RISK MEASURES VIA HEAVY TAILS

VLASTA KAŇKOVÁ

Abstract.
Economic and financial activities are often influenced simultaneously by a decision parameter and a random factor. Since mostly it is necessary to determine the decision parameter without knowledge of a random element realization, deterministic optimization problems depending on a probability measure correspond often to such situations. In applications very often the problem has to be solved on the data basis. Great effort has been paid to investigate properties of these (empirical) estimates; mostly under assumptions of "thin" tails and a linear dependence on the probability measure. The aim of this contribution is to focus on the cases when these assumptions are not fulfilled. This happens usually in economic and financial applications (see e.g. [10], [12], [14],[18]).

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1 INTRODUCTION

Let (Ω, S, P) be a probability space; $\xi (:= \xi(\omega) = [\xi_1(\omega), \ldots, \xi_s(\omega)])$ an s–dimensional random vector defined on (Ω, S, P); $F (:= F(z), z \in R^s)$ the distribution function of ξ; P_F the probability measure corresponding to F. Let, moreover, $g_0 (:= g_0(x, z))$ be a function defined on $R^n \times R^s$; $X_F \subset R^n$ a nonempty set generally depending on F, $X \subset R^n$ a nonempty "deterministic" set. If E_F denotes the operator of mathematical expectation corresponding to F, then static "classical" stochastic optimization problem can be introduced in the form:

Find

$$\phi(F, X_F) = \inf\{\mathrm{E}_F g_0(x, \xi) \mid x \in X_F\}. \tag{1}$$

The objective function in (1) depends linearly on the probability measure P_F. Recently appear problems that can be covered only by more general type of problems:

Find

$$\bar{\phi}(F, X_F) = \inf\{\mathrm{E}_F \bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi)) \mid x \in X_F\}, \tag{2}$$

where $h := h(x, z) = (h_1(x, z), \ldots, h_{m_1}(x, z))$ is m_1–dimensional vector function defined on $R^n \times R^s$, $\bar{g}_0 (:= \bar{g}_0(x, z, y))$ is a real–valued function defined on $R^n \times R^s \times R^{m_1}$.

Let us recall and analyze some simple examples that can appear:

1. If $L := (L(x,z))$ (defined on $R^n \times R^s$) represents a loss function, then $VaR_\alpha(x) := \min_u \{P\{\omega : L(x,\xi) \le u\} \ge \alpha\}, \alpha \in (0,1)$ can be considered as a risk measure, known as "Value–at –Risk" (see e.g. [3]).
   Setting $X_F := \{x \in X : [\min_u P\{\omega : L(x,\xi) \le u\} \ge \alpha] \le u_0\}$, u_0 constant; we can obtain the problem with risk measure in constraints.
2. $CVaR_\alpha(x) = \min_{v \in R}[v + \frac{1}{1-\alpha} \mathrm{E}_F(L(x,\xi) - v)^+]$ is another risk measure known as "Conditional Value–at–Risk". Setting $\bar{g}_0(x,z,y) := CVaR_\alpha(x)$ we obtain function not depending linearly on the probability measure. However, since according to [15]

$$\min_{x \in X} CVaR_\alpha(x) = \min_{(v,x) \in R^1 \times X} \{v + \frac{1}{1-\alpha} \mathrm{E}_F(L(x,\xi) - v)^+\}, \tag{3}$$

   the dependence of the objective on the probability measure is already linear.
3. Employing Markowitz approach to very simple portfolio problem:

$$\text{Find} \quad \max \sum_{k=1}^{n} \xi_k x_k \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1,\ldots,n, \quad s = n,$$

   with x_k a fraction of the unit wealth invested in the asset k, ξ_k the return of the asset, we can introduce the Markowitz problem (see e.g. [2]):

Find

$$\begin{gathered} \phi^M(F) = \max\{\sum_{k=1}^{n} \mu_k x_k - K \sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,j} x_j\} \quad \text{s. t.} \quad \sum_{k=1}^{n} x_k \le 1, \\ x_k \ge 0, \quad k = 1,\ldots,n, \quad K > 0 \text{ constant}, \end{gathered} \tag{4}$$

where $\mu_k = \mathrm{E}_F \xi_k$, $c_{k,j} = \mathrm{E}_F(\xi_k - \mu_k)(\xi_j - \mu_j)$, $k, j = 1,\ldots n$. The dependence on the probability measure in (4) is not linear. $\sum_{j=1}^{n} x_k c_{k,j} x_j$ can be considered as a risk measure that can be replaced, for example by $\mathrm{E}_F |\sum_{k=1}^{n} \xi_k x_k - \mathrm{E}_F[\sum_k \xi_k x_k]|$ (see [9]). The dependence on the probability measure is again nonlinear.

In applications often we have to replace the measure P_F by an empirical measure P_{F^N}.

Consequently, (instead of the problems (1) and (2)) the following problems are solved:

Find

$$\phi(F^N, X_{F^N}) = \inf\{\mathrm{E}_{F^N} g_0(x,\xi) \mid x \in X_{F^N}\}. \tag{5}$$

Find

$$\bar{\phi}(F^N, X_{F^N}) = \inf\{\mathrm{E}_{F^N} \bar{g}_0(x,\xi,\mathrm{E}_{F^N} h(x,\xi)) \mid x \in X_{F^N}\}. \tag{6}$$

Solving (5) and (6) we obtain estimates of the optimal values and optimal solutions. Their investigation started in [20], followed by many papers (see e.g. [1], [5], [6], [7], [17]). There consistency, the convergence rate and asymptotic distribution have been studied under the assumptions of "weak" tails distributions, $X_F = X$ and linear dependence of objective function on the probability measure. The exception are e.g. papers [4], [8] and [14]. We focus on the problem (2), the case of "heavy" tails and $X_F := X$.

2 SOME DEFINITIONS AND AUXILIARY ASSERTIONS

Let $F_i, i = 1,\ldots s$ denote one–dimensional marginal distribution functions corresponding to F; $P(R^s)$ the set of Borel probability measures on $R^s, s \geq 1$;

$M_1(R^s) = \{P \in P(R^s) : \int_{R^s} \|z\|_s^1 P(dz) < \infty\}$; $\|\cdot\|_s^1$ denote L_1 norm in R^s. We introduce the assumptions:

B. 1. $P_F, P_G \in M_1(R^s)$, there exist $\varepsilon > 0$ such that

- $\bar{g}_0(x,z,y)$ is for $x \in X(\varepsilon), z \in R^s$ a Lipschitz function of $y \in Y(\varepsilon)$ with a Lipschitz constant L^y;

  $Y(\varepsilon) = \{y \in R^{m_1} : y = h(x,z) \quad \text{for some} \quad x \in X(\varepsilon), z \in R^s\}$, $\mathrm{E}_F h(x,\xi)$, $\mathrm{E}_G h(x,\xi) \in Y(\varepsilon)$,
- for every $x \in X(\varepsilon), y \in Y(\varepsilon)$ there exist finite mathematical expectations,

  $\mathrm{E}_F \bar{g}_0(x,\xi,\mathrm{E}_F h(x,\xi))$, $\mathrm{E}_F g_0^1(x,\xi,\mathrm{E}_G h(x,\xi))$, $\mathrm{E}_G g_0^1(x,\xi,\mathrm{E}_F h(x,\xi))$, $\mathrm{E}_G g_0^1(x,\xi,\mathrm{E}_G h(x,\xi))$,
- $h_i(x,z), i = 1,\ldots, m_1$ are for every $x \in X(\varepsilon)$ Lipschitz functions of z with the Lipschitz constants L_h^i (corresponding to L_1 norm),
- $\bar{g}_0(x,z,y)$ is for every $x \in X(\varepsilon), y \in Y(\varepsilon)$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L^z (corresponding to L_1 norm).

B. 2. $\bar{g}_0(x, z, y), h(x, z)$ are uniformly continuous functions on $X(\varepsilon)\times R^s \times Y(\varepsilon)$,

B. 3. X is a convex set and $\bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi))$ a convex function on $X(\varepsilon)$.

($X(\varepsilon), \varepsilon > 0$ denotes $\varepsilon-$neighbourhood of X.)

Proposition 1. [8] Let $P_F, P_G \in M_1(R^s)$, the assumptions B.1 be fulfilled, then there exist $\hat{C} > 0$ such that it holds for $x \in X$

$$|\mathrm{E}_F \bar{g}_0(x, \xi, \mathrm{E}_F h(x, \xi)) - \mathrm{E}_G \bar{g}_0(x, \xi, \mathrm{E}_G h(x, \xi))| \leq \hat{C} \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| dz_i. \quad (7)$$

Proposition 1 reduces s–dimensional case to one dimensional. Of course a stochastic dependence between components of the random vector is there neglected. The idea to reduce s–dimensional case to one dimensional appeared already in [11].

3 PROBLEM ANALYSIS

To employ the Proposition 1 to empirical estimates we introduce the assumptions:

A.2. - $\{\xi^i\}_{i=1}^{\infty}$ is independent random sequence corresponding to F ,

- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$,

A.3. $P_{F_i}, i = 1, \ldots s$ are absolutely continuous w. r. t. the Lebesguemeasure on the R^1.

Lemma 1 [19] Let $s = 1$, $P_F \in M_1(R^1)$ and A.2 be fulfilled. Then

$$P\{\omega : \int_{-\infty}^{\infty} |F(z) - F^N(z)| dz \xrightarrow[N \to \infty]{} 0\} = 1.$$

Proposition 2. [4], [8] Let $s = 1, t > 0$ and A.2, A.3 be fulfilled, $\aleph$ denotes the set of natural numbers. If there exists $\beta > 0, R := R(N) > 0$ defined on $\aleph$ such that $R(N) \xrightarrow[N \to \infty]{} \infty$ and, moreover,

$$N^{\beta} \int_{-\infty}^{-R(N)} F(z) dz \xrightarrow[N \to \infty]{}, \qquad N^{\beta} \int_{R(N)}^{\infty} [1 - F(z)] \xrightarrow[N \to \infty]{} 0,$$

$$2NF(-R(N)) \xrightarrow[N \to \infty]{} 0, \qquad 2N[1 - F(R(N))] \xrightarrow[N \to \infty]{} 0, \quad (8)$$

$$(\frac{12 N^{\beta} R(N)}{t} + 1) \exp\{-2N(\frac{t}{12 R(N) N^{\beta}})^2\} \xrightarrow[N \to \infty]{} 0,$$

$$\text{then} \quad P\{\omega : N^{\beta} \int_{-\infty}^{\infty} |F(z) - F^N(z)| dz > t\} \xrightarrow[N \to \infty]{} 0. \quad (9)$$

Evidently, the validity of the relation (9) depends on the tails behaviour.

Proposition 3. [4] Let $s=1, t>0, r>0$ and A.2, A.3 be fulfilled. Let moreover ξ be a random variable such that $\mathrm{E}_F |\xi|^r < \infty$. If constants $\beta, \gamma > 0$ fulfil the inequalities $0<\beta+\gamma<1/2, \quad \gamma>1/r, \quad \beta+(1-r)\gamma<0$, then the relation (9) is valid.

4. MAIN RESULTS

Applying the assertions of former parts we obtain.

Theorem 1. Let the assumptions B.1, A.2, A.3 and either B.2 or B.3 be fulfilled, X be a compact set and $P_F \in M_1(R^s)$. Then

$$P\{\omega : |\bar{\phi}(F^N, X) - \bar{\phi}(F, X)| \xrightarrow[N\to\infty]{} 0\} = 1.$$

Proof. The assertion of Theorem 1 follows from Proposition 1 and Lemma 1.

If $f_i, i=1,\ldots,s$ denote the probability densities corresponding to F_i, then it holds.

Theorem 2. Let the assumptions B.1, A.2, A.3 be fulfilled, $P_F \in M_1(R^s), t>0$. If

- for some $r>2$ it holds that $\mathrm{E}_{F_i} |\xi_i|^r < +\infty, \quad i=1,\ldots,s$,
- $\beta, \gamma > 0$ fulfil the inequalities $0<\beta+\gamma<1/2, \quad \gamma>1/r, \quad \beta+(1-r)\gamma<0$,

then

$$P\{\sup_{x\in X} N^{\beta} |\mathrm{E}_{F^N} \bar{g}_0(x,\xi,\mathrm{E}_{F^N} h(x,\xi)) - \mathrm{E}_F \bar{g}_0(x,\xi,\mathrm{E}_F h(x,\xi))| > t\} \xrightarrow[N\to\infty]{} 0. \qquad (10)$$

If moreover either B.2 or B.3 is valid and X is a compact set , then also

$$P\{\omega : N^{\beta} |\bar{\phi}(F, X) - \bar{\phi}(F^N, X)| > t\} \xrightarrow[N\to\infty]{} 0. \qquad (11)$$

Proof The first assertion follows from Propositions 1,2,3. The second assertion follows from first one and from the properties of the convex functions and the integrals. (See a similar proof for the problem (1) in [4]).

Evidently, the convergence rate $\beta := \beta(r)$ introduced by Theorem 2 depends on the absolute moments existence; it holds that $\beta(r) \xrightarrow[r\to\infty]{} 1/2, \quad \beta(r) \xrightarrow[r\to 2^+]{} 0$. Consequently, the best convergence rate is valid not only for exponential tails but also for every distribution with finite all absolute moments (e.g. Weibull and lognormal); even in the case when finite moment generating function does not exist. Unfortunately we can not obtain (by this approach) any results in the case when there exist only finite $\mathrm{E}_F \, | \xi_i |^r, i = 1, \ldots, s$ for $r < 2$. This is the case of stable distributions (with exception of normal distribution) or the case of Pareto distribution with a shape parameter $\alpha \leq 2$.

5. CONCLUSION

The paper generalizes the results concerning the rate convergence of empirical estimates of static stochastic optimization problems depending linearly on probability measure [4] to the case when this assumption is not fulfilled. The corresponding simulation results presented in [4] can be employed in this more generalized case also. Employing some growth conditions (see e.g. [16]) the introduced results can be transformed to the estimates of the optimal solution. However the investigation in this direction as the investigation in the case of stable distributions is beyond the scope of this paper.

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RNDr. Vlasta Kaňková, CSc.

Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic

Department of Econometrics

Pod Vodárenskou věží 4, 182 08 Praha 8

Czech Republic

email: kankova@utia.cas.cz