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On some advanced type inequalities for Sugeno integral and T-(S-)evaluators

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ABSTRACT

In this paper strengthened versions of the Minkowski, Chebyshev, Jensen and Hölder inequalities for Sugeno integral and T-(S-)evaluators are given. As an application, some equivalent forms and some particular results have been established.

1. Introduction

The theory of nonadditive measures and integrals was a powerful tool in several fields [6,13]. Sugeno integral [29] is a useful tool in several theoretical and applied statistics. For instance, in decision theory, the Sugeno integral is a median, which is indeed a qualitative counterpart to the averaging operation underlying expected utility [7].

In most decision-making problems a global preference functional is used to help the decision-maker make the "best" decision. Of course, the choice of such a global preference functional is dictated by the behavior of the decision-maker but also by the nature of the available information, hence by the scale type on which it is represented. The use of the Sugeno integral can be envisaged from two points of view: decision under uncertainty and multi-criteria decision-making [8]. Sugeno integral is analogous to Lebesgue integral which has been studied by many authors, including Pap [11,23], Ralescu and Adams [24] and Wang and Klir [30], among others.

Integral inequalities play important roles in classical probability and measure theory. These are useful tools in several theoretical and applied fields. For instance, integral inequalities play a role in the development of a time scales calculus [22]. In general, any integral inequality can be a very powerful tool for applications. In particular, when we think of an integral operator as a predictive tool then an integral inequality can be very important in measuring and dimensioning such process. The study of inequalities for Sugeno integral was initiated by Román-Flores et al. [9,25–28], and then followed by the

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authors [1,3,4,15–17]. Recently, the authors generalized several classical inequalities, including Minkowski's, Chebyshev's and Hölder's inequalities, to the frame of Sugeno integral [1,2,15,18].

The aim of this paper is strengthened versions of the Minkowski, Chebyshev and Hölder type inequalities for Sugeno integral and relate them to T-evaluators and S-evaluators. As an application, some equivalent forms and some particular results have been established.

The paper is arranged as follows. For convenience of the reader, in the next section, we review some basic concepts and summarization of some previous known results. In Sections 3 and 4, we construct strengthened versions of the Minkowski, Chebyshev and Hölder type inequalities for Sugeno integral and relate them to T-evaluators and S-evaluators. Finally, some conclusions are given.

2. Preliminaries

In this section, we are going to review some well known results from the theory of nonadditive measures, Sugeno's integral and T-(S-)evaluators. For details, we refer to [24,29,30,12,5].

As usual we denote by **R** the set of real numbers. Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Let **N** denote the set of all positive integers and $\overline{\mathbf{R}_+}$ denote $[0,+\infty]$. Throughout this paper, we fix the measurable space $(X,\mathcal{F})$, and all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 2.1 ([24]). A set function $\mu:\mathcal{F}\to\overline{\mathbf{R}_+}$ is called a nonadditive measure if the following properties are satisfied:

(FM1) $\mu(\emptyset)=0$;
(FM2) $A\subset B$ implies $\mu(A)\leqslant\mu(B)$;
(FM3) $A_1\subset A_2\subset\cdots$ implies $\mu(\bigcup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$; and
(FM4) $A_1\supset A_2\supset\cdots$, and $\mu(A_1)<+\infty$ imply $\mu(\bigcap_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$.

When μ is a nonadditive measure, the triple $(X,\mathcal{F},\mu)$ then is called a nonadditive measure space.

Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space, by $\mathcal{F}_+(X)$ we denote the set of all nonnegative measurable functions $f:X\to[0,\infty)$ with respect to $\mathcal{F}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. Let f be a nonnegative real-valued function defined on X, we will denote the set $\{x\in X|f(x)\geqslant\alpha\}$ by F_α for $\alpha\geqslant 0$. Clearly, F_α is nonincreasing with respect to α, i.e., $\alpha\leqslant\beta$ implies $F_\alpha\supseteq F_\beta$. Moreover, for any fixed k in $(0,\infty)$ denote by $\mathcal{F}_k(X)$ the set of all measurable functions $f:X\to[0,k]$. Observe that the system $(\mathcal{F}_k(X))$ is strictly increasing and $\bigcup\mathcal{F}_k(X)\subsetneqq\mathcal{F}_+(X)$.

Definition 2.2 ([23,30]). Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $A\in\mathcal{F}$, the Sugeno integral of f on A, with respect to the nonadditive measure μ, is defined as

$$(S)\int_A f\,d\mu=\bigvee_{\alpha\geqslant 0}(\alpha\wedge\mu(A\cap F_\alpha)).$$

When $A=X$, then

$$(S)\int_X f\,d\mu=(S)\int f\,d\mu=\bigvee_{\alpha\geqslant 0}(\alpha\wedge\mu(F_\alpha)).$$

It is well known that Sugeno integral is a type of nonlinear integral [14]. I.e., for general case,

$$(S)\int(af+bg)d\mu=a(S)\int f\,d\mu+b(S)\int g\,d\mu$$

does not hold. Some basic properties of Sugeno integral are summarized in [23,30], we cite some of them in the next theorem.

Theorem 2.3 ([23,30]). *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space, then*

(i) $\mu(A\cap F_\alpha)\geqslant\alpha\Rightarrow(S)\int_A f\,d\mu\geqslant\alpha$;
(ii) $\mu(A\cap F_\alpha)\leqslant\alpha\Rightarrow(S)\int_A f\,d\mu\leqslant\alpha$;
(iii) $(S)\int_A f\,d\mu<\alpha\iff$ there exists $\gamma<\alpha$ such that $\mu(A\cap F_\gamma)<\alpha$;
(iv) $(S)\int_A f\,d\mu>\alpha\iff$ there exists $\gamma>\alpha$ such that $\mu(A\cap F_\gamma)>\alpha$;
(v) *If* $\mu(A)<\infty$, *then* $\mu(A\cap F_\alpha)\geqslant\alpha\iff(S)\int_A f\,d\mu\geqslant\alpha$;
(vi) *If* $f\leqslant g$, *then* $(S)\int f\,d\mu\leqslant(S)\int g\,d\mu$.

In [16], Ouyang and Fang proved the following result which generalized the corresponding one in [27].

Lemma 2.4. *Let m be the Lebesgue measure on* **R** *and let f: $[0,\infty) \to [0,\infty)$ be a nonincreasing function. If $(S)\int_0^a f\,dm = p$, then*

$$f(p-) \geqslant (S)\int_0^a f\,dm = p$$

for all $a \geqslant 0$, where $f(p-) = \lim_{x\to p^-} f(x)$.

Moreover, if $p < a$ and f is continuous at p, then $f(p-) = f(p) = p$.

Notice that if m is the Lebesgue measure and f is nonincreasing, then $f(p-) \geqslant p$ implies $(S)\int_0^a f\,dm \geqslant p$ for any $a \geqslant p$. In fact, the monotonicity of f and the fact $f(p-) \geqslant p$ imply that $[0,p) \subset F_p$. Thus, $m([0,a]\cap F_p) \geqslant m([0,a]\cap[0,p)) = m([0,p)) = p$. Now, by Theorem 2.3(i), we have $(S)\int_0^a f\,dm \geqslant p$.

Based on Lemma 2.4, Ouyang et al. proved some Chebyshev type inequalities [17] and their united form [15]. Notice that when proving these Theorems, the following lemma, which is derived from the transformation theorem for Sugeno integral (see [30]), plays a fundamental role.

Lemma 2.5. *Let $(S)\int_A f\,d\mu = p$. Then $\forall r \geqslant p,\ (S)\int_A f\,d\mu = (S)\int_0^r \mu(A\cap F_\alpha)dm$, where m is the Lebesgue measure.*

In this contribution, we will prove strengthened versions of the Minkowski, Chebyshev and Hölder inequalities for Sugeno integral and T-(S-)evaluators of comonotone functions. Recall that two functions $f,g: X \to \mathbf{R}$ are said to be comonotone if for all $(x,y) \in X^2$, $(f(x) - f(y))(g(x) - g(y)) \geqslant 0$. Clearly, if f and g are comonotone, then for all non-negative real numbers p, q either $F_p \subset G_q$ or $G_q \subset F_p$. Indeed, if this assertion does not hold, then there are $x \in F_p\backslash G_q$ and $y \in G_q\backslash F_p$. That is,

$$f(x) \geqslant p, g(x) < q \quad \text{and} \quad f(y) < p, g(y) \geqslant q,$$

and hence $(f(x) - f(y))(g(x) - g(y)) < 0$, contradicts with the comonotonicity of f and g. Notice that comonotone functions can be defined on any abstract space.

In [15], Mesiar and Ouyang proved the following Chebyshev type inequalities for Sugeno integral.

Theorem 2.6. *Let $f,g \in \mathcal{F}_+(X)$ and μ be an arbitrary nonadditive measure such that $(S)\int_A f d\mu$ and $(S)\int_A g d\mu$ are finite. Let $\star: [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$(S)\int_A f \star g\,d\mu \geqslant \left((S)\int_A f\,d\mu\right) \star \left((S)\int_A g\,d\mu\right) \tag{2.1}$$

holds.

It is known that

$$(S)\int_A f \star g\,d\mu \leqslant \left((S)\int_A f\,d\mu\right) \star \left((S)\int_A g\,d\mu\right), \tag{2.2}$$

where f, g are comonotone functions whenever $\star \geqslant \max$ (for a similar result, see [19]), it is of great interest to determine the operator $\star$ such that

$$(S)\int_A f \star g\,d\mu = \left((S)\int_A f\,d\mu\right) \star \left((S)\int_A g\,d\mu\right) \tag{2.3}$$

holds for any comonotone functions f, g, and for any nonadditive measure μ and any measurable set A. Ouyang et al. [21,20] proved that there are only 18 operators such that (2.3) holds, including the four well-known operators: minimum, maximum, PF (called the first projection, PF for short, if $x \star y = x$ for each pair (x,y)) and PL (called the last projection, PL for short, if $x \star y = y$ for each pair (x,y)).

Now, we give the following definitions which will be used later.

Definition 2.7 [5]. For a complete lattice $(X, \leqslant, \bot, \top)$ with the least and the greatest elements $\bot$ and $\top$, respectively, a function $\varphi: X \to [0,1]$ is said to be an evaluator on X iff it satisfies the following properties:

(1) $\varphi(\bot) = 0$, $\varphi(\top) = 1$.
(2) for all $a, b \in L$, if $a \leqslant b$ then $\varphi(a) \leqslant \varphi(b)$.

Definition 2.8 [12]. A binary operation $T: [0,1]\times[0,1] \to [0,1]$ is said to be a t-norm iff it satisfies the following properties:

(i) for each $y \in [0,1]$ $T(1,y) = y$,
(ii) for all $x,y \in [0,1]$ $T(x,y) = T(y,x)$,
(iii) for all $x,y_1,y_2 \in [0,1]$ if $y_1 \leqslant y_2$ then $T(x,y_1) \leqslant T(x,y_2)$,
(iv) for all $x,y,z \in [0,1]$ $T(x,T(y,z)) = T(T(x,y),z)$.

The four basic t-norms are:

- the minimum t-norm, $T_M(x,y) = \min\{x,y\}$,
- the product t-norm, $T_P(x,y) = x \cdot y$,
- the Łukasiewicz t-norm, $T_L(x,y) = \max\{0, x + y - 1\}$,
- the drastic product,

$$T_D(x,y) = \begin{cases} 0 & \text{if } \max\{x,y\} < 1, \\ \min\{x,y\} & \text{if } \max\{x,y\} = 1. \end{cases}$$

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm [12], if there is a t-norm T such that $S(x,y) = 1 - T(1-x, 1-y)$. Evidently, a t-conorm S satisfies:

(i′) $S(x,0) = S(0,x) = x$, $\forall x \in [0,1]$ as well as conditions (ii), (iii) and (iv). The basic t-conorms (dual of four basic t-norms) are:

- the maximum t-conorm, $S_M(x,y) = \max\{x,y\}$,
- the probabilistic sum, $S_P(x,y) = x + y - xy$,
- the Łukasiewicz t-conorm, $S_L(x,y) = \min\{1, x + y\}$,
- the drastic sum,

$$S_D(x,y) = \begin{cases} 1 & \text{if } \min\{x,y\} > 0, \\ \max\{x,y\} & \text{if } \min\{x,y\} = 0. \end{cases}$$

Definition 2.9 [5]. Consider a complete lattice $(X, \leqslant, \bot, \top)$, a t-norm T and a t-conorm S. An evaluator on X is called a T-evaluator iff for all $a, b \in X$

$T(\varphi(a), \varphi(b)) \leqslant \varphi(\min(a,b))$,
and it is called an S-evaluator iff
$S(\varphi(a), \varphi(b)) \geqslant \varphi(\max(a,b))$.

3. On some advanced type inequalities

The present section aims to provide some advanced type inequalities for Sugeno integral.

Theorem 3.1. *Let $(X, \mathcal{F}, \mu)$ be a nonadditive measure space and let $U, V : [0,\infty] \to [0,\infty]$ be continuous strictly increasing functions such that $U(x) \leqslant V(x)$ for all $x \in [0,\infty]$. If $f \in \mathcal{F}_+(X)$ is a measurable function, then the inequality*

$$U^{-1}\left((S)\int_A U(f)d\mu\right) \geqslant V^{-1}\left((S)\int_A V(f)d\mu\right) \tag{3.1}$$

holds for any $A \in \mathcal{F}$.

Proof. If $(S)\int_A U(f)d\mu = \infty$, then for any $M > 0$, $U^{-1}((S)\int_A U(f)d\mu) \geqslant U^{-1}(U(M)) = M$. Thus $U^{-1}((S)\int_A U(f)d\mu) = \infty$ and the right-hand side equals to ∞, hence (3.1) holds. If $(S)\int_A V(f)d\mu = \infty$, then for any M, we have $\mu(A \cap F_M\}) = \infty$. Then

$$\mu(A \cap \{x | U(f(x)) \geqslant U(M)\}) = \mu(A \cap F_M) \geqslant U(M).$$

Thus $U^{-1}((S)\int_A U(f)d\mu) \geqslant M$. Letting $M \to \infty$, we get the result as desired.

So, we can assume that both $(S)\int_A U(f)d\mu$ and $(S)\int_A V(f)d\mu$ are finite. Let $U^{-1}((S)\int_A U(f)d\mu) = a$ and $V^{-1}((S)\int_A V(f)d\mu) = b$. Then, Theorem 2.3(v), implies that

$$\mu(A \cap \{x | U(f) \geqslant U(a)\}) = \mu(A \cap F_a) \geqslant U(a)$$

and

$$\mu(A \cap \{x | V(f) \geqslant V(b)\}) = \mu(A \cap F_b) \geqslant V(b).$$

Since $U(x) \leqslant V(x)$ for all $x \in [0,\infty]$, then

$$U^{-1}\left((S)\int_A U(f)d\mu\right) \geqslant U^{-1}(U(b) \wedge \mu(A \cap F_b)) \geqslant U^{-1}(U(b) \wedge V(b)) \geqslant U^{-1}(U(b)) = b = V^{-1}\left((S)\int_A V(f)d\mu\right),$$

and the proof is completed. □

Corollary 3.2 [28]. *Let $(X, \mathcal{F}, \mu)$ be a nonadditive measure space and let $U : [0,\infty] \to [0,\infty]$ be continuous strictly increasing function such that $U(x) \leqslant x$ for all $x \in [0,\infty]$. If $f \in \mathcal{F}_+(X)$ is a measurable function, then the inequality*

$$(S)\int_A U(f)d\mu \geqslant U\left((S)\int_A f\,d\mu\right) \tag{3.2}$$

holds for any $A \in \mathcal{F}$.

Corollary 3.3 [28]. *Let $(X, \mathcal{F}, \mu)$ be a nonadditive measure space and let $V:[0,\infty] \to [0,\infty]$ be continuous strictly increasing function such that $x \leqslant V(x)$ for all $x \in [0,\infty]$. If $f \in \mathcal{F}_+(X)$ is a measurable function, then the inequality*

$$V\left((S)\int_A f\,d\mu\right) \geqslant (S)\int_A V(f)d\mu \tag{3.3}$$

holds for any $A \in \mathcal{F}$.

Theorem 3.4. *Fix a nonadditive measurable space $(X, \mathcal{F}, \mu)$. Let a continuous non-decreasing $\varphi:[0,\infty] \to [0,\infty]$ satisfying $\varphi(x) \geqslant x$ (or equivalently, composite $\varphi(\varphi(x)) \geqslant \varphi(x)$) for all $x \in [0,\infty]$ and a non-decreasing n-place function $H:[0,\infty]^n \to [0,\infty]$ such that H be continuous and bounded from below by maximum be given. Then for any system $U_1,\dots,U_n:[0,\infty] \to [0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1,f_2,\dots,f_n$ from $\mathcal{F}_+(X)$ it holds*

$$U^{-1}\left((S)\int_A U(H(\varphi(f_1),\dots,\varphi(f_n)))d\mu\right) \leqslant H\left(\varphi(U_1^{-1}((S)\int_A U_1(f_1)d\mu)),\dots,\varphi\left(U_n^{-1}((S)\int_A U_n(f_n)d\mu)\right)\right), \tag{3.4}$$

where $U = \max(U_1,U_2,\dots,U_n)$.

Proof. Let $(S)\int_A U_k(f_k)d\mu = p_k$ for any $k = 1,\dots,n$. Two cases can be considered:

(Case 1) Suppose that there exist

$$\{k|p_k = \infty,\ k = 1,\dots,n\}.$$

For example, let $(S)\int_A U_1(f_1)d\mu = \infty$. Then for any M, $U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right) \geqslant U_1^{-1}(U_1(M)) = M$. Therefore $U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right) = \infty$. Since $\varphi:[0,\infty] \to [0,\infty]$ is continuous and non-decreasing such that $\varphi(x) \geqslant x$ for all $x \in [0,\infty]$ and $H:[0,\infty]^n \to [0,\infty]$ is continuous and nondecreasing and bounded from below by maximum, there holds

$$H\left(\varphi\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right)\right),\dots,\varphi\left(U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right)\right) = \infty,$$

and (3.4) holds.

(Case 2) Suppose that $p_k < \infty$ for any $k = 1,\dots,n$. Let

$$U_k^{-1}\left((S)\int_A U_k(f_k)d\mu\right) = a_k \quad \text{for all } k = 1,\dots,n.$$

And $r = \max\{a_1,a_2,\dots,a_n\}$. Denote $A_k(\alpha) = \mu(A \cap \{x|U_k(f_k)(x) \geqslant \alpha\})$ and $B(\alpha) = \mu(A \cap \{x|U(H(\varphi(f_1),\varphi(f_2),\dots,\varphi(f_n)))(x) \geqslant \alpha\})$. By Lemma 2.5 we have

$$(S)\int_A U_k(f_k)d\mu = (S)\int_0^r A_k(\alpha)dm = U_k(a_k) \quad \text{for all } k = 1,\dots,n.$$

For each $\varepsilon > 0$, we have $A_k(U_k(a_k) + \varepsilon) \leqslant U_k(a_k)$. Then

$$\mu\left(A \cap \{x|f_k(x) \geqslant U_k^{-1}(U_k(a_k) + \varepsilon)\}\right) \leqslant U_k(a_k).$$

Since $\varphi:[0,\infty] \to [0,\infty]$ is continuous and non-decreasing such that $\varphi(x) \geqslant x$ for all $x \in [0,\infty]$, by the monotonicity of H and comonotonicity of $f_1,f_2,\dots,f_n$ as well as the fact that $H \geqslant \max$ we have

$$\begin{aligned}
&\mu\left(A \cap \left\{x|H(\varphi(f_1),\dots,\varphi(f_n)) \geqslant H\left(\varphi\left(U_1^{-1}(U_1(a_1)+\varepsilon)\right),\dots,\varphi\left(U_n^{-1}(U_n(a_n)+\varepsilon)\right)\right)\right\}\right) \\
&\quad \leqslant \mu\left(A \cap \left\{x|f_1 \geqslant U_1^{-1}(U_1(a_1)+\varepsilon)\right\} \cup \dots \cup \left\{x|f_n \geqslant \left(U_n^{-1}(U_n(a_n)+\varepsilon)\right)\right\}\right) \\
&\quad \leqslant \mu(A \cap \left\{x|f_1 \geqslant U_1^{-1}(U_1(a_1)+\varepsilon)\right\}) \vee \dots \vee \mu\left(A \cap \left\{x|f_n \geqslant \left(U_n^{-1}(U_n(a_n)+\varepsilon)\right)\right\}\right) \\
&\quad \leqslant A_1(U_1(a_1)+\varepsilon) \vee \dots \vee A_n(U_n(a_n)+\varepsilon) \leqslant U_1(a_1) \vee \dots \vee U_n(a_n) \leqslant U(a_1) \vee \dots \vee U(a_n) \leqslant U(H(a_1,\dots,a_n)) \\
&\quad \leqslant U(H(\varphi(a_1),\dots,\varphi(a_n))).
\end{aligned}$$

Letting $\varepsilon \to 0$, by the continuity of H and φ we have

$$\mu(A \cap \{x | H(\varphi(f_1), \ldots, \varphi(f_n)) \geqslant H(\varphi(a_1), \ldots, \varphi(a_n)) + \}) \leqslant U(H(\varphi(a_1), \ldots, \varphi(a_n))),$$

and hence $B(U(H(\varphi(a_1), \ldots, \varphi(a_n)))+) \leqslant U(H(\varphi(a_1), \ldots, \varphi(a_n)))$ and

$$\mu(A \cap \{x | U(H(\varphi(f_1), \ldots, \varphi(f_n))) \geqslant U(H(\varphi(a_1), \ldots, \varphi(a_n))) + \}) \leqslant U(H(\varphi(a_1), \ldots, \varphi(a_n))).$$

Then, Theorem 2.3(ii) implies that

$$\begin{aligned} U^{-1}\left((S)\int_A U(H(\varphi(f_1), \ldots, \varphi(f_n)))d\mu\right) &\leqslant H(\varphi(a_1), \ldots, \varphi(a_n)) \\ &= H\left(\varphi\left(U_{1'}^{-1}\left((S)\int_A U_1(f_1)d\mu\right)\right), \ldots, \varphi\left(U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right)\right). \end{aligned}$$

Hence, (3.4) is valid and the theorem is proved. □

Remark 3.5. Let $n = 2$, $\varphi(x) = x$ and $U(x) = U_1(x) = U_2(x) = x$. Then, we can use the same examples in [1] to show the necessities of $H \geqslant \max$ and the comonotonicity of f_1, f_2, and so we omit them here.

The following example shows that $\varphi(x) \geqslant x$ (or equivalently, composite $\varphi(\varphi(x)) \geqslant \varphi(x)$) for all $x \in [0, \infty]$ in Theorem 3.4 is inevitable.

Example 3.6. Let $X = [0,1]$, $f_1(x) = f_2(x) = x$, $\varphi(x) = x^2$, $U(x) = U_1(x) = U_2(x) = x$ and $H(x,y) = \min\{1, x + y\}$. If the nonadditive measure μ is defined as $\mu(A) = m^2(A)$, where m denotes the Lebesgue measure on R, then

$$(S)\int_0^1 H(\varphi(f_1), \varphi(f_2))d\mu = \bigvee_{\alpha \in [0,1]} \left(\alpha \wedge \left(1 - \frac{\sqrt{2\alpha}}{2}\right)\right) = 6 - 4\sqrt{2} = 0.34315.$$

But

$$\varphi\left((S)\int_0^1 f_1\, d\mu\right) = \varphi\left((S)\int_0^1 f_2\, d\mu\right) = \left(\frac{3 - \sqrt{5}}{2}\right)^2 = 0.1459.$$

Then

$$(S)\int_0^1 H(\varphi(f_1), \varphi(f_2))d\mu = 0.34315 > H\left(\varphi((S)\int_0^1 f_1\, d\mu), \varphi\left((S)\int_0^1 f_2\, d\mu\right)\right) = 0.2918,$$

which violates Theorem 3.4.

Corollary 3.7. *Fix a nonadditive measurable space $(X, \mathcal{F}, \mu)$. Let a non-decreasing n-place function $H : [0,\infty]^n \to [0,\infty]$ such that H be continuous and bounded from below by maximum be given. Then for any system $U_1, \ldots, U_n : [0,\infty] \to [0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_+(X)$ it holds*

$$U^{-1}\left((S)\int_A U(H(f_1, \ldots, f_n))d\mu\right) \leqslant H\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right), \ldots, U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right), \tag{3.5}$$

where $U = \max(U_1, U_2, \ldots, Un)$.

Corollary 3.8. *Let μ be an arbitrary nonadditive measure and $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. And let U_1, $U_2 : [0,\infty] \to [0,\infty]$ be continuous strictly increasing functions. If $f, g \in \mathcal{F}_+(X)$ are comonotone, then the inequality*

$$U^{-1}\left((S)\int_A U(f \star g)d\mu\right) \leqslant U_1^{-1}\left((S)\int_A U_1(f)d\mu\right) \star U_2^{-1}\left((S)\int_A U_2(f)d\mu\right) \tag{3.6}$$

holds where $U = \max(U_1, U_2)$.

Corollary 3.9 [2]. *Let μ be an arbitrary nonadditive measure and $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. And let $\varphi : [0,\infty] \to [0,\infty]$ be continuous and strictly increasing function. If $f, g \in \mathcal{F}_+(X)$ are comonotone, then the inequality*

$$\varphi^{-1}\left((S)\int_A \varphi(f \star g)d\mu\right) \leqslant \varphi^{-1}\left((S)\int_A \varphi(f)d\mu\right) \star \varphi^{-1}\left((S)\int_A \varphi(g)d\mu\right) \tag{3.7}$$

holds.

Remark 3.10. In [2], it also requires that $(S)\int_A \varphi(f \star g)d\mu < \infty$. But, as is shown in Theorem 3.4, this condition can be abandoned.

Corollary 3.11 [1]. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary nonadditive measure. And let $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality*

$$\left((S)\int_A (f \star g)^s d\mu\right)^{\frac{1}{s}} \leqslant \left((S)\int_A f^s\, d\mu\right)^{\frac{1}{s}} \star \left((S)\int_A g^s\, d\mu\right)^{\frac{1}{s}} \tag{3.8}$$

holds for all $0 < s < \infty$.

Corollary 3.12 [19]. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary nonadditive measure. And let $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality*

$$(S)\int_A (f \star g)d\mu \leqslant \left((S)\int_A f\, d\mu\right) \star \left((S)\int_A g\, d\mu\right) \tag{3.9}$$

holds.

When $V(x) = x$, $U_1(x) = x^p$ and $U_2(x) = x^q$ for all $p, q \geqslant 1$, by Corollary 3.8 and Theorem 3.1, we have the Hölder inequality.

Corollary 3.13. *Let $f,g : X \to [0,1]$ and μ be an arbitrary nonadditive measure. And let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality*

$$(S)\int_A (f \star g)d\mu \leqslant \left((S)\int_A f^p d\mu\right)^{\frac{1}{p}} \star \left((S)\int_A g^q d\mu\right)^{\frac{1}{q}} \tag{3.10}$$

holds for all $p,q \geqslant 1$.

Remark 3.14. If $\star : [0,1]^2 \to [0,1]$ is a t-conorm [12], then In Eq. (3.10) works for any comonotone functions f, g with $(S)\int_A f\, d\mu \leqslant 1$ and $(S)\int_A g\, d\mu \leqslant 1$.

Now, we construct some extensions of Minkowski, Chebyshev and Hölder type inequalities for Sugeno integral and relate them to S-evaluators.

Theorem 3.15. *Let a fixed $k \in (0,\infty)$. And let a continuous non-decreasing $\varphi : [0,k] \to [0,k]$ satisfying $\varphi(x) \geqslant x$ (or equivalently, composite $\varphi(\varphi(x)) \geqslant \varphi(x)$) for all $x \in [0,k]$ and a non-decreasing n-place function $H : [0,\infty]^n \to [0,\infty]$ such that H be continuous and bounded from below by maximum be given. Then for any system $U_1, \ldots, U_n : [0,\infty] \to [0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any nonadditive measure μ it holds*

$$U^{-1}\left((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu\right) \leqslant H\left(\varphi\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right)\right),\ldots,\varphi\left(U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right)\right), \tag{3.11}$$

where $U = \max(U_1, U_2, \ldots, U_n)$.

Proof. This is similar to the proof of Theorem 3.4 (case 2). □

Theorem 3.16. *Let a fixed $k \in (0,\infty)$. And let a non-decreasingn-place function $H : [0,\infty]n \to [0,\infty]$ such that H be continuous and bounded from below by maximum be given. Then for any system $U_1, \ldots, U_n : [0,\infty] \to [0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any nonadditive measure μ it holds*

$$U^{-1}\left((S)\int_A U(H(f_1,\ldots,f_n))d\mu\right) \leqslant H\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right),\ldots,U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right), \tag{3.12}$$

where $U = \max(U_1, U_2, \ldots, U_n)$.

Corollary 3.17 [3]. *Let a fixed $k \in (0,\infty)$. For any continuous and non-decreasing $\varphi : [0,k] \to [0,k]$ satisfying $\varphi(x) \geqslant x$ for all $x \in [0,k]$ and any non-decreasing n-place function $H : [0,\infty)^n \to [0,\infty)$ such that H be continuous and bounded from below by maximum and any comonotone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any nonadditive measure μ it holds*

$$(S)\int_A H(\varphi(f_1),\varphi(f_2)\ldots,\varphi(f_n))d\mu \leqslant H\left(\varphi\left((S)\int_A f_1\, d\mu\right),\varphi\left((S)\int_A f_2\, d\mu\right),\ldots,\varphi\left((S)\int_A f_n\, d\mu\right)\right). \tag{3.13}$$

Remark 3.18. If $H(0,\ldots,0)$ for a function H required in Corollary 3.17 holds, then H is a disjunctive (continuous) aggregation function on $[0,k]$, see [10]. Typical examples in the case $k = 1$ of such aggregation functions are continuous t-conorms, copulas, etc.

Corollary 3.19. *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $f,g:X\to[0,1]$ two comonotone measurable functions. And let $U_1,U_2:[0,\infty]\to[0,\infty]$ be continuous strictly increasing functions. If S is a continuous t-conorm and φ is continuous S-evaluator on X such that $\varphi(x)\geqslant x$, then the inequality*

$$U^{-1}\left((S)\int_A U(S(\varphi(f),\varphi(g)))d\mu\right)\leqslant S\left(\varphi\left(U_1^{-1}\left((S)\int_A U_1(f)d\mu\right)\right),\varphi\left(U_2^{-1}\left((S)\int_A U_2(g)d\mu\right)\right)\right) \tag{3.14}$$

holds for any $A\in\mathcal{F}$, where $U = \max(U_1,U_2)$.

Specially, when $U_1(x) = U_2(x) = x^s$ for all $s > 0$, we have the Minkowski inequality for S-evaluator.

Corollary 3.20. *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $f,g:X\to[0,1]$ two comonotone measurable functions. If S is a continuous t-conorm and φ is continuous S-evaluator on X such that $\varphi(x)\geqslant x$, then the inequality*

$$\left((S)\int_A (S(\varphi(f),\varphi(g)))^s d\mu\right)^{\frac{1}{s}}\leqslant S\left(\varphi\left(\left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}}\right),\varphi\left(\left((S)\int_A g^s\,d\mu\right)\right)^{\frac{1}{s}}\right) \tag{3.15}$$

holds for any $A\in\mathcal{F}$ and $0 < s < \infty$.

Again, we get an inequality related to the Chebyshev type for S-evaluator whenever $s = 1$.

Corollary 3.21 [3]. *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $f,g:X\to[0,1]$ two comonotone measurable functions. If S is a continuous t-conorm and φ is continuous S-evaluator on X such that $\varphi(x)\geqslant x$, then the inequality*

$$\left((S)\int_A S(\varphi(f),\varphi(g))d\mu\right)\leqslant S\left(\varphi\left((S)\int_A f\,d\mu\right),\varphi\left((S)\int_A g\,d\mu\right)\right) \tag{3.16}$$

holds for any $A\in\mathcal{F}$.

And, when $V(x) = x, U_1(x) = x^p$ and $U_2(x) = x^q$ for all $p,q\geqslant 1$, by Corollary 3.19 and Theorem 3.1, we have the Hölder inequality for S-evaluator.

Corollary 3.22. *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $f,g:X\to[0,1]$ two comonotone measurable functions. If S is a continuous t-conorm and φ is continuous S-evaluator on X such that $\varphi(x)\geqslant x$, then the inequality*

$$\left((S)\int_A S(\varphi(f),\varphi(g))d\mu\right)\leqslant S\left(\varphi\left(\left((S)\int_A f^p d\mu\right)^{\frac{1}{p}}\right),\varphi\left(\left((S)\int_A g^q d\mu\right)\right)^{\frac{1}{q}}\right) \tag{3.17}$$

holds for any $A\in\mathcal{F}$ and $p,q\geqslant 1$.

4. On reverse previous inequalities

In this section, we provide reverse previous inequalities for Sugeno integral.

Theorem 4.1. *Fix a nonadditive measurable space $(X,\mathcal{F},\mu)$. Let a continuous non-decreasing $\varphi:[0,\infty]\to[0,\infty]$ satisfying $\varphi(x)\leqslant x$ (or equivalently, composite $\varphi(\varphi(x))\leqslant\varphi(x)$) for all $x\in[0,\infty]$ and a non-decreasing n-place function $H:[0,\infty]^n\to[0,\infty]$ such that H be continuous and bounded from above by minimum be given. Then for any system $U_1,\ldots,U_n:[0,\infty]\to[0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1,f_2,\ldots,f_n$ from $\mathcal{F}_+(X)$ it holds*

$$U^{-1}\left((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu\right)\geqslant H\left(\varphi\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right)\right),\ldots,\varphi\left(U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right)\right), \tag{4.1}$$

where $U = \min(U_1,U_2,\ldots,U_n)$.

Proof. Let $(S)\int_A U_k(f_k)d\mu = p_k$ for any $k = 1,\ldots,n$ and let $T = \{k|p_k = \infty,\ i = 1,\ldots,n\}$. Three cases can be considered: (Case 1) Suppose that $T = n$, then $p_k = \infty$ for any $k = 1,\ldots,n$. Then for any M

$$\mu(A\cap\{x|f_k(x)\ \geqslant\ M\}) = \infty.$$

Since $\varphi:[0,\infty]\to[0,\infty]$ is continuous and non-decreasing such that $\varphi(x)\leqslant x$ for all $x\in[0,\infty]$, by comonotonicity of $f_1,f_2,\ldots,f_n$ and the monotonicity of H we have

$$\begin{aligned}\mu(A\cap\{x|U(H(\varphi(f_1),\ldots,\varphi(f_n)))\geqslant U(H(\varphi(M),\ldots,\varphi(M)))\})&\geqslant\mu(A\cap\{x|H(\varphi(f_1),\ldots,\varphi(f_n))\geqslant H(\varphi(M),\ldots,\varphi(M))\})\\&\geqslant\mu(A\cap\{x|f_1\geqslant M\})\wedge\mu(A\cap\{x|f_2\geqslant M\})\wedge\cdots\wedge\mu(A\cap\{x|f_n\geqslant M\})\geqslant U(H(M,\ldots,M))\\&\geqslant U(H(\varphi(M),\ldots,\varphi(M)))\}.\end{aligned}$$

Then, Theorem 2.3(i) implies that

$$U^{-1}\left((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu\right)\geqslant H(\varphi(M),\ldots,\varphi(M)).$$

Letting $M\to\infty$, by the continuity of H and φ, we get the desired inequality (4.1).

(Case 2) Suppose that $0<T<n$, then there exist

$$\{k|p_k=\infty,\ k=1,\ldots,n\}.$$

Without loss of generality, in this case we can assume that, $T=1$ and the other subcases can be proved similarly. For example, let $(S)\int_A U_1(f_1)d\mu=\infty$ and $(S)\int_A U_r(f_r)d\mu=U_r(p_r)<\infty,\ r=2,\ldots,n$, for some p_r then

$$\mu(A\cap\{x|f_r\geqslant p_r\})\geqslant U_r(p_r)\quad\text{and}\quad\mu(A\cap\{x|f_1(x)\geqslant M\})=\infty\text{ for any }M.$$

Thus the monotonicity of H and the comonotonicity of $f_1,f_2,\ldots,f_n$ imply that

$$\begin{aligned}\mu(A\cap\{x|H(\varphi(f_1),\varphi(f_2),\ldots,\varphi(f_n))\geqslant H(\varphi(M),\varphi(p_2),\ldots,\varphi(p_n))\})&\geqslant\mu(A\cap\{x|f_1(x)\geqslant M\}\cap\{x|f_2(x)\\&\geqslant p_2\}\cap\cdots\cap\{x|f_n(x)\geqslant p_n\})=\mu(A\cap\{x|f_1(x)\geqslant M\})\wedge\mu(\{x|f_2(x)\\&\geqslant p_2\})\wedge\cdots\wedge\mu(\{x|f_n(x)\geqslant p_n\})\geqslant U_2(p_2)\wedge U_3(p_3)\wedge\cdots\wedge U_n(p_n).\end{aligned}$$

Therefore

$$\begin{aligned}((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu)&\geqslant\begin{pmatrix}U(H(\varphi(M),\varphi(p_2),\varphi(p_3),\ldots,\varphi(p_n)))\\\wedge U_2(p_2)\wedge U_3(p_3)\wedge\cdots\wedge U_n(p_n)\end{pmatrix}\\&\geqslant\begin{pmatrix}U(H(\varphi(M),\varphi(p_2),\varphi(p_3),\ldots,\varphi(p_n)))\\\wedge U(p_2)\wedge U(p_3)\wedge\cdots\wedge U(p_n)\end{pmatrix}\\&\geqslant\begin{pmatrix}U(H(\varphi(M),\varphi(p_2),\varphi(p_3),\ldots,\varphi(p_n)))\\\wedge U(\varphi(p_2))\wedge U(\varphi(p_3))\wedge\cdots\wedge U(\varphi(p_n))\end{pmatrix}\\&=U(H(\varphi(M),\varphi(p_2),\varphi(p_3),\ldots,\varphi(p_n))),\end{aligned}$$

i.e.,

$$U^{-1}\left((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu\right)\geqslant H(\varphi(M),\varphi(p_2),\varphi(p_3),\ldots,\varphi(p_n)).$$

Letting $M\to\infty$, by the continuity of H and φ, we get the desired inequality (4.1).

(Case 3) Suppose that $T=0$, then $p_k<\infty$ for any $k=1,\ldots,n$. Let $(S)\int_A U_k(f_k)d\mu=U_k(p_k)<\infty$ for any $k=1,\ldots,n$ and some p_k. By Theorem 2.3(v) we have

$$\mu(A\cap\{x|f_k(x)\geqslant p_k\})\geqslant U_k(p_k)\quad\text{for all }k=1,\ldots,n.$$

Since $\varphi:[0,\infty]\to[0,\infty]$ is continuous and non-decreasing such that $\varphi(x)\leqslant x$ for all $x\in[0,\infty]$, by the monotonicity of H and comonotonicity of $f_1,f_2,\ldots,f_n$ as well as the fact that $H\leqslant\min$ we have

$$\begin{aligned}\mu(A\cap\{x|H(\varphi(f_1),\ldots,\varphi(f_n))\geqslant H(\varphi(p_1),\ldots,\varphi(p_n))\})&\geqslant\mu(A\cap\{x|f_1\geqslant p_1\}\cap\{x|f_2\geqslant p_2\}\cap\cdots\cap\{x|f_n\geqslant p_n\})\\&=\mu(A\cap\{x|f_1\geqslant p_1\})\wedge\mu(A\cap\{x|f_2\geqslant p_2\})\wedge\cdots\wedge\mu(A\cap\{x|f_n\geqslant p_n\})\\&\geqslant U_1(p_1)\wedge U_2(p_2)\wedge\cdots\wedge U_n(p_n)\geqslant U(p_1)\wedge U(p_2)\wedge\cdots\wedge U(p_n)\geqslant U(H(p_1,\ldots,p_n))\\&\geqslant U(H(\varphi(p_1),\ldots,\varphi(p_n))).\end{aligned}$$

Therefore

$$\begin{aligned}U^{-1}\left((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu\right)&\geqslant U^{-1}[U(H(\varphi(p_1),\ldots,\varphi(p_n)))\wedge\mu(A\cap\{x|H(\varphi(f_1),\ldots,\varphi(f_n))\\&\geqslant H(\varphi(p_1),\ldots,\varphi(p_n))\})]\geqslant U^{-1}[U(H(\varphi(p_1),\ldots,\varphi(p_n)))\wedge U(H(\varphi(p_1),\ldots,\varphi(p_n)))\\&=H(\varphi(p_1),\ldots,\varphi(p_n))=H\left(\varphi\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right)\right),\ldots,\right.\\&\left.\times\varphi\left(U_n^{-1}\left((S)\int_A U_{n+1}(f_n)d\mu\right)\right)\right).\end{aligned}$$

Hence, (4.1) is valid and the theorem is proved. □

Remark 4.2. Let $n = 2$, $\varphi(x) = x$ and $U(x) = U_1(x) = U_2(x) = x$. Then, we can use the same examples in [15] to show the necessities of $H \leqslant \min$ and the comonotonicity of f_1, f_2, and so we omit them here.

The following example shows that $\varphi(x) \leqslant x$ (or equivalently, composite $\varphi(\varphi(x)) \leqslant \varphi(x)$) for all $x \in [0,\infty]$ in Theorem 4.1 is inevitable.

Example 4.3. Let $X \in [0, \frac{1}{2}]$, $f_1(x) = x$, $f_2(x) = \frac{1}{2}$, $\varphi(x) = \sqrt{x}$, $U(x) = U_1(x) = U_2(x) = x$ and $H(x,y) = x \cdot y$. If the nonadditive measure μ is defined as $\mu(A) = m(A)$, where m denotes the Lebesgue measure on R, then

$$(S)\int_0^1 (\varphi(f_1) \times \varphi(f_2))d\mu = 0.30902, \quad \varphi\left((S)\int_0^1 f_1\, d\mu\right) = \frac{1}{2} \quad \text{and} \quad \varphi\left((S)\int_0^1 f_2\, d\mu\right) = \sqrt{\frac{1}{2}}.$$

But

$$0.30902 = (S)\int_0^1 H(\varphi(f_1), \varphi(f_2))d\mu < H\left(\varphi\left((S)\int_0^1 f_1 d\mu\right), \varphi\left((S)\int_0^1 f_2\, d\mu\right)\right) = 0.35355,$$

which violates Theorem 4.1.

Corollary 4.4. *Fix a nonadditive measurable space $(X, \mathcal{F}, \mu)$. Let a non-decreasing n-place function $H:[0,\infty]^n \to [0,\infty]$ such that H be continuous and bounded from above by minimum be given. Then for any system $U_1, \ldots, U_n : [0,\infty] \to [0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_+(X)$ it holds*

$$U^{-1}\left((S)\int_A U(H(f_1, \ldots, f_n))d\mu\right) \geqslant H\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right), \ldots, U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right), \tag{4.2}$$

where $U = \min(U_1, U_2, \ldots, U_n)$.

Corollary 4.5. *Let μ be an arbitrary nonadditive measure and $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. And let $U_1, U_2 : [0,\infty] \to [0,\infty]$ be continuous strictly increasing functions. If $f, g \in \mathcal{F}_+(X)$ are comonotone, then the inequality*

$$U^{-1}\left((S)\int_A U(f \star g)d\mu\right) \geqslant U_1^{-1}\left((S)\int_A U_1(f)d\mu\right) \star U_2^{-1}\left((S)\int_A U_2(f)d\mu\right) \tag{4.3}$$

holds where $U = \min(U_1, U_2)$.

Corollary 4.6 [2]. *Let μ be an arbitrary nonadditive measure and $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. And let $\varphi : [0,\infty) \to [0,\infty)$ be continuous and strictly increasing functions. If $f, g \in \mathcal{F}_+(X)$ are comonotone, then the inequality*

$$\varphi^{-1}\left((S)\int_A \varphi(f \star g)d\mu\right) \geqslant \varphi^{-1}\left((S)\int_A \varphi(f)d\mu\right) \star \varphi^{-1}\left((S)\int_A \varphi(g)d\mu\right) \tag{4.4}$$

holds.

Remark 4.7. In [2], it also requires that $(S)\int_A \varphi(f)d\mu < \infty$ and $(S)\int_A \varphi(g)d\mu < \infty$. But, as is shown in Theorem 4.1, this condition can be abandoned.

Corollary 4.8 [18]. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary nonadditive measure. And let $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$\left((S)\int_A (f \star g)^s d\mu\right)^{\frac{1}{s}} \geqslant \left((S)\int_A f^s\, d\mu\right)^{\frac{1}{s}} \star \left((S)\int_A g^s\, d\mu\right)^{\frac{1}{s}} \tag{4.5}$$

holds for all $0 < s < \infty$.

Corollary 4.9 [15]. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary nonadditive measure. And let $\star : [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$(S)\int_A (f \star g)d\mu \geqslant \left((S)\int_A f\,d\mu\right) \star \left((S)\int_A g\,d\mu\right) \tag{4.6}$$

holds.

By Corollary 4.5 and Theorem 3.1, we have the following corollary.

Corollary 4.10. *Let f, $g : X \to [0,1]$ and μ be an arbitrary nonadditive measure. And let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$(S)\int_A (f \star g)d\mu \geqslant \left((S)\int_A f^p d\mu\right)^{\frac{1}{p}} \star \left((S)\int_A g^q d\mu\right)^{\frac{1}{q}} \tag{4.7}$$

holds for all p, $q \leqslant 1$.

Now, we construct some extensions of reverse previous integral inequalities for Sugeno integral and relate them to T-evaluators.

Theorem 4.11. *Let a fixed $k \in (0,\infty)$. And let a continuous non-decreasing $\varphi : [0,k] \to [0,k]$ satisfying $\varphi(x) \leqslant x$ (or equivalently, composite $\varphi(\varphi(x)) \leqslant \varphi(x)$) for all $x \in [0,k]$ and a non-decreasing n-place function $H : [0,\infty]^n \to [0,\infty]$ such that H be continuous and bounded from above by minimum be given. Then for any system $U_{1,\ldots,}U_n : [0,\infty] \to [0,\infty]$ of continuous strictly increasing functions and any comontone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any nonadditive measure μ it holds*

$$U^{-1}\left((S)\int_A U(H(\varphi(f_1),\ldots,\varphi(f_n)))d\mu\right) \geqslant H\left(\varphi\left(U_1^{-1}\left((S)\int_A U_1(f_1)d\mu\right)\right),\ldots,\varphi\left(U_n^{-1}\left((S)\int_A U_n(f_n)d\mu\right)\right)\right), \tag{4.8}$$

where $U = \min(U_1, U_2, \ldots, U_n)$.

Proof. This is similar to the proof of Theorem 4.1 (case 3). □

Corollary 4.12 [3]. *Let a fixed $k \in (0,\infty)$. For any continuous and non-decreasing $\varphi : [0,k] \to [0,k]$ satisfying $\varphi(x) \leqslant x$ for all $x \in [0,k]$ and any non-decreasing n-place function $H : [0,\infty)^n \to [0,\infty)$ such that H be continuous and bounded from above by minimum and any comonotone system $f_1, f_2, \ldots, f_n$ from $\mathcal{F}_k(X)$ and any nonadditive measure μ it holds*

$$(S)\int_A H(\varphi(f_1),\varphi(f_2),\ldots,\varphi(f_n))d\mu \geqslant H\left(\varphi\left((S)\int_A f_1\,d\mu\right),\varphi\left((S)\int_A f_2\,d\mu\right),\ldots,\varphi\left((S)\int_A f_n\,d\mu\right)\right). \tag{4.9}$$

Remark 4.13. If $H(k,\ldots,k) = k$, then the function H required in Corollary 4.12 is a conjunctive (continuous) aggregation function on $[0,k]$, compare [10]. Typical examples of such functions on [0,1] interval, i.e., if $k = 1$, are (continuous) t-norms, copulas, quasi-copulas, etc. Note also that the function φ required in Corollary 4.12 can be seen as a (contracting) transformation of the scale $[0,k]$.

Corollary 4.14. *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. And let $U_1, U_2 : [0,\infty] \to [0,\infty]$ be continuous strictly increasing functions. If T is a continuous t-norm and φ is continuous T-evaluator on X such that $\varphi(x) \leqslant x$, then the inequality*

$$U^{-1}\left((S)\int_A U(T(\varphi(f),\varphi(g)))d\mu\right) \geqslant T\left(\varphi\left(U_1^{-1}((S)\int_A U_1(f)d\mu)\right),\varphi\left(U_2^{-1}((S)\int_A U_2(g)d\mu)\right)\right) \tag{4.10}$$

holds for any $A \in \mathcal{F}$, where $U = \min(U_1, U_2)$.

Specially, when $U_1(x) = U_2(x) = x^s$ for all $s > 0$, we have the Minkowski inequality for T-evaluator.

Corollary 4.15. *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. If T is a continuous t-norm and φ is continuous T-evaluator on X such that $\varphi(x) \leqslant x$, then the inequality*

$$\left((S)\int_A (T(\varphi(f),\varphi(g)))^s d\mu\right)^{\frac{1}{s}} \geqslant T\left(\varphi\left(((S)\int_A f^s d\mu)^{\frac{1}{s}}\right),\varphi\left(\left((S)\int_A g^s d\mu\right)\right)^{\frac{1}{s}}\right) \tag{4.11}$$

holds for any $A \in \mathcal{F}$ and $0 < s < \infty$.

Again, we get an inequality related to the Chebyshev type for T-evaluator whenever $s = 1$.

Corollary 4.16 [3]. *Let $(X, \mathcal{F}, \mu)$ be a nonadditive measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. If T is a continuous t-norm and φ is continuous T-evaluator on X such that $\varphi(x) \leqslant x$, then the inequality*

$$\left((S)\int_A T(\varphi(f), \varphi(g))d\mu\right) \geqslant T\left(\varphi\left((S)\int_A f d\mu\right), \varphi\left((S)\int_A g d\mu\right)\right) \tag{4.12}$$

holds for any $A \in \mathcal{F}$.

And, by Corollary 4.14 and Theorem 3.1, we have the following corollary.

Corollary 4.17. *Let $(X, \mathcal{F}, \mu)$ be a nonadditive measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. If T is a continuous t-norm and φ is continuous T-evaluator on X such that $\varphi(x) \leqslant x$, then the inequality*

$$\left((S)\int_A T(\varphi(f), \varphi(g))d\mu\right) \geqslant T\left(\varphi\left(\left((S)\int_A f^p d\mu\right)^{\frac{1}{p}}\right), \varphi\left(\left((S)\int_A g^q d\mu\right)\right)^{\frac{1}{q}}\right) \tag{4.13}$$

holds for any $A \in \mathcal{F}$ and $p, q \leqslant 1$.

5. Conclusion

In this paper, we have investigated strengthened versions of the Minkowski, Chebyshev, Jensen and Hölder type inequalities for Sugeno integrals and have related them to T-evaluators and S-evaluators. As an interesting open problem for further investigation we pose the generalization of equality (2.3) for n-ary case. To be more precise, it is worth studying the case when the inequalities (3.4) and/or (4.1) became equalities, independently of incoming functions $f_1, \ldots, f_n$.

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