# PENALTY FUNCTIONS OVER A CARTESIAN PRODUCT OF LATTICES

**H. Bustince, J. Fernandez**
Dep. Automatica y Computacion
Universidad Publica de Navarra
Pamplona, Spain
bustince@unavarra.es

**R. Mesiar**
Dep. Mathematics
Fac. Civil Engineering
Slovak University of Technology
Bratislava, Slovakia
and UTIA, CAS
Prague, Czech Republic
mesiar@math.sk

**G. Beliakov**
School of Information Technology
Deakin University
Burwood, Australia
gleb@deakin.edu.au

**T. Calvo**
Dep. Ciencias de la Computacion
Universidad de Alcala
Alcala de Henares, Spain
tomasa.calvo@uah.es

## Summary

In this work we present the concept of penalty function over a Cartesian product of lattices. To build these mappings, we make use of restricted dissimilarity functions and distances between fuzzy sets. We also present an algorithm that extends the weighted voting method for a fuzzy preference relation.

**Keywords:** Penalty function, lattices, weighted voting method

## 1 INTRODUCTION

A multi-expert decision making problem can be described as follows: we have a set of $n$ alternatives $X = \{x_1, \cdots, x_p\}, (p \geq 2)$, and a set of $n$ experts $U = E = \{e_1, \cdots, e_n\}, (n > 2)$ and each of the latter provides his/her preferences on the former set of alternatives. Find the alternative or set of alternatives that is (are) the most accepted by the experts ([7]).

This kind of problems can be solved by trying to determine, for each pair of the alternatives, a valuation that is the least dissimilar with those provided by the experts. That is, if we assume that the preference of expert $k$ of alternative $i$ over alternative $j$ is expressed by a numerical value $r_{ij}^k$, we can try to find a single numerical value that is the least dissimilar to the $n$ values $\{r_{ij}^1, \ldots, r_{ij}^n\}$ . In this way we arrive at a single preference relation (matrix) from which the best alternative can finally be chosen.

In this work we focus in the extension of penalty functions ([5]) to a lattice setting in order to carry on the selection of the least dissimilar value. In particular, we are going to consider Cartesian products of lattices and extend the idea of faithful penalty functions as presented in [9] to lattices by means of the concepts of restricted dissimilarity function ([3]) and distances between fuzzy sets. In order to show the usefulness of our theoretical approach, we present an algorithm that, starting from a normalized fuzzy preference relation, extends the weighted voting method by allowing the use of aggregation functions other than the arithmetic for the evaluation of each of the alternatives.

The structure of this work is the following. In section 2 we give some preliminary definitions and results. In Section 3 we introduce the concept of penalty function over a Cartesian product of lattices and we relate it to restricted dissimilarity functions. In Section 4 we present a construction of penalty functions based on distances. Section 5 is devoted to the Algorithm making use of our theoretical developments. We finish with some conclusions and references.

## 2 PRELIMINARIES

**Definition 1** *([1, 4]) A mapping* $M : [a, b]^n \to [a, b]$ *is an aggregation function if it is monotone non-decreasing in each of its components and satisfies* $M(\mathbf{a}) = M(a, a, \cdots, a) = a$ *and* $M(\mathbf{b}) = M(b, b, \cdots, b) = b$.

**Definition 2** *An aggregation function* $M$ *is called averaging or a mean if*

$$\min(x_1, \cdots, x_n) \leq M(x_1, \cdots, x_n) \leq \max(x_1, \cdots, x_n)$$

Any averaging aggregation function is idempotent, and also the converse is true.

We introduce now the concept of penalty function.

**Definition 3** *A penalty function is a mapping* $P : [a, b]^{n+1} \to \mathbb{R}^+ = [0, \infty]$ *such that:*

1. $P(\mathbf{x}, y) = 0$ *if* $x_i = y$ *for all* $i = 1, \cdots, n$;

2. $P(\mathbf{x}, y)$ *is quasi-convex in* $y$ *for any* $\mathbf{x}$*; that is*

$$P(\mathbf{x}, \lambda \cdot y_1 + (1-\lambda) \cdot y_2) \leq \max(P(\mathbf{x}, y_1), P(\mathbf{x}, y_2))$$

*for any* $\lambda \in [0,1]$ *and* $y_1, y_2 \in [a, b]$.

The penalty based function is

$$f(\mathbf{x}) = \arg\min_y P(\mathbf{x}, y),$$

if $y$ is the only minimum and $y = \frac{c+d}{2}$ if the set of minimums is given by the interval $[c, d]$.

**Theorem 1** *[5] Any averaging aggregation function can be represented as a penalty based function in the sense of Definition 3.*

Finally we also introduce the concept of restricted dissimilarity function.

**Definition 4** *[3] A mapping* $d_R : [0,1]^2 \to [0,1]$ *is a restricted dissimilarity function if:*

1. $d_R(x, y) = d_R(y, x)$ *for every* $x, y \in [0, 1]$*;*
2. $d_R(x, y) = 1$ *if and only if* $x = 0$ *and* $y = 1$ *or* $x = 1$ *and* $y = 0$*; that is,* $\{x, y\} = \{0, 1\}$*;*
3. $d_R(x, y) = 0$ *if and only if* $x = y$*;*
4. *For any* $x, y, z \in [0, 1]$*, if* $x \leq y \leq z$*, then* $d_R(x, y) \leq d_R(x, z)$ *and* $d_R(y, z) \leq d_R(x, z)$.

## 2.1 CARTESIAN PRODUCT OF LATTICES

**Definition 5** *A poset* $(P, \leq)$ *is a set* $P$ *with a relation* $\leq$ *which is reflexive, antisymmetric and transitive. A chain in a poset is a totally ordered set. The length of a chain is given by the cardinality of the chain minus one.*

**Definition 6** *A lattice* $\mathcal{L} = \{L, \leq, \wedge, \vee\}$ *is a poset with the partial ordering* $\leq$ *in L and operations* $\wedge$ *and* $\vee$ *which satisfy the properties of absortion, idempotency, commutativity, and associativity. That is, a poset such that any two elements have a unique minimal upper bound and a unique maximal lower bound in L.*

In this work we only deal with bounded lattices, that is, lattices for which there exist a maximal or greatest element and a minimal or smallest element.

**Proposition 1** *Let* $\mathcal{L}_1 = \{L_1, \leq_1, \wedge_1, \vee_1\}$ *and* $\mathcal{L}_2 = \{L_2, \leq_2, \wedge_2, \vee_2\}$ *be two lattices. The Cartesian product*

$$\mathcal{L}_1 \times \mathcal{L}_2 = \{L_1 \times L_2, \leq, \wedge, \vee\}$$

*with* $\leq$ *defined by*

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2$$

*and*

$$\wedge((x_1, y_1), (x_2, y_2)) = (\wedge_1(x_1, x_2), \wedge_2(y_1, y_2))$$
$$\vee((x_1, y_1), (x_2, y_2)) = (\vee_1(x_1, x_2), \vee_2(y_1, y_2))$$

*is a lattice.*

In this work we consider the Cartesian product of finite chains $\mathcal{C}$ or the Cartesian product of intervals. We must point out that if we make the Cartesian product of $m$ copies of lattice $\mathcal{L}$, each element $x = (x_1, \cdots, x_m) \in \mathcal{L} \times \mathcal{L} \cdots \times \mathcal{L}$ is such that $x_i \in \mathcal{L}$. Moreover, all the finite chains of the same length are isomorphic to each other. So we can always assume that we are working with chains of the type $\mathcal{C} = 0 \leq 1 \leq 2 \leq \cdots \leq n-1$.

**Theorem 2** *Let* $\mathcal{L}_k = \{\mathcal{C}_1 \times \cdots \times \mathcal{C}_k, \leq, \wedge, \vee\}$*. Let* $a, b$ *be two elements in* $\mathcal{L}_k$ *such that* $a \leq b$*. Then all the maximal chains joining* $a$ *and* $b$ *have the same length.*

**Proof.** See [2] □

**Corollary 1** *Take* $a, b \in \mathcal{L}_k = \{\mathcal{C}_1 \times \cdots \times \mathcal{C}_k, \leq, \wedge, \vee\}$*. Then all the maximal chains joining* $\wedge(a, b)$ *and* $\vee(a, b)$ *are of the same length.*

Taking into account the previous results, we have that if $\mathcal{L}$ is the Cartesian product of $m$ chains, then the distance between $x, y \in \mathcal{L}$ can be defined as the length of the chain $\mathcal{C}$ with minimal element $a = \wedge(x, y)$ and maximal element $b = \vee(x, y)$, minus one. That is,

$$d(x, y) = length(\mathcal{C}) - 1.$$

This definition is equivalent to the following.

$$d(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i) = \sum_{i=1}^{m} |x_i - y_i| \quad (1)$$

where $d_i$ is the distance in the i-th chain. Observe that, in the case of a finite chain, the absolute value in the last term corresponds to the usual absolute value taking into account the aforementioned isomorphism between a finite chain of $n$ elements and the chain $\mathcal{C} = 0 \leq 1 \leq 2 \leq \cdots \leq n-1$. It is easy to see that Eq (1) is a distance. It is called the natural distance.

## 2.2 PENALTY FUNCTIONS OVER A CARTESIAN PRODUCT OF LATTICES FROM LATTICE DISSIMILARITY FUNCTIONS AND DISTANCES

Consider the lattice $\mathcal{L}_m = \{\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee\}$. We denote

$$1_{\mathcal{L}_m} = (1_{\mathcal{C}_1)}, \cdots, 1_{\mathcal{C}_m)}),$$
$$0_{\mathcal{L}_m} = (0_{\mathcal{C}_1)}, \cdots, 0_{\mathcal{C}_1)}).$$

**Definition 7** *Take $\mathcal{L}_m = \{\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee\}$. A mapping*

$$\delta_R : \mathcal{L}_m \times \mathcal{L}_m \to \mathcal{L}_m$$

*is a lattice restricted dissimilarity function if*

1. $\delta_R(x, y) = \delta_R(y, x)$ *for any* $x, y \in \mathcal{L}_m$*;*
2. $\delta_R(x, y) = 1_{\mathcal{L}_m}$ *if and only if for any* $i = 1, \cdots, m$,
$$x_i = 1_{\mathcal{C}_i} \text{ and } y_i = 0_{\mathcal{C}_i}, \quad \text{or} \quad x_i = 0_{\mathcal{C}_i} \text{ and } y_i = 1_{\mathcal{C}_i};$$
3. $\delta_R(x, y) = 0_{\mathcal{L}_m}$ *if and only if* $x = y$*;*
4. *If* $x \leq y \leq z$ *then* $\delta_R(x, y) \leq \delta_R(x, z)$ *and* $\delta_R(y, z) \leq \delta_R(x, z)$.

From Def. 7 we can prove Proposition 2.

**Proposition 2** *Let each $\delta_{R_i} : \mathcal{C}_i^m \to \mathcal{C}_i$ be a lattice restricted dissimilarity function. Then the mapping defined as*

$$\delta_R(x, y) = (\delta_{R_1}(x_1, y_1), \cdots, \delta_{R_m}(x_m, y_m)) \quad (2)$$

*for every $x, y \in \mathcal{L}_m$ is a lattice restricted dissimilarity function.*

**Proof.** Direct from the Definition □

In this work we denote by $\mathcal{FS}(U)^m$ the set of sets $\mathbf{A} = (A_1, \cdots, A_m)$ with $A_i : U \to \mathcal{C}_i$ such that $\mathbf{A}(u_i) = (A_1(u_i), \cdots, A_m(u_i))$ for every $u_i \in U$. Notice that each of the $A_i$ is an L-fuzzy set in the sense of Goguen [8]; i.e., each $A_i$ is a fuzzy set defined over the lattice $\{\mathcal{C}_i, \leq_i, \wedge_i, \vee_i\}$.

The construction methods for restricted dissimilarity functions described in [3] can be easily adapted to lattice restricted dissimilarity functions, so we do not develop them here.

**Definition 8** *Take $\mathcal{L}_m = \{\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee\}$. A mapping*

$$\Omega : \mathcal{FS}(U)^m \times \mathcal{FS}(U)^m \to \mathcal{L}_m$$

*is a lattice distance in $\mathcal{FS}(U)^m$ if*

1. $\Omega(\mathbf{A}, \mathbf{B}) = \Omega(\mathbf{B}, \mathbf{A})$ *for every* $\mathbf{A}, \mathbf{B} \in \mathcal{FS}(U)^m$*;*
2. $\Omega(\mathbf{A}, \mathbf{B}) = 0_{\mathcal{L}_m}$ *if and only if* $A_i = B_i$ *for every* $i = 1, \cdots, m$*;*
3. $\Omega(\mathbf{A}, \mathbf{B}) = 1_{\mathcal{L}_m}$ *if and only if for every* $i = 1, \cdots, m$, $A_i$ *and* $B_i$ *are sets such that for every* $u_j$
$$A_i(u_j) = 1_{\mathcal{C}_i} \text{ and } B_i(u_j) = 0_{\mathcal{C}_i}$$
*or*
$$A_i(u_j) = 0_{\mathcal{C}_i} \text{ and } B_i(u_j) = 1_{\mathcal{C}_i};$$
4. *If* $\mathbf{A} \leq \mathbf{A}' \leq \mathbf{B}' \leq \mathbf{B}$, *then* $\Omega(\mathbf{A}, \mathbf{B}) \geq \Omega(\mathbf{A}', \mathbf{B}')$ *where* $\mathbf{A} = (A_1, \cdots, A_m) \leq (A'_1, \cdots, A'_m) = \mathbf{A}'$ *if* $A_i \leq A'_i$ *for every* $i$.

**Definition 9** *Let $\mathcal{L}$ be a bounded lattice. An aggregation function over the lattice $\mathcal{L}$ is a mapping:*

$$M : \mathcal{L}^m \to \mathcal{L} \quad (3)$$

*such that*

*i)* $M(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$ *and* $M(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$*;*

*ii)* $M$ *is increasing with respect to* $\leq$.

**Proposition 3** *Let $M_1, \cdots, M_m$ be aggregation functions*

$$M_i : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$$

*Then the mapping*

$$F : \mathcal{L}_m \times \mathcal{L}_m \to \mathcal{L}_m \text{ given by}$$
$$F(\mathbf{x}, \mathbf{y}) = (M_1(x_1, y_1), \cdots, M_m(x_m, y_m))$$

*is an aggregation function over $\mathcal{L}_m$.*

**Proof.** Direct □

**Proposition 4** *Let $\delta_{R_1}, \cdots, \delta_{R_m}$ be a lattice restricted dissimilarity function $\delta_{R_i} : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$. Let $M_1, \cdots, M_m$ be aggregation functions $M_i : \mathcal{C}_i^n \to \mathcal{C}_i$ such that*

*(L1)* $M_i(x_1, \cdots, x_n) = 1_{\mathcal{C}_i}$ *if and only if* $x_i = 1_{\mathcal{C}_i}$ *for every* $i = 1, \cdots, n$

*(L2)* $M_i(x_1, \cdots, x_n) = 0_{\mathcal{C}_i}$ *if and only if* $x_i = 0_{\mathcal{C}_i}$ *for every* $i = 1, \cdots, n$

*Then $\Omega(\mathbf{A}, \mathbf{B})$ defined as*

$$\left( \underset{i=1}{\overset{n}{M_1}}(\delta_{R_1}(A_1(u_i), B_1(u_i))), \cdots, \underset{i=1}{\overset{n}{M_m}}(\delta_{R_m}(A_m(u_i), B_m(u_i))) \right)$$

*is a lattice distance in $\mathcal{FS}(U)^m$.*

**Proof.** Direct □

## 3 PENALTY FUNCTIONS FROM LATTICE DISTANCES BETWEEN SETS OVER CARTESIAN PRODUCTS OF LATTICES

In this subsection we present a construction method of penalty functions in a Cartesian product of lattices from lattice distances between fuzzy sets.

We know that the arithmetic mean of convex functions is also a convex function. Next, we consider aggregation functions such that applied to convex functions we obtain another convex function, as in the arithmetic mean case. Observe that here and in the following, whenever we talk of convexity, we are dealing with chains that are in fact real intervals.

**Theorem 3** *Let $Y = (y_1, \cdots, y_m) \in \mathcal{L}_m$. For each $y_i$ $(i = 1, \cdots, m)$ we consider the set*

$$B_{y_i}(u_j) = y_i \text{ for all } u_j \in U \tag{4}$$

*and let $\mathbf{B}_Y = (B_{y_1}, \cdots, B_{y_m}) \in \mathcal{FS}(U)^m$. Let $M_1, \cdots, M_m$ be aggregation functions $M_i : \mathcal{C}_i^n \to \mathcal{C}_i$ such that each of them when composed with convex functions is also convex. Take the lattice restricted dissimilarity function $\delta_R(x, y) = (\delta_{R_1}(x_1, y_1), \cdots, \delta_{R_m}(x_m, y_m))$ such that each $\delta_{R_i}$ with $i = 1, \cdots, m$ is convex in one variable. Then*

*$P_\Omega : \mathcal{FS}(U)^{m+1} \to \mathcal{L}_m$ given by*

$$P_\Omega(\mathbf{A}, Y) = \Omega(\mathbf{A}, \mathbf{B}_Y)$$
$$= \left( \overset{n}{\underset{i=1}{M_1}}(\delta_{R_1}(A_1(u_i), y_1)), \cdots, \overset{n}{\underset{i=1}{M_m}}(\delta_{R_m}(A_m(u_i), y_m)) \right) \tag{5}$$

*satisfies:*

1. *$P_\Omega(\mathbf{A}, Y) \geq 0_{\mathcal{L}_m}$;*
2. *$P_\Omega(\mathbf{A}, Y) = 0_{\mathcal{L}_m}$ if $A_k(u_j) = y_k$ for every $k$ and for every $j$;*
3. *Each of its components is convex with respect to the corresponding $y_k$ $(k = 1, \cdots, m)$.*

**Proof.** Direct □

**Corollary 2** *In the setting of Theorem 3 if $M_1, \cdots, M_m$ satisfy (L1) and (L2), then*

1. *$P_\Omega(\mathbf{A}, Y) = 0_{\mathcal{L}_m}$ if and only if $A_k(u_j) = y_k$ for every $k$ and for every $j$;*
2. *$P_\Omega(\mathbf{A}, Y) = 1_{\mathcal{L}_m}$ if and only if $\{A_k(u_i), y_k\} = \{0_\mathcal{L}, 1_\mathcal{L}\}$.*

Analogously to the real case (see [9, 6]), we use the terminology **lattice faithful restricted dissimilarity functions** to denote the following lattice restricted dissimilarity functions:

$$\delta_R(x, y) = K(d(x, y)) = K(\sum_{i=1}^{m} |x_i - y_i|) \tag{6}$$

with $K : \mathcal{C} \to \mathcal{C}$ a convex with a unique minimum at $K(0) = 0$.

**Theorem 4** *In the setting of Theorem 3, if $\delta_{R_1}, \cdots, \delta_{R_m}$ are lattice faithful restricted dissimilarity functions, then the mapping*

*$F_{\mathcal{L}_m} : \mathcal{FS}(U)^m \to \mathcal{L}_m$ given by*

$$F_{\mathcal{L}_m}(\mathbf{A}) = \arg\min_Y P_\Omega(\mathbf{A}, Y) = \arg\min_Y \Omega(\mathbf{A}, \mathbf{B}_Y)$$
$$= \left( \arg\min_{y_j} (\overset{n}{\underset{i=1}{M_j}}(K_j(d(A_j(u_i), y_j)))) \right)_{j=1,\ldots,m}$$
$$= \left( \arg\min_{y_j} (\overset{n}{\underset{i=1}{M}}(K_j(|A_j(u_i) - y_|)))) \right)_{j=1,\ldots,m}$$

*is such that each of its components is an averaging aggregation function over $\mathcal{FS}(U)$ and $F_{\mathcal{L}_m}(\mathbf{A})$ is an averaging aggregation function over the Cartesian product $\mathcal{FS}(U)^m$.*

**Proof.** Apply Theorem 1 for each component □

From now on we will denote by $B_{y_q}$ the fuzzy set over $U$ such that all its membership values are equal to $y_q \in [0, 1]$; that is, $B_{y_q}(u_i) = y_q \in [0, 1]$ for all $u_i \in U$.

Let $Y = (y_1, \cdots, y_m)$ and $\mathbf{B}_Y = (B_{y_1}, \cdots, B_{y_m}) \in \mathcal{FS}(U)^m$. We will denote by $\mathcal{C}^*$ a chain whose elements belong to $[0, 1]$ and by $\mathcal{L}_m^*$ the product such that $\mathcal{L}_m^* = (\mathcal{C}^*)^m$.

**Theorem 5** *Let $K_i : \mathbb{R} \to \mathbb{R}^+$ be convex functions with a unique minimum at $K_i(0) = 0$ $(i = 1, \cdots, m)$, and take the distance between fuzzy sets defined as*

$$\mathcal{D}(A, B) = \sum_{i=1}^{n} |A(u_i) - B(u_i)| \tag{7}$$

*where $A, B \in \mathcal{FS}(U)$ and $Cardinal(U) = n$. Then the mapping*

*$P_\nabla : \mathcal{FS}(U)^m \times \mathcal{L}_m^* \to \mathbb{R}^+$ given by*

$$P_\nabla(\mathbf{A}, Y) = \mathcal{D}(\mathbf{A}, \mathbf{B}_Y)$$
$$= \sum_{q=1}^{m} K_q(\mathcal{D}(A_q, B_{y_q})) = \sum_{q=1}^{m} K_q \left( \sum_{p=1}^{n} |A_q(u_p) - y_q| \right) \tag{8}$$

*satisfies*

1. *$P_\nabla(\mathbf{A}, Y) \geq 0$;*

2. *$P_\nabla(\mathbf{A}, Y) = 0$ if and only if $A_q = y_q$ for every $q = 1, \cdots, m$;*

3. *is convex in $y_q$ for every $q = 1, \cdots, m$.*

**Proof.** Direct since the sum of convex functions is convex □

Observe that $P_\nabla$ is a penalty function defined over the Cartesian product of lattices $\mathcal{L}_m^{*n+1}$.

**Example 1** • *If we take $K_q(x) = x^2$ for all $q \in \{1, \cdots, m\}$, then*

$$P_\nabla(\mathbf{A}, Y) = \sum_{q=1}^{m}\left(\sum_{p=1}^{n}|A_q(u_p) - y_q|\right)^2 \tag{9}$$

• *If $K_q(x) = x$ for all $q \in \{1, \cdots, m\}$, then*

$$P_\nabla(\mathbf{A}, Y) = \sum_{q=1}^{m}\sum_{p=1}^{n}|A_q(u_p) - y_q| \tag{10}$$

**Theorem 6** *In the setting of Theorem 5, the mapping*

$$F(\mathbf{A}) = \mu = \arg\min_Y P_\nabla(\mathbf{A}, Y) \tag{11}$$

*where $\mu$ is the rounding to the smallest closest element, is an averaging aggregation function.*

**Proof.** Just observe that

$$\begin{aligned}
&\arg\min_{(y_1,\cdots y_m)} P_\nabla(\mathbf{A}, (y_1, \cdots, y_m)) \\
&= \arg\min_{(y_1,\cdots y_m)} \sum_{q=1}^{m} K_q\left(\sum_{p=1}^{n}|A_q(u_p) - y_q|\right) \\
&= \sum_{q=1}^{m} \arg\min_y K_q\left(\sum_{p=1}^{n}|A_q(u_p) - y_q|\right)
\end{aligned} \tag{12}$$

so it is enough to consider each of the mappings

$$\arg\min_y K_q\left(\sum_{p=1}^{n}|A_q(u_p) - y_q|\right) \tag{13}$$

but each of these functions is an aggregation function and since $K_q$ is convex, the result follows. □

*Remark* Notice that $\mathcal{FS}(U)^m$ with Zadeh's order is a bounded lattice.

# 4 AN APPLICATION TO DECISION MAKING PROBLEMS

In this section we present a simple algorithm that shows a possible application of our previous theoretical developments to a decision making problem.

Assume that we have to choose between a set of $p$ alternatives. Suppose that the normalized preference relation provided by an expert (or the collective normalized preference relation in case we have several experts) is given by the following matrix:

$$r = \begin{pmatrix} - & r_{12} & \cdots & r_{1p} \\ r_{21} & - & \cdots & r_{2p} \\ \cdots & \cdots & - & \cdots \\ r_{p1} & \cdots & \cdots & - \end{pmatrix} \tag{14}$$

The problem of how to obtain this matrix is not trivial. Nevertheless, we will consider that it has ben given in some way or another. Then, a widely used method to determine the best alternative is the weighted voting method, where the chosen alternative is arg $\underset{i=1,\cdots,p}{max} \sum_{1\leq j\neq i\leq p} r_{ij}$. That is, the arithmetic mean of each of the rows is considered, and the row providing the highest output (vote) is selected.

The algorithm that we propose is the following:

1. Select a penalty function $P_\nabla$ defined over the product of $p$ lattices.

2. Take a set of $q \leq p$ averaging aggregation functions: $\{M_1, \cdots, M_q\}$.

3. Build all the variations with repetition of the $q$ aggregation functions taken in groups of $p$ elements: $M_{\sigma(i)} = \{M_{(\sigma(i),1)}, \cdots, M_{(\sigma(i),p)}\}$.

4. Build

   $$\mathbf{A} = ((r_{12}, \cdots, r_{1p}), \cdots, (r_{p1}, \cdots, r_{p(p-1)}))$$

   (with $U = \{u_1, \cdots, u_p\}$ and $r_{jl}$ such that $j \neq l$)

5. FOR i:=1 to $q^p$ DO

   Take: $M_{\sigma(i)} = \{M_{(\sigma(i),1)}, \cdots, M_{(\sigma(i),p)}\}$

   FOR j:=1 to p DO

   Calculate: $M_{(\sigma(i),j)}(r_{j1}, \cdots, r_{jp}) = y_{(\sigma(i),j)}$ with $r_{jl}$ such that $j \neq l$

   Build: $B_{(\sigma(i),j)}(u_k) = y_{(\sigma(i),j)}$ for all $k := 1, \cdots, p$

   ENDFOR

   ENDFOR

6. Between all the variations with repetition of of the $q$ aggregation functions in groups of $p$ elements, take: $\mathbf{B}_Y = (B^*_{(\sigma(i),1)}, \cdots, B^*_{(\sigma(i),p)})$ which minimizes:

$$\begin{aligned}
P_\nabla(\mathbf{A}, Y) &= \mathcal{D}(\mathbf{A}, \mathbf{B}_Y) = \sum_{k=1}^{p} K_k(\mathcal{D}(A_q, B_{y_k})) \\
&= \sum_{k=1}^{p} K_k\left(\sum_{\substack{j=1 \\ k\neq j}}^{p} |r_{kj} - y_{(\sigma(i),k)}|\right)
\end{aligned}$$

7. Take the alternative:

$$x_i := \arg \max_{j=1,\cdots,p} B^*_{(\sigma(i),j)}$$

That is, in our algorithm we propose to replace the arithmetic mean by other averaging aggregation functions. These functions have to be picked beforehand and, in order to select the best alternative, we use a penalty function over a product of lattices to determine for which of the rows the output is less dissimilar than the inputs in the row. Notice that since we have not fixed a single aggregation function for each row we have flexibility to represent each row in the most suitable way.

## 5 CONCLUSIONS

In this work we have presented a possible extension of the concept of penalty function to a Cartesian product of lattices. To do so, we have made use of restricted dissimilarity functions and distances between fuzzy sets. We have also presented an algorithm for decision making problems that, starting from a normalized fuzzy preference relation, generalizes the weighted voting method by allowing the use of aggregation functions other than the arithmetic mean and uses a penalty function over a Cartesian product of lattices to determine the best alternative.

A drawback of this algorithm is the need of selecting beforehand both the aggregation functions and the penalty functions. In future works we intend to deal with this problem.

**Acknowledgements**

The work on this paper was supported by project TIN 2010-15055, MTM 2009-10962 and TIN2009-07901 from the Government of Spain, by grants P402/11/0378, APVV–0073–10 and VEGA 1/0080/10.

## References

[1] G. Beliakov, A. Pradera and T. Calvo,*Aggregation Functions: A Guide for Practitioners*, Studies In Fuzziness and Soft Computing, Springer, 221, 2007.

[2] G. Birkhoff, *Lattice theory*, American Mathematical Society, Colloquium Publications, 25, 1967.

[3] H. Bustince, E. Barrenechea and M. Pagola, Relationship between restricted dissimilarity functions, restricted equivalence functions and normal EN-functions: Image thresholding invariant, *Pattern Recognition Letters*, **29**, (4), (2008) 525-536.

[4] T. Calvo, A. Kolesarova, M. Komornikova and R. Mesiar, Aggregation operators: properties, classes and construction methods. In T. Calvo, G. Mayor and R. Mesiar (Eds.): *Aggregation Operators New Trends and Applications* (Physica-Verlag, Heidelberg, 2002), 3104.

[5] T. Calvo and G. Beliakov, Aggregation functions based on penalties, *Fuzzy Sets and Systems*, **161** (10), (2010), 1420-1436.

[6] T. Calvo, R. Mesiar and R. Yager, A quantitative weights and aggregation, *IEEE Trans. Fuzzy Systems*, **12**, (2004), 6269.

[7] F. Chiclana, F. Herrera and E. Herrera-Viedma, Integrating three representation models in fuzzy multiporpuse decision making based on fuzzy preference relations, *Fuzzy Sets and Systems*, **97**, (1998), 33-48.

[8] J. Goguen: L-fuzzy sets, *Journal of mathematical analysis and applications*, **18**, (1967), 145–174.

[9] R. Mesiar, Fuzzy set approach to the utility, preference relations,and aggregation operators, *European Journal of Operational Research*, **176**, (2007), 414-422.