The Bipolar Universal Integral

Salvatore Greco[1], Radko Mesiar[2], and Fabio Rindone[1]

[1] Department of Economics and Business,
University of Catania, Corso Italia 55, 95129 Catania, Italy
{salgreco,frindone}@unict.it

[2] Department of Mathematics and Descriptive Geometry,
Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
and Institute of Theory of Information and Automation,
Czech Academy of Sciences, Prague, Czech Republic
radko.mesiar@stuba.sk

Abstract. The concept of universal integral, recently proposed, generalizes the Choquet, Shilkret and Sugeno integrals. Those integrals admit a bipolar formulation, useful in those situations where the underlying scale is bipolar. In this paper we propose the bipolar universal integral generalizing the Choquet, Shilkret and Sugeno bipolar integrals. To complete the generalization we also provide the characterization of the bipolar universal integral with respect to a level dependent bi-capacity.

Keywords: Choquet, Sugeno and Shilkret integrals, universal integral, bipolar integrals.

1 Introduction

Recently a concept of universal integral has been proposed [14]. The universal integral generalizes the Choquet integral [2], the Sugeno integral [18] and the Shilkret integral [17]. Moreover, in [12], [13] a formulation of the universal integral with respect to a level dependent capacity has been proposed, in order to generalize the level-dependent Choquet integral [9], the level-dependent Shilkret integral [1] and the level-dependent Sugeno integral [15]. The Choquet, Shilkret and Sugeno integrals admit a bipolar formulation, useful in those situations where the underlying scale is bipolar ([5], [6], [10], [8]). In this paper we introduce and characterize the bipolar universal integral, which generalizes the Choquet, Shilkret and Sugeno bipolar integrals. We introduce and characterize also the bipolar universal integral with respect to a level dependent capacity, which generalizes the level-dependent bipolar Choquet, Shilkret and Sugeno integrals proposed in [9], [8].

The paper is organized as follows. In section 2 we introduce the basic concepts. In section 3 we define and characterize the bipolar universal integral. In section 4 we give an illustrative example of a bipolar universal integral which is neither the Choquet nor Sugeno or Shilkret type. In section 5 we define and characterize the bipolar universal integral with respect to a level dependent bi-capacity. Finally, in section 6, we present conclusions.

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2 Basic Concepts

Given a set of criteria $N = \{1, \ldots, n\}$, an *alternative* $\boldsymbol{x}$ can be identified with a score vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-\infty, +\infty]^n$, being x_i the evaluation of $\boldsymbol{x}$ with respect to the i^{th} criterion. For the sake of simplicity, without loss of generality, in the following we consider the bipolar scale $[-1, 1]$ to expose our results, so that $\boldsymbol{x} \in [-1, 1]^n$. Let us consider the set of all disjoint pairs of subsets of N, i.e. $\mathcal{Q} = \{(A, B) \in 2^N \times 2^N \ : \ A \cap B = \emptyset\}$. With respect to the binary relation $\precsim$ on $\mathcal{Q}$ defined as $(A, B) \precsim (C, D)$ iff $A \subseteq C$ and $B \supseteq D$, $\mathcal{Q}$ is a lattice, i.e. a partial ordered set in which any two elements have a unique supremum $(A, B) \vee (C, D) = (A \cup C, B \cap D)$ and a unique infimum $(A, B) \wedge (C, D) = (A \cap C, B \cup D)$. For all $(A, B) \in \mathcal{Q}$ the indicator function $1_{(A,B)} : N \to \{-1, 0, 1\}$ is the function which attains 1 on A, -1 on B and 0 on $(A \cup B)^c$.

Definition 1. *A function $\mu_b : \mathcal{Q} \to [-1, 1]$ is a normalized bi-capacity ([5], [6], [10]) on N if*

– $\mu_b(\emptyset, \emptyset) = 0$, $\mu_b(N, \emptyset) = 1$ *and* $\mu_b(\emptyset, N) = -1$;
– $\mu_b(A, B) \leq \mu_b(C, D)$ $\forall$ $(A, B), (C, D) \in \mathcal{Q}$: $(A, B) \precsim (C, D)$.

Definition 2. *The bipolar Choquet integral of $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1, 1]^n$ with respect to a bi-capacity μ_b is given by ([5], [6], [10], [9]):*

$$Ch_b(\boldsymbol{x}, \mu_b) = \int_0^{\infty} \mu_b(\{i \in N : x_i > t\}, \{i \in N : x_i < -t\}) dt. \tag{1}$$

The bipolar Choquet integral of $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1, 1]^n$ with respect to the bi-capacity μ_b can be rewritten as

$$Ch_b(\boldsymbol{x}, \mu_b) = \sum_{i=1}^{n} \left(|x_{(i)}| - |x_{(i-1)}|\right) \mu_b(\{j \in N : x_j \geq |x_{(i)}|\}, \{j \in N : x_j \leq -|x_{(i)}|\}), \tag{2}$$

being $() : N \to N$ any permutation of index such that $0 = |x_{(0)}| \leq |x_{(1)}| \leq \ldots \leq |x_{(n)}|$. Let us note that to ensure that $(\{j \in N : x_j \geq |t|\}, \{j \in N : x_j \leq -|t|\}) \in \mathcal{Q}$ for all $t \in \mathbb{R}$, we adopt the convention - which will be maintained trough all the paper - that in the case of $t = 0$ the inequality $x_j \leq -|t| = 0$ must be intended as $x_j < -|t| = 0$.

In this paper we use the symbol $\bigvee$ to indicate the maximum and $\bigwedge$ to indicate the minimum. The *symmetric maximum* of two elements - introduced and discussed in [3], [4] - is defined by the following binary operation:

$$a \circledvee b = \begin{cases} -\left(|a| \vee |b|\right) \text{ if } b \neq -a \text{ and either } |a| \vee |b| = -a \text{ or } = -b \\ 0 \qquad\qquad\quad \text{if } b = -a \\ |a| \vee |b| \qquad\quad \text{else.} \end{cases}$$

In [16] it has been showed as on the domain $[-1, 1]$ the symmetric maximum coincides with two recent symmetric extensions of the Choquet integral, the *balancing Choquet integral* and the *fusion Choquet integral*, when they are computed with respect to the strongest capacity (i.e. the capacity which attains zero on the empty set and one elsewhere). However, the symmetric maximum of a set X cannot be defined, being $\circledcirc$ non associative. Suppose that $X = \{3, -3, 2\}$, then $(3 \circledcirc -3) \circledcirc 2 = 2$ or $3 \circledcirc (-3 \circledcirc 2) = 0$, depending on the order. Several possible extensions of the symmetric maximum for dimension $n, n > 2$ have been proposed (see [4], [7] and also the relative discussion in [16]). One of these extensions is based on the splitting rule applied to the maximum and to the minimum as described in the following. Let $X = \{x_1, \dots, x_m\} \subseteq \mathbb{R}$, the *bipolar maximum* of X, shortly $\bigvee^b X$, is defined as follow: if there exists an element $x_k \in X$ such that $|x_k| > |x_j|\ \forall j : x_j \neq x_k$ then $\bigvee^b X = x_k$; otherwise $\bigvee^b X = 0$. Clearly, the bipolar maximum of a set X is related to the symmetric maximum of two elements by means of

$$\bigvee^b X = \left(\bigvee X\right) \circledcirc \left(\bigwedge X\right). \tag{3}$$

In the same way and for an infinite set X, it is possible to define the concept of $\sup^{bip} X$ as the symmetric maximum applied to the supremum and the infimum of X.

Definition 3. *The bipolar Shilkret integral of* $\boldsymbol{x} = (x_1, \dots, x_n) \in [-1, 1]^n$ *with respect to a bi-capacity* μ_b *is given by [8]:*

$$Sh_b(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \tag{4}$$

Definition 4. *A bipolar measure on* N *with a scale* $(-\alpha, \alpha)$, $\alpha > 0$, *is any function* $\nu_b : Q \to (-\alpha, \alpha)$ *satisfying the following properties:*

1. $\nu_b(\emptyset, \emptyset) = 0$;
2. $\nu_b(N, \emptyset) = \alpha$, $\nu_b(\emptyset, N) = -\alpha$;
3. $\nu_b(A, B) \leq \nu_b(C, D)\ \forall\ (A, B), (C, D) \in \mathcal{Q}\ :\ (A, B) \precsim (C, D)$.

Definition 5. *The bipolar Sugeno integral of* $\boldsymbol{x} = (x_1, \dots, x_n) \in (-\alpha, \alpha)^n$ *with respect to the bipolar measure* ν_b *on* N *with scale* $(-\alpha, \alpha)$ *is given by [8]:*

$$Su_b(\boldsymbol{x}, \nu_b) = \bigvee_{i \in N}^{b} \Big\{ \mathrm{sign}\left(\nu_b\left(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}\right)\right) \cdot \cdot \bigwedge \{|\nu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})|, |x_i|\} \Big\}. \tag{5}$$

3 The Universal Integral and the Bipolar Universal Integral

In order to define the universal integral it is necessary to introduce the concept of pseudomultiplication. This is a function $\otimes : [0, 1] \times [0, 1] \to [0, 1]$, which is nondecreasing in each component (i.e. for all $a_1,\ a_2,\ b_1,\ b_2 \in [0,1]$ with $a_1 \leq a_2$ and $b_1 \leq b_2$, $a_1 \otimes b_1 \leq a_2 \otimes b_2$), has 0 as annihilator (i.e. for all $a \in [0,1]$, $a \otimes 0 = 0 \otimes a = 0$) and has a neutral element $e \in]0,1]$ (i.e. for all $a \in [0,1]$, $a \otimes e = e \otimes a = a$). If $e = 1$ then $\otimes$ is a *semicopula*, i.e. a binary operation $\otimes : [0,1]^2 \to [0,1]$ that is nondecreasing in both components and has 1 as neutral element. Observe that in the definition of semicopula it is not necessary to state that 0 is a annihilator, because this can be elicited. A semicopula satisfies $a \otimes b \leq \min\{a,b\}$ for all $(a,b) \in [0,1]^2$, indeed, suppose that $a = \min\{a,b\}$ then $a \otimes b \leq a \otimes 1 = a$. It follows that for all $a \in [0,1]$, $0 \leq 0 \otimes a \leq 0$ and $0 \leq a \otimes 0 \leq 0$, i.e. $a \otimes 0 = 0 \otimes a = 0$ and, then, 0 is a annihilator. A semicopula $\otimes : [0,1]^2 \to [0,1]$ which is associative and commutative is called a *triangular norm*.

A capacity [2] or fuzzy measure [18] on N is a non decreasing set function $m : 2^N \to [0,1]$ such that $m(\emptyset) = 0$ and $m(N) = 1$.

Definition 6. *[14] Let F be the set of functions $f : N \to [0,1]$ and M the set of capacities on N. A function $I : M \times F \to [0,1]$ is a universal integral on the scale $[0,1]$ (or fuzzy integral) if the following axioms hold:*

(I1) $I(m,f)$ is nondecreasing with respect to m and with respect to f;
(I2) there exists a semicopula $\otimes$ such that for any $m \in M$, $c \in [0,1]$ and $A \subseteq N$, $I(m, c \cdot 1_A) = c \otimes m(A)$;
(I3) for all pairs $(m_1, f_1), (m_2, f_2) \in M \times F$, such that for all $t \in [0,1]$, $m_1 \{i \in N : f_1(i) \geq t\} = m_2 \{i \in N : f_2(i) \geq t\}$, $I(m_1, f_1) = I(m_2, f_2)$.

We can generalize the concept of universal integral from the scale $[0,1]$ to the symmetric scale $[-1,1]$ by extending definition 6.

Definition 7. *Let F_b be the set of functions $f : N \to [-1,1]$ and M_b the set of bi-capacities on Q. A function $I_b : M_b \times F_b \to [-1,1]$ is a bipolar universal integral on the scale $[-1,1]$ (or bipolar fuzzy integral) if the following axioms hold:*

(I1) $I_b(m_b, f)$ is nondecreasing with respect to m_b and with respect to f;
(I2) there exists a semicopula $\otimes$ such that for any $m_b \in M_b$, $c \in [0,1]$ and $(A,B) \in Q$, $I(m_b, c \cdot 1_{(A,B)}) = sign(m_b(A,B))\ (c \otimes |m_b(A,B)|)$;
(I3) for all pairs $(m_{b_1}, f_1), (m_{b_2}, f_2) \in M_b \times F_b$, such that for all $t \in [0,1]$, $m_{b_1}(\{i \in N : f_1(i) \geq t\}, \{i \in N : f_1(i) \leq -t\}) =$ $= m_{b_2}(\{i \in N : f_2(i) \geq t\}, \{i \in N : f_2(i) \leq -t\})$, $I(m_{b_1}, f_1) = I(m_{b_2}, f_2)$.

Clearly, in definition 6, F can be identified with $[0,1]^n$ and in definition 7, F_b can be identified with $[-1,1]^n$, such that a function $f : N \to [-1,1]$ can be regarded as a vector $\boldsymbol{x} \in [-1,1]^n$. Note that the bipolar Choquet, Shilkret and Sugeno integrals are bipolar universal integrals in the sense of Definition 7. Observe that the underlying semicopula $\otimes$ is the standard product in the case of the bipolar Choquet and Shilkret integrals, while $\otimes$ is the minimum (with neutral element $\beta = 1$) for the Sugeno integral.

Now we turn our attention to the characterization of the bipolar universal integral. Due to axiom $(I3)$ for each universal integral I_b and for each pair $(m_b, \boldsymbol{x}) \in M_b \times F_b$, the value $I_b(m_b, \boldsymbol{x})$ depends only on the function $h^{(m_b,\boldsymbol{x})}$: $[0,1] \to [-1,1]$, defined for all $t \in [0,1]$ by

$$h^{(m,\boldsymbol{x})}(t) = m_b\left(\{i \in N : x_i \geq t\}, \{i \in N : x_i \leq -t\}\right). \tag{6}$$

Note that for each $(m_b, \boldsymbol{x}) \in M_b \times F_b$ such a function is not in general monotone but it is Borel measurable, since it is a step function, i.e. a finite linear combination of indicator functions of intervals. To see this, suppose that $() : N \to N$ is a permutation of criteria such that $|x_{(1)}| \leq \ldots \leq |x_{(n)}|$ and let us consider the following intervals decomposition of $[0,1]$: $A_1 = [0, |x_{(1)}|]$, $A_j =]|x_{(j)}|, |x_{(j+1)}|]$ for all $j = 1, \ldots, n-1$ and $A_{n+1} =]|x_{(n)}|, 1]$. Thus, we can rewrite the function h as

$$h^{(m,\boldsymbol{x})}(t) = \sum_{j=1}^{n} m_b\left(\left\{i \in N : x_i \geq |x_{(j)}|\right\}, \left\{i \in N : x_i \leq -|x_{(j)}|\right\}\right) \cdot 1_{A_j}(t). \tag{7}$$

Let $\mathcal{H}_n$ be the subset of all step functions with no more than n-values in $\mathcal{F}_{[-1,1]}^{([0,1],\mathcal{B}([0,1]))}$, the set of all Borel measurable functions from $[0,1]$ to $[-1,1]$.

Proposition 1. *A function $I_b : M_b \times F_b \to [-1,1]$ is a bipolar universal integral on the scale $[-1,1]$ related to some semicopula $\otimes$ if and only if there is a function $J : \mathcal{H}_n \to \mathbb{R}$ satisfying the following conditions:*

(J1) *J is nondecreasing;*
(J2) *$J(d \cdot 1_{[x,x+c]}) = sign(d)(c \otimes |d|)$ for all $[x, x+c] \subseteq [0,1]$ and for all $d \in [-1,1]$;*
(J3) *$I(m_b, f) = J\left(h^{(m_b,f)}\right)$ for all $(m_b, f) \in M_b \times F_b$.*

4 An Illustrative Example

The following is an example of a bipolar universal integral (which is neither the Choquet nor Sugeno or Shilkret type), and illustrates the interrelationship between the functions I, J and the semicopula $\otimes$. Let $I_b : M_b \times F_b \to \mathbb{R}$ be given by

$$I(m_b, f) = \sup{}^{bip}\left\{\frac{t \cdot m_b\left(\{f \geq t\}, \{f \leq -t\}\right)}{1 - (1-t)\left(1 - |m_b\left(\{f \geq t\}, \{f \leq -t\}\right)|\right)} \mid t \in]0,1]\right\}. \tag{8}$$

Note that (8) defines a bipolar universal integral, indeed if $m_b \geq m_b'$ and $f \geq f'$ then $h^{(m_b,f)} \geq h^{(m_b',f')}$ and being the function $t \cdot h/[1-(1-t)(1-|h|)]$ non decreasing in $h \in \mathbb{R}$, we conclude that $I(m_b, f) \geq I(m_b', f')$ using the monotonicity of the bipolar supremum. Moreover

$$\begin{aligned} I(m_b, c \cdot 1_{(A,B)}) &= sign(m_b(A,B))\frac{t \cdot |m_b\left(\{f \geq t\}, \{f \leq -t\}\right)|}{1 - (1-t)\left(1 - |m_b\left(\{f \geq t\}, \{f \leq -t\}\right)|\right)} = \\ &= sign(m_b(A,B))(c \otimes |m_b(A,B)|). \end{aligned} \tag{9}$$

This means that the semicopula underlying the bipolar universal integral (9) is the Hamacher product

$$a \otimes b = \begin{cases} 0 & \text{if } a = b = 0 \\ \frac{a \cdot b}{1-(1-a)(1-b)} & \text{if } |a| + |b| \neq 0. \end{cases}$$

Now let us compute this integral in the simple situation of $N = \{1, 2\}$. In this case the functions we have to integrate can be identified with two dimensional vectors $\boldsymbol{x} = (x_1, x_2) \in [-1, 1]^2$ and we should define a bi-capacity on $\mathcal{Q}$. For example

$$m_b(\{1\}, \emptyset) = 0.6, \quad m_b(\{2\}, \emptyset) = 0.2, \quad m_b(\{1\}, \{2\}) = 0.1,$$

$$m_b(\{2\}, \{1\}) = -0.3, \quad m_b(\emptyset, \{1\}) = -0.1 \quad \text{and} \quad m_b(\emptyset, \{2\}) = -0.5.$$

First let us consider the four cases $|x_1| = |x_2|$. If $x \geq 0$:

$$I(m_b, (x, x)) = x, \quad I(m_b, (x, -x)) = \frac{0.1x}{0.1 + 0.9x},$$

$$I(m_b, (-x, x)) = \frac{-0.3x}{0.3 + 0.7x} \quad \text{and} \quad I(m_b, (-x, -x)) = -x.$$

For all the other possible cases, we have the following formula

$$I(m_b, (x, y)) = \begin{cases} \bigvee^b \left\{ y\ ,\ \frac{0.6x}{0.6+0.4x} \right\} & x > y \geq 0 \\ \bigvee^b \left\{ \frac{0.1|y|}{0.1+0.9|y|}\ ,\ \frac{0.6x}{0.6+0.4x} \right\} & x \geq 0 > y > -x \\ \bigvee^b \left\{ \frac{0.1x}{0.1+0.9x}\ ,\ \frac{-0.5|y|}{0.5+0.5|y|} \right\} & x \geq 0 \geq -x > y \\ \bigvee^b \left\{ x\ ,\ \frac{-0.5|y|}{0.5+0.5|y|} \right\} & 0 > x > y \\ \bigvee^b \left\{ x\ ,\ \frac{0.2y}{0.2+0.8y} \right\} & y > x \geq 0 \\ \bigvee^b \left\{ \frac{-0.3|x|}{0.3+0.7|x|}\ ,\ \frac{0.2y}{0.2+0.8y} \right\} & y \geq 0 > x > -y \\ \bigvee^b \left\{ \frac{-0.3y}{0.3+0.7y}\ ,\ \frac{-0.1|x|}{0.1+0.9|x|} \right\} & y \geq 0 \geq -y > x \\ \bigvee^b \left\{ y\ ,\ \frac{-0.1|x|}{0.1+0.9|x|} \right\} & 0 > y > x. \end{cases} \tag{10}$$

5 The Bipolar Universal Integral with Respect to a Level Dependent Bi-capacity

All the bipolar fuzzy integrals (1), (4) and (5) as well as the universal integral, admit a further generalization with respect to a *level dependent capacity* ([9],

[8], [13]). Next, after remembering previous definitions, we will give the concept of *bipolar universal integral with respect to a level dependent capacity.*

Definition 8. *[9] A bipolar level dependent bi-capacity is a function* $\mu_{bLD} : \mathcal{Q} \times [0,1] \to [-1,1]$ *satisfying the following properties:*

1. *for all* $t \in [0,1]$, $\mu_{bLD}(\emptyset,\emptyset,t) = 0, \mu_{bLD}(N,\emptyset,t) = 1, \mu_{bLD}(\emptyset,N,t) = -1$*;*
2. *for all* $(A,B,t),(C,D,t) \in \mathcal{Q} \times [0,1]$ *such that* $(A,B) \precsim (C,D)$, $\mu_{bLD}(A,B,t) \leq \mu_{bLD}(C,D,t)$*;*
3. *for all* $(A,B) \in \mathcal{Q}$, $\mu_{bLD}(A,B,t)$ *considered as a function with respect to* t *is Borel measurable.*

Definition 9. *[9] The bipolar Choquet integral of a vector* $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$ *with respect to the level dependent bi-capacity* μ_{bLD} *is given by*

$$Ch_{bLD}(\boldsymbol{x}) = \int_0^{max_i|x_i|} \mu_{bLD}(\{i \in N : x_i \geq t\},\{i \in N : x_i \leq -t\},t)dt. \quad (11)$$

A level dependent bi-capacity μ_{bLD} is said Shilkret compatible if for for all $t,r \in [-1,1]$ such that $t \leq r$, and $(A,B),(C,D) \in \mathcal{Q}$ with $(A,B) \precsim (C,D)$, $t\mu_{bLD}((A,B),t) \leq r\mu_{bLD}((C,D),r)$.

Definition 10. *[8] The bipolar level dependent Shilkret integral of* $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$ *with respect to a Shilkret compatible bi-capacity level dependent,* μ_{bLD}*, is given by*

$$Sh_{bLD}(\boldsymbol{x},\mu_{bLD}) = \bigvee_{i\in N}^{b} \left\{ \sup_{t\in\,]0,|x_i|\,]} \{t \cdot \mu_{bLD}(\{j \in N : x_j \geq t\},\{j \in N : x_j \leq -t\},t)\} \right\}. \quad (12)$$

Definition 11. *[8] A bipolar level dependent measure on* N *with a scale* $[-\alpha,\alpha]$ *with* $\alpha > 0$*, is any function* $\nu_{bLD} : \mathcal{Q} \times [-\alpha,\alpha] \to [-\alpha,\alpha]$ *satisfying the following properties:*

1. $\nu_{bLD}(\emptyset,\emptyset,t) = 0$ *for all* $t \in [-\alpha,\alpha]$*;*
2. $\nu_{bLD}(N,\emptyset,t) = \alpha$, $\nu_{bLD}(\emptyset,N,t) = -\alpha$ *for all* $t \in (\alpha,\beta)$*;*
3. *for all* $(A,B),(C,D) \in \mathcal{Q}$ *such that* $(A,B) \precsim (C,D)$*, and for all* $t \in [-\alpha,\alpha]$, $\nu_{bLD}(A,B,t) \leq \nu_{bLD}(C,D,t)$.

Definition 12. *[8] The bipolar level dependent Sugeno integral of* $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-\alpha,\alpha]^n$ *with respect to the bipolar measure* ν_{bLD} *is given by*

$$\bigvee_{i\in N}^{b} \Big\{ \sup{}^{bip}_{t\in\,]0,|x_i|\,]} \{\text{sign}\,[\nu_{bLD}(\{j \in N : x_j \geq t\},\{j \in N : x_j \leq -t\},t)]$$
$$\cdot \min\{|\nu_{bLD}(\{j \in N : x_j \geq t\},\{j \in N : x_j \leq -t\},t)|\,,t\}\}\Big\} = Su_{bLD}(\boldsymbol{x},\nu_{bLD}). (13)$$

A level dependent bi-capacity can be, also, indicated as $M_b^t = (m_{b,t})_{t\in]0,1]}$ where $m_{b,t}$ is a bi-capacity. Given a level dependent bi-capacity $M_b^t = (m_{b,t})_{t\in]0,1]}$ for each alternative $\boldsymbol{x} \in [-1,1]^n$ we can define the function $h_{M_b^t,f} : [0,1] \to [-1,1]$, which accumulates all the information contained in M_b^t and f, by:

$$h_{M_b^t,f}(t) = m_{b,t}\left(\{j \in N : x_j \geq t\}, \{j \in N : x_j \leq -t\}\right) \tag{14}$$

In general, the function $h_{M_b^t,f}$ is neither monotone nor Borel measurable. Following the ideas of inner and outer measures in Caratheodory's approach [11], we introduce the two functions $\left(h_{M_b^t,f}\right)^* : [0,1] \to [-1,1]$ and $\left(h_{M_b^t,f}\right)_* : [0,1] \to [-1,1]$ defined by

$$\begin{aligned} \left(h_{M_b^t,f}\right)^* &= \inf\left\{h \in \mathcal{H} \mid h \geq h_{M_b^t,f}\right\}, \\ \left(h_{M_b^t,f}\right)_* &= \sup\left\{h \in \mathcal{H} \mid h \geq h_{M_b^t,f}\right\}. \end{aligned} \tag{15}$$

Clearly, both functions (15) are non increasing and, therefore, belong to $\mathcal{H}$. If the level dependent bi-capacity M_b^t is constant, then the three functions considered in (14), (15) coincide.

Let $\mathcal{M}_b$ the set of all level dependent bi-capacities on Q, for a fixed $M_b^t \in \mathcal{M}_b$ a function $f : N \to [-1,1]$ is M_b^t-measurable if the function $h_{M_b^t,f}$ is Borel measurable. Let $F_{[-1,1]}^{M_b^t}$ be the set of all M_b^t measurable functions. Let us consider

$$\mathcal{L}_{[-1,1]} = \bigcup_{M_b^t \in \mathcal{M}_b} M_b^t \times F_{[-1,1]}^{M_b^t}$$

Definition 13. *A function $L_b : \mathcal{L}_{[-1,1]} \to [-1,1]$ is a level-dependent bipolar universal integral on the scale $[-1,1]$ if the following axioms hold:*

(I1) $I_b(m,f)$ is nondecreasing in each component;
(I2) there is a bipolar universal integral $I_b : M_b \times F_b \to \mathbb{R}$ such that for each bipolar capacity $m_b \in M_b$, for each $\boldsymbol{x} \in [-1,1]^n$ and for each level dependent bipolar capacity $M_b^t \in \mathcal{M}_b$, satisfying $m_{b,t} = m_b$ for all $t \in]0,1]$, we have

$$L_b\left(M_b^t, \boldsymbol{x}\right) = I_b\left(m_b, \boldsymbol{x}\right);$$

(I3) for all pairs (M_{b_1}, f_1), $(M_{b_2}, f_2) \in \mathcal{L}_{[-1,1]}$ with $h_{M_{b_1},f_1} = h_{M_{b_2},f_2}$ we have

$$L_b\left(M_{b_1}, f_1\right) = L_b\left(M_{b_2}, f_2\right).$$

Obviously the bipolar Choquet, Shilkret and Sugeno integrals with respect to a level dependent capacity are level-dependent bipolar universal integrals in the sense of Definition 13.

Finally, we present the representation theorem which gives necessary and sufficient conditions to be a function $L_b : \mathcal{L}_{[-1,1]} \to [-1,1]$ a level-dependent bipolar universal integral.

Proposition 2. *A function $L_b : \mathcal{L}_{[-1,1]} \to [-1,1]$ is a level-dependent bipolar universal integral related to some semicopula $\otimes$ if and only if there is a semicopula $\otimes : [0,1]^2 \to [0,1]$ and a function $J : \mathcal{H} \to \mathbb{R}$ satisfying the following conditions:*

(J1) J *is nondecreasing;*
(J2) $J(d \cdot 1_{]0,c]}) = sign(d)(c \otimes |d|)$ *for all* $[x, x+c] \subseteq [0,1]$ *and for all* $d \in [-1,1]$*;*
(J3) $L_b\,(M_b, f) = J\,(h_{M_b,f})$ *for all* $(M_b^t, f) \in \mathcal{L}_{[-1,1]}$.

6 Conclusions

The concept of universal integral generalizes, over all, the Choquet, Shilkret and Sugeno integrals. Those integrals admit a bipolar formulation, helpful for the case in which the underlying scale is bipolar. In this paper we have defined and characterized the bipolar universal integral, thus providing a common frame including the bipolar Choquet, Shilkret and Sugeno integrals. Moreover, we have also defined and characterized the bipolar universal integral with respect to a level dependent bi-capacity, which includes, as notable examples, the bipolar level dependent Choquet, Shilkret and Sugeno integrals.

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