Copula-Based Integration of Vector-Valued Functions

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Abstract. A copula-based method to integrate a real vector-valued function, obtaining a single real number, is discussed. Special attention is paid to the case when the underlying universe is finite. The integral considered here is shown to be an extension of $[0, 1]$-valued copula-based universal integrals.

Keywords: Capacity, Copula, Universal integral, Vector-valued function.

1 Introduction

The concept of universal integrals was proposed in [4]. As a particular case, $[0, 1]$-valued universal integrals were considered: these integrals assign to a measurable function $f\colon X \to [0, 1]$ a value from $[0, 1]$, and the measure under consideration is a capacity on the measurable space $(X, \mathcal{A})$. The case of $[0, 1]$-valued universal integrals based on some special(two-dimensional) copulas was proposed first in [2] in an attempt to find a natural link between Choquet and Sugeno integral. General (two-dimensional) copulas were considered in [3] (see also [4]).

We propose a copula-based integral for measurable functions with values in $[0, 1]^n$, i.e., for real vector-valued functions, with respect to some capacity, considering particularly the case of a finite universe.

2 Copulas and $[0, 1]$-Valued Integrals

Copulas were introduced in [6] in an attempt to describe the stochastic dependence within random vectors. Recall that, for a fixed $n \geq 2$, an n-dimensional copula $C\colon [0, 1]^n \to [0, 1]$ provides a link between the joint probability distribution $F_Z : \mathbb{R}^n \to [0, 1]$ of a random vector $Z = (X_1, X_2, \ldots, X_n)$ and the marginal probability distributions $F_{X_1}, F_{X_2}, \ldots, F_{X_n} : \mathbb{R} \to [0, 1]$ of the random variables $X_1, X_2, \ldots, X_n$ via

$$F_Z(x_1, x_2, \ldots, x_n) = C\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right).$$

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Definition 1. An (*n-dimensional*) *copula* is a function $C\colon [0,1]^n \to [0,1]$ which is n-increasing, i.e., for each n-dimensional interval $[\mathbf{u},\mathbf{v}] \subseteq [0,1]^n$ we have

$$V_C([\mathbf{u},\mathbf{v}]) = \sum_{\mathbf{a}\in\{0,1\}^n} (-1)^{\sum\limits_{i=1}^{n} a_i} \cdot C(\mathbf{w_a}) \geq 0,$$

where

$$(\mathbf{w_a})_i = \begin{cases} v_i & \text{if } a_i = 0, \\ u_i & \text{if } a_i = 1, \end{cases}$$

and which satisfies the following two boundary conditions:

(i) 1 is a neutral element of C in the sense that $C(u_1, u_2, \ldots, u_n) = u_i$ whenever $u_j = 1$ for all $j \neq i$,
(ii) 0 is an annihilator of C in the sense that $C(u_1, u_2, \ldots, u_n) = 0$ whenever $0 \in \{u_1, u_2, \ldots, u_n\}$.

As a consequence, each copula is non-decreasing in each coordinate and 1-Lipschitz (with respect to the L_1-norm). The set of n-dimensional copulas is convex.

Prototypical examples are the greatest copula M given by $M(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$ describing comonotone dependence, and the copula Π given by $\Pi(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i$ describing independence. Note that, in the case $n = 2$, the function $W\colon [0,1]^2 \to [0,1]$ given by $W(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ is the smallest copula, describing countermonotone dependence, i.e., for each two-dimensional copula C we have $W \leq C \leq M$. If $n > 2$, no smallest copula exists, but still the n-ary extension of the associative copula W provides a greatest lower bound for the set of all n-dimensional copulas.

Note that n-dimensional copulas are in a one-to-one correspondence with probability measures on the Borel subsets of $[0,1]^n$ with uniform margins. This correspondence is fully described by

$$P_C([0,u_1] \times [0,u_2] \times \cdots \times [0,u_n]) = C(u_1, u_2, \ldots, u_n).$$

For more details about copulas see [5].

Denote by $\mathcal{S}$ the class of all measurable spaces $(X, \mathcal{A})$. For a given measurable space $(X, \mathcal{A})$, let $\mathcal{F}_{(X,\mathcal{A})}$ be the set of all $\mathcal{A}$-measurable functions from X to $[0,1]$, and $\mathcal{M}_{(X,\mathcal{A})}$ the set of all capacities $m\colon \mathcal{A} \to [0,1]$, i.e., the set of all monotone set functions m satisfying the boundary conditions $m(\emptyset) = 0$ and $m(X) = 1$. Following [4], we can define $[0,1]$-valued universal integrals.

Definition 2. A function $\mathbf{I}\colon \bigcup_{(X,\mathcal{A})\in\mathcal{S}} (\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}) \to [0,1]$ is called a $[0,1]$-*valued universal integral* if it satisfies the following axioms:

(I1) $\mathbf{I}$ is non-decreasing in each component;
(I2) $\mathbf{I}(m, \mathbf{1}_E) = m(E)$ for each $(X, \mathcal{A}) \in \mathcal{S}$, $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $E \in \mathcal{A}$;

(I3) $\mathbf{I}(m, c \cdot \mathbf{1}_X) = c$ for each $(X, \mathcal{A}) \in \mathcal{S}$, $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $c \in [0,1]$;

(I4) $\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2)$ whenever $(m_1, f_1) \in \mathcal{M}_{(X_1,\mathcal{A}_1)} \times \mathcal{F}_{(X_1,\mathcal{A}_1)}$ and $(m_2, f_2) \in \mathcal{M}_{(X_2,\mathcal{A}_2)} \times \mathcal{F}_{(X_2,\mathcal{A}_2)}$ satisfy $m_1(\{f_1 \geq t\}) = m_2(\{f_2 \geq t\})$ for all $t \in [0,1]$.

The special class of *copula-based* $[0,1]$-*valued universal integrals* was proposed in [3] (see also [4]).

Proposition 1. *Let* $C\colon [0,1]^2 \to [0,1]$ *be a two-dimensional copula. Then the function* $\mathbf{I}_C\colon \bigcup_{(X,\mathcal{A})\in\mathcal{S}}(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}) \to [0,1]$ *given by*

$$\mathbf{I}_C(m, f) = P_C(\{(u_1, u_2) \in [0,1]^2 \mid u_2 \leq m(\{f \geq u_1\})\})$$

is a $[0,1]$-*valued universal integral.*

Observe that $\mathbf{I}_\Pi$ coincides with the *Choquet integral* [1], and that $\mathbf{I}_M$ is the *Sugeno integral* [7].

3 Vector-Valued Functions and Copula-Based $[0,1]$-Valued Universal Integrals

For fixed $n \in \mathbb{N}$ and $(X, \mathcal{A}) \in \mathcal{S}$, let $\mathcal{F}^{(n)}_{(X,\mathcal{A})}$ be the set of all $\mathcal{A}$-measurable functions from X to $[0,1]^n$.

Definition 3. A function $\mathbf{I}^{(n)}\colon \bigcup_{(X,\mathcal{A})\in\mathcal{S}}\left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}^{(n)}_{(X,\mathcal{A})}\right) \to [0,1]$ is called a $[0,1]$-*valued* n-*universal integral* if it satisfies the following axioms:

(In1) $\mathbf{I}^{(n)}$ is non-decreasing in each component;

(In2) $\mathbf{I}^{(n)}(m, \mathbf{1}^{(n)}_E) = m(E)$ for each $(X, \mathcal{A}) \in \mathcal{S}$, $m \in \mathcal{M}_{(X,\mathcal{A})}$ and $E \in \mathcal{A}$, where $\mathbf{1}^{(n)}_E\colon X \to [0,1]^n$ is given by

$$\mathbf{1}^{(n)}_E(x) = \begin{cases} (1, 1, \ldots, 1) & \text{if } x \in E, \\ (0, 0, \ldots, 0) & \text{otherwise;} \end{cases}$$

(In3) $\mathbf{I}^{(n)}(m, c^{(i,n)}) = c$ for each $(X, \mathcal{A}) \in \mathcal{S}$, $m \in \mathcal{M}_{(X,\mathcal{A})}$, $i \in \{1, 2, \ldots, n\}$ and $c \in [0,1]$, where $c^{(i,n)} \in \mathcal{F}^{(n)}_{(X,\mathcal{A})}$ is given by $c^{(i,n)}(x) = (c_{1,i}, \ldots, c_{n,i})$ with $c_{i,i} = c$ and $c_{j,i} = 1$ whenever $j \neq i$;

(In4) $\mathbf{I}^{(n)}(m_1, f_1) = \mathbf{I}^{(n)}(m_2, f_2)$ whenever $(m_1, f_1) \in \mathcal{M}_{(X_1,\mathcal{A}_1)} \times \mathcal{F}^{(n)}_{(X_1,\mathcal{A}_1)}$ and $(m_2, f_2) \in \mathcal{M}_{(X_2,\mathcal{A}_2)} \times \mathcal{F}^{(n)}_{(X_2,\mathcal{A}_2)}$ satisfy $m_1(\{f_1 \geq \mathbf{u}\}) = m_2(\{f_2 \geq \mathbf{u}\})$ for all $\mathbf{u} \in [0,1]^n$.

Evidently, this generalizes the concept of $[0,1]$-valued universal integrals given in Definition 2 which are obtained here if $n = 1$.

Theorem 1. *For each* $n \in \mathbb{N}$ *and each* $(n+1)$*-dimensional copula* C *the function* $\mathbf{I}_C^{(n)}\colon \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}^{(n)}\right) \to [0,1]$ *given by*

$$\mathbf{I}_C^{(n)}(m,f) = P_C(\{(u_1,\dots,u_n,v) \in [0,1]^{n+1} \mid v \le m(\{f \ge (u_1,\dots,u_n)\})\}) \quad (1)$$

is a $[0,1]$*-valued* n*-universal integral.*

Observe that, because of the $\mathcal{A}$-measurability of f, the set

$$\{(u_1,\dots,u_n,v) \in [0,1]^{n+1} \mid v \le m(\{f \ge (u_1,\dots,u_n)\})\}$$

is a Borel subset of $[0,1]^{n+1}$, implying that $\mathbf{I}_C^{(n)}$ is well-defined.

Proposition 2. *Assume that for* $f = (f_1, f_2, \dots, f_n) \in \mathcal{F}_{(X,\mathcal{A})}^{(n)}$ *the set*

$$\{\{f_i \ge t\} \mid i \in \{1,2,\dots,n\}, t \in [0,1]\}$$

forms a chain. Then for each $m \in \mathcal{M}_{(X,\mathcal{A})}$ *we have*

$$\mathbf{I}_\Pi^{(n)}(m,f) = \mathbf{I}_\Pi\left(m, \prod_{i=1}^n f_i\right),$$
$$\mathbf{I}_M^{(n)}(m,f) = \mathbf{I}_M\left(m, \bigwedge_{i=1}^n f_i\right).$$

4 Discrete Copula-Based $[0,1]$-Valued n-Universal Integrals

Given an $(n+1)$-dimensional copula C, the function $\mathbf{I}_C^{(n)}$ in (1) is a copula-based $[0,1]$-valued n-universal integral and, therefore, can be defined on arbitrary measurable spaces $(X,\mathcal{A}) \in \mathcal{S}$. In this section we consider finite sets $X = \{1,2,\dots,k\}$ only, and $\mathcal{A} = 2^X$. Then the function $h_{m,f}\colon [0,1]^n \to [0,1]$ given by $h_{m,f}(\mathbf{u}) = m(\{f \ge \mathbf{u}\})$ is a piecewise constant function, with constant values on some n-dimensional intervals determined by the function $f\colon X \to [0,1]^n$. The additivity of the probability measure P_C allows us to obtain the following simplification of (1) in this discrete situation.

Theorem 2. *For each* $n \in \mathbb{N}$*, for each* $X = \{1,2,\dots,k\}$*, for each capacity* $m\colon 2^X \to [0,1]$*, for each* $(n+1)$*-dimensional copula* $C\colon [0,1]^{n+1} \to [0,1]$*, and for each* $f = (f_1, f_2, \dots, f_n) \in \mathcal{F}_{(X,\mathcal{A})}^{(n)}$ *we have*

$$\mathbf{I}_C^{(n)}(m,f) = \sum_{\mathbf{i}\in X^n} V_{D_{\mathbf{i}}^{(m,f)}}([f_1(\sigma_1(i_1-1)),\dots,f_n(\sigma_n(i_n-1))] \times [f_1(\sigma_1(i_1)),\dots,f_n(\sigma_n(i_n))]),$$

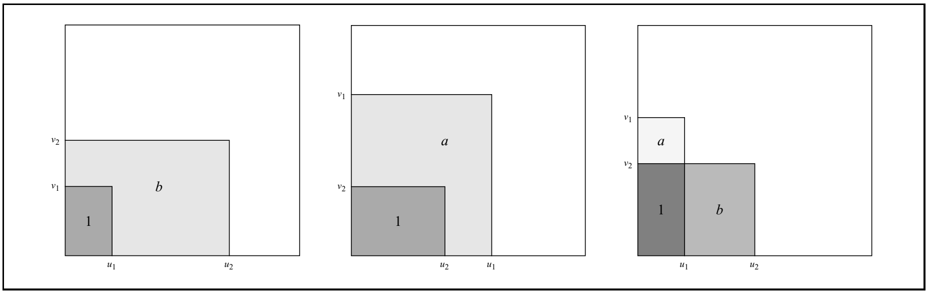

Fig. 1. The three cases in Example1: $f(1) \leq f(2)$ (left), $f(2) \leq f(1)$ (center), and $f(1), f(2)$ incomparable

where the function $D_{\mathbf{i}}^{(m,f)} \colon [0,1]^n \to [0,1]$ *is given by*

$$D_{\mathbf{i}}^{(m,f)}(\mathbf{u}) = C(u_1, \ldots, u_n, h_{m,f}(f_1(\sigma_1(i_1)), \ldots, f_n(\sigma_n(i_n)))),$$

and, for each $j \in \{1, 2, \ldots, n\}$*,* $\sigma_j \colon X \to X$ *is a permutation satisfying*

$$f_j(\sigma_j(1)) \leq f_j(\sigma_j(2)) \leq \cdots \leq f_j(\sigma_j(n)),$$

using the convention $\sigma_j(0) = 0$*.*

Observe that, in the case $n = 1$, the "vector" $\mathbf{i} = (i)$ has one column only, i.e., $D_i^{(m,f)}(u) = C(u, h_{m,f}(f(\sigma(i))))$. Subsequently, we get

$$\mathbf{I}_C^{(n)}(m, f) \\ = \sum_{i \in X} (C(f(\sigma(i)), h_{m,f}(f(\sigma(i)))) - C(f(\sigma(i-1)), h_{m,f}(f(\sigma(i))))),$$

which is exactly the formula for a discrete copula-based $[0,1]$-valued universal integral as discussed in [4].

Example 1. Consider $n = k = 2$, i.e., $X = \{1, 2\}$, a capacity $m \colon 2^X \to [0,1]$ determined by $m(\{1\}) = a$ and $m(\{2\}) = b$, and the product copula $\Pi \colon [0,1]^2 \to [0,1]$. For an $f = (f_1, f_2) \in \mathcal{F}_{(X,\mathcal{A})}^{(2)}$ the two values $f(1) = (u_1, v_1)$ and $f(2) = (u_2, v_2)$ can be either comparable or incomparable. In Figure 1 all three cases ($f(1) \leq f(2)$, $f(2) \leq f(1)$, and $f(1), f(2)$ incomparable) are visualized, the values inside the areas indicating the value of the corresponding function.

For the Π-based $[0,1]$-valued 2-universal integral $\mathbf{I}_\Pi^{(2)}$ we obtain

$$\mathbf{I}_\Pi^{(2)}(m, f) \\ = \begin{cases} u_1 v_1 + b(u_2 v_2 - u_1 v_1) & \text{if } f(1) \leq f(2), \\ u_2 v_2 + a(u_1 v_1 - u_2 v_2) & \text{if } f(2) \leq f(1), \\ a u_1 v_1 + b u_2 v_2 + (1 - a - b)(u_1 \wedge u_2)(v_1 \wedge v_2) & \text{otherwise.} \end{cases}$$

Observe that f_1 and f_2 are comonotone whenever $f(1) \leq f(2)$ or $f(2) \leq f(1)$, and then we have $\mathbf{I}_{\Pi}^{(2)}(m, f) = \mathbf{I}_{\Pi}(m, f_1 \cdot f_2)$, i.e., the standard Choquet integral of the product of the component functions of f (see Proposition 2).

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