulas play an important role in the representation of supermodular binary aggregation functions. In fact, if a binary aggregation function $A:[0,1]^2 \to [0,1]$ is supermodular, so is the function $B:[0,1]^2 \to [0,1]$ given by $B(x,y) = A(x,y) - A(x,0) - A(0,y)$. Moreover, B is non-decreasing and 0 is an annihilator of B. Then, because of [10, Theorem 17], we have the following result:

Proposition 2.5. *An aggregation function* $A:[0,1]^2 \to [0,1]$ *is supermodular if and only if there are non-decreasing functions* $g_1, g_2, g_3, g_4:[0,1] \to [0,1]$ *with* $g_i(0) = 0$ *and* $g_i(1) = 1$ *for* $i \in \{1,2,3,4\}$, *a copula* $C:[0,1]^2 \to [0,1]$, *and numbers* $a, b, c \in [0,1]$ *with* $a + b + c = 1$ *such that, for all* $(x,y) \in [0,1]^2$,

$$A(x,y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)). \tag{7}$$

If 0 is an annihilator of the aggregation function $A:[0,1]^2 \to [0,1]$, i.e., if $A(x,0) = A(0,x) = 0$ for all $x \in [0,1]$, then (7) reduces to

$$A(x,y) = C(f(x), g(y)), \tag{8}$$

where $f,g:[0,1] \to [0,1]$ are non-decreasing functions with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Note that then we have $f(x) = A(x,1)$ and $g(x) = A(1,x)$ for all $x \in [0,1]$.

Proposition 2.5 can be read also in this way: each binary supermodular aggregation function is a convex combination of a modular aggregation function and a distorted copula.

In general, the composition of (super-)modular functions is not necessarily (super-)modular: the functions $A,B:[0,1]^2 \to [0,1]$ given by $A(x,y) = \sqrt{x}$ and $B(x,y) = \frac{x+y}{2}$ are both modular and, therefore, supermodular. However, the composition $A(B,B):[0,1]^2 \to [0,1]$ given by $A(B,B)(x,y) = \sqrt{\frac{x+y}{2}}$ is not supermodular.

The following definition restricts the notion of ultramodularity of real functions, as considered in [20], to the class of aggregation functions.

Definition 2.6. An n-ary aggregation function $A:[0,1]^n \to [0,1]$ is called *ultramodular* if, for all $\mathbf{x},\mathbf{y},\mathbf{z} \in [0,1]^n$ with $\mathbf{x}+\mathbf{y}+\mathbf{z} \in [0,1]^n$,

$$A(\mathbf{x}+\mathbf{y}+\mathbf{z}) - A(\mathbf{x}+\mathbf{y}) \geqslant A(\mathbf{x}+\mathbf{z}) - A(\mathbf{x}). \tag{9}$$

Ultramodularity implies supermodularity of aggregation functions. To see this, for arbitrary $\mathbf{x},\mathbf{y} \in [0,1]^n$ put first $\mathbf{u} = \mathbf{y} - \mathbf{x} \wedge \mathbf{y}$ and $\mathbf{v} = \mathbf{x} - \mathbf{x} \wedge \mathbf{y}$. Then we get

$$\mathbf{x} \vee \mathbf{y} = \mathbf{x} + \mathbf{y} - \mathbf{x} \wedge \mathbf{y} = \mathbf{x} \wedge \mathbf{y} + \mathbf{u} + \mathbf{v}$$

and, because of (9),

$$A(\mathbf{x} \vee \mathbf{y}) + A(\mathbf{x} \wedge \mathbf{y}) = A(\mathbf{x} \wedge \mathbf{y} + \mathbf{u} + \mathbf{v}) + A(\mathbf{x} \wedge \mathbf{y}) \geqslant A(\mathbf{x} \wedge \mathbf{y} + \mathbf{v}) + A(\mathbf{x} \wedge \mathbf{y} + \mathbf{u}) = A(\mathbf{x}) + A(\mathbf{y}).$$

In the case of one-dimensional aggregation functions, ultramodularity (9) is just standard convexity. Therefore, ultramodularity can also be seen as an extension of one-dimensional convexity. The following result (Corollary 4.1 of [20]) states the exact relationship between ultramodular and supermodular functions $f:[0,1]^n \to [0,1]$:

Proposition 2.7. *A function* $f:[0,1]^n \to [0,1]$ *is ultramodular if and only if f is supermodular and each of its one-dimensional sections is convex, i.e., for each* $\mathbf{x} \in [0,1]^n$ *and each* $i \in \{1,\ldots,n\}$ *the function* $f_{\mathbf{x},i}:[0,1] \to [0,1]$ *given by* $f_{\mathbf{x},i}(u) = f(\mathbf{y})$, *where* $y_i = u$ *and* $y_j = x_j$ *whenever* $j \neq i$, *is convex.*

If, additionally, some smoothness conditions are satisfied, then we get the following consequence of Propositions 2.3 and 2.7:

Corollary 2.8. *Let* $n \geqslant 2$ *and assume that all partial derivatives of order 2 of the function* $f:[0,1]^n \to [0,1]$ *exist. Then f is ultramodular if and only if all partial derivatives of order* 2 *are non-negative.*

Remark 2.9.

(i) Because of Propositions 2.3 and 2.7, for an n-ary aggregation function $A:[0,1]^n \to [0,1]$ the following are equivalent:
  (a) A is ultramodular;
  (b) each two-dimensional section of A is ultramodular;
  (c) each two-dimensional section of A is supermodular and each one-dimensional section of A is convex.

(ii) Another equivalent condition to the ultramodularity (9) of an n-ary aggregation function $A:[0,1]^n \to [0,1]$ is the validity of

$$A(\mathbf{x}+\mathbf{u}) + A(\mathbf{x}-\mathbf{u}) \geqslant A(\mathbf{x}+\mathbf{v}) + A(\mathbf{x}-\mathbf{v}) \tag{10}$$

for all $\mathbf{x},\mathbf{u} \in [0,1]^n$, $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| \leqslant \mathbf{u}$ and $\mathbf{x}+\mathbf{u}, \mathbf{x}-\mathbf{u}, \mathbf{x}+\mathbf{v}, \mathbf{x}-\mathbf{v} \in [0,1]^n$ (indeed, it is sufficient to put $\mathbf{y} = \mathbf{u}+\mathbf{v}$ and $\mathbf{z} = \mathbf{u}-\mathbf{v}$). Relaxing the requirement $\mathbf{u} \in [0,1]^n$ and $|\mathbf{v}| \leqslant \mathbf{u}$ into $\mathbf{u} \in \mathbb{R}^n$ and $|\mathbf{v}| \leqslant |\mathbf{u}|$ we get the definition of *symmetrically monotone functions* given in [28]. Note that symmetrically monotone aggregation functions $A:[0,1]^n \to [0,1]$ are exactly ultramodular aggregation functions which are modular, i.e., $A(\mathbf{x}) = \sum_{i=1}^n f_i(x_i)$ with $f_i:[0,1] \to [0,1]$ being convex for each $i \in \{1,\ldots,n\}$ (compare Propositions 2.2 and 2.7).

Remark 2.10.

(i) For $n = 2$, the ultramodularity (9) of an aggregation function $A : [0,1]^2 \to [0,1]$ is equivalent to A being *P-increasing* (see [12]), i.e., to

$$A(u_1, v_1) + A(u_4, v_4) \geqslant \max\left(A(u_2, v_2) + A(u_3, v_3), A(u_3, v_2) + A(u_2, v_3)\right)$$

for all $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \in [0,1]$ satisfying $u_1 \leqslant u_2 \wedge u_3 \leqslant u_2 \vee u_3 \leqslant u_4$, $v_1 \leqslant v_2 \wedge v_3 \leqslant v_2 \vee v_3 \leqslant v_4$, $u_1 + u_4 \geqslant u_2 + u_3$, and $v_1 + v_4 \geqslant v_2 + v_3$.

(ii) Fix two non-decreasing functions $f, g : [0,1] \to [0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then the smallest supermodular aggregation function $A_* : [0,1]^2 \to [0,1]$ satisfying $A_*(x,1) = f(x)$ and $A_*(1,y) = g(y)$ for all $x, y \in [0,1]$ is given by $A_*(x,y) = \max(f(x) + g(y) - 1, 0)$. Observe that A_* is ultramodular (and, subsequently, the smallest ultramodular aggregation function with fixed margins f and g) if and only if both f and g are convex. In particular, if $f = g = \mathrm{id}_{[0,1]}$ then $A_* = W$, the smallest binary copula. On the other hand, the greatest supermodular aggregation function $A^* : [0,1]^2 \to [0,1]$ satisfying $A^*(x,1) = f(x)$ and $A^*(1,y) = g(y)$ for all $x, y \in [0,1]$ is given by $A^*(x,y) = \min(f(x), g(y))$. However, A^* is ultramodular only if $f = g = \mathbf{1}_{\{1\}}$, i.e., if A^* coincides with the smallest binary aggregation function $\mathbf{1}_{\{(1,1)\}}$.

(iii) Given a copula $C : [0,1]^2 \to [0,1]$, for each $c \in [0,1]$ the horizontal section $h_c : [0,1] \to [0,1]$ given by $h_c(x) = C(x,c)$ obviously satisfies $h_c(0) = 0$ and $h_c(1) = c$. Then the greatest possible convex horizontal section h_c is given by $h_c(x) = c \cdot x$, corresponding to the product copula Π (hence we have $C(x,c) \leqslant c \cdot x = \Pi(x,c)$). It is easy to verify that Π is an ultramodular copula, and hence Π is the greatest ultramodular copula. From a statistical point of view, ultamodular copulas describe the dependence structure of stochastically decreasing random vectors (X,Y) (see also [23, Corollary 5.2.11]), and thus each ultramodular copula is *Negative Quadrant Dependent* (NQD).

(iv) Fig. 1 visualizes the modularity, supermodularity and ultramodularity, respectively, of a real function $f : [0,1]^2 \to [0,1]$, indicating the sign of the values of f in the vertices of the corresponding shaded area in the sum to be computed, and the relationship of this sum to 0.

3. Some constructions

We now show that ultramodularity is preserved by the composition of aggregation functions (here the monotonicity of the aggregation functions is crucial). First of all, we give the following result which extends [9, Theorem 5.2]:

Theorem 3.1. *Let $A : [0,1]^n \to [0,1]$ be an aggregation function and $k \geqslant 2$. Then the following are equivalent:*

(i) *A is ultramodular.*

(ii) *If $B_1, \ldots, B_n : [0,1]^k \to [0,1]$ are non-decreasing supermodular functions then the composite $D : [0,1]^k \to [0,1]$ given by $D(\mathbf{x}) = A(B_1(\mathbf{x}), \ldots, B_n(\mathbf{x}))$ is a supermodular function.*

Proof. To show that (i) implies (ii), let A be an ultramodular aggregation function and $B_1, \ldots, B_n$ be non-decreasing supermodular functions. Evidently, D is an aggregation function. Choose $\mathbf{x}, \mathbf{y} \in [0,1]^k$ and denote, for each $i \in \{1, \ldots, n\}$, $a_i = B_i(\mathbf{x}) - B_i(\mathbf{x} \wedge \mathbf{y})$ and $b_i = B_i(\mathbf{y}) - B_i(\mathbf{x} \wedge \mathbf{y})$, $\mathbf{u} = (a_1, \ldots, a_n)$, $\mathbf{v} = (b_1, \ldots, b_n)$, and $\mathbf{z} = (B_1(\mathbf{x} \wedge \mathbf{y}), \ldots, B_n(\mathbf{x} \wedge \mathbf{y}))$. The monotonicity of the B_i's implies $\mathbf{u}, \mathbf{v} \in [0,1]^n$, and due to their supermodularity we get $(B_1(\mathbf{x} \vee \mathbf{y}), \ldots, B_n(\mathbf{x} \vee \mathbf{y})) \geqslant \mathbf{u} + \mathbf{v} + \mathbf{z}$. Now, the monotonicity and the ultramodularity of A yield

$$D(\mathbf{x} \vee \mathbf{y}) \geqslant A(\mathbf{z} + \mathbf{u} + \mathbf{v}) \geqslant A(\mathbf{z} + \mathbf{u}) + A(\mathbf{z} + \mathbf{v}) - A(\mathbf{z}) = D(\mathbf{x}) + D(\mathbf{y}) - D(\mathbf{x} \wedge \mathbf{y}),$$

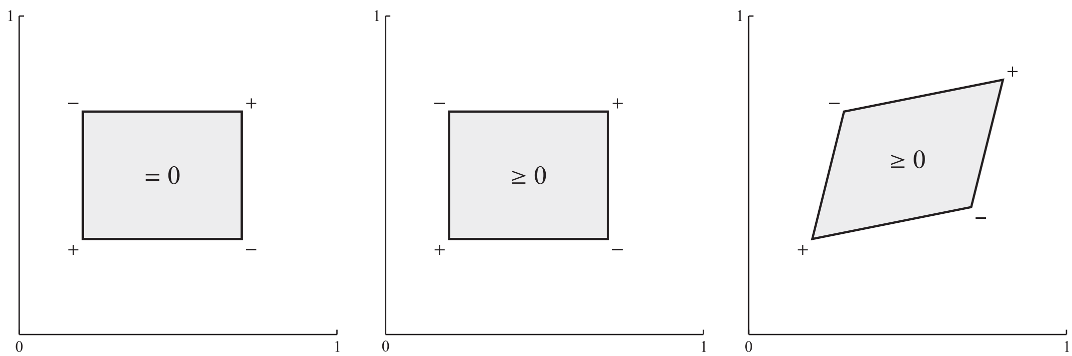

Fig. 1. Modularity (left), supermodularity (center), and ultramodularity of a function $f : [0,1]^2 \to [0,1]$.

i.e., D is supermodular.

Now suppose that (ii) holds. To show that the one-dimensional sections of A are convex, consider, without loss of generality, the function $f:[0,1]\to[0,1]$ given by $f(x) = A(x,u_2,\ldots,u_n)$, where $u_2,\ldots,u_n\in[0,1]$ are fixed. Define the functions $B_1,\ldots,B_n:[0,1]^k\to[0,1]$ by $B_1(\mathbf{x})=\frac{x_1+x_2}{2}$ and $B_i(\mathbf{x}) = u_i$ for $i > 1$. If, for arbitrary $x,y\in[0,1]$, we put $\mathbf{x} = (x,y,0,\ldots,0)$ and $\mathbf{y} = (y,x,0,\ldots,0)$ then we obtain $D(\mathbf{x})=D(\mathbf{y})=f\left(\frac{x+y}{2}\right)$, $D(\mathbf{x}\wedge\mathbf{y})=f(x\wedge y)$, and $D(\mathbf{x}\vee\mathbf{y}) = f(x\vee y)$. Since $B_1,\ldots,B_n$ are non-decreasing supermodular functions, also D is supermodular, proving $\frac{f(x)+f(y)}{2}\geqslant f\left(\frac{x+y}{2}\right)$, i.e., the convexity of f. Note that, in the case $n = 1$, this means that A is ultramodular. If $n > 1$, because of Proposition 2.3 it suffices to show the supermodularity of the two-dimensional sections of A. This can be seen by defining $g:[0,1]^2\to[0,1]$ by $g(x) = A(x,y,u_3,\ldots,u_n)$, $B_1(\mathbf{x}) = x_1$, $B_2(\mathbf{x}) = x_2$, and $B_i(\mathbf{x}) = u_i$ for $i > 2$, and by using similar arguments as above. □

Now we are ready to show that the class of ultramodular aggregation functions is closed under composition, i.e., ultramodular aggregation functions form a clone [19].

Theorem 3.2. *Let $A:[0,1]^n\to[0,1]$ and $B_1,\ldots,B_n:[0,1]^k\to[0,1]$ be ultramodular aggregation functions. Then the composite function $D:[0,1]^k\to[0,1]$ given by $D(\mathbf{x}) = A(B_1(\mathbf{x}),\ldots,B_n(\mathbf{x}))$ is also an ultramodular aggregation function.*

Proof. Because of Theorem 3.1, D is supermodular (this holds also if $k = 1$; indeed, the first part of the proof of Theorem 3.1 works also in the case $k = 1$), and thus only the convexity of its one-dimensional sections needs to be shown. Let $g:[0,1]\to[0,1]$ be a one-dimensional section of the composite function D, i.e., there are one-dimensional sections $f_1,\ldots,f_n:[0,1]\to[0,1]$ of $B_1,\ldots,B_n$, respectively, (which are convex because of Proposition 2.7) such that $g(x) = A(f_1(x),\ldots,f_n(x))$. Of course, g is non-decreasing and therefore its convexity is equivalent to the validity of the Jensen inequality (2) in the form

$$g(x+a)-g(x)\leqslant g(x+2a)-g(x+a) \tag{11}$$

for all $x,a\in[0,1]$ with $x + 2a \leqslant 1$. From the convexity of $f_1,\ldots,f_n$ we obtain, for each $i\in\{1,\ldots,n\}$, $0\leqslant f_i(x + a) - f_i(x)\leqslant f_i(x + 2a) - f_i(x + a)$. Putting $a_i = f_i(x + a) - f_i(x)$ and $b_i = f_i(x + 2a) - f_i(x + a)$, we have

$$\begin{aligned} g(x+2a) &= A(f_1(x+2a),\ldots,f_n(x+2a)) = A(f_1(x)+a_1+b_1,\ldots,f_n(x)+a_n+b_n) \\ &\geqslant A(f_1(x)+a_1,\ldots,f_n(x)+a_n)+A(f_1(x)+b_1,\ldots,f_n(x)+b_n)-A(f_1(x),\ldots,f_n(x)) \geqslant 2g(x+a)-g(x), \end{aligned}$$

which proves (11). Here the first inequality follows from the ultramodularity and the second one from the monotonicity of A. □

Theorem 3.2 has several important consequences (some of them can be found in [20, Proposition 4.1]).

Corollary 3.3. *Let $A_1,\ldots,A_j:[0,1]^n\to[0,1]$ be n-ary ultramodular aggregation functions and $f:[0,1]\to[0,1]$ a non-decreasing function with $f(0) = 0$ and $f(1) = 1$. Then we have:*

(i) *Each convex combination of $A_1,\ldots,A_j$ is an n-ary ultramodular aggregation function.*
(ii) *The product of $A_1,\ldots,A_j$ is an n-ary ultramodular aggregation function.*
(iii) *If $A:[0,1]^n\to[0,1]$ is an n-ary ultramodular aggregation function and f is convex then the composition $f\circ A$ is an n-ary ultramodular aggregation function.*
(iv) *If $A:[0,1]^2\to[0,1]$ is a binary associative ultramodular aggregation function (i.e., $A(x,A(y,z)) = A(A(x,y),z)$ for all $x,y,z\in[0,1]$) then, for each $k > 2$, the k-ary extension of A to $[0,1]^k$ defined by*

$$A(x_1,x_2,\ldots,x_k)=A(x_1,A(x_2,\ldots,A(x_{k-1},x_k),\ldots))$$

is a k-ary ultramodular aggregation function.

Proof. Statements (i)–(iii) follow from Theorem 3.2 taking into account that the weighted arithmetic mean, the product $\prod$ (which is a copula with linear, i.e., convex one-dimensional sections) and the function f in (iii) (for non-decreasing functions in one variable convexity means ultramodularity) are ultramodular aggregation functions. The proof of (iv) is done by induction: if the k-ary extension $A^{(k)}$ of A is ultramodular then also $A^{(k+1)}:[0,1]^{k+1}\to[0,1]$ given by $A^{(k+1)}(x_1,\ldots,x_k,x_{k+1}) = A(A^{(k)}(x_1,\ldots,x_k),x_{k+1})$ is also ultramodular as a consequence of the ultramodularity of the functions $B_1,B_2:[0,1]^{k+1}\to[0,1]$ given by $B_1(x_1,\ldots,x_k,x_{k+1}) = A^{(k)}(x_1,\ldots,x_k)$ and $B_2(x_1,\ldots,x_k,x_{k+1}) = x_{k+1}$, respectively. □

When constructing ultramodular aggregation functions, we can focus on special types of aggregation functions. However, in some cases the ultramodularity can be a contradictory or rather restrictive requirement. For instance, *disjunctive aggregation functions* (such as *triangular conorms* [18]) cannot be ultramodular. As an example of the second type we recall the *Choquet integral* [5,8] and present the necessary details.

If $n\in\mathbb{N}$ and $X = \{1,\ldots,n\}$ then, for a capacity m on X, i.e., a non-decreasing function $m:2^X\to[0,1]$ with $m(\emptyset) = 0$ and $m(X) = 1$, and $\mathbf{x}\in[0,1]^n$ the Choquet integral [5] is given by

$$\mathrm{Ch}(m,\mathbf{x}) = \int_0^1 m(\{x_i \geqslant u\})du = \sum_{i=1}^{n} x_{\pi(i)}(m(\{\pi(i),\ldots,\pi(n)\}) - m(\{\pi(i+1),\ldots,\pi(n)\})),$$

where $\pi: X \to X$ is a permutation of X with $x_{\pi(1)} \leqslant x_{\pi(2)} \leqslant \cdots \leqslant x_{\pi(n)}$ and, by convention, $\{\pi(n+1), \pi(n)\} = \emptyset$.

For a fixed capacity m, the function $\mathrm{Ch}_m: [0,1]^n \to [0,1]$ given by $\mathrm{Ch}_m(\mathbf{x}) = \mathrm{Ch}(m,\mathbf{x})$ is an aggregation function, a so-called *Choquet integral-based aggregation function.*

Proposition 3.4. *Let $Ch_m: [0,1]^n \to [0,1]$ be a Choquet integral-based aggregation function based on a capacity m on $X = \{1,\ldots,n\}$. Then we have:*

(i) Ch_m *is superadditive, i.e., for all* $\mathbf{x},\mathbf{y} \in [0,1]^n$ *with* $\mathbf{x}+\mathbf{y} \in [0,1]^n$ *we have*

$$\mathrm{Ch}_m(\mathbf{x}+\mathbf{y}) \geqslant \mathrm{Ch}_m(\mathbf{x}) + \mathrm{Ch}_m(\mathbf{y}),$$

if and only if the capacity m is supermodular.

(ii) Ch_m *is ultramodular if and only if the capacity m is modular, i.e.,* Ch_m *is a weighted arithmetic mean.*

Proof. Statement (i) follows from [8]. If m is modular (i.e., a probability measure) then Ch_m is a weighted arithmetic mean and, thus, ultramodular. Conversely, if Ch_m is ultramodular, then Ch_m is also superadditive (indeed, it suffices to put $\mathbf{x} = \mathbf{0}$ in (9)), and thus m is supermodular because of (i), and each one-dimensional section of Ch_m is convex. This means in particular that, for an arbitrary permutation σ of $X\backslash\{1\}$, the function $f: [0,1] \to [0,1]$ given by

$$f(x) = \mathrm{Ch}_m\left(x, \frac{\sigma(2)-1}{n}, \ldots, \frac{\sigma(n)-1}{n}\right)$$

is convex. Clearly, f is a continuous piecewise linear function which is linear on each interval $\left[\frac{i-1}{n}, \frac{i}{n}\right],\ i \in \{1,\ldots,n\}$. If τ denotes the inverse permutation of σ then the corresponding slopes of the restrictions $f|_{\left[\frac{i-1}{n},\frac{i}{n}\right]},\ i \in \{1,\ldots,n\}$, are given by $m(\{1,\tau(2),\ldots,\tau(i)\}) - m(\{\tau(2),\ldots,\tau(i)\})$ whenever $i < n$, and by $m(\{1\})$ for $i = n$. Therefore, the convexity of f is equivalent to $h: X \to [0,1]$, where $h(i)$ is the slope of f on the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, being non-decreasing. A similar claim holds for each other coordinate $j \in X$, i.e., for all $i,j \in X$ with $i \neq j$ and all $B \subseteq X\backslash\{i,j\}$ we have

$$m(B \cup \{i,j\}) - m(B \cup \{i\}) \leqslant m(B \cup \{j\}) - m(B). \tag{12}$$

Because of the supermodularity of m, the converse inequality of (12) holds, too, i.e., we have $m(B \cup \{i,j\}) - m(B \cup \{i\}) = m(B \cup \{j\}) - m(B)$. For $B = \emptyset$ this means that $m(\{i,j\}) = m(\{i\}) + m(\{j\})$, and for $B = \{k\}$ with $k \in X\backslash\{i,j\}$ we obtain

$$m(\{i,j,k\}) + m(\{j\}) = m(\{i,k\}) + m(\{j,k\}) = m(\{i\}) + m(\{k\}) + m(\{j\}) + m(\{k\}),$$

i.e., $m(\{i,j,k\}) = m(\{i\}) + m(\{j\}) + m(\{k\})$. By induction, $m(B) = \sum_{i\in B} m(\{i\})$ for each $B \subseteq X$, i.e., m is modular. □

Remark 3.5. Observe that each $\{0,1\}$-valued supermodular capacity on X has the form m_B, B being some non-empty subset of X, where $m_B(A) = 1$ if $B \subseteq A$, and $m_B(A) = 0$ otherwise. Then $\mathrm{Ch}_{m_B}(\mathbf{x}) = \min\{x_i | i \in B\}$ for each $\mathbf{x} \in [0,1]^n$. Moreover, a general supermodular capacity on X is a convex combination of $\{0,1\}$-valued supermodular capacities on X, and thus each superadditive Choquet integral-based aggregation function $\mathrm{Ch}_m: [0,1]^n \to [0,1]$ has the form

$$\mathrm{Ch}_m(\mathbf{x}) = \sum_{j=1}^{k} \lambda_j \cdot \min\{x_i | i \in B_j\},$$

where $k \in \mathbb{N},\ \lambda_j > 0$ and $\emptyset \subset B_j \subseteq X$ for $j \in \{1,\ldots,k\}$, and $\sum_{j=1}^{k} \lambda_j = 1$.

We shall identify the function $\min: [0,1]^k \to [0,1]$ and the greatest lower bound $\min: 2^{[0,1]} \to [0,1]$, i.e., both $\min(x_1,\ldots,x_n)$ and $\min\{x_1,\ldots,x_n\}$ mean the same, namely, the smallest of the numbers $x_1,\ldots,x_n \in [0,1]$. Since min is supermodular for each arity, this implies that each superadditive Choquet integral-based aggregation function $\mathrm{Ch}_m: [0,1]^n \to [0,1]$ is supermodular (this result can be found in [24, Theorem 7.17]).

There are several modifications and constructions of aggregation functions which preserve the supermodularity. However, only few of them preserve also ultramodularity. We recall some of the modifications and constructions in the framework of copulas (i.e., which preserve supermodularity).

Remark 3.6. For each binary copula $C: [0,1]^2 \to [0,1]$ we have $W \leqslant C \leqslant M$, where the lower and upper Fréchet-Hoeffding bounds W and M are given by $W(x,y) = \max(x+y-1, 0)$ and $M(x,y) = \min(x,y)$, respectively.

(i) *Ordinal sum*: If $(C_k)_{k\in K}$ is a family of copulas and if $(]a_k, b_k[)_{k\in K}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the ordinal sum $C = (\langle a_k, b_k, C_k\rangle)_{k\in K}$ is given by

$$C(x,y) = \begin{cases} a_k + (b_k - a_k)C_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in]a_k, b_k[^2, \\ M(x,y) & \text{otherwise.} \end{cases}$$

(ii) *W-ordinal sum*: If $(C_k)_{k\in K}$ is a family of copulas and if $(]a_k,b_k[)_{k\in K}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the W-ordinal sum $C = W\text{-}(\langle a_k,b_k,C_k\rangle)_{k\in K}$ is given by (see [6,11,17])

$$C(x,y)=\begin{cases}(b_k-a_k)C_k\left(\frac{x-a_k}{b_k-a_k},\frac{y-1+b_k}{b_k-a_k}\right) & \text{if } (x,y)\in]a_k,b_k[\times]1-b_k,1-a_k[,\\ W(x,y) & \text{otherwise.}\end{cases}$$

(iii) *g-ordinal sum*: If $(C_k)_{k\in K}$ is a family of copulas and if $(]a_k,b_k[)_{k\in K}$ is a family of pairwise disjoint open subintervals of $[0,1]$ then the g-ordinal sum $C = \mathrm{g}-(\langle a_k,b_k,C_k\rangle)_{k\in K}$ is given by (see [22])

$$C(x,y)=\begin{cases}a_k y+(b_k-a_k)\cdot C_k\left(\frac{x-a_k}{b_k-a_k},y\right) & \text{if } x\in]a_k,b_k[,\\ xy & \text{otherwise.}\end{cases}$$

(iv) *Flipping*: If C is a copula then the flippings C^- and C_- are given by (see [23])

$$C^-(x,y)=x-C(x,1-y), \tag{13}$$
$$C_-(x,y)=y-C(1-x,y), \tag{14}$$

(v) *Survival copula*: If C is a copula then the survival copula $\widehat{C}$ is given by (see [23])

$$\widehat{C}(x,y)=x+y-1+C(1-x,1-y).$$

(vi) *Cycle shifting*: If C is a copula and $s\in]0,1[$ then the cycle shiftings $C_s^{x\text{-cshift}}$ and $C_s^{y\text{-cshift}}$ are given by (see [7])

$$C_s^{x\text{-cshift}}(u,v)=\begin{cases}C(u+s,v)-C(s,v) & \text{if } u\in[0,1-s],\\ C(u+s-1,v)+v-C(s,v) & \text{otherwise,}\end{cases}$$
$$C_s^{y\text{-cshift}}(u,v)=\begin{cases}C(u,v+s)-C(u,s) & \text{if } v\in[0,1-s],\\ C(u,v+s-1)+u-C(u,s) & \text{otherwise.}\end{cases}$$

The only copula whose ultramodularity is preserved by flipping or cycle shifting is the product copula Π. From Remark 2.9(iv) we know that a necessary condition for a copula C to be ultramodular is that $C\leqslant\Pi$ holds, i.e., C is Negative Quadrant Dependent (NQD). However, flipping changes the property NQD into PQD (Positive Quadrant Dependent, see [23]), i.e., if $C\leqslant\Pi$ then $C^-\geqslant\Pi$ and $C_-\geqslant\Pi$, implying that the only ultramodular copula remaining ultramodular after flipping is Π. If C is a copula and h_c a horizontal section thereof, then the corresponding horizontal section h_c^* of the copula $C_s^{x\text{-cshift}}$ is given by

$$h_c^*(u)=\begin{cases}h_c(s+u)-h_c(s) & \text{if } u\in[0,1-s],\\ h_c(u+s-1)+c-h_c(s) & \text{otherwise.}\end{cases}$$

Then, if h_c is convex, h_c^* is convex only if h_c is linear, i.e., if $h_c(u) = c\cdot u$, implying $C = \Pi$. Analogous reasoning based on vertical sections can be used for y-cycle shifting.

Moreover, based on Proposition 2.7, no non-trivial ordinal sum or g-ordinal sum of copulas can be ultramodular. On the other hand, a W-ordinal sum of copulas is ultramodular if and only if each summand copula is ultramodular. Similarly, a survival copula $\widehat{C}$ is ultramodular if and only if C is ultramodular.

4. Structure of ultramodular aggregation functions

Denote, for $n\in\mathbb{N}$, by $\mathcal{U}_n$ the set of n-ary ultramodular aggregation functions, and put $\mathcal{U}=\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$. Because of Theorem 3.2, the set $\mathcal{U}$ is closed under composition of functions. Moreover, this means that each $\mathcal{U}_n$ is a convex set (it is even a compact subset of the set of all functions from $[0,1]^n$ to $[0,1]$, equipped with the topology of pointwise convergence). The set $\mathcal{U}_1$ consists of all convex, nondecreasing functions $f:[0,1]\to[0,1]$ satisfying $f(0) = 0$ and $f(1) = 1$, and its smallest and greatest elements are $\mathbf{1}_{\{1\}}$ and $\mathrm{id}_{[0,1]}$, respectively. Since for each $f\in\mathcal{U}_1$ its restriction $f|_{[0,1[}$ is continuous, it is possible to write f as a convex combination of a continuous element g of $\mathcal{U}_1$ and $\mathbf{1}_{\{1\}}$: indeed, $f = \lambda g + (1-\lambda)\mathbf{1}_{\{1\}}$ where $\lambda = f(1^-)$ and λg is the continuous extension of $f|_{[0,1[}$.

We start with showing that for an $A\in\mathcal{U}_n$ to be continuous it is sufficient to show that it is continuous at the point $\mathbf{1}$:

Lemma 4.1. *An n-ary ultramodular aggregation function* $A:[0,1]^n\to[0,1]$ *is continuous if and only if* $\sup\{A(\mathbf{x})|\mathbf{x}\in[0,1[^n\} = 1$.

Proof. Suppose that $A\in\mathcal{U}_n$ is non-continuous, but continuous at the point $\mathbf{1}$. Because of the monotonicity of A, there is some non-continuous one-dimensional section of A. From the convexity of this section we know that this non-continuity can occur only in its right endpoint. This means that there is some $i\in\{1,\dots,n\}$ and some $\mathbf{x}\in[0,1]^n$ such that for all $s\in[0,1[$

$$A(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n)-A(x_1,\dots,x_{i-1},s,x_{i+1},\dots,x_n)\geqslant\varepsilon>0.$$

Since A is continuous in $\mathbf{1}$ there is an $\alpha\in[0,1[$ such that

$$A(\mathbf{1}) - A(1,\ldots,1,\alpha,1,\ldots,1) < \varepsilon.$$

Putting $\mathbf{u} = (x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n)$ and $\mathbf{v} = (1,\ldots,1,\alpha,1,\ldots,1)$ we obtain $A(\mathbf{u} \vee \mathbf{v}) - A(\mathbf{v}) < \varepsilon$ and $A(\mathbf{u}) - A(\mathbf{u} \wedge \mathbf{v}) \geqslant \varepsilon$, contradicting the supermodularity of A. The converse implication is obvious. □

Based on that, we have the following decomposition of elements of $\mathcal{U}_n$ for $n > 1$:

Proposition 4.2. *Each function $A \in \mathcal{U}_n$ can be written as a convex combination $A = \lambda A^* + (1-\lambda)A^{**}$ where $\lambda = \sup\{A(\mathbf{x}) | \mathbf{x} \in [0,1[^n\}$, $A^* \in \mathcal{U}_n$ is continuous and A^{**} is an n-ary aggregation function with $A^{**}(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^n$.*

Proof. The monotonicity of A and the continuity of each of its one-dimensional sections imply that $A|_{[0,1[^n}$ is continuous (compare Remark 1.3(ii) in [18] for the case $n = 2$). Let $B : [0,1]^n \to [0,1]$ be the (unique) continuous extension of $A|_{[0,1[^n}$. If $\lambda = B(\mathbf{1}) = 0$ then $B = 0$, and A^* can be chosen arbitrarily and $A^{**} = A$. If $\lambda > 0$ then $A^* = \frac{1}{\lambda}B$ is a continuous element of $\mathcal{U}_n$. Now, if $\lambda = 1$ then $A^* = B = A$, and A^{**} can be chosen arbitrarily. If $0 < \lambda < 1$ then $A^{**} = \frac{A-B}{1-\lambda}$, and we have $A^{**}(\mathbf{1}) = 1$. Because of $A|_{[0,1[^n} = B|_{[0,1[^n}$ we get $A^{**}(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^n$. The monotonicity of A^{**} is equivalent to the monotonicity of its one-dimensional sections which is non-trivial only if, for some fixed $\mathbf{a} \in [0,1]^n$, one of its coordinates equals 1. Without loss of generality, consider the section $h : [0,1] \to [0,1]$ given by $h(x) = A(x,1,a_3,\ldots,a_n)$. For each $\varepsilon \in]0,1[$, the ultramodularity of A implies

$$A(y,1,a_3,\ldots,a_n) - A(x,1,a_3,\ldots,a_n) \geqslant A(y,1-\varepsilon,a_3,\ldots,a_n) - A(x,1-\varepsilon,a_3,\ldots,a_n) \tag{15}$$

for all $x,y \in [0,1]$ with $x < y$. Taking the limit $\varepsilon \to 0$, (15) turns into

$$A(y,1,a_3,\ldots,a_n) - A(x,1,a_3,\ldots,a_n) \geqslant B(y,1,a_3,\ldots,a_n) - B(x,1,a_3,\ldots,a_n),$$

implying $h(x) \leqslant h(y)$. □

Remark 4.3.

(i) The aggregation function A^{**} mentioned in Proposition 4.2 is not ultramodular, in general. Indeed, define $A \in \mathcal{U}_2$ by

$$A(x,y) = \begin{cases} \max\left(5xy - \frac{9}{2}, 0\right) & \text{if } (x,y) \in [0,1[^2, \\ \max\left(\frac{5}{9}y, 5y-4\right) & \text{if } x = 1, \\ \max\left(\frac{5}{9}x, 5x-4\right) & \text{if } y = 1. \end{cases}$$

Then $\lambda = \frac{1}{2}$, and A^* and A^{**} are given by $A^*(x,y) = \max(10xy - 9, 0)$ and

$$A^{**}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min\left(\frac{10}{9}y, 1\right) & \text{if } x = 1, \\ \min\left(\frac{10}{9}x, 1\right) & \text{if } y = 1. \end{cases}$$

The section $h : [0,1] \to [0,1]$ given by $h(x) = A^{**}(x,1) = \min\left(\frac{10}{9}x, 1\right)$ is not convex, i.e., A^{**} is not ultramodular.

(ii) Because of Proposition 4.2, the ultramodularity of an n-ary aggregation function implies its continuity up to the right boundary of $[0,1]^n$, extending a similar fact for non-decreasing functions to dimension n.

Given a fixed $\mathbf{v} \in [0,1]^n$ and $\mathbf{t}_1,\ldots,\mathbf{t}_k \in [0,\infty[^n$ we define

$$E_{\mathbf{v};\mathbf{t}_1,\ldots,\mathbf{t}_k} = \left\{ \mathbf{x} \in [0,1]^n \,\middle|\, \mathbf{x} = \mathbf{v} + \sum_{j=1}^{k} \alpha_j \cdot \mathbf{t}_j \text{ and } \alpha_1,\ldots,\alpha_k \in [0,1] \right\}.$$

Evidently, to cover all possible k-dimensional sections of $[0,1]^n$, it is enough to consider $k \leqslant n$ and independent vectors $\mathbf{t}_1,\ldots,\mathbf{t}_k$. As special cases (with $k = 1$) we mention the ith one-dimensional section with arbitrary $\mathbf{v}$ and $\mathbf{t} = \mathbf{e}_i = (0,\ldots,0,1,0,\ldots,0)$, and the diagonal section with $\mathbf{v} = (0,\ldots,0)$ and $\mathbf{t} = (1,\ldots,1)$. As a consequence of Definition 2.6, for a given ultramodular aggregation function $A : [0,1]^n \to [0,1]$, also the restriction $A|_{E_{\mathbf{v};\mathbf{t}_1,\ldots,\mathbf{t}_k}}$ of A to $E_{\mathbf{v};\mathbf{t}_1,\ldots,\mathbf{t}_k}$ is ultramodular.

Theorem 4.4. *Let $A : [0,1]^n \to [0,1]$ be an aggregation function with $A(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^n$. Then A is ultramodular if and only if the following hold:*

(i) *all $(n-1)$-dimensional sections $B_i = A|_{E_i}$ of A, $i \in \{1,\ldots,n\}$, are ultramodular, where $E_i = E_{\mathbf{e}_i;\mathbf{e}_1,\ldots,\mathbf{e}_{i-1},\mathbf{e}_{i+1},\ldots,\mathbf{e}_n}$.*

(ii) *for all $i,j \in \{1,\ldots,n\}$ with $i \neq j$ and all $\mathbf{x} \in E_i \cap E_j$ we have*

$$A(\mathbf{x}) \geqslant \sup\{B_i(\mathbf{y}) | \mathbf{y} \in E_i, \mathbf{y} < \mathbf{x}\} + \sup\{B_j(\mathbf{z}) | \mathbf{z} \in E_j, \mathbf{z} < \mathbf{x}\}.$$

Proof. Since the last inequality follows from the ultramodularity of A, the necessity is obvious. Conversely, evidently each one-dimensional section of A is either constant zero up to the endpoint (and thus convex) or it coincides with some one-dimensional section of some B_i, again showing its convexity. The validity of (ii) is trivial if $\mathbf{x} \in [0,1[^n$ or $\mathbf{y} \in [0,1[^n$. Suppose that $\mathbf{x},\mathbf{y} \in [0,1]^n \setminus [0,1[^n$. Then there are $i,j \in \{1,\ldots,n\}$ with $\mathbf{x} \in E_i$ and $\mathbf{y} \in E_j$. If $\mathbf{x} \wedge \mathbf{y} \in [0,1]^n \setminus [0,1[^n$ then we may suppose $i = j$, and (ii) follows from the supermodularity of B_i. If $\mathbf{x} \wedge \mathbf{y} \in [0,1[^n$ then $i \neq j$ and $\mathbf{x} \vee \mathbf{y} \in E_i \cap E_j$, in which case (ii) follows from the fact that $\mathbf{x} < \mathbf{x} \wedge \mathbf{y}$ and $\mathbf{y} < \mathbf{x} \wedge \mathbf{y}$. □

Example 4.5. An aggregation function $A : [0,1]^2 \to [0,1]$ is non-continuous, ultramodular and satisfies $A(\mathbf{x}) = 0$ for all $\mathbf{x} \in [0,1[^2$ if and only if there are numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0,1]$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leqslant 1$ and continuous, non-decreasing convex functions $f,g : [0,1] \to [0,1]$ such that

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \lambda_1 + \lambda_2 \cdot f(x) & \text{if } (x,y) \in [0,1[\times \{1\}, \\ \lambda_3 + \lambda_4 \cdot g(y) & \text{if } (x,y) \in \{1\} \times [0,1[, \\ 1 & \text{otherwise.} \end{cases}$$

The smallest non-continuous ultramodular binary aggregation function vanishing on $[0,1[^2$ is $\mathbf{1}_{\{(1,1)\}}$, and there is no greatest aggregation function of this type. However, for each $\alpha \in [0,1]$, the function A_α given by

$$A_\alpha(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \alpha & \text{if } (x,y) \in [0,1[\times \{1\}, \\ 1-\alpha & \text{if } (x,y) \in \{1\} \times [0,1[, \\ 1 & \text{otherwise} \end{cases}$$

is a maximal non-continuous ultramodular binary aggregation function vanishing on $[0,1[^2$.

We have the following characterization of maximal continuous binary ultramodular aggregation functions:

Proposition 4.6. *A function $A : [0,1]^2 \to [0,1]$ is a maximal continuous ultramodular aggregation function (i.e., there is no continuous ultramodular aggregation function $B : [0,1]^2 \to [0,1]$ with $B(x,y) \geqslant A(x,y)$ for all $(x,y) \in [0,1]^2$ and $B(x_0,y_0) > A(x_0,y_0)$ for some $(x_0,y_0) \in [0,1]^2$) if and only if A is a weighted arithmetic mean, i.e., $A(x,y) = \lambda \cdot x + (1-\lambda) \cdot y$ for some $\lambda \in [0,1]$.*

Proof. Note that for each supermodular aggregation function $A : [0,1]^2 \to [0,1]$ we necessarily have $A(1,0) + A(0,1) \leqslant 1$. Moreover, if $A(1,0) + A(0,1) = 1$, i.e., $A(1,1) - A(1,0) - A(0,1) + A(0,0) = 0$, then for each rectangle $[x,x^*] \times [y,y^*]$ necessarily $A(x^*,y^*) - A(x^*,0) - A(x,y^*) + A(x,y) = 0$, and the aggregation function is additive, i.e., $A(x,y) = A(x,0) + A(0,y)$. On the other hand, since the one-dimensional sections of the ultramodular aggregation function A are convex, we get $A(x,0) \leqslant x \cdot A(1,0) = \lambda \cdot x$ and $A(0,y) \leqslant y \cdot A(0,1) = (1-\lambda) \cdot y$, where $\lambda = A(1,0)$. Then clearly each weighted arithmetic mean given by $A(x,y) = \lambda \cdot x + (1-\lambda) \cdot y$ is a maximal element of the set of all continuous elements of $\mathcal{U}_2$. Moreover, these facts also prove that each element of $\mathcal{U}_2$ satisfying $A(1,0) + A(0,1) = 1$ which is different from the weighted arithmetic mean is bounded from above by a corresponding weighted arithmetic mean (with coinciding values at the corner points of the unit square). On the other hand, if for each continuous element A of $\mathcal{U}_2$ we put $a = A(1,0)$ and $b = A(0,1)$ and if $a + b < 1$ then, as already mentioned, $A(x,0) \leqslant x \cdot a$, and evidently $A(0,y) \leqslant b \cdot y < (1-a) \cdot y$. Moreover, due to the convexity of the one-dimensional sections of A, we have $A(1,y) \leqslant a + (1-a) \cdot y$ and $A(x,1) \leqslant (1-a) + a \cdot x$, implying $A(x,y) \leqslant a \cdot x + (1-a) \cdot y$ for all $(x,y) \in [0,1]^2$. Hence the weighted arithmetic mean B given by $B(x,y) = a \cdot x + (1-a) \cdot y$ satisfies $B \geqslant A$ and $B(0,1) > A(0,1)$, i.e., A is not a maximal element of the set of all continuous members of $\mathcal{U}_2$. □

Propositions 2.5 and 2.7 imply the following representation for binary ultramodular aggregation functions:

Corollary 4.7. *If $A \in \mathcal{U}_2$ then we have*

$$A = \lambda \cdot A_1 + (1-\lambda) \cdot A_2, \tag{16}$$

where A_1 is a modular element of $\mathcal{U}_2$, A_2 is a supermodular binary aggregation function with annihilator 0, and $\lambda = 1 - A(1,0) - A(0,1) \in [0,1]$.

Proof. If $A \in \mathcal{U}_2$ then, as a consequence of Proposition 2.5, there exist non-decreasing functions $g_1,g_2,g_3,g_4 : [0,1] \to [0,1]$ with $g_i(0) = 0$ and $g_i(1) = 1$ for all $i \in \{1,2,3,4\}$, a copula C and numbers $a,b,c \in [0,1]$ with $a + b + c = 1$ such that, for all $(x,y) \in [0,1]^2$,

$$A(x,y) = a \cdot g_1(x) + b \cdot g_2(y) + c \cdot C(g_3(x), g_4(y)) = \lambda \cdot A_1 + (1-\lambda) \cdot A_2,$$

where $\lambda = a + b$, $A_1(x,y) = \frac{1}{a+b} \cdot (a \cdot g_1(x) + b \cdot g_2(y))$ whenever $\lambda > 0$ (and A_1 an arbitrary modular element of $\mathcal{U}_2$ if $\lambda = 0$), and $A_2(x,y) = C(g_3(x),g_4(y))$ (if $\lambda = 1$ then A_2 can be chosen arbitrarily). The supermodularity of A_2 is a consequence of the supermodularity of C. □

A full characterization of the elements of the set $\mathcal{U}_2$ is still missing. Obviously, if $A_2 \in \mathcal{U}_2$ has annihilator 0 then (16) yields an element $A \in \mathcal{U}_2$ for each modular $A_1 \in \mathcal{U}_2$ and each $\lambda \in [0,1]$.

Remark 4.8.

(i) The supermodular aggregation function A_2 mentioned in Corollary 4.7 is not ultramodular, in general. Take, e.g., $A \in \mathcal{U}_2$ given by

$$A(x,y) = \begin{cases} \frac{4}{3}xy & \text{if } (x,y) \in \left[0,\frac{1}{2}\right]^2, \\ \frac{2x+2y-1}{3} & \text{otherwise.} \end{cases}$$

Then $\lambda = \frac{2}{3}$, $A_1(x,y) = \frac{f(x)+f(y)}{2}$ with $f(x) = \max(2x-1,0)$ and $A_2(x,y) = g(x) \cdot g(y)$ with $g(x) = \min(2x,1)$, and A_2 is supermodular. However, A_2 is not ultramodular, since the section $h:[0,1] \to [0,1]$ given by $h(x) = A(x,1) = g(x)$ is not convex.

(ii) If A_2 is a supermodular binary aggregation function with annihilator 0 which is not ultramodular then the set $[\lambda_0,1]$ of all λ such that, for some modular $A_1 \in \mathcal{U}_2$, the convex combination $\lambda \cdot A_1 + (1-\lambda) \cdot A_2$ is ultramodular, is a proper subset of $[0,1]$. In other words, $\lambda_0 = 0$ if and only if A_2 is ultramodular. It is not difficult to show that, for A_1 and A_2 considered in (i), we have $\lambda_0 = \frac{2}{3}$.

(iii) There are supermodular binary aggregation functions A_2 with annihilator 0 such that the set $[\lambda_0,1]$ in (ii) is trivial, i.e., $\lambda_0 = 1$ (in which case A_2 is necessarily non-ultramodular). An example for such an A_2 is the geometric mean, i.e., $A_2(x,y) = \sqrt{x \cdot y}$ (note that it has unbounded partial derivatives).

Proposition 4.9. *A binary aggregation function $A \in \mathcal{U}_2$ can be written as in (16) with $A_1 \in \mathcal{U}_2$ being modular and $A_2 \in \mathcal{U}_2$ having annihilator 0 if and only if, for all $r,s \in [0,1]$, the functions $h_r, v_s:[0,1] \to [0,1]$ given by $h_r(x) = A(x,r) - A(x,0)$ and $v_s(y) = A(s,y) - A(0,y)$, respectively, are convex.*

Proof. Assume that A can be written as in (16). If $\lambda < 1$ then

$$h_r(x) = \lambda \cdot (A_1(x,r) - A_1(x,0)) + (1-\lambda) \cdot A_2(x,r),$$
$$v_s(y) = \lambda \cdot (A_1(s,y) - A_1(0,y)) + (1-\lambda) \cdot A_2(s,y).$$

Note that $(1-\lambda) \cdot A_2(\cdot,r)$ and $(1-\lambda) \cdot A_2(s,\cdot)$ are multiples of sections of $A_2 \in \mathcal{U}_2$ and, therefore, convex. Similarly, $\lambda \cdot (A_1(\cdot,r) - A_1(\cdot,0))$ and $\lambda \cdot (A_1(s,\cdot) - A_1(0,\cdot))$ are convex, implying that h_r and v_s convex. If $\lambda = 1$ then $A_1 = A$, implying that $h_r = A(0,r)$ and $v_s = A(s,0)$ are constant and, therefore, convex.

Conversely, suppose that all the functions h_r and v_s are convex. Because of Corollary 4.7, $A = \lambda \cdot A_1 + (1-\lambda) \cdot A_2$ with $A_1 \in \mathcal{U}_2$ being modular and A_2 being supermodular with annihilator 0. Then $v_s(y) = \lambda \cdot A_1(s,y) + (1-\lambda) \cdot A_2(s,y)$, and the convexity of v_s implies that either $\lambda = 1$ (in which case $A = A_1$ and A_2 can be chosen arbitrarily) or $g:[0,1] \to [0,1]$ given by $g(y) = A_2(s,y)$ is a convex section of A_2. Thus, if A is not modular then A_2 is necessarily ultramodular. □

Now we present a way to construct continuous ultramodular aggregation operators from (possibly) non-ultramodular ones.

Proposition 4.10. *Let $C:[0,1]^2 \to [0,1]$ be a copula, $f,g:[0,1] \to [0,1]$ non-decreasing surjections, and assume that all sections of $A_2:[0,1]^2 \to [0,1]$ given by $A_2(x,y) = C(f(x),g(y))$ have bounded second derivatives. Then there is a $\lambda \in [0,1[$ and a modular aggregation function $A_1 \in \mathcal{U}_2$ such that $A:[0,1]^2 \to [0,1]$ given by $A = \lambda \cdot A_1 + (1-\lambda) \cdot A_2$ is an ultramodular aggregation function.*

Proof. Define the functions $\alpha, \beta : [0,1] \to \mathbb{R}$ by

$$\alpha(x) = \inf\left\{ \frac{\partial^2}{\partial x^2} A_2(x,y) \,\middle|\, y \in [0,1] \right\},$$
$$\beta(y) = \inf\left\{ \frac{\partial^2}{\partial y^2} A_2(x,y) \,\middle|\, x \in [0,1] \right\},$$

respectively, and the functions $\gamma, \delta, \varepsilon, \zeta : [0,1] \to \mathbb{R}$ by

$$\gamma(x) = \int_0^x \max(-\alpha(u),0)du, \quad \varepsilon(x) = \int_0^x \max(-\beta(u),0)du,$$
$$\delta(x) = \int_0^x \gamma(u)du, \quad \zeta(x) = \int_0^x \varepsilon(u)du,$$

respectively. Then it is not difficult to check that, for $\lambda_0 = \frac{\delta(1)+\zeta(1)}{\delta(1)+\zeta(1)+1}$, for each $\lambda \in [\lambda_0, 1]$ and for all $a, b \in [0,1]$ with $a + b = \frac{\lambda-\lambda_0}{\lambda\cdot(1-\lambda_0)}$, the function $A_1 : [0,1]^2 \to [0,1]$ given by

$$A_1(x,y) = a\cdot x + \frac{1-\lambda}{\lambda}\cdot\delta(x) + b\cdot y + \frac{1-\lambda}{\lambda}\cdot\zeta(y)$$

is modular, implying that $A = \lambda \cdot A_1 + (1-\lambda)\cdot A_2$ is an ultramodular aggregation operator. □

Example 4.11. Using the notations of Proposition 4.10 and of its proof, put $C = \Pi$ and define the functions f and g by $f(x) = 2x - x^2$ and $g(x) = \frac{3x-x^3}{2}$. Note that the function A_2 is not ultramodular. Then we get $\alpha(x) = -2,\ \beta(x) = -3x,\ \delta(x) = x^2,\ \zeta(x) = \frac{x^3}{2}$, and $\lambda_0 = \frac{3}{5}$. Finally, we obtain the two-parametric family $(A_{\lambda,a})_{\lambda\in[\frac{3}{5},1],a\in[0,\frac{5\lambda-3}{2\lambda}]}$ of ultramodular aggregation functions given by

$$A_{\lambda,a}(x,y) = \lambda\left(ax + \frac{1-\lambda}{\lambda}x^2 + \left(\frac{5\lambda-3}{2\lambda} - a\right)y + \frac{1-\lambda}{\lambda}\frac{y^3}{2}\right) + (1-\lambda)\frac{(2x-x^2)\cdot(3y-y^3)}{2}.$$

5. Concluding remarks

We have discussed the structure and some construction methods for ultramodular aggregation functions. These functions play an important role in the areas of supermodular measures (compare, e.g., [4]) and of bivariate copulas, among others. In our future research we will study ultramodular binary copulas as well as an application of ultramodular aggregation functions in the areas mentioned above. Copulas of higher dimensions describe the stochastic dependence structure of k-dimensional random vectors with $k > 2$, and they are linked to a stronger form of convexity of one-dimensional functions. For example, in the case of Archimedean copulas, the corresponding additive generator has a derivative of $(k-2)$th order which is convex [21]. Following the idea of ultramodular functions given in (9), in a next step we will propose and study stronger versions of ultramodularity related to that, leading to the stronger forms of convexity mentioned above in the case of functions in one variable. We expect that this approach will be of help in constructing copulas of higher dimensions—so far, only few such methods are known in the literature.

Acknowledgments

The third author was supported by the Grants APVV-0012-07, VEGA 1/0080/10 and GACR P402/11/0378. The authors also thank the three anonymous referees whose valuable comments helped to improve the paper.

References

[1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Heidelberg, 2007.
[2] G. Birkhoff, Lattice Theory, American Mathematical Society, Providence, 1973.
[3] H.W. Block, W.S. Griffith, T.H. Savits, L-superadditive structure functions, Adv. Appl. Prob. 21 (1989) 919–929.
[4] A.G. Bronevich, On the closure of families of fuzzy measures under eventwise aggregations, Fuzzy Sets Syst. 153 (2005) 45–70.
[5] G. Choquet, Theory of capacities, Ann. Inst. Fourier (Grenoble) 5 (1953/1954) 131–292.
[6] B. De Baets, H. De Meyer, Orthogonal grid constructions of copulas, IEEE Trans. Fuzzy Syst. 15 (2007) 1053–1062.
[7] B. De Baets, H. De Meyer, J. Kalická, R. Mesiar, Flipping and cyclic shifting of binary aggregation functions, Fuzzy Sets Syst. 160 (2009) 752–765.
[8] D. Denneberg, Non-Additive Measure and Integral, Kluwer Academic Publishers, Dordrecht, 1994.
[9] F. Durante, R. Mesiar, P.L. Papini, C. Sempi, 2-increasing binary aggregation operators, Inform. Sci. 177 (2007) 111–129.
[10] F. Durante, S. Saminger-Platz, P. Sarkoci, On representations of 2-increasing binary aggregation functions, Inform. Sci. 178 (2008) 4534–4541.
[11] F. Durante, S. Saminger-Platz, P. Sarkoci, Rectangular patchwork for bivariate copulas and tail dependence, Commun. Stat. Theory Methods 38 (2009) 2515–2527.
[12] F. Durante, C. Sempi, On the characterization of a class of binary operations on bivariate distribution functions, Publ. Math. Debrecen 69 (2006) 47–63.
[13] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, Cambridge, 2009.
[14] H. Joe, Multivariate models and dependence concepts, Monographs on Statistics and Applied Probability, vol. 73, Chapman & Hall, London, 1997.
[15] J.H.B. Kemperman, On the FKG-inequality for measures on a partially ordered space, Nederl. Akad. Wetensch. Proc. Ser. A 39 (1977) 313–331.
[16] C. Kimberling, On a class of associative functions, Publ. Math. Debrecen 20 (1973) 21–39.
[17] E.P. Klement, A. Kolesárová, R. Mesiar, C. Sempi, Copulas constructed from horizontal sections, Commun. Stat. Theory Methods 36 (2007) 2901–2911.
[18] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[19] D. Lau, Function Algebras on Finite Sets. A Basic Course on Many-Valued Logic and Clone Theory, Springer, Berlin, 2006.
[20] M. Marinacci, L. Montrucchio, Ultramodular functions, Math. Oper. Res. 30 (2005) 311–332.
[21] A.J. McNeil, J. Nešlehová, Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions, Ann. Stat. 37 (2009) 3059–3097.
[22] R. Mesiar, V. Jágr, M. Juráňová, M. Komorníková, Univariate conditioning of copulas, Kybernetika (Prague) 44 (2008) 807–816.
[23] R.B. Nelsen, An introduction to Copulas, Lecture Notes in Statistics, second ed., vol. 139, Springer, New York, 2006.
[24] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht, 1995.
[25] A.W. Roberts, D.E. Varberg, Convex Functions, Academic Press, New York, 1973.
[26] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[27] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.
[28] K. Sundaresan, Monotone gradients on Banach lattices, Proc. Am. Math. Soc. 98 (1986) 448–454.
[29] D.M. Topkis, Minimizing a submodular function on a lattice, Oper. Res. 26 (1978) 305–321.
[30] J. van Tiel, Convex Analysis, John Wiley & Sons, New York, 1984.
[31] R. Webster, Convexity, Oxford University Press, Oxford, 1994.