Sufficiency conditions for pole assignment in column-regularizable implicit linear systems

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***Abstract*— The paper is devoted to the problem of pole assignment by state feedback in non-square implicit linear systems. In particular, the proof of Theorem 4.6 in [5] (here Theorem 1) is completed by a proof of sufficiency conditions, providing a complete solution to the problem of pole assignment in the case of column regularizable systems.**

I. Introduction

The main subject of the study is the implicit linear system

$$E\dot{x}(t) \;=\; Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

where $E, A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times m}$ are matrices over $\mathbb{R}$, the field of real numbers, and $x(t)$, $u(t)$ are the state and control input of the system, respectively. The system (1) will frequently be referred to as the triple (E, A, B). System (1) is considered in the general case, when q is not necessary equal to n, and such a system is called *non-square*. Non-square systems arise for example in networks modeling, signal flow graphs, Petri nets, and can be applied to circuit systems, composite systems [1], [2].

Applying the linear and proportional state feedback

$$u(t) \;=\; Fx(t) + v(t), \tag{2}$$

where $F \in \mathbb{R}^{m\times n}$ and $v(t)$ is a new control input, to the system (1), gives rise to the closed-loop system

$$E\dot{x}(t) \;=\; [A + BF]x(t) + Bv(t). \tag{3}$$

By choosing different state feedback gains F, we alter the response of the closed-loop system. In particular, to shape the desired system response one assigns the prescribed pole structure to the closed-loop system by choosing the appropriate matrix F in (2). Such a problem is called the pole structure assignment by state feedback [7]. A simpler version of this version is called pole assignment problem. In particular, it deals with the assignment of the prescribed (finite and infinite) poles to the system (3) using a control of the form (2). These problems belong to the most important ones in control and are of great practical interest. For example, they are used or the design of controller as well as observer [4].

In [5] the problem of pole assignment is considered in the non-square systems and there are given necessary conditions of its solvabiity. Here, the proof is presented for the sufficiency of that conditions in the so-called column regularizable systems, which is defined below.

II. Background

The symbol $\lhd$ stands for the divisibility of the polynomials $\alpha(s), \beta(s) \in \mathbb{R}[s]$, i.e. $\alpha(s) \lhd \beta(s)$ ($\beta(s) \rhd \alpha(s)$) means $\alpha(s)$ divides $\beta(s)$, and the degree, $\deg x(s)$, of a polynomial vector $x(s) \in \mathbb{R}^k[s]$ is the greatest degree of all its entries $x_i(s)$. Accordingly, the degree of column i of a polynomial matrix $M(s) \in \mathbb{R}^{p\times m}[s]$ is denoted by $\deg_{ci} M(s)$. Such a matrix is called *column reduced* if it can be written in the form $M(s) = M_{lc}\,\mathrm{diag}\,\{s^{c_i}\}_{i=1}^{m} + \bar{M}(s)$, where $M_{lc} \in \mathbb{R}^{p\times m}$ is of full column rank and $\bar{M}(s) \in \mathbb{R}^{p\times m}[s]$ is such that $\deg_{ci} \bar{M}(s) < c_i := \deg_{ci} M(s)$. Two polynomial matrices $A(s)$ and $B(s)$ are said to be *equivalent*, we then write $A(s) \cong B(s)$, if there exist unimodular matrices $U(s)$ and $V(s)$ over $\mathbb{R}[s]$ such that $A(s) = U(s)B(s)V(s)$. A polynomial matrix of degree 1 is called a *matrix pencil*.

The system (1) is called *regular* if the pencil $sE - A$ is regular, i.e. E and A are square, and $\det[sE{-}A]$ is not identically equal to zero. The system (1) is called *regularizable* by state feedback if there exists an F such that the pencil $sE - A - BF$ is regular. In the case of non-square systems an analogous concept, *weak regularizability*, is defined in [5]. The system (1) is called *weakly (row or column) regularizable* if the pencil $sE{-}A{-}BF$ is of full row or column rank for some $F \in \mathbb{R}^{m\times n}$. The weak regularizability seems to be a pertinent property of system (1) since it guaranties the existence of a transfer function, possibly non-unique.

The *pole structure* of the system (E, A, B) is defined by the zero structure of the pencil $sE - A$. The *finite zero structure* of $sE - A$ is given by the invariant polynomials of $sE - A$,

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say $\psi_i(s) \triangleright \psi_{i+1}(s),\ i = 1, \ldots, r-1,\ r := \operatorname{rank}[sE - A]$. The *infinite zero structure* is defined [8] by the terms s^{-d_i}, $d_i > 0,\ i = 1, \ldots, k_d$, occurring in the Smith-McMillan form at infinity of $sE-A$. The integers d_i are called the infinite zero orders. The *finite poles* are given by the roots of the invariant polynomial $\psi_i(s)$ of $sE - A$, including the multiplicities, and the *pole at infinity* is described by its multiplicity

$$d := \sum_{i=1}^{k_d} d_i \ .$$

The *problem of pole assignment* by state feedback lies in finding conditions (necessary and sufficient, if possible) under which there exists an $F \in \mathbb{R}^{m\times n}$ such that the roots of a prescribed monic polynomial, say $\psi(s)$, and a positive integer, say d, will define the finite and infinite zeros of $sE-A-BF$.

The main concepts and tools used for solving the considered problem are given in [7], [5] (see also references therein) and briefly recalled below.

A. Feedback Canonical Form

Under the action of the feedback group, which consists of quadruples (P, Q, G, F), where $P,\ Q,\ G,\ F \in \mathbb{R}^{m\times n}$ are matrices over $\mathbb{R}$, $P,\ Q,\ G$ invertible, each system (E, A, B) can be brought into the *feedback canonical form* [6],

$$\begin{aligned}(P, Q, G, F) \circ (E, A, B) &= (PEQ, P[A + BF]Q, PBG) \\ &=: (E_C,\ A_C,\ B_C).\end{aligned}$$

The feedback canonical form consists of a pencil $sE_C - A_C$ and a matrix B_C that are block-partitioned, in the form

$$sE_C - A_C := \text{blockdiag } \{sE_j - A_j\}\,,\ j \in \{\epsilon, \sigma, q, p, l, \eta\},$$

where $sE_j - A_j$ is again a block diagonal matrix pencil consisting of the blocks, non-increasingly ordered by size, of types (b_j), for $j \in \{\epsilon, \sigma, q, p, l, \eta\}$,

$$(b_\epsilon)\ \left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}}^{\epsilon_i+1}\right\}\epsilon_i \qquad (b_\sigma)\ \left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s \end{bmatrix}}^{\sigma_i}\right\}\sigma_i$$

$$(b_q)\ \left.\overbrace{\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}}^{q_i}\right\}q_i{+}1 \qquad (b_p)\ \left.\overbrace{\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}}^{p_i+1}\right\}p_i{+}1$$

$$(b_l)\ \left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -a_{i0} & -a_{i1} & \cdots & s-a_{il_i} \end{bmatrix}}^{l_i}\right\}l_i \qquad (b_r)\ \left.\overbrace{\begin{bmatrix} s & & \\ -1 & \ddots & \\ & \ddots & s \\ & & -1 \end{bmatrix}}^{\eta_i}\right\}\eta_i{+}1\ ,$$

$i = 1, \ldots, k_t$, with k_t denoting the number of the corresponding blocks. The values describing these blocks are called:

- the nonproper controllability indices, $\epsilon_1 \geq \ldots \geq \epsilon_{k_\epsilon} \geq 0$;
- the proper controllability indices, $\sigma_1 \geq \ldots \geq \sigma_{k_\sigma} > 0$;
- the almost proper controllability indices, $q_1 \geq \ldots \geq q_{k_q} \geq 0$;
- the almost nonproper controllability indices, $p_1 \geq \ldots \geq p_{k_p} \geq 0$;
- the fixed invariant polynomials of $[sE_C - A_C,\ \ -B_C]$ represented by the polynomials $\alpha_i(s) = s^{l_i} + a_{il_i}s^{l_i-1} + \cdots + a_{i1}s + a_{i0}$, $l_i > 0$, $\alpha_1(s) \triangleright \alpha_2(s) \triangleright \cdots \triangleright \alpha_{k_l}(s)$;
- the row minimal indices of $[sE_C - A_C,\ \ -B_C]$, $\eta_1 \geq \ldots \geq \eta_{k_\eta} \geq 0$.

Similarly, B_C takes the form

$$B_C := \begin{bmatrix} 0 & 0 \\ B_\sigma & 0 \\ 0 & B_q \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{aligned} B_\sigma &:= \text{blockdiag}\left\{[0\cdots 01]^T \in \mathbb{R}^{\sigma_i}\right\}_{i=1}^{k_\sigma} \\ B_q &:= \text{blockdiag}\left\{[0\cdots 01]^T \in \mathbb{R}^{q_i+1}\right\}_{i=1}^{k_q} \end{aligned}$$

As the main subject of the paper is a study of the influence of state feedback (2) upon (1), the system is already assumed to be in the feedback canonical form and the index C will therefore be omitted in the sequel.

Proposition 1: [7], [5] The following holds:

(a) (E, A, B) is regularizable if $k_\epsilon = k_q$ and $k_\eta = 0$.
(b) (E, A, B) is row regularizable if $k_\epsilon \geq k_q$ and $k_\eta = 0$.
(c) (E, A, B) is column regularizable if $k_\epsilon \leq k_q$.

B. Normal External Description

Definition 1: Polynomial matrices $N(s)$, $D(s)$ are said to form a *normal external description* (NED) of the system (E, A, B) if they satisfy the following conditions:

- $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ forms a minimal polynomial basis for $\operatorname{Ker}[sE-A,\ \ -B]$, i.e.
$$[sE - A,\ -B] \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0 \qquad (4)$$
- $N(s)$ forms a minimal polynomial basis for $\operatorname{Ker}\Pi[sE-A]$, where Π is a maximal left annihilator of B.

An NED is not unique unless it is in the canonical polynomial basis [3], which will be assumed hereafter.

It should be noted that the NED reflects just those parts of $[sE - A, -B]$ that are given by the $\epsilon-$ and $\sigma-$blocks. Let $\bar{B}$ be such that $[B\ \bar{B}]$ is of full column rank and

$$\operatorname{rank}[sE - A,\ \ -[B\ \bar{B}]] = q,\ \forall s \in \mathbb{C} \cup \infty.$$

The system $(E, A, [B\ \bar{B}])$ defined in such a way is called an extended system of (1). Its NED, say $\begin{bmatrix} N_E(s) \\ D_E(s) \end{bmatrix}$, reveals the same information as the pencil $[sE - A,\ \ -[B\ \bar{B}]]$.

To handle the finite and infinite poles of (1) in a unified way, the conformal mapping $s = \frac{1+aw}{w}$, where $a \in \mathbb{R}$, and is not a pole of (E, A, B), is used. Then, the point $s = \infty$ is moved to $w = 0$, while all the finite points except $s = a$

are kept in finite positions. Applying the conformal mapping to the equation

$$\big[sE-A,\ -[B\ \bar{B}]\big]\begin{bmatrix} N_E(s) \\ D_E(s) \end{bmatrix}=0, \qquad (5)$$

premultiplying it by $\operatorname{diag}\{w^{\nu_i}\}$, $\nu_i := \deg_{ri}\big[sE-A,\ -[B\ \bar{B}]\big]$, and postmultiplying by $\operatorname{diag}\{w^{\mu_i}\}$, $\mu_i := \deg_{ci}\big[\begin{smallmatrix} N_E(s) \\ D_E(s)\end{smallmatrix}\big]$, gives

$$\big[w\tilde{E}-\tilde{A},\ -[\tilde{B}(w)\ \tilde{\bar{B}}(w)]\big]\begin{bmatrix} \tilde{N}_E(w) \\ \tilde{D}_E(w) \end{bmatrix}=0, \qquad (6)$$

which can be viewed as a w-analogue of (5). Then the action of the state feedback upon (6), and hence the extended system of (1), is described by the following relationship

$$\big[w\tilde{E}-\tilde{A}-\tilde{B}(w)F,\ -[\tilde{B}(w)\ \tilde{\bar{B}}(w)]\big]\begin{bmatrix} \tilde{N}_E(w) \\ \tilde{D}_{EF}(w) \end{bmatrix}=0$$

where

$$\tilde{D}_{EF}(w) := \tilde{D}_E(w) - \begin{bmatrix} F \\ 0 \end{bmatrix}\tilde{N}_E(w).$$

In particular, both $\big[w\tilde{E}-\tilde{A}-\tilde{B}(w)F\big]$ and $\tilde{D}_{EF}(w)$ have the same (non-unit) invariant polynomials

$$\begin{aligned}\psi_i(w) &:= w^{d_i+\deg\psi_i(s)}\psi_i\left(\frac{1+aw}{w}\right)\\ &:= w^{d_i}\tilde{\psi}_i(w),\end{aligned}$$

where d_i ($d_i := 0, i > k_d$) and $\tilde{\psi}_i(w)$ are the infinite zero orders and w-analogues of (non-unit) invariant polynomials $\psi_i(s)$ of the pencil $sE-A-BF$, respectively. So, the zero structure of the polynomial matrix $\tilde{D}_{EF}(w)$ will be investigated instead of that of the pencil $sE-A-BF$.

The matrix $\tilde{D}_{EF}(w)$ is of the form

$$\tilde{D}_{EF}(w)=\left[\begin{array}{cccccc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} & \tilde{D}_{1q} & \tilde{D}_{1p} & \tilde{D}_{1l} & \tilde{D}_{1\eta} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q+\tilde{D}_{2q} & \tilde{D}_{2p} & \tilde{D}_{2l} & \tilde{D}_{2\eta} \\ \hline 0 & 0 & Z_q & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{S}_\eta \end{array}\right] \qquad (7)$$

where

$$\tilde{S}_\sigma := \operatorname{diag}\{(1+aw)^{\sigma_i}\}_{i=1}^{k_\sigma},\ \tilde{S}_q := \operatorname{diag}\{(1+aw)^{q_i}\}_{i=1}^{k_q},$$

$$Z_q := \operatorname{diag}\{-w^{q_i}\}_{i=1}^{k_q},\quad Z_p := \operatorname{diag}\{-w^{p_i}\}_{i=1}^{k_p},$$

$$\tilde{S}_\alpha := \operatorname{diag}\{\tilde{\alpha}_i(w)\}_{i=1}^{k_l}, \tilde{S}_\eta := \operatorname{blockdiag}\left\{\begin{bmatrix}(1+aw)^{\eta_i} \\ -w^{\eta_i}\end{bmatrix}\right\}_{i=1}^{k_\eta},$$

and $\tilde{D}_{ij}$ are polynomial matrices satisfying the conditions:

- $\deg_{ci}\begin{bmatrix}\tilde{D}_{1j} \\ \tilde{D}_{2j}\end{bmatrix} \le j_i,\ \ j\in\{\epsilon,\sigma,q,p,l,\eta\},\ i=1,2,\ldots,$ (8)
- $\tilde{D}_{ij},\ j\in\{\sigma,q,l,\eta\}$ consists of the polynomials with zero constant terms, and
- $\big[\begin{smallmatrix}\tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma}\end{smallmatrix}\big]$ (or at least one of its $(k_\sigma+k_q)\times(k_q+k_\sigma)$ submatrices $\big[\begin{smallmatrix}\tilde{D}'_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} \\ \tilde{D}'_{2\epsilon} & \tilde{D}_{2\sigma}\end{smallmatrix}\big]$, $k_\epsilon > k_q$) is column reduced with the degrees equal to $j_i,\ i=1,2,\ldots, j\in\{\epsilon,\sigma\}$.

C. Problem Formulation

Given a weakly regularizable system (1), a monic polynomial $\psi(s)$, and integer $d>0$, find conditions under which there exists a matrix $F\in\mathbb{R}^{m\times n}$ such that (in w-notation) $\tilde{\psi}(w)w^d$ will be a $\operatorname{gcddm}\big[w\tilde{E}-\tilde{A}-\tilde{B}(w)F\big]$

Using the concept of NED, it follows that $\tilde{\psi}(w)w^d$ is also $\operatorname{gcddm} D_{EF}(w)$. Thus, $\operatorname{gcddm}\big[w\tilde{E}-\tilde{A}-\tilde{B}(w)F\big]$ can be replaced by $\operatorname{gcddm} D_{EF}(w)$ in the above formulation.

D. Previous Results

The results known in the case of regularizable systems are now recalled.

Proposition 2: [7] Given a regularizable system (1) ($k_\epsilon = k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d\ge 0$, then there exists a matrix F in (2) such that $\det[sE-A-BF]=\psi(s)$ and the sum of the infinite zero orders of $sE-A-BF$ equals d if and only if the conditions (9)-(11) (and (12) if $k_\epsilon=0$) are satisfied:

$$\begin{aligned}
\deg\psi(s)+d &= \sum_{i=1}^{k_\epsilon}\epsilon_i+\sum_{i=1}^{k_\sigma}\sigma_i+\sum_{i=1}^{k_q}q_i+\sum_{i=1}^{k_p}p_i+\sum_{i=1}^{k_l}l_i \quad (9)\\
\psi(s) &\rhd \alpha_1(s)\alpha_2(s)\ldots\alpha_{k_l}(s) \quad (10)\\
d &\ge \sum_{i=1}^{k_q}q_i+\sum_{i=1}^{k_p}p_i \quad (11)\\
\deg\psi(s) &= \sum_{i=1}^{k_\sigma}\sigma_i+\sum_{i=1}^{k_l}l_i \quad (12)
\end{aligned}$$

III. Main Results

Consider a column regularizable system (1) and the corresponding matrix $\tilde{D}_{EF}(w)$, see (7), with $k_q\ge k_\epsilon$. Bringing the matrix $\tilde{S}_\eta$, by elementary operations, to the form, $S_{\tilde{\eta}}\cong\big[\begin{smallmatrix}I_{k_\eta} \\ 0\end{smallmatrix}\big]$, the matrix $\tilde{D}_{EF}(w)$ will further be simplified. Particularly, the matrices $\tilde{D}_{1\eta},\tilde{D}_{2\eta}$ can be zeroed, which means that we can study just a submatrix of $\tilde{D}_{EF}(w)$, denoted as $P(w)$, that does not contain rows and columns corresponding to the η-blocks. It should also be clear that $\operatorname{gcddm}\tilde{D}_{EF}(w)=\operatorname{gcddm}P(w)$ since the only nonzero dominant minors of $\tilde{D}_{EF}(w)$ are those of $P(w)$. Thus, we will investigate the matrix $P(w)$ instead of $\tilde{D}_{EF}(w)$. To that end, let $\mathbb{S}_t^k$ denote the set of all k-tuples $\{j_1,j_2,\ldots,j_k\}$, $j_1<j_2<\ldots<j_k$, $j_i\le t$, $j_i,t\in\mathbb{N}$, the set of natural numbers, $i=1,2,\ldots,k$, $k\le t$. Let further $P^{[\boldsymbol{\alpha}]}$ and $P_{[\boldsymbol{\beta}]}$, $\boldsymbol{\alpha}\in\mathbb{S}_m^j$, $\boldsymbol{\beta}\in\mathbb{S}_n^k$, denote submatrices of an $m\times n$ matrix P consisting of rows $i_1,i_2,\ldots,i_j$ and columns $j_1,j_2,\ldots,j_k$ of P, respectively. For example, $P_{[\boldsymbol{\beta}]}^{[/\boldsymbol{\alpha}]}$, $\boldsymbol{\alpha}\in\mathbb{S}_m^j$, $\boldsymbol{\beta}\in\mathbb{S}_n^k$, where $/\boldsymbol{\alpha}:=\{1,2,\ldots,m\}-\boldsymbol{\alpha}$, denotes a submatrix of P obtained by eliminating rows $i_1,i_2,\ldots,i_j$ of P and having columns $j_1,j_2,\ldots,j_k$ of P.

Lemma 1: Let $P(s) = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ be an $(m+p)\times(n+p)$, $m-n \leq p$, polynomial matrix of full column rank with Z nonsingular and diagonal. Then

$$\operatorname{dm} P(s) = \det[X\ Y_{[/\boldsymbol{k}]}]^{[\boldsymbol{j}]} \det Z^{[\boldsymbol{k}]}_{[\boldsymbol{k}]}, \tag{13}$$

where

$$\boldsymbol{k} \in \mathbb{S}_p^{p-i},\ \boldsymbol{j} \in \mathbb{S}_m^{n+i},\ i = 0, 1, \ldots, m-n. \tag{14}$$

Proof: Clearly, the dominant minors of $P(s)$ are determinants of $(n+p)\times(n+p)$ submatrices of P, i.e.

$$\operatorname{dm} P(s) = \det P^{[\boldsymbol{j}]}(s),\ \boldsymbol{j} \in \mathbb{S}_{m+p}^{n+p}.$$

More particularly,

$$\operatorname{dm} P(s) = \det \begin{bmatrix} X^{[\boldsymbol{j}]} & Y^{[\boldsymbol{j}]} \\ 0 & Z^{[\boldsymbol{k}]} \end{bmatrix},$$

where $\boldsymbol{j}, \boldsymbol{k}$ are as in (14). Then (13) follows as a consequence of the diagonal form of Z. ∎

Theorem 1: Let a column regularizable system (1) ($k_q \geq k_\epsilon$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$ be given. Then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $\tilde{\psi}(w)w^d = \operatorname{gcddm}\left[w\tilde{E} - \tilde{A} - F\tilde{B}(w)\right]$ if and only if the conditions (15)-(19) (and (20) if $k_\epsilon = 0$) are satisfied.

$$\deg\psi(s) + d \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_\epsilon}q_i + \sum_{i=1}^{k_p}p_i + \sum_{i=1}^{k_l}l_i \tag{15}$$

$$\psi(s) \rhd \prod_{i=k_q-k_\epsilon+1}^{k_l}\alpha_i(s) \tag{16}$$

$$d \geq \sum_{i=1}^{k_\epsilon+k_p} z_i, \tag{17}$$

$$\deg\psi(s) \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_l}l_i \tag{18}$$

$$d \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_\epsilon}q_i + \sum_{i=1}^{k_p}p_i \tag{19}$$

$$d \leq \sum_{i=1}^{k_p}p_i \tag{20}$$

where equality holds in (15) for $k_\epsilon = k_q$, $\{z_i\}_{i=1}^{k_\epsilon+k_p}$ denotes the set of the first $k_\epsilon + k_p$ indices of the non-decreasingly ordered set $\{q_i\}_{i=1}^{k_q} \cup \{p_i\}_{i=1}^{k_p}$, and $\alpha_i(s) := 1$ for $k_l \leq k_q - k_\epsilon$.

Proof: A proof of necessity is given in [5].
Sufficiency. When $k_\epsilon = k_q$ the conditions of Theorem 1 turn out to be those of Proposition 2, which means that just the case $k_q > k_\epsilon$ is to be proved.

Let $k_p^\star$ and $k_q^\star$ denote the numbers of indices p_i and q_i in $\{z_i\}_{i=1}^{k_\epsilon+k_p}$ such that $k_p^\star + k_q^\star = k_p + k_\epsilon$. Let further $k_l^\star := k_l - k_q k_\epsilon$ for $k_l > k_q - k_\epsilon$ and $k_l^\star := 0$ for $k_l \leq k_q - k_\epsilon$.

To prove that the conditions (15)-(20) are sufficient, a matrix $P(w)$ will be constructed such that

$$\operatorname{gcddm} P(w) = w^{\sum z_i} \prod_{i=k_l-k_l^\star+1}^{k_l} \tilde{\alpha}_i(w) w^{d'} \psi'(w) \tag{21}$$

where

$$0 \leq d' \leq A_1 + \sum_{i=1}^{k_\epsilon} q_i - \sum_{i=k_q-k_q^\star+1}^{k_q} q_i + \sum_{i=1}^{k_p-k_p^\star} p_i \tag{22}$$

$$0 \leq \deg\psi'(w) \leq A_2 + \sum_{i=1}^{k_l-k_l^\star} l_i \tag{23}$$

$$A_1 + A_2 \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i \tag{24}$$

with $A_1 = 0$ for $k_\epsilon = 0$.

Let further $\boldsymbol{k_p^\star} \in \mathbb{S}_{k_p}^{k_p^\star}$ and $\boldsymbol{k_l^\star} \in \mathbb{S}_{k_l}^{k_l^\star}$ be such that

$$\begin{aligned} \boldsymbol{k_p^\star} &= \{k_p - k_p^\star + 1, \ldots, k_p\},\ k_p - k_p^\star = k_q^\star - k_\epsilon \\ \boldsymbol{k_l^\star} &= \{k_l - k_l^\star + 1, \ldots, k_l\} \end{aligned}$$

and let the matrix $P(w)$ be partitioned as follows:

$$\left[\begin{array}{cc|ccccc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} & \tilde{D}_{1q} & \tilde{D}_{1p[/\boldsymbol{k_p^\star}]} & \tilde{D}_{1l[/\boldsymbol{k_l^\star}]} & \tilde{D}_{1p[\boldsymbol{k_p^\star}]} & \tilde{D}_{1l[\boldsymbol{k_l^\star}]} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q + \tilde{D}_{2q} & \tilde{D}_{2p[/\boldsymbol{k_p^\star}]} & \tilde{D}_{2l[/\boldsymbol{k_l^\star}]} & \tilde{D}_{2p[\boldsymbol{k_p^\star}]} & \tilde{D}_{2l[\boldsymbol{k_l^\star}]} \\ \hdashline 0 & 0 & Z_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_{p[/\boldsymbol{k_p^\star}]}^{[/\boldsymbol{k_p^\star}]} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_{\alpha[/\boldsymbol{k_l^\star}]}^{[/\boldsymbol{k_l^\star}]} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_{p[\boldsymbol{k_p^\star}]}^{[\boldsymbol{k_p^\star}]} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{S}_{\alpha[\boldsymbol{k_l^\star}]}^{[\boldsymbol{k_l^\star}]} \end{array}\right]$$

It can be seen that if

$$\tilde{D}_{1p[\boldsymbol{k_p^\star}]} = \tilde{D}_{1l[\boldsymbol{k_l^\star}]} = \tilde{D}_{2p[\boldsymbol{k_p^\star}]} = \tilde{D}_{2l[\boldsymbol{k_l^\star}]} := 0, \tag{25}$$

then

$$\begin{aligned} \operatorname{gcddm} P(w) &= \operatorname{gcddm} P_1(w) \det Z_{p[\boldsymbol{k_p^\star}]}^{[\boldsymbol{k_p^\star}]} \det \tilde{S}_{\alpha[\boldsymbol{k_l^\star}]}^{[\boldsymbol{k_l^\star}]} = \\ &= \operatorname{gcddm} P_1(w) w^{\sum_{i=k_p-k_p^\star+1}^{k_p} p_i} \prod_{i=k_l-k_l^\star+1}^{k_l} \tilde{\alpha}_i(w) \end{aligned} \tag{26}$$

where

$$P_1(w) := \left[\begin{array}{cc|ccc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} & \tilde{D}_{1q} & \tilde{D}_{1p[/\boldsymbol{k_p^\star}]} & \tilde{D}_{1l[/\boldsymbol{k_l^\star}]} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q + \tilde{D}_{2q} & \tilde{D}_{2p[/\boldsymbol{k_p^\star}]} & \tilde{D}_{2l[/\boldsymbol{k_l^\star}]} \\ \hdashline 0 & 0 & Z_q & 0 & 0 \\ 0 & 0 & 0 & Z_{p[/\boldsymbol{k_p^\star}]}^{[/\boldsymbol{k_p^\star}]} & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_{\alpha[/\boldsymbol{k_l^\star}]}^{[/\boldsymbol{k_l^\star}]} \end{array}\right]$$

In what follows it will be shown that the matrices $\tilde{D}_{ij}$ satisfying (8) can always be chosen such that

$$\operatorname{gcddm} P_1(w) = w^{\sum_{i=k_q-k_q^\star+1}^{k_q} q_i} w^{d'} \psi'(w), \tag{27}$$

where $d', \psi'(w)$ satisfies (22), (23), respectively.

Put

$$\tilde{D}_{1q} := 0,\ \tilde{D}_{2q} := I_{k_q} - \tilde{S}_q, \tag{28}$$

which is always possible by (8), subtract the second block of rows multiplied by Z_q from the third block of rows of $P_1(w)$, and zero the matrices in the second block of rows by the third block of columns. Finally permute the second block of rows and the fifth one and the third block of columns and fifth one. The matrix $P_1(w)$ will be in the form

$$\begin{bmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} & | & \tilde{D}_{1p[/\boldsymbol{k_p^\star}]} & \tilde{D}_{1l[/\boldsymbol{k_l^\star}]} & 0 \\ A_\epsilon & A_\sigma & | & A_p & A_l & 0 \\ - & - & - & - & - & - \\ 0 & 0 & | & Z_{p[/\boldsymbol{k_p^\star}]}^{[/\boldsymbol{k_p^\star}]} & 0 & 0 \\ 0 & 0 & | & 0 & \tilde{S}_{\alpha[/\boldsymbol{k_l^\star}]}^{[/\boldsymbol{k_l^\star}]} & 0 \\ 0 & 0 & | & 0 & 0 & I_{k_q} \end{bmatrix} =: \begin{bmatrix} X & | & Y \\ - & - & - \\ 0 & | & Z \end{bmatrix}, \tag{29}$$

where

$$A_\epsilon := Z_q\ \tilde{D}_{2\epsilon}, \quad A_p := Z_q\ \tilde{D}_{2p[/\boldsymbol{k_p^\star}]}$$
$$A_\sigma := Z_q\ \tilde{D}_{2\sigma}, \quad A_l := Z_q\ \tilde{D}_{2l[/\boldsymbol{k_l^\star}]}$$

Denote

$$Z_\alpha := \begin{bmatrix} \tilde{S}_{\alpha[/\boldsymbol{k_l^\star}]}^{[/\boldsymbol{k_l^\star}]} & 0 \\ 0 & I_{k_q} \end{bmatrix}$$

The matrix $P_1(w)$ is now of the form of the matrix $P(s)$ in Lemma 1 with X, Y, Z defined by (29), $m := k_\sigma + k_q$, $n := k_\epsilon + k_\sigma$, $p := k_p - k_p^\star + k_l - k_l^\star + k_q$, and its dominant minors satisfy (13).

In view of (8), put

$$[\tilde{D}_{1p[/\boldsymbol{k_p^\star}]}\ \ \tilde{D}_{1l[/\boldsymbol{k_l^\star}]}] := 0,$$

$$\tilde{D}_{2p[/\boldsymbol{k_p^\star}]} := \begin{bmatrix} 0_{(k_q-(k_p-k_p^\star))\times(k_p-k_p^\star)} \\ \operatorname{diag}\{\beta_i(w)\} \end{bmatrix} \tag{30}$$

$$\tilde{D}_{2l[/\boldsymbol{k_l^\star}]} := \begin{bmatrix} 0_{(k_q-(k_l-k_l^\star))\times(k_l-k_l^\star)} \\ \operatorname{diag}\{\gamma_i(w)\} \end{bmatrix}$$

where $\beta_i(w)$ are polynomials that satisfy $\deg \beta_i(w) \le p_i$, $i = 1, 2, \ldots, k_p - k_p^\star$, i.e. $\forall i \in /\boldsymbol{k_p^\star}$, and $\gamma_i(w)$ are polynomials with constant terms equal to zero satisfying $\deg \gamma_i(w) \le l_i$, $i = 1, 2, \ldots, k_l - k_l^\star$, i.e. $\forall i \in /\boldsymbol{k_l^\star}$. Notice that $k_q - (k_p - k_p^\star) \ge k_\epsilon$, $k_q - (k_l - k_l^\star) \ge k_\epsilon$.

The dominant minors of the matrix $P_1(w)$ can be written in the following form.

$$\operatorname{dm} P_1(w) = \det X^{[\boldsymbol{j_1}]} \det Y_{[/\boldsymbol{k}]}^{[\boldsymbol{j_2}]} \det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]} =$$
$$\det X^{[\boldsymbol{j_1}]} \det \left[\begin{bmatrix} 0 \\ A_p \end{bmatrix}_{[/\boldsymbol{k_p'}]} \begin{bmatrix} 0 & 0 \\ A_l & 0 \end{bmatrix}_{[/\boldsymbol{k_l'}]} \right]^{[\boldsymbol{j_2}]} \det Z_{p[\boldsymbol{k_p'}]}^{[\boldsymbol{k_p'}]} \det Z_{\alpha[\boldsymbol{k_l'}]}^{[\boldsymbol{k_l'}]}$$

where

$$\begin{aligned} &\boldsymbol{j_1} \in \mathbb{S}_{k_\sigma+k_q}^{k_\epsilon+k_\sigma}, && \boldsymbol{k_p'} \in \mathbb{S}_{k_p-k_p^\star}^{k_p-k_p^\star-i_1} \\ &\boldsymbol{j_1} \cup \boldsymbol{j_2} = \boldsymbol{j}, && \boldsymbol{k_l'} \in \mathbb{S}_{k_l-k_l^\star+k_q}^{k_l-k_l^\star+k_q-i_2} \\ &i,\ \boldsymbol{j},\ \boldsymbol{k} \text{ are as in (13)}, && i_1 + i_2 = i \end{aligned} \tag{31}$$

More explicitly,

$$\operatorname{dm} P_1(w) = \det X^{[\boldsymbol{j_1}]} \left\{ \prod_{\forall i \in /\boldsymbol{k_p'}} \left(w^{q_{k_q-(k_p-k_p^\star)+i}} \beta_i(w) \right) \prod_{\forall i \in \boldsymbol{k_p'}} w^{p_i} \right.$$
$$\left. \prod_{\forall i \in /\boldsymbol{k_l'}} \left(w^{q_{k_q-(k_l-k_l^\star)+i}} \gamma_i(w) \right) \prod_{\forall i \in \boldsymbol{k_l'}} \tilde{\alpha}_i(w) \right\} \tag{32}$$

where $\gamma_i(w) := 0$ and $\tilde{\alpha}_i(w) := 1$ for $i > k_l - k_l^\star$. The relationship (32) implies that

$$\begin{aligned} \operatorname{gcddm} P_1(w) &= \gcd(\det X^{[\boldsymbol{j_1}]},\ \boldsymbol{j_1} \in \mathbb{S}_{k_\sigma+k_q}^{k_\epsilon+k_\sigma})\ G \\ &= \operatorname{gcddm} X\ G, \end{aligned}$$

where G denotes the $\gcd$ of the bracketed expression in (32) for all $\boldsymbol{k_p'}, \boldsymbol{k_l'}$.

Let

$$\operatorname{gcddm} X := w^{\bar{d}} \bar{\psi}(w),$$
$$G := w^{\hat{d}}\ \hat{\psi}(w),$$

where $\bar{\psi}(w)$ and $\hat{\psi}(w)$ are coprime polynomials.

Now G will be investigated in more detail. First, the following holds for all $\boldsymbol{k_p'}, \boldsymbol{k_l'}$ satisfying (31)

$$\boldsymbol{k_p'} \cup /\boldsymbol{k_p'} = \{1, 2, \ldots, k_p - k_p^\star\}$$
$$\boldsymbol{k_l'} \cup /\boldsymbol{k_l'} = \{1, 2, \ldots, k_l - k_l^\star, \ldots k_l - k_l^\star + k_q\}$$

Next, consider the "boundary" subsets of $\mathbb{S}_{k_p-k_p^\star}^{k_p-k_p^\star-i_1}$ and $\mathbb{S}_{k_l-k_l^\star+k_q}^{k_l-k_l^\star+k_q-i_2}$ and the corresponding parts of dominant minors of $P_1(w)$ that contribute to G, say $\operatorname{dm} P_{1G}(w)$. Let $\boldsymbol{k_l'} = \{1, 2, \ldots k_l - k_l^\star, \ldots, k_l - k_l^\star + k_q\}$. Then,

- if $i_1 = 0$, $\boldsymbol{k_p'} = \{1, 2, \ldots k_p - k_p^\star\}$, $/\boldsymbol{k_p'} = \emptyset$ and
$$\operatorname{dm} P_{1G}(w) = \prod_{i=1}^{k_p-k_p^\star} w^{p_i} \prod_{\forall i \in \boldsymbol{k_l'}} \tilde{\alpha}_i(w)$$
- if $i_1 = k_p - k_p^\star$, $\boldsymbol{k_p'} = \emptyset$, $/\boldsymbol{k_p'} = \{1, 2, \ldots, k_p - k_p^\star\}$ and
$$\operatorname{dm} P_1(w)_G = \prod_{i=1}^{k_p-k_p^\star} \left(w^{q_{k_q-(k_p-k_p^\star)+i}} \beta_i(w) \right) \prod_{\forall i \in \boldsymbol{k_l'}} \tilde{\alpha}_i(w)$$

Evidently, the value $\hat{d}$ is constrained by these two $\operatorname{dm} P_{1G}(w)$. In particular, if all $\beta_i(w)$ are divisible by w, then the smallest $\hat{d} = \sum_{i=k_q-k_q^\star+k_\epsilon+1}^{k_q} q_i$ (recall that $k_p - k_p^\star = k_q^\star - k_\epsilon$). If $\beta_i(w)$ is not divisible by w, the value of $\hat{d}$ can be increased up to $\sum_{i=1}^{k_p-k_p^\star} p_i$. To sum up, for all i_1, $0 \le i_1 \le k_p - k_p^\star$, the inequalities

$$\sum_{i=k_q-k_q^\star+k_\epsilon+1}^{k_q} q_i \ \le\ \hat{d} \le \sum_{i=1}^{k_p-k_p^\star} p_i.$$

are satisfied.

Analogously, to estimate the value of $\deg \hat{\psi}(w)$, it is sufficient to consider the below subsets of $\mathbb{S}_{k_l-k_l^\star+k_q}^{k_l-k_l^\star+k_q-i_2}$ and the corresponding $\operatorname{dm} P_{1G}(w)$ with $\boldsymbol{k_p'} = \{1, 2, \ldots k_p - k_p^\star\}$. Then,

- if $i_2 = 0$, $\boldsymbol{k_l'} = \{1, 2, \dots k_l - k_l^\star, \dots, k_l - k_l^\star + k_q\}$, $/\boldsymbol{k_l'} = \emptyset$ and $\mathrm{dm}\, P_{1G}(w) = \prod_{i=1}^{k_l - k_l^\star} \tilde{\alpha}_i(w) \prod_{\forall i \in \boldsymbol{k_p'}} w^{p_i}$
- if $i_2 = k_l - k_l^\star$, $\boldsymbol{k_l'} = \{k_l - k_l^\star + 1, k_l - k_l^\star + 2, \dots, k_l - k_l^\star + k_q\}$, $/\boldsymbol{k_l'} = \{1, 2, \dots, k_l - k_l^\star\}$ and $\mathrm{dm}\, P_{1G}(w) = \prod_{i=1}^{k_l - k_l^\star} \left(w^{q_{k_q - (k_l - k_l^\star) + i}} \gamma_i(w) \right) \prod_{\forall i \in \boldsymbol{k_p'}} w^{p_i}$

It can be seen that the degree of the non-divisible part by w that can be assigned to G cannot exceed $\sum_{i=1}^{k_l - k_l^\star} l_i$ and reaches its maximal value if the polynomials $\gamma_i(w)$ are zero. On the other this degree can reach zero if the polynomials $\gamma_i(w)$ and $\tilde{\alpha}_i(w)$ are coprime. At the end, there always exist matrices $\tilde{D}_{ij}$ such that G satisfies the following set of inequalities.

$$\sum_{i=k_q - k_q^\star + k_\epsilon + 1}^{k_q} q_i \ \le\ \hat{d} \le \sum_{i=1}^{k_p - k_p^\star} p_i, \qquad (33)$$

$$0 \ \le\ \deg \hat{\psi}(w) \ \le\ \sum_{i=1}^{k_l - k_l^\star} l_i, \qquad (34)$$

Consider now the $\mathrm{gcddm}\, X$. It follows that the matrices $\tilde{D}_{ij}$ satisfying (8) can be chosen such that

$$\sum_{i=k_q - k_\epsilon + 1}^{k_q} q_i \ \le \bar{d} \ \le A_1 + \sum_{i=1}^{k_\epsilon} q_i \qquad (35)$$

$$0 \ \le \deg \bar{\psi}(w) \ \le A_2 \qquad (36)$$

where

$$A_1 + A_2 \ \le \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i$$

When $k_\epsilon = 0$, the $\mathrm{gcddm}\, X$ is not divisible by w, which implies

$$\bar{d} = 0 \ \Leftrightarrow\ A_1 = 0. \qquad (37)$$

The conditions (33)-(37) then directly lead to

$$\mathrm{gcddm}\, P_1(w) = w^{\hat{d} + \bar{d}} \hat{\psi}(w) \bar{\psi}(w)$$

where

$$\sum_{i=k_q - k_q^\star + 1}^{k_q} q_i \le \hat{d} + \bar{d} \le A_1 + \sum_{i=1}^{k_p - k_p^\star} p_i + \sum_{i=1}^{k_\epsilon} q_i \ ,$$

$$0 \le \deg \left(\hat{\psi}(w) \bar{\psi}(w) \right) \le A_2 + \sum_{i=1}^{k_l - k_l^\star} l_i,$$

and $A_1 = 0$ if $k_\epsilon = 0$, which shows that (27) holds. Taking into the account (26), the relationship (21) follows. Then, having the matrix $\tilde{D}_{NF}(w)$, a state feedback gain can be calculated using the relationship $\tilde{D}_{NF}(w) = -F\tilde{N}_E(w)$. ■

IV. Conclusion

The problem of pole assignment by state feedback to the column regularizable systems (1) is considered in the paper. Necessary conditions of its solvability established in [5] are extended by proving their sufficiency. The results are stated in Theorem 1 that gives necessary and sufficient conditions for pole placement.

Acknowledgment

The work was supported by the Grant Agency of the Czech Republic under contract No. 103/12/2431(P. Zagalak) and the Technology Agency of the Czech Republic, Competence Centres Programme, project TE01020197 Centre for Applied Cybernetics 3 (V.Kučera).

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